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A Note on Equivalence of Bipartite States under Local Unitary Transformations

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Abstract

The equivalence of arbitrary dimensional bipartite states under local unitary transformations (LUT) is studied. A set of invariants and ancillary invariants under LUT is presented. We show that two states are equivalent under LUT if and only if they have the same values for all of these invariants.

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The quantum entangled states have been used as the key resources in quantum information processing and quantum computation [1]. An important property of quantum entanglement is that the entanglement of a bipartite quantum state remains invariant under local unitary transformations on the subsystems. Therefore invariants of local unitary transformations have special importance. For instance the trace norms of realigned or partial transposed density matrices in entanglement measure, separability criteria are some of these invariants [2]. Two quantum states are locally equivalent if and only if all the invariants have equal values for these two states. For bipartite mixed states, a generally non-operational method has been presented to compute all the invariants of local unitary transformations in [3, 4]. In [5], the invariants for general two-qubit systems are studied and a complete set of 18 polynomial invariants is presented. In [6], the invariants for three qubits states have been discussed. A complete set of invariants for generic mixed states are presented. In [7] the invariants for a class of non-generic three-qubit states have been investigated. In [8], complete sets of invariants for some classes of density matrices have been presented. The invariants for tripartite pure states have been also studied [9].

In [10] a complete set of invariants for generic density matrices with full rank has been presented. In this note we extend the results to generalized generic density matrices with arbitrary rank, by taking into account the vector space corresponding to the zero eigenvalues.

Let H be an N -dimensional complex Hilbert space, with $|i\rangle$, $i = 1, \dots, N$, as an orthonormal basis. Let ρ be a density matrix defined on $H \otimes H$ with $\text{rank}(\rho) = n \leq N^2$. It can be written as

$$\rho = \sum_{i=1}^n \lambda_i |v_i\rangle\langle v_i|,$$

where $|v_i\rangle$ is the eigenvector with respect to the nonzero eigenvalue λ_i . $|v_i\rangle$ has the form:

$$|v_i\rangle = \sum_{k,l=1}^N a_{kl}^i |kl\rangle, \quad a_{kl}^i \in \mathcal{C}, \quad \sum_{k,l=1}^N a_{kl}^i a_{kl}^{i*} = 1, \quad i = 1, \dots, n.$$

Let A_i denote the matrix given by $(A_i)_{kl} = a_{kl}^i$. We introduce $\{\rho_i\}$, $\{\theta_i\}$,

$$\rho_i = \text{Tr}_2 |v_i\rangle\langle v_i| = A_i A_i^\dagger, \quad \theta_i = \text{Tr}_1 |v_i\rangle\langle v_i| = A_i^t A_i^*, \quad i = 1, 2, \dots, n, \quad (1)$$

where Tr_1 and Tr_2 stand for the traces over the first and second Hilbert spaces, A^t , A^* are the transpose and the complex conjugation of A respectively.

Two density matrices ρ and ρ' are said to be equivalent under local unitary transformations if there exist unitary operators U_1 (resp. U_2) on the first (resp. second) space of $H \otimes H$ such that

$$\rho' = (U_1 \otimes U_2) \rho (U_1 \otimes U_2)^\dagger. \quad (2)$$

Let $\Omega(\rho)$ and $\Theta(\rho)$ be two ‘‘metric tensor’’ matrices, with entries given by

$$\Omega(\rho)_{ij} = \text{Tr}(\rho_i \rho_j), \quad \Theta(\rho)_{ij} = \text{Tr}(\theta_i \theta_j), \quad \text{for } i, j = 1, \dots, n. \quad (3)$$

We call a mixed state ρ a generic one if Ω , Θ satisfy

$$\det(\Omega(\rho)) \neq 0 \quad \text{and} \quad \det(\Theta(\rho)) \neq 0. \quad (4)$$

In [10] it has been shown that two full-ranked bipartite states ($n = N^2$) satisfying (4) are equivalent under local unitary transformations if and only if they have the same values of the following invariants: Ω , Θ ,

$$X(\rho)_{ijk} = \text{Tr}(\rho_i \rho_j \rho_k), \quad Y(\rho)_{ijk} = \text{Tr}(\theta_i \theta_j \theta_k), \quad i, j, k = 1, \dots, n, \quad (5)$$

together with the condition

$$J^s(\rho) = \text{Tr}(\rho^s), \quad s = 1, 2, \dots, N^2, \quad (6)$$

which guarantee that the density matrices have the same set of eigenvalues.

For the case $n < N^2$, the invariants (3), (5) and (6) are no longer enough to verify the equivalence of two generic states under local unitary transformations, and some ancillary invariants are needed.

From the generic condition $\det(\Omega(\rho)) \neq 0$ and $\det(\Theta(\rho)) \neq 0$, we have that $\{\rho_i, i = 1, \dots, n\}$ and $\{\theta_i, i = 1, \dots, n\}$ are two sets of linear independent matrices. One can always find some $N \times N$ matrices ρ_i , θ_i (we call them ancillary matrices), $i = n + 1, \dots, N^2$, such

that $\{\rho_i, i = 1, \dots, N^2\}$ and $\{\theta_i, i = 1, \dots, N^2\}$ span the $N^2 \times N^2$ matrix space respectively. Therefore the $N^2 \times N^2$ matrices $\tilde{\Omega}(\rho)$ and $\tilde{\Theta}(\rho)$,

$$\tilde{\Omega}(\rho)_{ij} = Tr(\rho_i \rho_j), \quad \tilde{\Theta}(\rho)_{ij} = Tr(\theta_i \theta_j), \quad i, j = 1, \dots, N^2, \quad (7)$$

satisfy

$$\det(\tilde{\Omega}(\rho)) \neq 0 \text{ and } \det(\tilde{\Theta}(\rho)) \neq 0. \quad (8)$$

Set

$$\tilde{X}(\rho)_{ijk} = Tr(\rho_i \rho_j \rho_k), \quad \tilde{Y}(\rho)_{ijk} = Tr(\theta_i \theta_j \theta_k), \quad i, j, k = 1, \dots, N^2. \quad (9)$$

We call $\tilde{\Omega}(\rho)_{ij}$, $\tilde{\Theta}(\rho)_{ij}$, $\tilde{X}(\rho)_{ijk}$, $\tilde{Y}(\rho)_{ijk}$, with at least one of their sub-indices taking values from $n + 1$ to N^2 , the ancillary invariants.

[Theorem]: Two generic density matrices are equivalent under local unitary transformations if and only if there exists a ordering of the corresponding eigenstates such that the following invariants have the same values for both density matrices:

$$J^s(\rho) = Tr(\rho^s), \quad s = 1, 2, \dots, N^2, \quad \tilde{\Omega}(\rho), \quad \tilde{\Theta}(\rho), \quad \tilde{X}(\rho), \quad \tilde{Y}(\rho). \quad (10)$$

[Proof]: Suppose that ρ and ρ' are equivalent under local unitary transformation $\mu \otimes \omega$, $\rho' = \mu \otimes \omega \rho \mu^\dagger \otimes \omega^\dagger$. Correspondingly, we have $|\nu'_i\rangle = \mu \otimes \omega |\nu_i\rangle$, i.e. $A'_i = \mu A_i \omega^\dagger$ for $i = 1, \dots, n$. As to the ancillary invariants, if $\rho_i, \theta_i, i = n + 1, \dots, N^2$ are the ancillary matrices associated to ρ , we can choose $\rho'_i = \mu \rho_i \mu^\dagger$, $\theta'_i = \omega \theta_i \omega^\dagger$, $i = n + 1, \dots, N^2$ for ρ' . Therefore $\rho'_i = A'_i A_i^\dagger = \mu \rho_i \mu^\dagger$, $\theta'_i = A'_i A_i^* = \omega \theta_i \omega^\dagger$, $i = 1, \dots, N^2$. It is straightforward to verify that the quantities in (10) are invariants under local unitary transformations, e.g. $\Omega(\rho')_{ij} = Tr(\rho'_i \rho'_j) = Tr(\mu \rho_i \rho_j \mu^\dagger) = Tr(\rho_i \rho_j) = \Omega(\rho)_{ij}$, $\Theta(\rho')_{ij} = Tr(\theta'_i \theta'_j) = Tr(\omega \theta_i \theta_j \omega^\dagger) = \Theta(\rho)_{ij}$, $i, j = 1, \dots, N^2$.

Conversely we suppose that the states $\rho = \sum_{i=1}^n \lambda_i |\nu_i\rangle \langle \nu_i|$ and $\rho' = \sum_{i=1}^n \lambda'_i |\nu'_i\rangle \langle \nu'_i|$ give the same values to the invariants in (3) and (5). And there exist ancillary matrices $\rho_i, \rho'_i, \theta_i, \theta'_i, i = n + 1, \dots, N^2$, such that they have the same values of (6) and the ancillary invariants in (7) and (9). ρ and ρ' can be proved to be equivalent under local unitary transformations by using the method in [10]. Having the same values of (6) implies that ρ' and ρ have the same nonzero eigenvalues, $\lambda'_i = \lambda_i$, $i = 1, \dots, n$. The condition (8), $\det(\tilde{\Omega}(\rho)) \neq 0$ implies that $\{\rho_i\}$, $i = 1, \dots, N^2$, span the space of $N \times N$ matrices and therefore

$$\rho_i \rho_j = \sum_{k=1}^n C_{ij}^k \rho_k, \quad C_{ij}^k \in \mathbb{C}, \quad (11)$$

which gives rise to $\tilde{\Omega}_{ij} = \sum_{k=1}^n C_{ij}^k$. Hence $\tilde{X}_{ijk} = \sum_{l=1}^n C_{ij}^l \tilde{\Omega}_{lk}$ and

$$C_{ij}^l = \sum_{k=1}^n \tilde{X}_{ijk} \tilde{\Omega}^{lk}, \quad (12)$$

where the matrices $\tilde{\Omega}^{ij}$ is the corresponding inverses of the matrices $\tilde{\Omega}_{ij}$. We have that $\{\rho_i\}$ form an irreducible N -dimensional representation of the algebra $gl(N, \mathbb{C})$ with structure constants $C_{ij}^k - C_{ji}^k$. Similarly $\{\rho'_i\}$, $i = 1, \dots, N^2$, also form an irreducible N -dimensional representation of the algebra $gl(N, \mathbb{C})$ with same structure constants. These two sets of representations of the algebra $gl(N, \mathbb{C})$ are equivalent, $\rho'_i = u \rho_i u^\dagger$, for some unitary u .

Similarly, from $\tilde{\Theta}(\rho) = \tilde{\Theta}(\rho')$ and $\tilde{Y}_{ijk}(\rho) = \tilde{Y}_{ijk}(\rho')$ we can deduce that $\theta'_i = w^\dagger \theta_i w$, for some unitary w . From the singular value decomposition of matrices, we have $|\nu'_i\rangle = u \otimes w |\nu_i\rangle$, $i = 1, \dots, n$, and $\rho' = u \otimes w \rho u^\dagger \otimes w^\dagger$. Hence ρ' and ρ are equivalent under local unitary transformations. ■

Remark The invariants in (10) could be redundant. For example when ρ and ρ' are 2×2 pure states, one only needs $Tr \rho_1 \rho_1 = Tr \rho'_1 \rho'_1$ for verifying the equivalence of them. But for higher dimensional systems, all these invariants are needed for generic states. Moreover when ρ and ρ' has equal nonzero eigenvalues, if they are equivalent under local unitary transformations we can always find a set of eigenvectors suitably labeled such that they have the same invariants (10), as seen from the proof.

As an example, let

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\psi'_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

and

$$|\psi_2\rangle = |01\rangle, \quad |\psi'_2\rangle = |00\rangle.$$

We consider

$$\rho = \frac{1}{3}|\psi_1\rangle\langle\psi_1| + \frac{2}{3}|\psi_2\rangle\langle\psi_2|,$$

$$\rho' = \frac{1}{3}|\psi'_1\rangle\langle\psi'_1| + \frac{2}{3}|\psi'_2\rangle\langle\psi'_2|.$$

These matrices have the same eigenvalues. The corresponding eigenvectors give rise to

$$\rho_1 = \theta_1 = \rho'_1 = \theta'_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

$$\rho_2 = \rho'_2 = \theta'_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

By directly calculations we have,

$$\Omega(\rho) = \Omega(\rho') = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = \Theta(\rho) = \Theta(\rho')$$

$$X(\rho)_{ijk} = X(\rho')_{ijk} = Y(\rho)_{ijk} = Y(\rho')_{ijk}$$

$$= \begin{cases} 1, & \text{if } i = j = k = 2 \\ \frac{1}{4}, & \text{if } ijk \in \{111, 112, 121, 211\} \\ \frac{1}{2}, & \text{for the rest.} \end{cases}$$

So $\det(\Omega) = \det(\Theta) = \frac{1}{2} \neq 0$ for both ρ , ρ' , and they are generic states. We choose the ancillary matrices as:

$$\rho_3 = \rho'_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \theta_3 = \theta'_3, \quad \rho_4 = \rho'_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \theta_4 = \theta'_4.$$

We have

$$\tilde{\Omega}(\rho) = \tilde{\Omega}(\rho') = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \tilde{\Theta}(\rho) = \tilde{\Theta}(\rho')$$

and

$$\tilde{X}(\rho)_{ijk} = \tilde{X}(\rho')_{ijk} = \begin{cases} X(\rho)_{ijk}, & i, j, k = 1, 2 \\ \frac{1}{2}, & ijk \in \{134, 341, 413, 143, 431, 314\} \\ 1, & ijk \in \{234, 342, 423\} \\ 0, & \text{for else} \end{cases},$$

$$\tilde{Y}(\rho)_{ijk} = \tilde{Y}(\rho')_{ijk} = \begin{cases} X(\rho)_{ijk}, & i, j, k = 1, 2 \\ \frac{1}{2}, & ijk \in \{134, 341, 413, 143, 431, 314\} \\ 1, & ijk \in \{243, 432, 324\} \\ 0, & \text{for else} \end{cases}.$$

From the theorem we can conclude that ρ and ρ' are equivalent under local unitary transformations.

We have studied the equivalence of two bipartite states with arbitrary dimensions by using some ancillary invariants under the unitary transformation. This method applies to all the bipartite generic density matrices. As for the nongeneric states, i.e. the states satisfying $\det(\Omega(\rho)) = 0$ or $\det(\Theta(\rho)) = 0$ or both, we can deal with the problem in the following way. If $\det(\Omega(\rho)) = 0$, i.e. ρ_i , $i = 1, \dots, n$, are linear dependent, we choose the maximal linear independent subset (denote as S) of $\{\rho_i, i = 1, \dots, n\}$. For two states ρ and ρ' with the same invariants in (10), we only need to find a unitary matrix μ satisfying $\rho'_i = \mu\rho_i\mu^\dagger$ for $\rho_i \in S$ and $\rho'_i \in S'$. According to the subset S , we can get submatrices of $\Omega(\rho)$ and $X(\rho)$. We denote these submatrices as $\bar{\Omega}(\rho)$, $\bar{X}(\rho)$. We extend them to matrices $\tilde{\Omega}(\rho)$, $\tilde{\Theta}(\rho)$ instead of $\Omega(\rho)$, $X(\rho)$. Then using the theorem and the relation between S (resp. S') and $\{\rho_i, i = 1, \dots, n\}$ (resp. $\{\rho'_i, i = 1, \dots, n\}$), we can get $\rho'_i = \mu\rho_i\mu^\dagger, i = 1, \dots, n$ for some unitary matrix μ . The case of $\det(\Theta(\rho)) = 0$ can be similarly treated.

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