Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

A notion of Euler characteristic for fractals

by

Marta Llorente and Steffen Winter

Preprint no.: 31 2004

A notion of Euler characteristic for fractals

Marta Llorente *and Steffen Winter †

May 11, 2004

Abstract

A notion of (average) fractal Euler number for subsets of $\mathbb{R}^d$ with infinite singular complexes is introduced by means of rescaled Euler numbers of infinitesimal ϵ-neighbourhoods. For certain classes of self-similar sets we calculate the associated Euler exponent and the (average) fractal Euler number with the help of the Renewal theorem. Examples like the Sierpinski gasket or carpet are provided.

1 Introduction

The Euler characteristic of a finite cell complex K in the Euclidean space $\mathbb{R}^d$ is given by

$$\chi(K) = \sum_{k=0}^{d} (-1)^k \alpha_k$$

where α_k denotes the number of k-cells of K. If β_k is the k-th Betti number of K, i.e. the rank of the k-th homology group of K, the Euler-Poincaré formula

$$\chi(K) = \sum_{k=0}^{d} (-1)^k \beta_k$$

shows that $\chi(K)$ does not depend on the special decomposition of the complex K (for more details see, e.g. Dold [1]). In this way the Euler characteristic is a homological, homotopic and topological invariant. For appropriate classes of sets from convex geometry, differential geometry and geometric measure theory χ is well defined and special calculation methods are known

*Max-Planck-Institute for Mathematics in the Sciences, llorente@mis.mpg.de
†Uni Jena, winter@minet.uni-jena.de

(for one of them compare Section 4). For fractal subsets of $\mathbb{R}^d$, however, the associated complexes are typically infinite and the Euler number in the classical sense is not determined.

Until now, very little is known about the topological structure of self-similar sets, which have been extensively studied in other respects (see, for example, [2] and the references therein). In the present paper we introduce the notion of *limit* or *fractal Euler number* for some classes of self-similar sets. In particular, we obtain self-similar sets with equal fractal dimensions and different fractal Euler numbers (see Subsection 2.3).

The main idea is the following: for nice classical singular sets M like (compact) Lipschitz submanifolds the Euler number of parallel sets M_ϵ coincides with that of M for sufficiently small $\epsilon > 0$ and may be calculated by a generalised Gauss-Bonnet formula (cf. [11]). It turns out that for more general compact sets $F \subset \mathbb{R}^d$ the ϵ-neighbourhoods

$$F_\epsilon := \{x \in \mathbb{R}^d \ : \ \inf_{y \in F} \mathrm{d}(x, y) \leq \epsilon\}$$

admit the classical Euler characteristic $\chi(F_\epsilon)$. (Here $\mathrm{d}(\cdot, \cdot)$ denotes the Euclidean metric.) We suppose now that $\chi(F_\epsilon)$ is determined for all $\epsilon > 0$. In this case the following notions are well defined. Let b be the diameter of the compact set $F \subset \mathbb{R}^d$.

Definition 1.1. The *Euler exponent* of F is the number

$$s = \inf\{t \geq 0 : \epsilon^t \, |\chi(F_\epsilon)| \text{ is bounded}\}.$$

Definition 1.2. If the limit

$$\chi_f(F) := \lim_{\epsilon \searrow 0} \left(\frac{\epsilon}{b}\right)^s \chi(F_\epsilon)$$

exists, then it is called the *fractal Euler number* of the set F.

The diameter b in the definition is a normalisation to ensure scaling invariance of the limit (compare Corollary 2.1). Unfortunately, the existence of such limits is a rare event. From the study of local quantities of self-conformal sets like densities or tangent measure distributions it is well-known that, in general, average limits provide better results. It turns out that this is a useful tool for our purposes, too.

Definition 1.3. If the limit

$$\chi_f^a(F) := \lim_{\delta \searrow 0} \frac{1}{|\log \delta|} \int_\delta^1 \left(\frac{\epsilon}{b}\right)^s \chi(F_\epsilon) \frac{d\epsilon}{\epsilon}$$

exists, then it is called the *average fractal Euler number* of the set F.

In both definitions we could work with upper and lower limits leading to the quantities $\overline{\chi}_f(F)$, $\underline{\chi}_f(F)$ and $\overline{\chi}_f^a(F)$, $\underline{\chi}_f^a(F)$, respectively, for all sets $F \subset \mathbb{R}^d$ with unbounded $\chi(F_\epsilon)$ as $\epsilon \searrow 0$. In the present paper, however, we are interested in finding sufficient conditions for the existence of the (average) limits. We restrict our considerations to the special case of self-similar sets F and assume that the Euler numbers of parallel sets F_ϵ exist. Moreover, if $S_1, \ldots, S_N$ are the similarity mappings generating F, we assume that the *overlap function*

$$R(\epsilon) = \chi(F_\epsilon) - \sum_{i=1}^{N} \chi((S_i F)_\epsilon)$$

of the set F satisfies

$$\epsilon^\gamma \, |R(\epsilon)| \leq c, \tag{1.1}$$

for some positive constants $\gamma < s$ and c. This and the scaling behaviour of F admit the application of the Renewal theorem from probability theory to our problem. Appropriate calculations show that in this situation the Euler exponent s coincides with the similarity dimension of F. Moreover, $\chi_f^a(F)$ is calculated in terms of the contraction ratios of the maps $S_1, \ldots, S_N$. Under additional conditions on the contraction ratios also the fractal Euler limit $\chi_f(F)$ exists (see Theorem 1). More tractable sufficient conditions for the existence of (average) fractal Euler numbers are discussed in the second part of the paper where we specify the results to the convex ring (Theorem 2). In this setting direct calculations of limits are possible. Examples are provided. For self-similar sets on the real line fractal Euler numbers essentially coincide with the rescaled limit of the classical notion of gap counting function (cf. [2]). As a corollary of Theorem 1 we obtain the existence of average limits for the gap counting function.

The paper is organised as follows. In Section 2 the main results are formulated and illustrated with some examples. First a general existence theorem of fractal Euler numbers for self-similar sets is discussed, while in Subsection 2.2 we restrict to self-similar sets with neighbourhoods in the convex ring. After some examples illustrating the concept and the calculation methods, in 2.4 we treat the special case of self-similar sets in $\mathbb{R}$. Section 3 is devoted to the proof of Theorem 1. For this purpose an appropriate version of the Renewal theorem is presented and its consequences for average limits are discussed. In Section 4 we recall the notion of Euler characteristic in the convex ring setting and discuss the consequences for neighbourhoods of self-similar sets to be in the convex ring. In Section 5 Theorem 2 is proved and extensions to a larger class of self-similar sets are discussed.

Acknowledgment The authors would like to thank Prof. Martina Zähle for

pointing out these interesting problems to us and for many fruitful discussions and invaluable suggestions.

2 Main results and examples

2.1 Existence theorem

Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, \ldots, N$, be contracting similarities. Denote the contraction ratio of S_i by $r_i \in (0,1)$ and let $r_{\max} := \max_{i=1,\ldots,N} r_i$. It is a well known fact in fractal geometry (cf. Hutchinson [5]), that for such a system $\{S_1, \ldots, S_N\}$ of similarities there is a unique, non-empty, compact subset F of $\mathbb{R}^d$ such that $\mathbf{S}(F) = F$, where $\mathbf{S}$ is the set mapping defined by

$$\mathbf{S}(A) = \bigcup_{i=1}^{N} S_i A, \quad A \subseteq \mathbb{R}^d.$$

F is called the *self-similar set* associated with $\{S_1, \ldots, S_N\}$. Moreover, the unique solution σ of $\sum_{i=1}^N r_i^\sigma = 1$ is called the *similarity dimension* of F. Throughout the paper, whenever we refer to a self-similar set, we keep these notations. The system $\{S_1, \ldots, S_N\}$ is said to satisfy the *open set condition* (OSC) if there exists an open, non-empty, bounded subset $U \subset \mathbb{R}^d$ with $\bigcup_i S_i U \subseteq U$ and $S_i U \cap S_j U = \varnothing$ for all $i \neq j$. $\{S_1, \ldots, S_N\}$ is said to satisfy the *strong separation condition* (SSC) if $S_i F \cap S_j F = \varnothing$ for all $i \neq j$.

To formulate our main result the following concept is needed. Let $h > 0$. A finite set of positive real numbers $\{y_1, ..., y_N\}$ is called *h-arithmetic* if h is the largest number such that $y_i \in h\mathbb{Z}$ for $i = 1, \ldots, N$. If no such number h exists for $\{y_1, ..., y_N\}$, the set is called *non-arithmetic.*

From now on we assume that the Euler characteristic $\chi(F_\epsilon)$ of the ϵ-neighbourhoods is defined for $\epsilon > 0$, i.e. we restrict to those self-similar sets F for which all Betti numbers of the singular complex of the parallel sets F_ϵ are finite for all $\epsilon > 0$. Note that this assumption is the base of the whole concept. Without it the definitions of fractal Euler numbers would not make much sense. In this situation the geometric invariance of the Euler characteristic and the fact that

$$(S_i F)_\epsilon = S_i(F_{\epsilon/r_i}) \tag{2.1}$$

imply that also $\chi((S_i F)_\epsilon)$ is well defined. This allows to define the *overlap function* $R : (0, +\infty) \to \mathbb{Z}$ of F by

$$R(\epsilon) = \chi(F_\epsilon) - \sum_{i=1}^{N} \chi((S_i F)_\epsilon). \tag{2.2}$$

The name is motivated by the fact that $R(\epsilon)$ is closely related to the set $O(\epsilon) = \bigcup_{i \neq j} (S_i F)_\epsilon \cap (S_j F)_\epsilon$, called the *overlap* of F_ϵ (compare Section 2.3 and [10]). It turns out that the overlap function is the crucial object to study for deciding whether (average) fractal Euler numbers exist or not.

The following theorem gives sufficient conditions for the existence of the (average) fractal Euler number of a self-similar set F. The stated formula shows that, in case it exists, the (average) fractal Euler number only depends on the similarity dimension, the contraction ratios and the overlap function of F.

Theorem 1. *Let F be a self-similar set with diameter b and similarity dimension s. Suppose that the Euler numbers of F_ϵ are defined for all $\epsilon > 0$. Furthermore, assume that the overlap function R has a discrete set of discontinuities and satisfies condition (1.1). Then the Euler exponent of F equals s and the following holds:*

(i) The average fractal Euler number $\chi_f^a(F)$ exists and

$$\chi_f^a(F) = \frac{1}{\mu} \left(\int_0^1 \epsilon^{s-1} R(b\epsilon)\ d\epsilon + \frac{N-1}{s} \right), \tag{2.3}$$

where $\mu = -\sum_{i=1}^N r_i^s \log r_i$.

(ii) If $\{-\log r_1, \dots, -\log r_N\}$ is non-arithmetic, the fractal Euler number $\chi_f(F)$ exists and equals $\chi_f^a(F)$.

We prove this theorem in Section 3. In general, the calculation of Euler numbers is difficult. Also, condition (1.1) on the overlap function might be difficult to verify. In order to show that Theorem 1 allows explicit calculations of fractal Euler numbers - at least for some classes of self-similar sets - we restrict the setting to a subclass of cell complexes, namely to the convex ring.

Before doing this we want to point out some general properties of fractal Euler numbers. First observe that whenever the fractal Euler number $\chi_f(F)$ of a set F exists, then the average counterpart $\chi_f^a(F)$ exists as well and coincides with $\chi_f(F)$. It is also immediate from the Definitions 1.2 and 1.3 that some invariance properties of the Euler characteristic carry over to the limits if they exist, namely the scaling invariance, which is due to the normalisation constant b in the definitions, and the invariance with respect to Euclidean motions. Corollary 2.1 states the invariance for χ_f^a implying the same result for χ_f. For $\lambda > 0$, define $\lambda F := \{\lambda x : x \in F\}$ and for an Euclidean motion g let $gF := \{gx : x \in F\}$.

Corollary 2.1. *(Scaling and motion invariance). Let F be a subset of $\mathbb{R}^d$. Assume that $\chi_f^a(F)$ exists. Then, for $\lambda > 0$ and g an Euclidean motion, $\chi_f^a(\lambda F)$ and $\chi_f^a(gF)$ exist and coincide with $\chi_f^a(F)$.*

Note that the above corollary is a direct consequence of the existence of the limit in Definition 1.3. It holds, in particular, if the conditions in Theorem 1 are satisfied.

2.2 Results for the convex ring

The *convex ring* $\mathcal{R}^d$ is the family of all sets that are finite unions of compact convex subsets of $\mathbb{R}^d$. $\mathcal{R}^d$ is closed with respect to unions and intersections. Moreover, the Euler characteristic χ is well defined for all elements of $\mathcal{R}^d$, which are also refered to as *polyconvex sets*, and the following additivity property holds for $A, B \in \mathcal{R}^d$:

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B). \tag{2.4}$$

Convex ring and Euler characteristic in this class of sets are discussed in more detail in Section 4. Here we just point out that the notion of Euler characteristic in the convex ring coincides with that for cell complexes for all polyconvex sets. One of the advantages of this setting is that it allows to give sufficient geometric conditions easy to verify for the existence of (average) fractal Euler numbers, as the following theorem shows. Let ∂A denote the (topological) boundary of a set $A \subset \mathbb{R}^d$. Note that by $A \subset B$ we mean that A is a *proper* subset of B.

Theorem 2. *Let F be a self-similar set. Assume there exists a set $M \in \mathcal{R}^d$ such that $\partial M \subset F \subset M$ and*

$$S_i(M) \cap S_j(M) = S_i(\partial M) \cap S_j(\partial M) \tag{2.5}$$

for all $i \neq j$. Then the following is true:

(i) The average fractal Euler number $\chi_f^a(F)$ exists and formula (2.3) holds.

(ii) If $\{-\log r_1, \dots, -\log r_N\}$ is non-arithmetic, the fractal Euler number $\chi_f(F)$ exists and equals $\chi_f^a(F)$.

Assume that for a self-similar set F and for some $\delta > 0$, the δ-neighbourhood of F is polyconvex. It turns out that this is equivalent to assuming that all its neighbourhoods are polyconvex (compare Proposition 4.5). This is a consequence of the self-similarity of F and the fact that neighbourhoods

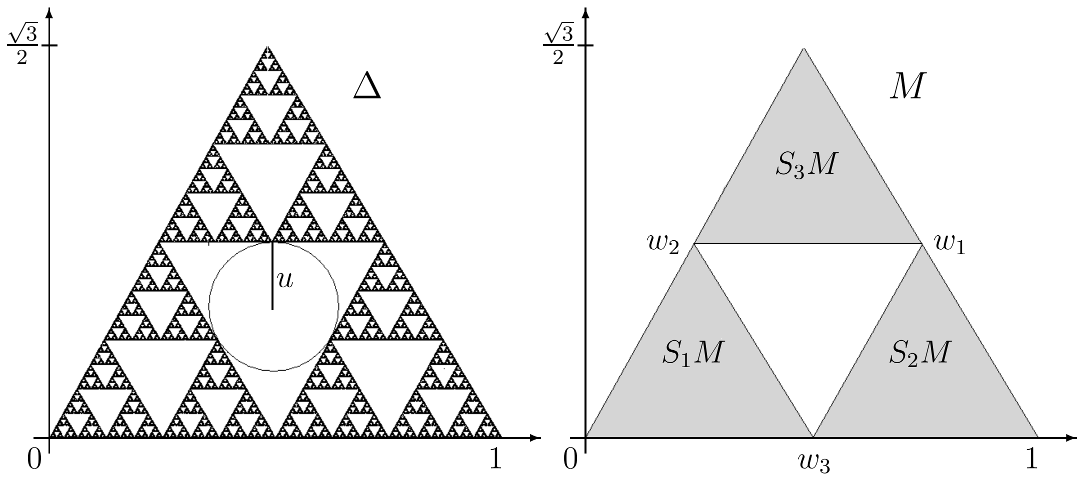

Figure 1: The Sierpinski gasket Δ is the self-similar set associated to the list $\{S_1, S_2, S_3\}$ of similarities each mapping the triangle M on the right hand side to one of the smaller triangles S_iM with contraction ratio $\frac{1}{2}$.

of polyconvex sets are polyconvex. Hence the Euler characteristic of neighbourhoods as well as the overlap function R of F are well defined in this situation. Moreover, it turns out that R has a discrete set of discontinuities (cf. Proposition 4.6). The only possible accumulation point of discontinuities of R is 0. Thus the assumption $F_\delta \in \mathcal{R}^d$ for some $\delta > 0$ already implies some of the assumptions in Theorem 1. If we assume additionally the remaining one, namely that the overlap function R satisfies condition (1.1), then the assertions of Theorem 1 hold. Therefore, the assumptions in Theorem 1 simplify in the polyconvex setting.

In Theorem 2 we assume the existence of a polyconvex set M being a "good approximation" of the set F and satisfying certain boundary conditions that prevent strong overlapping in F and make R bounded.

Sometimes, as for Sierpinski gasket and the Sierpinski carpet, the convex hull $[F]$ of a self-similar set F is a suitable set M for Theorem 2. Recall that the convex hull of a set $K \subset \mathbb{R}^d$ is defined as $[K] = \bigcap\{C : K \subset C, C \text{ convex}\}$.

Example 2.2. (Sierpinski gasket) The Sierpinski gasket Δ satisfies the assumptions of Theorem 2 for the set $M = [\Delta]$ (compare Figure 1). Each of the intersections $S_iM \cap S_jM$ is a single point w_k, $(k \neq i \neq j)$, of their boundary, implying equation (2.5). Thus $\chi_f^a(\Delta)$ exists.

If a self-similar set F in $\mathbb{R}^d$ can be described by a cut out procedure of an initial set $A \in \mathcal{R}^d$, then, as far as no point from the boundary of A is removed in the cut out process, A is a suitable set M for Theorem 2, compare for instance Example 2.5. Note that totally disconnected self-similar sets in $\mathbb{R}^d$ with $d \geq 2$ are naturally excluded by the conditions of Theorem 2. For self-similar sets in $\mathbb{R}$ we refer to Section 2.4. However, it is not difficult to find self-similar sets not fitting into the framework of Theorem 2 to which Theorem 1 still applies. Example 5.6 and Proposition 5.7 exhibit this phenomenon.

2.3 Examples

Here we present some examples of self-similar sets F, to which Theorem 2 can be applied to obtain their (average) fractal Euler numbers, including the Sierpinski gasket and the Sierpinski carpet. In all the examples an appropriate set M is easily determined and the conditions on M are easily verified, implying already the existence of $\chi_f^a(F)$. To calculate the actual value of $\chi_f^a(F)$ using formula (2.3), we need to determine the overlap function R of F.

Before turning to the examples we discuss the overlap function R in more detail. In the convex ring we have the following useful formula for R, which is due to the inclusion-exclusion principle (4.2) for the Euler characteristic (compare Section 4):

$$R(\epsilon) = \sum_{k=2}^{N} (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq N} \chi\left((S_{i_1}F)_\epsilon \cap \ldots \cap (S_{i_k}F)_\epsilon\right). \tag{2.6}$$

Observe that all sets $(S_iF)_\epsilon$ and their intersections are polyconvex such that all Euler numbers in this formula are defined. In examples often very few of the intersections in (2.6) are non-empty and have to be considered for $R(\epsilon)$. Since the number of non-empty intersections can not increase as $\epsilon \searrow 0$, this formula proves to be particularly useful for small $\epsilon > 0$.

Since the overlap function R assumes only integer values and since it has a discrete set of discontinuities, it is a piecewise constant function with integer steps at discontinuity points and constant integer values in between. In many examples the set of discontinuities is in fact finite, such that R is bounded and condition (1.1) is trivially satisfied. As we will see in the examples and, in general, in Section 5, the conditions in Theorem 2 imply this finiteness. In the examples we fix the diameter b of the self-similar sets to be 1, avoiding this constant in the formulas. By the scaling invariance (cf. Corollary 2.1) the diameter does not affect the limits.

Example 2.3. (Sierpinski gasket Δ - continued) We calculate $\chi_f^a(\Delta)$. The only discontinuity point of R is $u = \frac{\sqrt{3}}{12}$, being the radius of the incircle of the middle triangle of Δ (compare Figure 1). For $\epsilon \geq u$, the neighbourhood Δ_ϵ

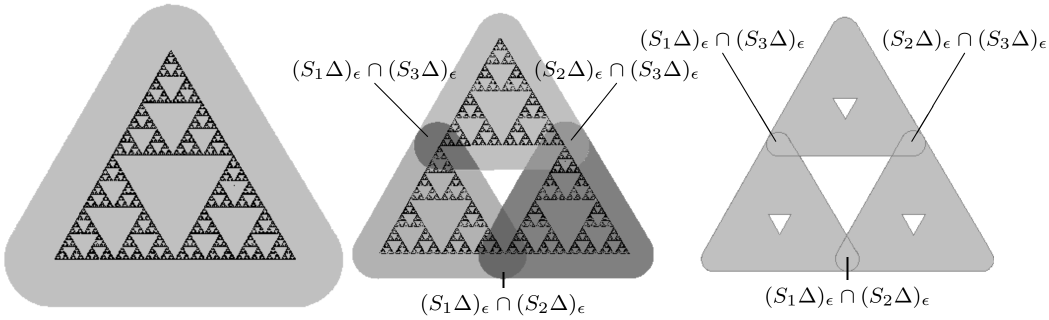

Figure 2: ϵ-neighbourhoods of the Sierpinski gasket for $\epsilon \geq u$ (left) and $\epsilon < u$ (middle and right). Note that for $\epsilon < u$, the sets $(S_i\Delta)_\epsilon \cap (S_j\Delta)_\epsilon$ remain convex and disjoint as $\epsilon \searrow 0$.

is convex as well as the sets $(S_i\Delta)_\epsilon$, implying $R(\epsilon) = \chi(\Delta_\epsilon) - \sum \chi((S_i\Delta)_\epsilon) = -2$. For $\epsilon < u$, Δ_ϵ is not convex anymore (cf. Figure 2). But now the intersection $(S_i\Delta)_\epsilon \cap (S_j\Delta)_\epsilon$ is a convex set, $i \neq j$, while the intersection of all three sets $(S_i\Delta)_\epsilon$ is empty. In equation (2.6) just the terms for $k = 2$ remain. Thus $R(\epsilon) = (-1)\big(\chi((S_1\Delta)_\epsilon \cap (S_2\Delta)_\epsilon) + \chi((S_1\Delta)_\epsilon \cap (S_3\Delta)_\epsilon) + \chi((S_2\Delta)_\epsilon \cap (S_1\Delta)_\epsilon)\big) = -3$. Now formula (2.3) yields the average fractal Euler number of Δ:

$$\chi_f^a(\Delta) = -\frac{u^s}{\mu s} = -\frac{u^s}{\log 3} \approx -0.098.$$

Example 2.4. (Sierpinski carpet) The Sierpinski carpet Q is the self-similar set associated to a system of 8 similarities S_i each mapping the square $H = [0, \frac{1}{\sqrt{2}}]^2$ to one of the small squares $1, \ldots, 8$ with contraction ratio $\frac{1}{3}$ (cf. Figure 3). Q has similarity dimension $s = \frac{\log 8}{\log 3}$. As for the Sierpinski gasket, Q satisfies Theorem 2 for the set $H = [Q]$, the convex hull of Q, and, since all contraction ratios are the same, the theorem ensures the existence of the average fractal Euler number $\chi_f^a(Q)$. To determine $\chi_f^a(Q)$ we look at the overlap function R. Again it has only one point of discontinuity, namely $u = \frac{1}{6\sqrt{2}}$. For $\epsilon \geq u$, Q_ϵ as well as $(S_iQ)_\epsilon$ are convex, implying $R(\epsilon) = 1 - 8 = -7$, by (2.2). For $\epsilon < u$ we consider the intersection structure of the sets $(S_iQ)_\epsilon$ and use formula (2.6). There are non-empty intersections of 2 and of 3 of these sets, all being convex, but no intersections of higher order. $(S_3Q)_\epsilon$, for instance, intersects $(S_2Q)_\epsilon$ and $(S_4Q)_\epsilon$ and also the set $(S_3Q)_\epsilon \cap (S_2Q)_\epsilon \cap (S_4Q)_\epsilon$ is non-empty. All together there are 12 intersection of two sets and 4 intersections of three sets, implying $R(\epsilon) = -12 + 4 = -8$.

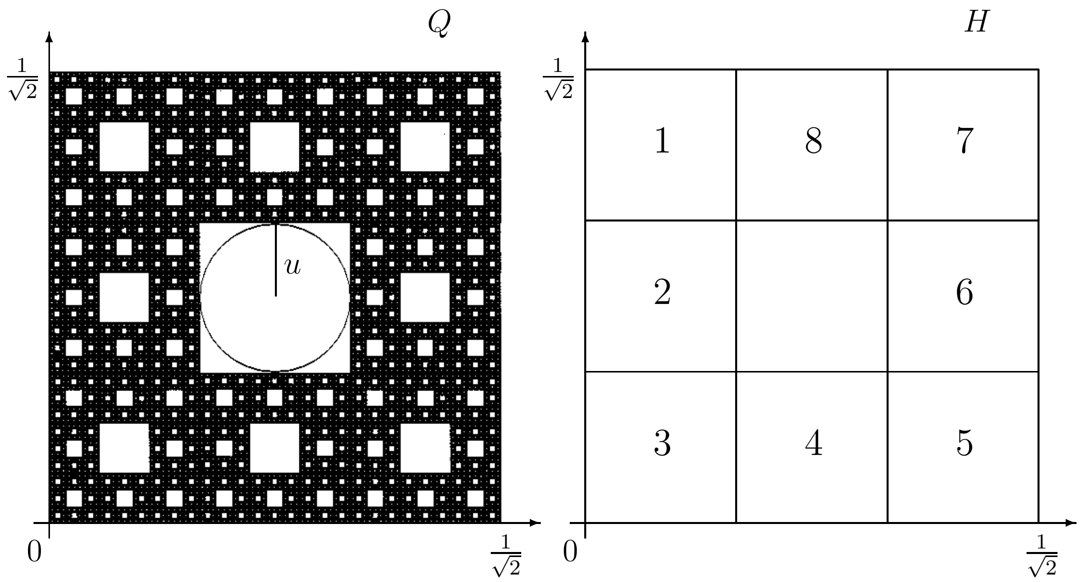

Figure 3: Sierpinski carpet Q and the square $H = [0, \frac{1}{\sqrt{2}}]^2$.

Integration according to formula (2.3) yields

$$\chi_f^a(Q) = -\frac{u^s}{\mu s} = -\frac{u^s}{\log 8} \approx -0.019.$$

In the following example we compare the average fractal Euler numbers of two self-similar sets with equal dimension but different topological structure.

Example 2.5. (U sets) We modify the construction of the Sierpinski carpet by considering only seven similarities, again each mapping the square $H = [0, \frac{1}{\sqrt{2}}]^2$ in Figure 3 (right) to one of the squares $1, \dots, 7$. But this time we include a rotation for some of the maps. For the set U_1 we rotate the squares 1 and 7 by angle π, 4 by $\frac{\pi}{2}$ and 3 by $\frac{3\pi}{2}$. For U_2 we rotate 1, 2, 6 and 7 by π, 4 and 5 by $\frac{\pi}{2}$ and 3 by $\frac{3\pi}{2}$ (compare Figure 4). The so defined self-similar sets U_1 and U_2 both have similarity dimension $s = \frac{\log 7}{\log 3}$. Their convex hull $[U_1] = [U_2] = H$ is not an appropriate set for Theorem 2, but U_1 and U_2 satisfy the conditions of this theorem for the set $U = \bigcup_i S_i^1 H = \bigcup_i S_i^2 H$, where the S_i^k, $i = 1, \dots, 7$, are the similarities generating the set U_k.

Since all contraction ratios are the same, the average fractal Euler numbers $\chi_f^a(U_1)$ and $\chi_f^a(U_2)$ exist. Their values differ, since the overlap functions R_1 of U_1 and R_2 of U_2 are different. Both have the same unique discontinuity point $u = \frac{1}{18\sqrt{2}}$ and their values coincide for $\epsilon \geq u$: $R_1(\epsilon) = R_2(\epsilon) = 1 - 7 = -6$, by equation (2.2). But for $\epsilon < u$, $R_1(\epsilon)$ and $R_2(\epsilon)$ assume different values. Analysing the intersection structure of the sets $(S_i^k U_k)_\epsilon$ for U_k, equation

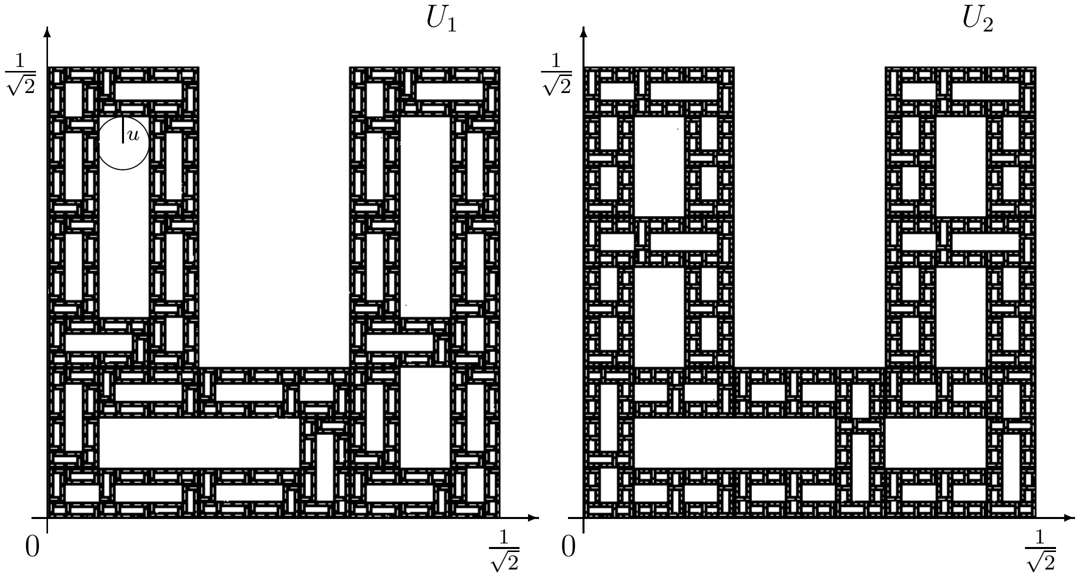

Figure 4: Self-similar sets with equal dimension but different fractal Euler numbers.

(2.6) yields $R_1(\epsilon) = -10$ and $R_2(\epsilon) = -12$. By (2.3) and with $\mu = \log 3$, the average fractal Euler numbers are:

$$\chi_f^a(U_1) = -4\frac{u^s}{\mu s} = -4\frac{u^s}{\log 7} \approx -0.015,$$

and

$$\chi_f^a(U_2) = \chi_f^a(U_1) - 2\frac{1}{\mu}\int_0^u \epsilon^{s-1} d\epsilon = -6\frac{u^s}{\log 7} \approx -0.023.$$

Note that U_1 and U_2 have different numbers of "holes of radius u", namely 4 and 6, respectively. This is reflected in the average fractal Euler numbers.

In all the examples considered so far we found a unique discontinuity point u for the overlap function R. We can give u a geometric meaning as the "radius of the largest holes" in the corresponding self-similar set F, or, more precisely, as the maximum distance to F of points in the union of the bounded connected components of the complement of F. But this unique discontinuity point is due to the simplicity of the examples, where all "holes" were either "largest holes" or iterated copies of them. The situation is not always as simple as the following example of a modified Sierpinski carpet shows. Moreover, in contrast to the previous examples in this one the fractal Euler number χ_f exists. (The non-existence of the the fractal Euler number in the previous examples can be easily seen by comparing $\epsilon_k^s \chi(F_{\epsilon_k})$ for different sequences $\epsilon_k \searrow 0$, also see Example 2.8.)

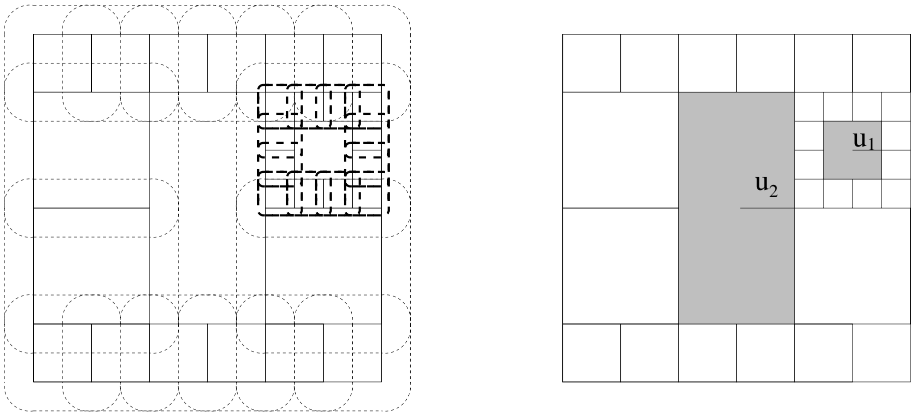

Figure 5: Modified Sierpinski carpet

Example 2.6. Starting from the square $M = [0, \frac{1}{\sqrt{2}}]^2$, the self-similar set F is generated by a system of 27 similarities each mapping M to one of the subsquares described in Figure 5. Three of the similarities have contraction ratio $\frac{1}{3}$, while twelve of them have ratio $\frac{1}{6}$ and the remaining twelve ratio $\frac{1}{12}$. Thus the similarity dimension s of F is the unique solution of $\frac{3}{3^s} + \frac{12}{6^s} + \frac{12}{12^s} = 1$ ($s \approx 1.818$), and the non-arithmetic case applies. Since F satisfies the conditions of Theorem 2 for M, $\chi_f(F)$ exists. The overlap function of F has two discontinuity points corresponding to the radii $u_1 = \frac{1}{6\sqrt{2}}$ and $u_2 = \frac{1}{12\sqrt{2}}$ of the two holes in $\mathbf{S}(M)$:

$$R(\epsilon) = \begin{cases} -28 & \text{if } 0 \leq \epsilon < u_1 \\ -27 & \text{if } \ u_1 \leq \epsilon < u_2 \\ -26 & \text{if } \ u_2 \leq \epsilon < 1 \end{cases}$$

Hence, by formula (2.3), $\chi_f(F) = -\frac{(u_1)^s + (u_2)^s}{\mu s} \approx -0.021$.

2.4 Self-similar sets in $\mathbb{R}$

We consider self-similar sets F in $\mathbb{R}$, generated by contracting similarities $S_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, N$, with similarity dimension s. Without loss of generality we assume that F has diameter 1. Suppose that the system $\{S_1, \ldots, S_N\}$ satisfies the SSC. Then there is a positive distance between each two of the sets S_iF and we can arrange the indices i such that S_iF is situated to the left of $S_{i+1}F$. For $i = 1, ..., N-1$, let b_i denote the length of the gap between S_iF and $S_{i+1}F$, that is, $b_i = \mathrm{d}(S_iF, S_{i+1}F) = \min\{|x-y| : x \in S_iF, y \in S_{i+1}F\}$.

Following Falconer [2], the *gap counting function* of F is defined as

$$G(\epsilon) = \#\{\text{complementary intervals of } F \text{ with length } > \epsilon\}.$$

Moreover, we define the *gap limit* of F by

$$\mathcal{G}(F) := \lim_{\epsilon \searrow 0} \epsilon^s G(\epsilon) \tag{2.7}$$

and the *average gap limit* of F by

$$\mathcal{G}^a(F) := \lim_{\delta \searrow 0} \frac{1}{|\log \delta|} \int_\delta^1 \epsilon^s G(\epsilon) \frac{d\epsilon}{\epsilon},$$

in case the limits exist. Falconer gave sufficient conditions for the existence of the gap limit (2.7) and provided a formula in terms of the gaps b_i (compare [2, Proposition 7.5]). Here we obtain the same result as a corollary of Theorem 1. Moreover we give sufficient conditions for the average gap limit to exist.

The main point to observe is the close relation between the gap counting function and the Euler characteristic of neighbourhoods of F. First note, that all neighbourhoods F_ϵ are finite unions of compact intervals, hence polyconvex. Moreover, their Euler characteristic, being the number of their connected components, is determined by the number of gaps of F of length greater than 2ϵ:

$$\chi(F_\epsilon) = 1 + G(2\epsilon). \tag{2.8}$$

This relation between both expressions results in a similar interdependence of their rescaled limits, as the following corollary to Theorem 1 shows.

Corollary 2.7. *Let F be a self-similar set in $\mathbb{R}$ with diameter 1 and similarity dimension s. Suppose that the system $\{S_1, \ldots, S_N\}$ generating F satisfies the SSC. Then the Euler exponent of F equals s and the following holds:*

(i) The average fractal Euler number and the average gap limit always exist and are given by

$$\mathcal{G}^a(F) = 2^s \chi_f^a(F) = \frac{1}{\mu s} \sum_{i=1}^{N-1} {b_i}^s, \tag{2.9}$$

where $\mu = -\sum_{i=1}^N r_i^s \log r_i$.

(ii) If $\{-\log r_1, \ldots, -\log r_N\}$ is non-arithmetic, both, fractal Euler number $\chi_f(F)$ and gap limit $\mathcal{G}(F)$, exist and coincide with their averaged counterparts.

Proof. Let $u_i = \frac{b_i}{2}$ denote the radius of the i-th first level gap ($i = 1, \ldots, N-1$). It is not difficult to see, that $-R(\epsilon)$ is the number of first level gaps of radius greater than ϵ, i.e.

$$R(\epsilon) = -\#\{u_i > \epsilon \ : \ i = 1, \ldots, N-1\} = -\sum_{i=1}^{N-1} 1_{[u_i,1]}(\epsilon).$$

In particular, this implies that $R(\epsilon) = 0$ for small ϵ (i.e. for $\epsilon < \min_i u_i$). Therefore, Theorem 1 can be applied. $\chi_f^a(F)$ exists and it holds

$$\chi_f^a(F) = \frac{1}{\mu}\left(-\sum_{i=1}^{N-1}\int_{u_i}^{1}\epsilon^{s-1}d\epsilon + \frac{N-1}{s}\right) = \frac{1}{\mu s}\sum_{i=1}^{N-1} u_i^s.$$

Now (2.8) implies the existence of the average gap limit $\mathcal{G}^a(F)$, since

$$\mathcal{G}^a(F) \;=\; \lim_{\delta\searrow 0}\frac{1}{|\log\delta|}\int_\delta^1 \epsilon^s G(\epsilon)\frac{d\epsilon}{\epsilon} = \lim_{\delta\searrow 0}\frac{1}{|\log\delta|}\int_\delta^1 \epsilon^s \chi(F_{\epsilon/2})\frac{d\epsilon}{\epsilon} = 2^s\chi_f^a(F),$$

proving (i). In case $\{-\log r_1, \ldots, -\log r_N\}$ is non-arithmetic, $\chi_f(F)$ exists and coincides with $\chi_f^a(F)$, implying that the gap limit $\mathcal{G}(F)$ exists and is given by

$$\mathcal{G}(F) = \lim_{\epsilon\searrow 0}\epsilon^s G(\epsilon) = 2^s \lim_{\epsilon\searrow 0}\epsilon^s\chi(F_\epsilon) = 2^s\chi_f(F).$$

This completes the proof. □

As with other lacunarity parameters, such as the Minkowski content (see [2],[3],[8]), for many interesting self-similar sets fractal Euler number and gap limit do not exist. This leads to considering their average counterparts, as in the following example of the middle-third Cantor set.

Example 2.8. Let C be the middle-third Cantor set, which is generated by the similarities $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$. Therefore, $r_1 = r_2 = \frac{1}{3}$, $b_1 = \frac{1}{3}$ and $s = \frac{\log 2}{\log 3}$. Since C is not h-arithmetic with $h = \log 3$, the existence of $\chi_f(C)$ and $\mathcal{G}(C)$ is not guaranteed by Corollary 2.7. In fact, both numbers do not exist: Since the gap counting function of C is

$$G(\epsilon) = 2^k - 1 \quad \text{for} \quad 3^{-(k+1)} < \epsilon \le 3^{-k},$$

the sequences $(\epsilon_k) = (\frac{1}{2}3^{-k})$ and $(\tilde{\epsilon}_k) = 3^{-k}$, $k \in \mathbb{N}$ provide different limits as $k \to \infty$, namely

$$\lim_{k\to\infty}\epsilon_k^s G(\epsilon_k) = 2^{-s} < 1 = \lim_{k\to\infty}\tilde{\epsilon}_k^s G(\tilde{\epsilon}_k)$$

and similarly

$$\lim_{k\to\infty}\epsilon_k^s \chi(C_{\epsilon_k}) = 2^{-2s} < 2^{-s} = \lim_{k\to\infty}\tilde{\epsilon}_k^s\chi(C_{\tilde{\epsilon}_k}) = 1.$$

On the other hand, formula (2.9) yields $\mathcal{G}^a(C) = \frac{1}{2\log 2} \approx 0.721$ and $\chi_f^a(C) = \frac{2^{-(s+1)}}{\log 2} \approx 0.466$.

3 Proof of Theorem 1

3.1 Renewal theorem

For proving Theorem 1 we need the Renewal theorem which we state and discuss now. Let P be a Borel probability measure with support contained in $[0, \infty)$ and $\mu = \int_0^\infty tP(dt) < \infty$. Let $z : \mathbb{R} \to \mathbb{R}$ be a function with a discrete set of discontinuities satisfying

$$|z(t)| \leq c_1 e^{-c_2|t|} \qquad \forall t \in \mathbb{R} \tag{3.1}$$

for some constants $0 < c_1, c_2 < \infty$. It is well known in probability theory that under the above conditions on z the equation

$$Z(t) = z(t) + \int_0^\infty Z(t-\tau)P(d\tau) \tag{3.2}$$

has a unique solution $Z(t)$ in the class of functions satisfying $\lim_{t\to-\infty} Z(t) = 0$. Equation (3.2) is called *renewal equation* and the asymptotic behaviour of its solution as $t \to \infty$ is given by the so-called Renewal theorem. We quote the following discrete version of the Renewal theorem from Levitin and Vassiliev [9, p. 198], which is adapted to the fractal setting and where a complete proof is provided (also compare Falconer [2, Corollary 7.3, p. 122]). A function $g : \mathbb{R} \to \mathbb{R}$ is said to be *asymptotic* to a function $f : \mathbb{R} \to \mathbb{R}$, $g \sim f$, if for all $\epsilon > 0$ there exists a number $D = D(\epsilon)$ such that

$$(1-\epsilon)f(t) \leq g(t) \leq (1+\epsilon)f(t) \text{ for all } t > D. \tag{3.3}$$

Theorem 3.1. (Renewal Theorem) *Let $0 < y_1 \leq y_2 \leq \ldots \leq y_N$ and $p_1, \ldots, p_N$ be positive real numbers such that $\sum_{i=1}^N p_i = 1$. For a function z as defined above, let $Z : \mathbb{R} \to \mathbb{R}$ be the unique solution of the renewal equation*

$$Z(t) = z(t) + \sum_{i=1}^N p_i Z(t - y_i) \tag{3.4}$$

satisfying $\lim_{t\to-\infty} Z(t) = 0$.

(i) If the set $\{y_1, ..., y_N\}$ is non-arithmetic, then

$$\lim_{t\to\infty} Z(t) = \frac{1}{\mu}\int_{-\infty}^\infty z(\tau)d\tau.$$

(ii) If $\{y_1, ..., y_N\}$ is h-arithmetic for some $h > 0$ then

$$Z(t) \sim \frac{h}{\mu}\sum_{k=-\infty}^\infty z(t-kh).$$

Moreover, Z is uniformly bounded in $\mathbb{R}$.

Theorem 3.1 implies that in the non-arithmetic case the limit $\lim_{t\to\infty} Z(t)$ exists, while in the h-arithmetic case Z is asymptotic to some periodic function of period h (i.e. to some function f with $f(t+h) = f(t)$ for all $t \in \mathbb{R}$). The latter is sufficient for the limit $\lim_{T\to\infty} \frac{1}{T}\int_0^T Z(t)dt$ to exist as the following lemma shows.

Lemma 3.2. *(i) Let $f \in \mathbf{L}^1_{loc}(\mathbb{R})$ be a periodic function with period $h > 0$, i.e. $f(t+h) = f(t)$ for all $t \in \mathbb{R}$, and let $L := \int_0^h f(t)dt$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)dt$$

exists and equals $h^{-1}L$.

(ii) If $g : \mathbb{R} \to \mathbb{R}$ is a function such that $g \sim f$, then the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g(t)dt$$

exists and equals $h^{-1}L$.

Proof. (i) For $T > 0$ choose $n \in \mathbb{N}$ such that $nh < T \leq (n+1)h$. Then

$$\frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\left(\sum_{i=0}^{n-1}\int_{ih}^{(i+1)h} f(t)dt + \int_{nh}^T f(t)dt\right).$$

By the periodicity of f, the right hand side is bounded from above by $\frac{1}{T}(nL + \int_0^h |f(t)|\, dt)$ and from below by $\frac{1}{T}(nL - \int_0^h |f(t)|\, dt)$. Since $f \in \mathbf{L}^1_{loc}(\mathbb{R})$, the stated limit follows by letting $T \to \infty$.
(ii) Fix some $\epsilon > 0$. On one hand (3.3) implies

$$\frac{1}{T}\int_0^T g(t)dt \leq \frac{1}{T}\left(\int_0^D g(t)dt + (1+\epsilon)\int_D^T f(t)dt\right)$$

for $T > D$ and thus

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T g(t)dt \leq (1+\epsilon)h^{-1}L.$$

On the other hand (3.3) yields

$$(1-\epsilon)h^{-1}L \leq \liminf_{T\to\infty} \frac{1}{T}\int_0^T g(t)dt$$

The statement follows by letting $\epsilon \searrow 0$. $\square$

As a direct consequence of the Renewal theorem and Lemma 3.2 we have the following

Corollary 3.3. *Under the assumptions of Theorem 3.1 the following limit always exists and equals to the expression on the right hand side:*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T Z(t)dt = \frac{1}{\mu}\int_{-\infty}^{\infty} z(\tau)d\tau.$$

Proof. For $\{y_1, ..., y_N\}$ being h-arithmetic, just note that the function $f(t) = \frac{h}{\mu}\sum_{k=-\infty}^{\infty} z(t-kh)$ in Theorem 3.1(ii) is uniformly bounded and periodic and apply Lemma 3.2(ii) to $g(t) = Z(t)$. In the non-arithmetic case the limit $\lim_{t\to\infty} Z(t)$ exists and the assertion follows by applying Lemma 3.2 to $g(t) = Z(t)$ which is asymptotic to the constant function $f(t) \equiv \lim_{t\to\infty} Z(t)$. □

3.2 Proof of Theorem 1

The following statement describes the structure of ϵ-neighbourhoods of general compact sets $K \subset \mathbb{R}^d$ for large ϵ. A consequence is that the Euler characteristic $\chi(K_\epsilon)$ of the neighbourhoods exists and equals 1 for these ϵ. We make use of this fact in the proof of formula (2.3).

Recall that a set $A \subset K$ is called a *strong deformation retract* of K if there is a homotopy $\theta : K \times [0,1] \to K$ such that $\theta(\cdot, 0) = id$, $\theta(K, 1) \subset A$ and $\theta(a,t) = a$ for all $a \in A$ and $t \in [0,1]$. The map $\theta(\cdot, 1) : K \to A$ is called a *retraction.*

Proposition 3.4. *Let b be the diameter of the compact set $K \subset \mathbb{R}^d$. Then, for $\epsilon \geq b$, the convex hull $[K]$ of K is a strong deformation retract of the parallel set K_ϵ and $\chi(K_\epsilon) = 1$.*

Proof. Fix $\epsilon \geq b$. By construction, $[K] \subset K_\epsilon$. Let $\Pi_{[K]}$ be the metric projection from $\mathbb{R}^d$ onto $[K]$. It is not difficult to see that for all $x \in K_\epsilon$ the line segment connecting x and $\Pi_{[K]}(x)$ is contained in K_ϵ. Therefore, the mapping $\theta : K_\epsilon \times [0,1] \to K_\epsilon$ defined by

$$\theta(x,t) := (1-t)x + t\Pi_{[K]}(x)$$

is a homotopy satisfying $\theta(\cdot, 0) = id$, $\theta(x,1) = \Pi_{[K]}(x) \in [K]$ for all $x \in K_\epsilon$, and $\theta(x,t) = x$ for all $x \in [K]$ and $t \in [0,1]$. Hence, $[K]$ is a strong deformation retract of K_ϵ. The latter implies that the homology groups of the singular complexes of $[K]$ and K_ϵ are isomorphic, and $\chi(K_\epsilon) = 1$ follows from the contractability of the convex set $[K]$. □

Considering the definition of the overlap function R and equation (2.1), which implies $\chi((S_iF)_\epsilon) = \chi(F_{\epsilon/r_i})$ for $i = 1, \ldots, N$, we can write the following for the Euler characteristic of F_ϵ:

$$\chi(F_\epsilon) = \sum_{i=1}^{N} \chi(F_{\epsilon/r_i}) + R(\epsilon) \tag{3.5}$$

Define

$$Z(t) = be^{-st}\chi(F_{be^{-t}}) \tag{3.6}$$

for $t \geq 0$ and $Z(t) = 0$ for $t < 0$. Using (3.5) we obtain

$$\begin{aligned} Z(t) &= be^{-st}\left(\sum_{i=1}^{N} \chi(F_{be^{-(t+\log r_i)}}) + R(be^{-t})\right) \\ &= \sum_{i=1}^{N} r_i^s be^{-s(t+\log r_i)}\chi(F_{be^{-(t+\log r_i)}}) + be^{-st}R(be^{-t}) \end{aligned}$$

The terms of the sum over i satisfying $t + \log r_i \geq 0$ can be replaced by $r_i^s Z(t + \log r_i)$. Then we have

$$Z(t) = \sum_{t+\log r_i \geq 0} r_i^s Z(t + \log r_i) + z(t) \tag{3.7}$$

where, for $t \geq 0$,

$$z(t) = be^{-st}R(be^{-t}) + \sum_{t+\log r_i < 0} be^{-st}\chi(F_{be^{-(t+\log r_i)}}) \tag{3.8}$$

and, for $t < 0$, $z(t) = 0$.

Since $Z(t) = 0$ for $t < 0$, we can add the zeros for $t + \log r_i < 0$ in the sum in (3.7) to obtain

$$Z(t) = \sum_{i=1}^{N} r_i^s Z(t + \log r_i) + z(t). \tag{3.9}$$

The function $z(t)$ can be simplified, to see that the term containing the overlap function really is the important one. Noting that $be^{-(t+\log r_i)} > b$ for $t + \log r_i < 0$, by Proposition 3.4, the Euler characteristic $\chi(F_{be^{-(t+\log r_i)}})$ equals 1 whenever it occurs in $z(t)$. Using indicator functions $1_{[r_i,1]}$ defined by $1_{[r_i,1]}(y) = 1$ if $y \in [r_i, 1]$ and $1_{[r_i,1]}(y) = 0$ else, we can write (3.8) as

$$z(t) = be^{-st}\left(R(be^{-t}) + \sum_{i=1}^{N} 1_{[r_i,1]}(e^{-t})\right). \tag{3.10}$$

Now observe that (3.9) is a renewal equation for Z and that, trivially, $\lim_{t\to-\infty} Z(t) = 0$. The assumptions on the overlap function R ensure that z has a discrete set of discontinuities. Moreover, condition (1.1) implies the existence of positive constants $c_1 := b(cb^{-\gamma} + N)$ and $c_2 := s - \gamma > 0$ such that

$$|z(t)| \le be^{-st}(\left|R(be^{-t})\right| + N) \le c_1 e^{-c_2 t}$$

for $t \ge 0$. Thus condition (3.1) is satisfied and we can apply Theorem 3.1 with $p_i = r_i^s$ and $y_i = -\log r_i$. There are two cases to consider.

The non-arithmetic case. If $\{-\log r_1, \dots, -\log r_N\}$ is non-arithmetic, the limit

$$\lim_{t\to\infty} Z(t) = b \lim_{t\to\infty} e^{-st}\chi(F_{be^{-t}}) = b \lim_{\epsilon\searrow 0} \left(\frac{\epsilon}{b}\right)^s \chi(F_\epsilon) = b\chi_f(F)$$

exists and equals the integral

$$\frac{1}{\mu}\int_0^\infty z(\tau)d\tau = \frac{1}{\mu}\int_0^1 z(-\log r)\frac{dr}{r}$$

where we substituted $r = e^{-\tau}$ to obtain the right hand side and $\mu = -\sum_{i=1}^N r_i^s \log r_i$. Since

$$z(-\log r) = br^s\left(R(br) + \sum_{i=1}^N 1_{[r_i,1]}(r)\right),$$

integration yields

$$\begin{aligned}
\chi_f(F) &= \frac{1}{\mu}\left(\int_0^1 r^{s-1}R(br)dr + \sum_{i=1}^N \int_0^1 r^{s-1}1_{[r_i,1]}(r)dr\right) \\
&= \frac{1}{\mu}\left(\int_0^1 r^{s-1}R(br)dr + \frac{N-1}{s}\right).
\end{aligned}$$

This completes the proof of (ii) of Theorem 1. For the non-arithmetic case, (i) easily follows from (ii), since $\chi_f^a(F) = \chi_f(F)$ in case $\chi_f(F)$ exists.

The h-arithmetic case. If $\{-\log r_1, \dots, -\log r_N\}$ is h-arithmetic, Corollary 3.3 states that the limit

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T Z(t)\ dt = b\lim_{\delta\searrow 0}\frac{1}{|\log\delta|}\int_\delta^1 \left(\frac{\epsilon}{b}\right)^s \chi(F_\epsilon)\frac{d\epsilon}{\epsilon} = b\chi_f^a(F)$$

exists and equals

$$\frac{1}{\mu}\int_0^\infty z(\tau)d\tau.$$

Hence, repeating the calculations from the non-arithmetic case, we obtain formula (2.3) for the h-arithmetic case. This completes the proof of Theorem 1.

4 Convex ring and fractal Euler number

4.1 Euler characteristic in $\mathcal{R}^d$

Here we briefly introduce the convex ring $\mathcal{R}^d$ and the Euler characteristic in this class of sets. We follow the lattice theoretic approach of Klee [7] and Rota [12]. For more detailed explanations compare [6] and the references therein.

A subset C of $\mathbb{R}^d$ is said to be *convex* if for any two points $x, y \in C$ the line segment connecting them is a subset of C. Denote by $\mathcal{K}^d$ the collection of all compact convex sets in $\mathbb{R}^d$. We call a set *polyconvex* if it is a finite union of compact convex sets. Since the intersection of two convex sets is convex, union and intersection of polyconvex sets are again polyconvex sets. That is, the family of polyconvex sets in $\mathbb{R}^d$ is closed with respect to unions and intersections. This family is usually referred to as the *convex ring* and denoted by $\mathcal{R}^d$. Note that $\varnothing \in \mathcal{K}^d \subseteq \mathcal{R}^d$.

Definition 4.1. A *valuation* on $\mathcal{R}^d$ is a function $\nu : \mathcal{R}^d \to \mathbb{R}$ such that $\nu(\varnothing) = 0$ and, for all $A, B \in \mathcal{R}^d$,

$$\nu(A \cup B) = \nu(A) + \nu(B) - \nu(A \cap B). \tag{4.1}$$

By iterating the identity (4.1) we obtain the so called *inclusion-exclusion principle* for the valuation ν:

$$\nu\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_k \le n} \nu(A_{i_1} \cap \ldots \cap A_{i_k}), \tag{4.2}$$

whenever $A_1, \ldots, A_n \in \mathcal{R}^d$.

The Euler characteristic should be a valuation that assigns the value 1 to non-empty convex sets $C \in \mathcal{K}^d$. Hadwiger [4] considered extensions of this trivial valuation on $\mathcal{K}^d$ to the convex ring $\mathcal{R}^d$ and showed the existence and uniqueness of such an extension, as the following theorem states.

Theorem 4.2. *[14, Thm. 3.4.12] There exists a unique valuation χ on the convex ring $\mathcal{R}^d$, such that $\chi(C) = 1$ whenever $C \in \mathcal{K}^d$.*

This valuation χ on $\mathcal{R}^d$ is called the *Euler characteristic*. We emphasize again that for polyconvex sets this notion of Euler characteristic coincides with the more general notion of Euler characteristic for cell complexes. For any set $A \in \mathcal{R}^d$, $\chi(A)$ can be determined by choosing a decomposition into compact convex sets and using the inclusion-exclusion principle. Note that, according to Theorem 4.2, $\chi(A)$ is independent of the choice of the decomposition of A.

Denote by E^d the *Euclidean group* on $\mathbb{R}^d$ generated by translations and rotations. A valuation ν on $\mathcal{R}^d$ is said to be *motion invariant* if

$$\nu(gA) = \nu(A)$$

for all $g \in E^d$ and $A \in \mathcal{R}^d$, where $gA = \{g(a) : a \in A\}$. A valuation ν on $\mathcal{R}^d$ is *scaling invariant* (or *homogeneous of degree 0*) if $\chi(\lambda A) = \chi(A)$ for all $\lambda > 0$ and $A \in \mathcal{R}^d$, where $\lambda A := \{\lambda a : a \in A\}$. Note that the convex ring is closed with respect to Euclidean motions and scaling, i.e. with A also gA and λA are polyconvex, and that χ is a motion and scaling invariant valuation.

The following lemma collects some basic properties of polyconvex sets and their ϵ-neighbourhoods, which we use frequently in the sequel. For convenience we sometimes use the "0-neighbourhood" A_0 to denote the set A itself. Observe that (iii) is a special case of Proposition 3.4, however, we include a very simple proof of this fact for the convex ring setting.

Lemma 4.3. *For a set $A \in \mathcal{R}^d$ the following holds:*

(i) $A_\epsilon \in \mathcal{R}^d$ *for all* $\epsilon > 0$.

(ii) $\chi(A_\epsilon)$*, as a function of* ϵ*, has a finite set of discontinuities in* $(0, \infty)$ *and* $\lim_{\epsilon \searrow 0} \chi(A_\epsilon) = \chi(A)$.

(iii) $\chi(A_\epsilon) = 1$ *for all* $\epsilon \geq b$*, where* b *denotes the diameter of* A.

Proof. Assertion (i) follows immediately from the fact that neighbourhoods of convex sets are convex. For a proof of (ii) and (iii), let $C^1, \ldots, C^n \in \mathcal{K}^d$ be sets such that $A = \bigcup_{k=1}^n C^k$. Then by the inclusion-exclusion principle,

$$\chi(A_\epsilon) = \chi\left(\bigcup_{k=1}^n C_\epsilon^k\right) = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \chi\left(C_\epsilon^{i_1} \cap \ldots \cap C_\epsilon^{i_k}\right). \quad (4.3)$$

Now observe that each of the sets $C_\epsilon^{i_1} \cap \ldots \cap C_\epsilon^{i_k}$ in the sum is convex and has either Euler characteristic 0 or 1. If $C^{i_1} \cap \ldots \cap C^{i_k} \neq \varnothing$ then $\chi\left(C_\epsilon^{i_1} \cap \ldots \cap C_\epsilon^{i_k}\right)$, as a function of ϵ, is constant equal to 1, while in case $C^{i_1} \cap \ldots \cap C^{i_k} = \varnothing$, $\chi\left(C_\epsilon^{i_1} \cap \ldots \cap C_\epsilon^{i_k}\right)$, as a function of ϵ, has exactly one discontinuity point $\alpha > 0$ where the Euler characteristic jumps from 0 to 1. Thus each of the terms in the finite sum (4.3) has at most one discontinuity point, implying that $\chi(A_\epsilon)$, as a function of ϵ, has a finite number of discontinuities in $(0, \infty)$. $\chi(A_\epsilon) \to \chi(A)$ as $\epsilon \searrow 0$ follows since $\chi(A_\epsilon)$ is continuous (from the right) at 0.

(iii) Observe that for $\epsilon \geq b$ the sets $C_\epsilon^{i_1} \cap \ldots \cap C_\epsilon^{i_k}$ are non-empty and convex and therefore have Euler characteristic one. Applying this to (4.3) and noting that $\sum_{1\leq i_1<\ldots<i_k\leq n} 1 = \binom{n}{k}$ and $\sum_{i=1}^{n}(-1)^{k-1}\binom{n}{k} = 1$, by the binomial formula, assertion (iii) follows. $\square$

Remark 4.4. Lemma 4.3 (i) implies that for any polyconvex set A, the Euler exponent s of A is 0, since $|\chi(A_\epsilon)|$ is bounded. Thus $\chi_f(A) = \lim_{\epsilon\searrow 0}\chi(A_\epsilon)$ and so, by Lemma 4.3 (ii),

$$\chi_f(A) = \chi(A).$$

Hence for polyconvex sets the fractal Euler number coincides with the classical Euler characteristic, a further justification of the concepts introduced in this paper.

4.2 Self-similar sets with ϵ-neighbourhoods in $\mathcal{R}^d$

Given a self-similar set F, it is very easy to decide, whether it fits into the convex ring setting, i.e. whether its ϵ-neighbourhoods are polyconvex sets, or not. Either all neighbourhoods are polyconvex or none.

Proposition 4.5. *Let F be a self-similar set. Then the following statements are equivalent:*

(i) For all $\epsilon > 0$, $F_\epsilon \in \mathcal{R}^d$.

(ii) There is an $\epsilon > 0$ such that $F_\epsilon \in \mathcal{R}^d$.

Proof. We prove (ii) $\Rightarrow$ (i), the other direction is trivial. Assume $F_\epsilon \in \mathcal{R}^d$ for some $\epsilon > 0$. By Lemma 4.3 (i), $F_\delta \in \mathcal{R}^d$ for all $\delta > \epsilon$.

Let now $\delta < \epsilon$ and $n \in \mathbb{N}$ such that $\delta > \epsilon {r_{\max}}^n$. We can write F as the union of its n-th level cylinder sets implying

$$F_\delta = \bigcup_{i_k\in\{1,\ldots,N\}} (S_{i_1}\circ\ldots\circ S_{i_n}F)_\delta. \tag{4.4}$$

Observe that $(S_{i_1}\circ\ldots\circ S_{i_n}F)_\delta = S_{i_1}\circ\ldots\circ S_{i_n}(F_{\delta/r_{i_1}\ldots r_{i_n}})$. Since $\frac{\delta}{r_{i_1}\ldots r_{i_n}} \geq \frac{\delta}{r_{\max}^n} > \epsilon$, we have $F_{\delta/r_{i_1}\ldots r_{i_n}} \in \mathcal{R}^d$. Thus F_δ, being the finite union of polyconvex sets in (4.4), is an element of $\mathcal{R}^d$. $\square$

According to Proposition 4.5, it is sufficient to investigate an arbitrarily chosen single neighbourhood F_ϵ of F. If it is polyconvex all neighbourhoods of F are, else none is. The value ϵ can be chosen large (i.e. $\epsilon \geq b$) such that the ϵ-neighbourhood is simply connected and has no "inner structure", simplifying the investigation. However, there are self-similar sets with non-polyconvex neighbourhoods, e.g. the von Koch curve or self-similar sets in $\mathbb{R}^d$, $d \geq 2$, satisfying the SSC.

In view of Proposition 4.5 it is clear that the overlap function R is well defined if we assume that $F_\epsilon \in \mathcal{R}^d$ for some $\epsilon > 0$. But with this assumption we can even say more about the overlap function, namely that R has a discrete set of discontinuities:

Proposition 4.6. *Let F be a self-similar set with polyconvex neighbourhoods. Then the overlap function $R : (0, \infty) \to \mathbb{R}$ has a discrete set of discontinuities.*

Proof. Let $\delta > 0$. Since F_δ and $(S_i F)_\delta$ are in $\mathcal{R}^d$, by Lemma 4.3 (ii), $\chi(F_{\delta+\epsilon})$ and $\chi((S_i F)_{\delta+\epsilon})$ have, as functions of ϵ, a finite set of discontinuities in $(0, \infty)$. Therefore, the overlap function $R(\epsilon) = \chi(F_\epsilon) - \sum_{i=1}^N \chi((S_i F)_\epsilon)$ has a finite set of discontinuities in (δ, ∞). Since this holds for each $\delta > 0$, R has a discrete set of discontinuities in $(0, \infty)$. □

Note that 0 is the only possible accumulation point of discontinuities of the overlap function R.

5 Classes of self-similar sets with (average) fractal Euler number

In this section we discuss the class of self-similar sets F satisfying the conditions of Theorem 2 and some extension of this class. In the examples in Section 2.3 one can see that the ϵ-neighbourhoods of F can be described in terms of the set M of Theorem 2. Depending on ϵ we can find a level n such that F_ϵ coincides with the union of the ϵ-neighbourhoods of all n-th level iterated images $S_{i_1} \circ \ldots \circ S_{i_n} M$ of the set M. Lemma 5.1 shows that this is true for the whole class of self-similar sets satisfying the conditions of Theorem 2, in fact, that it is even true for a larger class.

Let F be a self-similar set with diameter b. We define the function

$$n(\epsilon) := \min\{n \in \mathbb{N}_0 \ : \ r_{\max}^n b \leq \epsilon\}.$$

For $A \subset \mathbb{R}^d$, let $A^0 := A$ and for $n = 1, 2, \ldots$, define $A^n := \bigcup_{i=1}^N S_i(A^{n-1})$. With ∂A, $\overline{A}$ and $\overset{\circ}{A}$ we denote the (topological) boundary, closure and interior of the set A, respectively.

Lemma 5.1. *Assume there exists a compact set $M \subset \mathbb{R}^d$ such that $\partial M \subset F \subset M$. Then for all $\epsilon \geq b$*

$$F_\epsilon = M_\epsilon. \tag{5.1}$$

More generally,

$$F_\epsilon = (M^m)_\epsilon \tag{5.2}$$

for all integers $m \geq n(\epsilon)$ and $\epsilon > 0$. In particular, if $M \in \mathcal{R}^d$, then $F_\epsilon \in \mathcal{R}^d$ for all $\epsilon > 0$.

Proof. Let $M \subset \mathbb{R}^d$ be given as above and $\epsilon \geq b$. Then it is clear that $M \subset F_\epsilon$. On the other hand, for $x \in M_\epsilon \setminus M$ we have $\mathrm{d}(x, \partial M) \leq \epsilon$ and, since $\partial M \subset F$, $x \in F_\epsilon$. Therefore, $M_\epsilon = M \cup (M_\epsilon \setminus M) \subseteq F_\epsilon$. The reversed inclusion is clear since $F \subset M$, thus (5.1) holds.

To prove (5.2) we first note that, by definition of $n(\epsilon)$, $\frac{\epsilon}{r_{i_1} \ldots r_{i_m}} \geq \frac{\epsilon}{r_{\max}^{n(\epsilon)}} \geq b$ for any $m \geq n(\epsilon)$ and $i_j \in \{1, \ldots, N\}$ $(j = 1, \ldots, m)$. Therefore, by (5.1), $M_{\epsilon/(r_{i_1} \ldots r_{i_m})} = F_{\epsilon/(r_{i_1} \ldots r_{i_m})}$ implying

$$\begin{aligned}
(M^m)_\epsilon &= \bigcup_{i_1=1}^{N} \ldots \bigcup_{i_m=1}^{N} S_{i_1} \circ \ldots \circ S_{i_m}\big(M_{\epsilon/(r_{i_1} \ldots r_{i_m})}\big) \\
&= \bigcup_{i_1=1}^{N} \ldots \bigcup_{i_m=1}^{N} S_{i_1} \circ \ldots \circ S_{i_m}\big(F_{\epsilon/(r_{i_1} \ldots r_{i_m})}\big) = F_\epsilon.
\end{aligned}$$

Finally, if $M \in \mathcal{R}^d$, then $F_\epsilon \in \mathcal{R}^d$ follows directly from (5.1) and Proposition 4.5. □

Remark 5.2. Clearly, if the set M in Lemma 5.1 is convex, then $M = [F]$. As pointed out before, for many examples of self-similar sets the set $M = [F]$ satisfies $\partial M \subset F \subset M$. However, as Example 2.5 shows, there are self-similar sets satisfying the conditions of Lemma 5.1 with $M \neq [F]$.

If a set $M \in \mathcal{R}^d$ of Lemma 5.1 exists for a self-similar set F, then Proposition 4.6 implies that the overlap function R has a discrete set of discontinuities. The following lemma shows that for the class of self-similar sets satisfying the conditions of Theorem 2 the set of discontinuities of the (piecewise constant) function R is even finite implying that R is bounded.

Lemma 5.3. *If there exists a set $M \in \mathcal{R}^d$ such that $\partial M \subset F \subset M$ and*

$$S_i(M) \cap S_j(M) = S_i(\partial M) \cap S_j(\partial M) \tag{5.3}$$

for all $i \neq j$, then the overlap function R has a finite set of discontinuities and is bounded. Moreover,

$$\lim_{\epsilon \searrow 0} R(\epsilon) = \sum_{k=2}^{N} (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq N} \chi(S_{i_1} M \cap \ldots \cap S_{i_k} M). \tag{5.4}$$

Proof. It suffices to show that

$$(S_i F)_\epsilon \cap (S_j F)_\epsilon = (S_i M)_\epsilon \cap (S_j M)_\epsilon \tag{5.5}$$

for $\epsilon > 0$ and $i \neq j$. By the inclusion-exclusion principle, this implies, that the overlap function can be written as

$$\begin{aligned} R(\epsilon) &= \sum_{k=2}^{N} (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq N} \chi((S_{i_1} M)_\epsilon \cap \ldots \cap (S_{i_k} M)_\epsilon) \\ &= \chi(M_\epsilon) - \sum_{i=1}^{N} \chi((S_i M)_\epsilon) \end{aligned}$$

By Lemma 4.3 (ii), each term in the last sum has, as a function of ϵ, a finite set of discontinuities in $(0, \infty)$, implying the same for R. Therefore, since R is a piecewise constant function, we conclude that it is bounded. Formula (5.4) follows immediately.

To complete the proof we show equation (5.5). Since $S_i F \subset S_i M$, one of the inclusions follows immediately. To prove the remaining one, let $x \in (S_i M)_\epsilon \cap (S_j M)_\epsilon$. We show that $\mathrm{d}(x, S_i \partial M) \leq \epsilon$. Assume the contrary. Then, $x \in S_i M$ ($x \in (S_i M)_\epsilon \setminus S_i M$ implies $\mathrm{d}(x, S_i \partial M) \leq \epsilon$.) Since $x \in (S_j M)_\epsilon$ we can find $y \in S_j M$ such that $\mathrm{d}(x, y) \leq \epsilon$. Now the assumption implies that on one hand $y \in S_i M \cap S_j M$ since $\mathrm{d}(x, (S_i M)^c) > \epsilon$. On the other hand $y \notin S_i \partial M$, a contradiction to (5.3). Thus $\mathrm{d}(x, S_i \partial M) \leq \epsilon$, implying that $x \in (S_i \partial M)_\epsilon \subset (S_i F)_\epsilon$. The same arguments apply to j. Hence $x \in (S_i F)_\epsilon \cap (S_j F)_\epsilon$, completing the proof of (5.5). $\square$

Remark 5.4. Condition (5.3) in Lemma 5.3 cannot be removed. Consider, for example, a modified Sierpinski gasket with strong overlap as defined by the following system of similarities in $\mathbb{R}^2$: $S_1 x = \frac{2}{3} x$, $S_2 x = \frac{2}{3} x + (\frac{1}{3}, 0)$ and $S_3 x = \frac{1}{3} x + (\frac{1}{3}, \frac{1}{\sqrt{3}})$. The convex hull of this self-similar set satisfies the conditions of M in Lemma 5.3 except (5.3) but its overlap function does not have a finite set of discontinuities.

Remark 5.5. If, in addition to the conditions of Lemma 5.3, we have $\mathbf{S}(M) \subset M$ (which is always true for convex M), then $\{S_1, \ldots, S_N\}$ satisfies the OSC for the set $\overset{\circ}{M}$: condition (5.3) implies $S_i(\overset{\circ}{M}) \cap S_j(\overset{\circ}{M}) = \varnothing$ and $\mathbf{S}(M) \subset M$ obviously $\mathbf{S}(\overset{\circ}{M}) \subset \overset{\circ}{M}$.

Observe that the set M satisfying the conditions in Lemma 5.3 is not uniquely determined for a self-similar set F. In fact, if there is one, there are many such sets. It is not difficult to see that, for instance, with M, also the set $\mathbf{S}(M)$ satisfies the conditions. In the examples it is easy to find such sets M with the assumption $\mathbf{S}(M) \subset M$ not satisfied. For the Sierpinski gasket Δ, for example, the set M' being the union of the set $\mathbf{S} \circ \mathbf{S}[\Delta]$ and the triangle $w_1 w_2 w_3$ (compare Figure 1) satisfies the conditions of Lemma 5.3 but not $\mathbf{S}(M') \subset M'$. But this condition must be satisfied, if we want to use the interior of M' as the open set of the OSC. It is, however, not clear if this assumption is really needed. We conjecture that, whenever a set M of Lemma 5.3 exists, one can also find such a set M with $\mathbf{S}(M) \subset M$. Note that, if $\mathbf{S}(M) \subset M$, the fact $\overset{\circ}{M} \cap F \neq \varnothing$ implies that $\overset{\circ}{M}$ is also a suitable open set for the strong open set condition, which in $\mathbb{R}^d$ is equivalent to the OSC (cf. [13]).

Proof of Theorem 2. By Lemma 5.1, the existence of a set M for F as assumed in Theorem 2 implies that $F_\epsilon \in \mathcal{R}^d$ for $\epsilon > 0$. Moreover, Lemma 5.3 states that under the assumptions of Theorem 2 the overlap function R of F has a finite set of discontinuities and that R is bounded. Therefore, the conditions of Theorem 1 are satisfied and the assertions follow directly. $\square$

Although the conditions in Theorem 2 provide a reasonable class of self-similar sets for which the (average) fractal Euler number exist, the following example shows that there are still interesting self-similar sets satisfying the conditions of Theorem 1 that are not included in the class covered by Theorem 2. In particular, the condition $\partial M \subset F$ can be relaxed (cf. Proposition 5.7).

Example 5.6. Let U_3 be the self-similar set in Figure 6 (left) obtained in the same way as U_1 and U_2 in Example 2.5 with the obvious changes in the choice of the similarities. Proceeding as in the examples in Section 2.3, we obtain

$$R(\epsilon) = \begin{cases} 1 - 7 = -6 & \text{if } \epsilon \geq u \\ -12 + 7 = -10 & \text{if } 0 < \epsilon < u \end{cases}$$

where $u = \frac{1}{18\sqrt{2}}$. Therefore,

$$\chi_f^a(U_3) \quad = \quad \frac{1}{\mu}\Big(\int_0^1 \epsilon^{s-1} R(\epsilon)\, d\epsilon + \frac{N-1}{s} \Big) = -\frac{4u^s}{\mu s} = -\frac{4u^s}{\log 7} \approx -0.015.$$

The above example motivates the following Proposition that gives some sufficient geometric conditions for $R(\epsilon)$ to be bounded in the case when ∂M is not a subset of F.

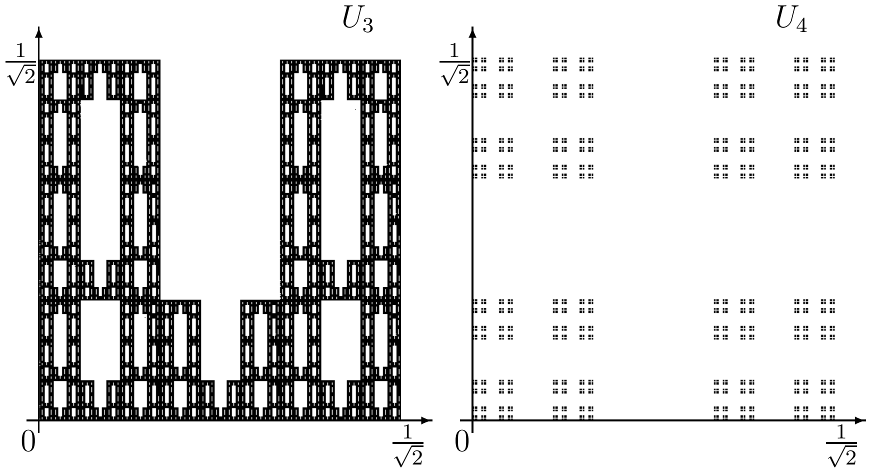

Figure 6: U-set U_3, which does not meet the assumptions of Theorem 2, and Cantor set U_4 with non-polyconvex neighbourhoods.

Proposition 5.7. *Let F be a self-similar set. Assume there is a set $M \in \mathcal{R}^d$ such that $F \subset M$ and*

$$S_iM \cap S_jM = S_iA \cap S_jA \tag{5.6}$$

for all $i \neq j$, where $A = \partial M \cap F$. Let $V = \{y \in \mathbb{R}^d \setminus M \ : \ \mathrm{d}(y, \partial M) < \mathrm{d}(y, A)\}$. If

$$(S_iM)_\epsilon \cap (S_jM)_\epsilon \cap S_iV = (S_iM \cap S_jM)_\epsilon \cap S_iV \tag{5.7}$$

for all $i \neq j$ and all $\epsilon > 0$, then the overlap function R has a finite set of discontinuities. Moreover, equation (5.4) holds.

Proof. First we show that the assumptions imply

$$(S_iF)_\epsilon \cap (S_jF)_\epsilon = (S_iM)_\epsilon \cap (S_jM)_\epsilon \tag{5.8}$$

for $\epsilon > 0$ and $i \neq j$.

Since $S_iF \subset S_iM$, one of the inclusions in (5.5) follows immediately. To prove the remaining one, let $x \in (S_iF)_\epsilon \cap (S_jF)_\epsilon$.
(i) For $x \in S_iV$, conditions (5.7) and (5.6) imply $x \in (S_iA \cap S_iA)_\epsilon$ and, since $A \subset F$, $x \in (S_iF \cap S_jF)_\epsilon \subseteq (S_iF)_\epsilon \cap (S_jF)_\epsilon$, proving (5.8) in this case.
(ii) For $x \notin S_iV$, we show that $\mathrm{d}(x, S_iA) \leq \epsilon$. Assume the contrary. Then, clearly, $x \in S_iM$. ($x \in (S_iM)_\epsilon \setminus S_iM$ implies $\mathrm{d}(x, S_iA) \leq \mathrm{d}(x, S_i\partial M) \leq \epsilon$ since $x \notin S_iV$.) Since $x \in (S_jM)_\epsilon$ we can find $y \in S_jM$ such that $\mathrm{d}(x, y) \leq \epsilon$. Now the assumption implies that on one hand $y \in S_iM \cap S_jM$ since $\mathrm{d}(x, (S_iM)^c) > \epsilon$, on the other hand $y \notin S_iA$, a contradiction to (5.6). Thus

$d(x, S_iA) \le \epsilon$, implying that $x \in (S_iA)_\epsilon \subset (S_iF)_\epsilon$. The same arguments apply to j (we can assume that $x \notin S_jV$, otherwise we are in case (i)). Hence $x \in (S_iF)_\epsilon \cap (S_jF)_\epsilon$, completing the proof of (5.8).

Since (5.8) holds, we can now proceed as in the proof of Lemma 5.3. □

Remark 5.8. Note that, as a consequence of equation (5.2), we can replace $\chi(F_\epsilon)$ with $\chi((M^{n(\epsilon)})_\epsilon)$ in the Definitions 1.2 and 1.3 of the (average) fractal Euler number, whenever the conditions of Lemma 5.1 are satisfied. Therefore, one can study the asymptotic behaviour of $\left(\frac{\epsilon}{b}\right)^s \chi(F_\epsilon) = \left(\frac{\epsilon}{b}\right)^s \chi((M^{n(\epsilon)})_\epsilon)$ by applying the Renewal theorem directly to the equation

$$\chi((M^{n(\epsilon)})_\epsilon) = \sum_{i=1}^{N} \chi((M^{n(\epsilon/r_i)})_{\epsilon/r_i}) + \tilde{R}(\epsilon)$$

where

$$\tilde{R}(\epsilon) = \chi((M^{n(\epsilon)})_\epsilon) - \sum_{i=1}^{N} \chi((M^{n(\epsilon)-1})_{\epsilon/r_i}).$$

On the other hand, it is clear that the conditions of Lemma 5.1 can not be satisfied by self-similar sets satisfying the SSC. Therefore one could extend the notion of (average) fractal Euler number to this class of self-similar sets by using the formula

$$\chi_f(F) = \lim_{\epsilon \searrow 0} \left(\frac{\epsilon}{b}\right)^s \chi((M^{n(\epsilon)})_\epsilon),$$

and its average counterpart, with $M = [F]$ as a definition.

We illustrate this possible extension with the following example. Consider the Cantor set in $\mathbb{R}^2$ described in Figure 6 (right). It is not difficult to determine the function $\tilde{R}$ of this set:

$$\tilde{R}(\epsilon) = \begin{cases} -3 & \text{for } \frac{1}{6} \le \epsilon \\ -4 & \text{for } \frac{1}{6\sqrt{2}} \le \epsilon < \frac{1}{6} \\ 0 & \text{for } \epsilon < \frac{1}{6\sqrt{2}} \end{cases}$$

Thus $\tilde{R}$ is a bounded function with only two discontinuities and the redefined "average fractal Euler number" of this self-similar set exists and can easily be calculated.

References

[1] A. Dold. *Lectures on Algebraic Topology.* (Springer-Verlag, 1980).

[2] K. J. Falconer. *Techniques in fractal geometry.* (New York: Wiley, 1997).

[3] D. Gatzouras. *Lacunarity of self-similar and stochastically self-similar sets.* Trans. Amer. Math. Soc. (**5**) 352 (2000) 1953-1983.

[4] H. Hadwiger, *Eulers Charakteristik und kombinatorische Geometrie.* J. Reine Angew. Math. **194** (1955) 101-110.

[5] J. E. Hutchinson. *Fractals and self-similarity.* Indiana Univ. Math. J. **30** (1981) 713-747.

[6] D. A. Klain and G-C. Rota. *Introduction to geometric probability.* Accademia Nazionale dei Lincei. (Cambridge University Press, 1997).

[7] V. Klee. *The Euler characteristic in combinatorial geometry.* Amer. Math. Monthly (**70**) (1963) 119-127.

[8] M. Lapidus and C. Pomerance. *The Riemann Zeta function and the one-dimensional Weyl-Berry conjecture for fractal drums.* Proc. London. Math. Soc. (**3**) 66 (1993) 41-69.

[9] M. Levitin and D. Vassiliev. *Spectral assymptotics, Renewal theorem, and the Berry conjecture for a class of fractals.* Proc. London. Math. Soc. (**3**) 72 (1996) 188-214.

[10] M. Morán. *Dynamical boundary of a self-similar set.* Fundamenta Mathematicae (**160**) (1999) 1-14.

[11] J. Rataj and M. Zähle. *Normal cycles of Lipschitz manifolds by approximation with parallel sets.* Differ. Geom. and its Appl. (**19**) (2003) 113-126.

[12] G-C. Rota. *On the combinatorics of the Euler characteristic.* Studies in Pure Mathematics, Academic Press, London (1971) 221-233.

[13] A. Schief. *Separation properties of self-similar sets.* Proc. Amer. Math. Soc. (**122**) (1994) 111-115.

[14] R. Schneider. *Convex bodies: The Brunn-Minkowski Theory.* (New York: Cambridge University Press, 1993).