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Analysis of Multiple Scattering Iterations for High-frequency Scattering Problems. I: The Two-dimensional Case

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ANALYSIS OF MULTIPLE SCATTERING ITERATIONS FOR HIGH-FREQUENCY SCATTERING PROBLEMS. I: THE TWO-DIMENSIONAL CASE

FATIH ECEVIT AND FERNANDO REITICH

Abstract. We present an analysis of a recently proposed integral-equation method for the solution of high-frequency electromagnetic and acoustic scattering problems that delivers error-controllable solutions in frequency-independent computational times. Within single scattering configurations the method is based on the use of an appropriate ansatz for the unknown surface densities and on suitable extensions of the method of stationary phase. The extension to multiple-scattering configurations, in turn, is attained through consideration of an iterative (Neumann) series that successively accounts for further geometrical wave reflections. As we show, for a collection of two-dimensional (cylindrical) convex obstacles, this series can be rearranged into a sum of *periodic orbits* (of increasing period), each corresponding to contributions arising from waves that reflect off a fixed subset of scatterers when these are transversed sequentially in a periodic manner. Here, we analyze the properties of these periodic orbits in the high-frequency regime, by deriving precise asymptotic expansions for the "currents" (i.e. the normal derivative of the fields) that they induce on the surface of the obstacles. As we demonstrate these expansions can be used to provide accurate estimates of the rate at which their magnitude decreases with increasing number of reflections, which defines the overall rate of convergence of the multiple-scattering series. Moreover, we show that the detailed asymptotic knowledge of these currents can be used to *accelerate* this convergence and, thus, to reduce the number of iterations necessary to attain a prescribed accuracy.

1. Introduction

The problem of simulating the behavior of wave-like processes has provided a particularly sustained, demanding and motivating challenge for the development of efficient and accurate numerical methods since the advent of computers. The classical issues present in most other applications, such as those related to the environmental and/or geometrical intricacies of the media in which quantities of interest are defined, are augmented in the context of wave propagation by the intrinsic complexities (i.e. oscillations) of the quantities themselves. In the realm of acoustic or electromagnetic waves, which concerns the present paper, very efficient methodologies have been devised, particularly in the last twenty years, to simulate their propagation in rather complicated settings. These techniques can be based, for instance, on finite elements (see e.g. [17, 16, 28] and the references therein), finite differences [25, 29] or boundary integral equations [8, 3, 10, 4], and they can, today, effectively address these problems, with a high degree of accuracy, in domains that can span tens or perhaps even a few hundred wavelengths. The very nature of these classical approaches, however, limits their applicability at higher frequencies since the numerical resolution of field oscillations translates in a commensurately higher number of degrees of freedom and this, in turn, can easily lead to impractical computational times. Recently, a new approach that bypasses these limitations was proposed in [6] to deal with single-scattering configurations, which was shown to deliver error-controllable solutions with a number of operations that is frequency independent. The method was extended in [7] to iteratively include multiple scattering effects and, again, each iteration (corresponding to a geometrical reflection in the limit of infinite frequency) was shown to be attainable to within any prescribed accuracy with a computational cost that does not increase with increasing frequency. Although every numerical experiment in [7] suggests that the iterative procedure does converge spectrally (with varying rates), this issue was not explicitly addressed and, perhaps more importantly, the dependence of the rate of convergence on the specific configuration remains unknown. Here, we take on this very question as we develop a framework for the analysis of the multiple-scattering iterations in the context of a set of smooth (convex) interacting scatterers.

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As we mentioned, our work is based on the recent developments of [6] which rely on a combination of a few ideas to efficiently resolve the integral equation formulation of the scattering problem. First, as in previous attempts (see e.g. [21, 20, 2, 13, 26]), the method resorts to high-frequency asymptotics (geometrical optics) to pre-determine the phase of the unknown field variables, which reduces the problem to one of evaluating slowly varying modulations. In contrast with prior work, however, the scheme rigorously accounts for the sharp transitions that may be present in these slow envelopes (e.g. across shadow boundaries) and it further uses novel quadrature formulas that result in frequency independent evaluations. More precisely, a second element in the procedure consists of the use of appropriately refined grids to capture boundary layers, with increasing densities in their vicinity but with a fixed, frequncy-independent overall number of nodes. The integration scheme, finally, is based on the realization that the extraction of the phase from the unknown fields has the additional implication of allowing for a complete determination of the phase of the integrands. Indeed, as shown in [6], this characteristic enables the design of a method of "localized integration" around *critical points* (i.e. singular and stationary points) which provides a mechanism for the evauation of the integrals to any desired accuracy using a number of quadrature points that remians constant for arbitrarily high wavenumbers k. Within single-scattering configurations (i.e. those wherein reflected rays do not return to the scatterer) the results in [6] show that these prescriptions lead to a most efficient scattering solver. An actual proof that provides a rigorous upper bound for the operation count (of $\mathcal{O}\left(k^{1/9}\right)$) to maintain a prescribed accuracy with increasing frequency in the case of circular/spherical boundaries was recently established in [14] for a p-version of a boundary element implementation of this approach. Also, the design and implementation of a related methodology (also based on an appropriate "ansatz" and on suitably refined discretizations) that is applicable to the treatment of convex polygons has been presented in [9]; in this case, the main theoretical result guarantees that, within the framework proposed therein, the error in best approximation of the surface current grows at most logarithmically in k.

Work on the rigorous incorporation of multiple-scattering effects, on the other hand, reduces to that in [7]. A basic observation that is exploited in this work concerns a relation between an iterative solution procedure for the integral equation and successive geometrical wave reflections in the limit of infinite frequency. More precisely, the m-th term in the Neumann series solution of a suitable re-formulation of the integral equation (cf. Sect. 2.3 below) can be shown to correspond to the contributions to the overall solution that arise from waves that have, in the infinite-frequency limit, reflected precisely m times. As demonstrated in [7], the relevance of this interpretation lies in the fact that it allows for the reduction of the full problem to the iterated solution of single-scattering problems, to which the methods described above can be readily applied. These guarantee that each successive reflection can be accounted for to any desired accuracy in times that do not increase with increasing wavenumber, but they do not elucidate the relative size of these contributions or their dependence on the geometrical configuration. Here we shed some light on precisely these issues. To this end, we begin by showing that the aforementioned series can be rearranged into a sum of *periodic orbits* (of increasing period), each corresponding to contributions arising from waves that reflect off a fixed subset of scatterers when these are transversed sequentially in a periodic manner. Thus, these orbits constitute a fundamental building block for the multiple-scattering effects, and we proceed to analyze their properties in the high-frequency regime. Our approach is based on a derivation of precise asymptotic expansions for the "currents" (i.e. the normal derivative of the total fields) that they induce on the surface of the obstacles. As we demonstrate these expansions can be used to provide accurate estimates of the rate at which their magnitude decreases with increasing number of reflections, which impacts the overall rate of convergence of the multiple-scattering series. Moreover, we show that the detailed asymptotic knowledge of these currents can be used to *accelerate* this convergence and, thus, to reduce the number of iterations necessary to attain a prescribed accuracy. Finally, our theoretical developments are complemented by a variety of numerical results that confirm the accuracy of the high-frequency expansions as well as the benefits of the proposed acceleration strategies.

The rest of the paper is organized as follows. In Sect. 2 we introduce the scattering problem and its integral equation formulations (Sect. 2.1), and we review the basic ideas behind the methods of [6, 7] to resolve these in the high-frequency regime (Sects. 2.2 and 2.3, respectively). Further, in Sect. 2.4, we derive a reformulation of the multiple scattering series in the form of a sum of contributions arising from periodic orbits which highlights the relevance of these paths. Periodic orbits are analyzed in Sects. 3 and 4, where we derive asymptotic expansions for the currents induced on the surface of the obstacles and for the rate of decrease of their amplitude as the number of reflections increases. The results of these analyses are exemplified in Sect. 5. Sample evaluations of convergence rates and comparisons with the derived asymptotic formulas are presented in Sect. 5.1. Finally, in Sect. 5.2 we introduce and illustrate a strategy to accelerate the evaluation of the contribution of periodic orbits by incorporating the knowledge garnered from the considerations in Sects. 3 and 4. These numerical results demonstrate the sharp characteristics of the analytical derivations, and the practical consequences that they may have in evaluating high-frequency scattering returns.

2. Preliminaries

In this section we collect some preliminary results that will provide the framework for the developments that follow. We begin with a statement of the scattering problem and recall its integral equation formulation. We then review some recently introduced methods for its solution at high frequencies that incorporate multiple scattering effects; finally, we show that these effects can be fully accounted for through consideration of periodic orbits.

2.1. **The scattering problem and integral equations.** We consider the problem of evaluating the scattering of an incident (acoustic / electromagnetic) plane wave $u^{\mathrm{inc}}(x) = e^{ik\alpha\cdot x}$, $|\alpha| = 1$, from a smooth impenetrable obstacle K. Throughout this paper we concentrate on two-dimensional configurations for which the relevant (frequency-domain) problem is modeled by the scalar Helmholtz equation

$$\Delta u(x) + k^2 u(x) = 0, \quad x \in \Omega = \mathbb{R}^2 \backslash \overline{K}, \tag{2.1}$$

where the scattered field u is required to satisfy the Sommerfeld radiation condition

$$\lim_{|x|\to\infty} |x|^{1/2} \left[\left(\frac{x}{|x|}, \nabla u(x) \right) - iku(x) \right] = 0. \tag{2.2}$$

For definiteness, we assume Dirichlet conditions on the boundary of the scatterer K (TE polarization in electromagnetics)

$$u(x) = -u^{\mathrm{inc}}(x) = -e^{ik\alpha\cdot x}, \quad x \in \partial K. \tag{2.3}$$

As will be clear from the derivations that follow, extensions to other boundary conditions are rather straightforward. The treatment of fully three-dimensional geometries and vector scattering models (e.g. Maxwell's equations) can be carried over with a similar strategy. For these latter cases, however, the actual derivations are significantly more involved and the results display some distinct characteristics; their detailed discussion is therefore left for future work.

The problem (2.1)–(2.3) can be recast in the form of an integral equation in a variety of ways (see e.g. [11]). For our purposes, a most convenient form is that derived from the Green identities

$$-u(x) = \int_{\partial K} \left(\frac{\partial u(y)}{\partial \nu(y)} \Phi(x,y) - u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right) ds(y) \tag{2.4}$$

and

$$0 = \int_{\partial K} \left(\frac{\partial u^{\mathrm{inc}}(y)}{\partial \nu(y)} \Phi(x,y) - u^{\mathrm{inc}}(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right) ds(y) \tag{2.5}$$

valid for all $x \in \Omega$, where $\nu(y)$ denotes the vector normal to ∂K and exterior to K, and

$$\Phi(x,y) = \frac{i}{4}\, H_0^{(1)}(k|x-y|)$$

is the *outgoing* Green function. Adding (2.4) and (2.5), and using (2.3), it follows that

$$u(x) = -\int_{\partial K} \Phi(x,y)\eta(y)\, ds(y), \quad x \in \Omega, \tag{2.6}$$

where

$$\eta(y) = \frac{\partial \left(u(y) + u^{\mathrm{inc}}(y) \right)}{\partial \nu(y)} \tag{2.7}$$

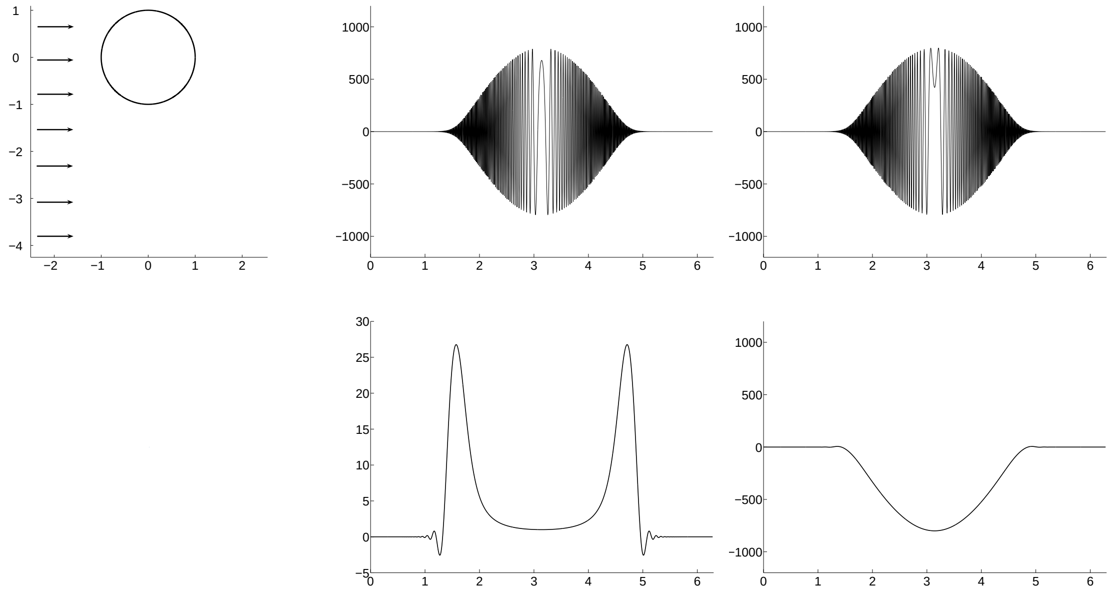


FIGURE 1. A single-scattering configuration: the incidence is a plane wave aligned with the x axis and with wavenumber $k = 400$. Top row: real and imaginary parts of the solution η to (2.8); bottom row: real and imaginary parts of η^{slow} as defined in (2.9).

represents the total induced current in the electromagnetic case. Using (2.6), (2.7) and the jump relations for the derivatives of single-layer potentials [11] we obtain the second-kind integral equation

$$\eta(x) - \int_{\partial K} \frac{\partial G(x,y)}{\partial \nu(x)}\, \eta(y)\, ds(y) = 2\, \frac{\partial u^{inc}(x)}{\partial \nu(x)}, \quad x \in \partial K \tag{2.8}$$

for the unknown current η, where we have set $G = -2\Phi$. The solution of the integral equation (2.8) is not unique when the wavenumber k is an internal resonance and thus, in practical implementations, a "combined field" integral equation (CFIE) formulation is traditionally used [11]. However, the ideas that follow clearly extend to the CFIE formulation and thus, for the sake of simplicity in presentation, we shall assume that the wave number k is not an internal resonance and work with the integral equation (2.8).

2.2. **High-frequency integral-equation method: single scattering** [6]**.** As recognized in [6, 7], the advantages of (2.8) over alternative formulations (e.g. such as those based on the "indirect approach" [11]) in the numerical simulation of high-frequency applications stem from the physical nature of the unknown density η. Indeed, in the absence of multiple scattering, physical considerations suggest that the actual current should oscillate in-sync with the incident radiation, which allows for the *pre-determination* of its phase. More precisely, in this case, the current admits a factorization

$$\eta(x) = \eta^{\text{slow}}(x)\; e^{ik\alpha\cdot x}, \tag{2.9}$$

where η^{slow} is "slowly oscillatory", that is, its variations do not accentuate with increasing frequency and, therefore, its numerical approximation demands a significantly reduced number of degrees of freedom (see Figure 1). In fact, in case K is convex, a very precise form of (2.9) has been shown to hold [23], which provides accurate descriptions for the behavior of the slow envelope in the illuminated and shadow regions

$$\partial K^{IL} = \{x \in \partial K : \alpha \cdot \nu(x) < 0\} \tag{2.10}$$

$$\partial K^{SR} = \{x \in \partial K : \alpha \cdot \nu(x) > 0\} \tag{2.11}$$

(see Figure 2), and for the transition between these through the shadow boundaries

$$\partial K^{SB} = \{x \in \partial K : \alpha \cdot \nu(x) = 0\}.$$

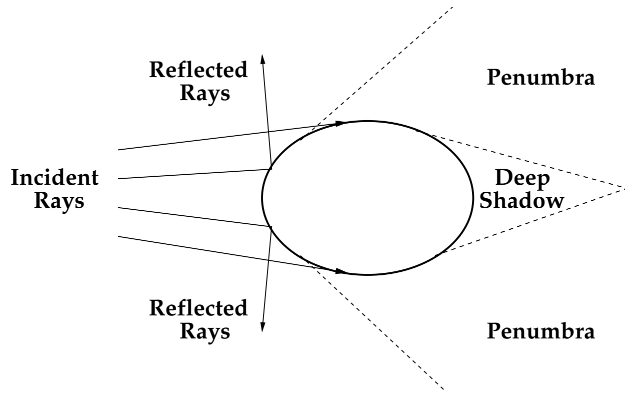


FIGURE 2. The regions outside a convex object determined by an incident field.

Theorem 2.1 ([23]). *If K is convex then, for all $P, Q \geq 0$, the current η^{slow} admits the representation*

$$\eta^{\text{slow}}(x) = \eta^{\text{slow}}(x, k, \alpha) = \sum_{p=0}^{P} \sum_{q=0}^{Q} k^{2/3-2p/3-q} b_{p,q}(\alpha, x) \Psi^{(p)}(k^{1/3} Z(\alpha, x)) + R_{P,Q}(k, \alpha, x) \tag{2.12}$$

where the complex-valued functions $b_{p,q}$ and the real-valued function Z are smooth, and Ψ is entire in the complex plane. Moreover, Z is positive in the illuminated region, negative in the shadow region, and vanishes precisely to first order at the shadow boundary (see Figure 2). The function Ψ behaves asymptotically as

$$\Psi(\tau) = \begin{cases} \sum_{p=0}^{n} a_p \tau^{1-3p} + \mathcal{O}(\tau^{1-3(n+1)}) & \text{as } \tau \to \infty, \\ c_0 e^{-i\tau^3/3 - i\tau\beta}(1 + \mathcal{O}(e^{\tau c_1})) & \text{as } \tau \to -\infty, \end{cases} \tag{2.13}$$

for some constants c_0 and $c_1 > 0$, where $\beta = e^{-2\pi i/3}\beta_1$ and β_1 is the right-most root of Ai. *And the remainder $R_{P,Q}$ satisfies*

$$|D_x^\gamma R_{P,Q}(k, \alpha, x)| \leq C_{P,Q,\gamma}\ (1+k)^{-\min\{2P/3, Q+1/3\}+1/3|\gamma|}$$

for some constants $C_{P,Q,\gamma}$.

As was shown in [6], the representations (2.9), (2.12) can be used as the basis for an efficient (spectral) numerical scheme for the solution of the scattering problem, which can deliver answers within any prescribed accuracy in frequency-independent computational times. The procedure is based on the determination of the slow envelope η^{slow} which, from (2.8) and (2.9) clearly solves

$$\eta^{\text{slow}}(x) - \int_{\partial K} \frac{\partial G(x, y)}{\partial \nu(x)} e^{ik\alpha\cdot(y-x)} \eta^{\text{slow}}(y)\, ds(y) = 2ik\, \alpha \cdot \nu(x), \quad x \in \partial K. \tag{2.14}$$

The method of [6] relies on the iterative solution of a discretized version of (2.14), which reduces the problem to 1) the determination of an appropriate finite-dimensional representation of the unknown η^{slow} and 2) the design of an effective quadrature formula for the integral in the left-hand side. The expansion (2.12) provides the theoretical grounds to resolve the first problem: the discretization is chosen to be equispaced, and frequency-independent, in the illuminated region and it is refined in a neighborhood of the shadow boundaries to capture the corresponding boundary layers. In accordance with (2.12), this neighborhood covers a region of size proportional to $k^{-1/3}$, where the constant of proportionality is chosen so as to allow for the neglect, to within a desired accuracy, of the (exponentially small) contributions arising from the remaining, deep shadow region (cf. $\tau \to -\infty$ in (2.13)). Moreover, equation (2.13) guarantees that a *fixed*, frequency-independent number of points can be placed in these transition regions to obtain uniformly accurate solutions.

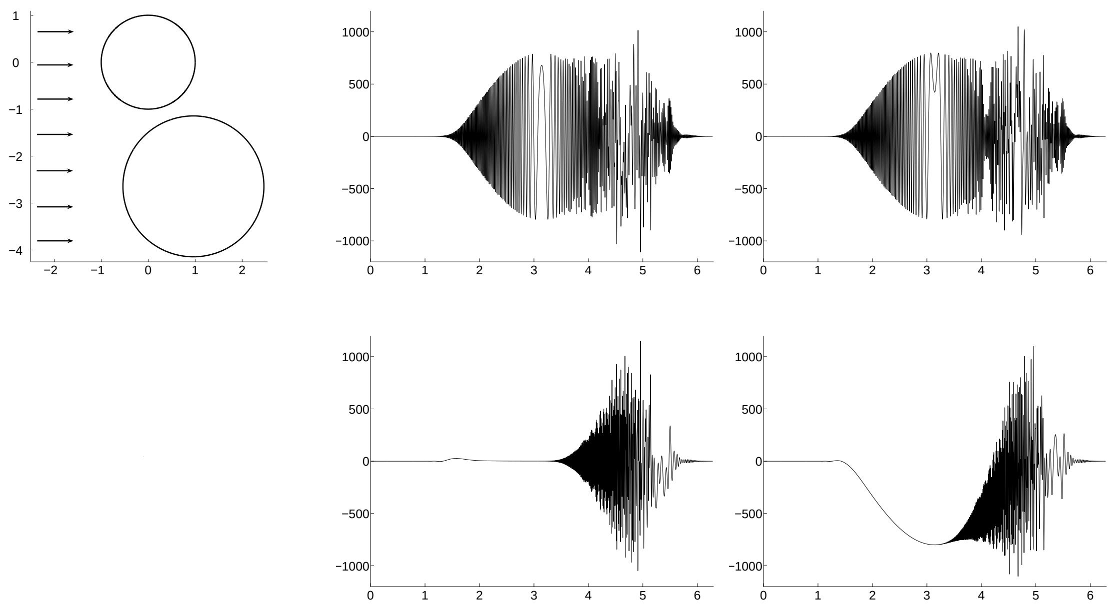


FIGURE 3. A multiple-scattering configuration: the incidence is a plane wave aligned with the x axis and with wavenumber $k = 400$. Top row: real and imaginary parts of the solution η to (2.8); bottom row: real and imaginary parts of $\eta(x)e^{-ik\alpha\cdot x}$ (cf. (2.9)).

The integration scheme, on the other hand, is based on an error controllable extension of the Method of Stationary Phase [5]. More precisely, the non-oscillatory nature of η^{slow} and the asymptotic behavior [15]

$$G(x,y) = -2\Phi(x,y) = -\frac{i}{2}\,H_0^{(1)}(k|x-y|) \sim -\frac{i}{2}\sqrt{\frac{2}{\pi k|x-y|}}e^{ik|x-y|-i\pi/4} \text{ as } k\to\infty,\ x\neq y, \tag{2.15}$$

allows for the complete determination of the *phase* of the integrand in (2.14) for each fixed value of the "target point" x. And, as shown in [6], this can be used to suitably *localize* the integral around *critical points* (i.e. singular points of the integrand, and stationary points of the phase), thereby enabling its evaluation in a fixed number of operations independently of k.

2.3. **High-frequency integral-equation method: multiple scattering** [7]**.** As is clear from the preceding discussion, a factorization of the form (2.9) is crucial in allowing for an efficient numerical solution of the integral equation (2.8) in the high-frequency regime. Evidently, in the presence of multiple-scattering the relation (2.9) is no longer valid (see Figure 3). However, as suggested in [7], this relation possesses a natural extension to this case in the form

$$\eta(x) = \eta^{\text{slow}}(x)\; e^{ik\varphi(x)}, \tag{2.16}$$

where φ corresponds to the solution of the asymptotic *geometrical optics* (GO) model, that is, to the solution of the eikonal equation.

Still, an additional problem arises in this case, as the solution $\varphi(x)$ will generally be multi-valued. On the other hand, these multiple values correspond precisely to successive wave reflections which suggests that they may be amenable to a sequential treatment. As was shown in [7], this is indeed possible if the integral equation (2.8) is suitably reformulated. To review this (with a view to our analysis of the iterations in Sects. 3 and 4), let us assume that the scatterer K is decomposed into a collection of finitely many disjoint sets $K = \bigcup_{\sigma\in\mathcal{I}} K_\sigma$. Then, the integral equation (2.8) can be written as

$$(I-R)\eta = f \tag{2.17}$$

where $\eta(x) = (\eta_{\sigma_1}(x), \ldots, \eta_{\sigma_{|\mathcal{I}|}}(x))^t$ and $f(x) = (f_{\sigma_1}(x), \ldots, f_{\sigma_{|\mathcal{I}|}}(x))^t$ with η_σ and f_σ defined on ∂K_σ and

$$f_\sigma(x) = 2ike^{ik\alpha\cdot x}\alpha \cdot \nu_\sigma(x) \qquad \sigma \in \mathcal{I} \ ,$$

and the operator R is defined as

$$(R_{\sigma\tau}\eta_\tau)(x) = \int_{\partial K_\tau} \frac{\partial G(x,y)}{\partial \nu_\sigma(x)}\, \eta_\tau(y)\, ds(y) \quad \text{for } x \in \partial K_\sigma. \tag{2.18}$$

Inverting the diagonal part of (2.17) yields the equivalent relation

$$(I - T)\eta = g \tag{2.19}$$

with

$$g_\sigma = (I - R_{\sigma\sigma})^{-1} f_\sigma, \qquad \sigma \in \mathcal{I} \tag{2.20}$$

and

$$T_{\sigma\tau} = \begin{cases} (I - R_{\sigma\sigma})^{-1} R_{\sigma\tau} & \text{if } \sigma \neq \tau \\ 0 & \text{otherwise.} \end{cases} \tag{2.21}$$

As described in [7], the formulation (2.19) provides a convenient mechanism to account for multiple scattering since the m-th term in its Neumann series solution

$$\eta = \sum_{m=0}^{\infty} \eta^m = \sum_{m=0}^{\infty} T^m g \tag{2.22}$$

corresponds exactly to contributions arising as a result of waves that (in the high-frequency regime) have undergone m geometrical reflections. More precisely, we have

$$\eta^m\big|_{\partial K_\sigma} = \sum_{\substack{\tau_0,\cdots\tau_{m-1}\in\mathcal{I} \\ \sigma\neq\tau_{m-1},\tau_j\neq\tau_{j-1}}} T_{\sigma\tau_{m-1}} T_{\tau_{m-1}\tau_{m-2}} \cdots T_{\tau_1\tau_0} g_{\tau_0}, \tag{2.23}$$

where each application of a $T_{\sigma\tau}$ entails an evaluation on ∂K_σ of a field generated by a current on ∂K_τ (cf. (2.18)), and its use as an incidence for a subsequent solution of a (single-)scattering problem on ∂K_σ (corresponding to the inversion of $I - R_{\sigma\sigma}$ in (2.21)). In particular, this interpretation guarantees that for every path $(\tau_0, \cdots, \tau_{m-1}, \tau_m)$ with $\tau_m = \sigma$ in (2.23) the geometrical phase is *uniquely* defined as

$$\varphi(x) = \varphi_m(x) = \begin{cases} \alpha \cdot x & \text{if } m = 0 \\ \alpha \cdot x_0^m(x) + \sum\limits_{j=0}^{m-1} |x_{j+1}^m(x) - x_j^m(x)| & \text{if } m \geq 1 \end{cases} \tag{2.24}$$

for $x \in \partial K_\sigma$, where the points

$$(x_0^m(x), \ldots, x_m^m(x)) \in \partial K_{\tau_0} \times \cdots \times \partial K_{\tau_m} \tag{2.25}$$

satisfy

$$\begin{cases} x_m^m(x) = x \\ \alpha \cdot \nu(x_0^m(x)) < 0 \\ (x_{j+1}^m(x) - x_j^m(x)) \cdot \nu(x_j^m(x)) > 0, \ 0 < j < m \\ \dfrac{x_1^m(x) - x_0^m(x)}{|x_1^m(x) - x_0^m(x)|} = \alpha - 2\alpha \cdot \nu(x_0^m(x))\, \nu(x_0^m(x)) \\ \dfrac{x_{j+1}^m(x) - x_j^m(x)}{|x_{j+1}^m(x) - x_j^m(x)|} = \dfrac{x_j^m(x) - x_{j-1}^m(x)}{|x_j^m(x) - x_{j-1}^m(x)|} - 2\dfrac{x_j^m(x) - x_{j-1}^m(x)}{|x_j^m(x) - x_{j-1}^m(x)|} \cdot \nu(x_j^m(x))\, \nu(x_j^m(x)), \ 0 < j < m \end{cases} \tag{2.26}$$

and $\nu(x_j^m(x)) = \nu_{\tau_j}(x_j^m(x))$. Thus, using (2.24) in (2.16), the numerical approximation of each term in (2.23) can be effected following the single-scattering prescriptions described in Sect. 2.2.

2.4. **Primitive periodic orbits and multiple scattering reformulation.** While, as we mentioned, the formulation (2.22), (2.23) of the multiple-scattering effects can be used to reduce the problem of their numerical evaluation to that of solving a sequence of single-scattering problems, it is not the one that is best suited to analyze their asymptotic properties. To this end, it is more convenient to re-arrange the sum (2.22) in a manner that makes it explicit that the multiple-scattering contributions to the induced currents can be viewed as arising from a superposition of fields corresponding to *infinite, periodic* ray paths since, as we shall see, these are amenable to an analysis that can determine their asymptotic behavior.

The precise definition of these paths, which we shall refer to as "primitive periodic orbits", is as follows:

Definition 2.2 (Primitive Periodic Orbits)**.** For $n \geq 2$, we call an infinite sequence $\{\sigma_m\}_{m\geq 0} \in \mathcal{I}^{\mathbb{N}}$ a "*primitive $n-$periodic orbit*" if

$$\begin{cases} \sigma_{n-1} \neq \sigma_0 \\ \sigma_m \neq \sigma_{m-1} \text{ for } m = 1, \ldots, n-1 \\ \nexists m < n \text{ with } l = \dfrac{n}{m} \in \mathbb{N} \text{ and } (\sigma_0, \ldots, \sigma_{m-1})^l = (\sigma_0, \ldots, \sigma_{n-1}) \\ \sigma_{m+jn} = \sigma_m \text{ for } m = 0, \ldots, n-1 \text{ and } j \geq 0; \end{cases}$$

and denote by $\mathcal{P}^n$ the collection of all primitive $n-$periodic orbits. For each $\sigma^n = \{\sigma_m^n\}_{m\geq 0} \in \mathcal{P}^n$, we define the corresponding "*primitive $n-$periodic orbit correction*"

$$\eta_{\sigma^n} = \left\{\eta_{\sigma_m^n}\right\}_{m\geq 0}$$

by

$$\eta_{\sigma_m^n} = \begin{cases} g_{\sigma_0^n} & \text{if } m = 0, \\ T_{\sigma_m^n \sigma_{m-1}^n} \eta_{\sigma_{m-1}^n} & \text{if } m > 0, \end{cases} \tag{2.27}$$

and we let

$$\overline{\eta}_{\sigma^n} = \{\overline{\eta}_{\sigma_m^n}\}_{m\geq n-1} = \{\eta_{\sigma_m^n}\}_{m\geq n-1}. \tag{2.28}$$

With this definition, the next result is now immediate.

Lemma 2.3 (Rearrangement into Primitive Periodic Orbits)**.** *If the Neumann series (2.22) converges absolutely, then*

$$\eta = g + \sum_{n=2}^{\infty} \sum_{\sigma^n \in \mathcal{P}^n} \overline{\eta}_{\sigma^n}. \tag{2.29}$$

Note that explicitly, from (2.20) and (2.21), the components of $g = (g_{\sigma_1}, \ldots, g_{\sigma_{\mathcal{I}}})^t$ in (2.29) are the solutions of the integral equations

$$g_\sigma(x) - \int_{\partial K_\sigma} \frac{\partial G(x,y)}{\partial \nu(x)}\, g_\sigma(y)\, ds(y) = 2\, \frac{\partial u^{\mathrm{inc}}(x)}{\partial \nu(x)}, \quad x \in \partial K_\sigma \tag{2.30}$$

while the functions $\eta_{\sigma_m^n}$ in (2.28) that contribute to $\overline{\eta}_{\sigma^n}$ solve

$$\eta_{\sigma_m^n}(x) - \int_{\partial K_{\sigma_m^n}} \frac{\partial G(x,y)}{\partial \nu(x)}\, \eta_{\sigma_m^n}(y)\, ds(y) = \int_{\partial K_{\sigma_{m-1}^n}} \frac{\partial G(x,y)}{\partial \nu(x)}\, \eta_{\sigma_{m-1}^n}(y)\, ds(y), \quad x \in \partial K_{\sigma_m^n}. \tag{2.31}$$

Equivalently, the equations for the slow envelopes read

$$g_\sigma^{\mathrm{slow}}(x) - \int_{\partial K_\sigma} \frac{\partial G(x,y)}{\partial \nu(x)}\, e^{ik(\varphi_0(y)-\varphi_0(x))}\, g_\sigma^{\mathrm{slow}}(y)\, ds(y) = e^{-ik\varphi_0(x)} \left(2\, \frac{\partial u^{\mathrm{inc}}(x)}{\partial \nu(x)}\right), \quad x \in \partial K_\sigma \tag{2.32}$$

and

$$\begin{aligned} \eta_{\sigma_m^n}^{\mathrm{slow}}(x) - \int_{\partial K_{\sigma_m^n}} \frac{\partial G(x,y)}{\partial \nu(x)}\, e^{ik\left(\varphi_{\sigma_m^n}(y)-\varphi_{\sigma_m^n}(x)\right)}\, \eta_{\sigma_m^n}^{\mathrm{slow}}(y)\, ds(y) \\ = e^{-ik\varphi_{\sigma_m^n}(x)} \int_{\partial K_{\sigma_{m-1}^n}} \frac{\partial G(x,y)}{\partial \nu(x)}\, e^{ik\varphi_{\sigma_{m-1}^n}(y)}\, \eta_{\sigma_{m-1}^n}^{\mathrm{slow}}(y)\, ds(y), \end{aligned} \tag{2.33}$$

for $x \in \partial K_{\sigma_m^n}$, where the phase $\varphi_{\sigma_m^n}(x)$ is defined as in (2.24) on the path $(x_0^m(x), \ldots, x_m^m(x)) \in \partial K_{\sigma_0^n} \times \ldots \times \partial K_{\sigma_m^n}$ given by (2.26).

3. Asymptotic expansions of currents on periodic orbits

In this section we derive expressions for the asymptotic behavior of the currents $\eta_{\sigma_m^n}(x)$ in (2.31) with arbitrary period n. As we shall show in the following sections these formulas can be used to derive asymptotic convergence rates as the number of reflections increases and, moreover, they can also serve as the basis for acceleration strategies that allow for the attainment of accurate solutions with a reduced number of iterations.

For the derivations that follow we shall assume that the obstacles $\{K_\sigma : \sigma \in \mathcal{I}\}$ are *convex*, that they possess regular parameterizations

$$x_\sigma(t_\sigma) = (x_\sigma^1(t_\sigma), x_\sigma^2(t_\sigma)), \quad 0 \le t \le 2\pi \tag{3.1}$$

(in counterclockwise orientation), and that they satisfy

(a) the *visibility* condition

$$\forall \sigma, \tau, \rho \in \mathcal{I} \ : \quad \overline{K_\rho} \cap \overline{co}(K_\sigma, K_\tau) \neq \varnothing \ \Rightarrow \ \rho \in \{\sigma, \tau\}$$

and

(b) the *no-occlusion* condition

$$\forall \sigma, \tau \in \mathcal{I} \ : \quad \overline{\{x + t\alpha : x \in K_\sigma, t \in \mathbb{R}\}} \cap \overline{K_\tau} \neq \varnothing \Rightarrow \ \sigma = \tau.$$

These conditions guarantee that, for any given $x \in \partial K_{\sigma_m^n}$, the ray path $(x_0^m(x), \ldots, x_m^m(x)) \in \partial K_{\sigma_0^n} \times \ldots \times \partial K_{\sigma_m^n}$ determined by the conditions (2.26) is well-defined. For brevity, we shall henceforth refer to this path as the "broken $(m+1)$-ray terminating at $x \in \partial K_{\sigma_m^n}$". Further, the calculations below on the asymptotic behavior of the induced currents are independent of the periodicity of the path σ^n and we shall therefore simply write K_m, η_m, x_m, t_m, …, for $K_{\sigma_m^n}$, $\eta_{\sigma_m^n}$, $x_{\sigma_m^n}$, $t_{\sigma_m^n}$, …, to simplify the notation.

The main result in this section is summarized in the following theorem.

Theorem 3.1. *For any $m \ge 0$, the iterated current η_m satisfies*

$$\eta_m^{\mathrm{slow}}(x) = \left(1 + \mathcal{O}\left(k^{-1}\right)\right) \times \begin{cases} 2ik\, \alpha \cdot \nu(x) & \text{if } m = 0 \\ \\ Q_m(x)\ [R_m^m(x)]^{-1/2}\ \eta_{m-1}^{\mathrm{slow}}(x_{m-1}^m(x)) & \text{if } m \ge 1 \end{cases} \tag{3.2}$$

as $k \to \infty$, on any compact subset of the illuminated region ∂K_m^{IL} (cf. (2.10)). Here

$$Q_m(x) = \frac{x - x_{m-1}^m(x)}{|x - x_{m-1}^m(x)|} \cdot \nu(x) \left(\frac{x - x_{m-1}^m(x)}{|x - x_{m-1}^m(x)|} \cdot \nu(x_{m-1}^m(x)) \right)^{-1}$$

and $R_j^m(x)$ is defined recursively as

$$R_j^m(x) = \begin{cases} b_1^m(x) & \text{if } j = 1 \\ b_j^m(x) + c_j^m(x) \left(1 - \dfrac{1}{R_{j-1}^m(x)}\right) & \text{if } 2 \le j \le m \end{cases} \tag{3.3}$$

where

$$b_j^m(x) = 1 + \frac{2\kappa(x_{j-1}^m(x))\, |x_j^m(x) - x_{j-1}^m(x)|}{\dfrac{x_j^m(x) - x_{j-1}^m(x)}{|x_j^m(x) - x_{j-1}^m(x)|} \cdot \nu(x_{j-1}^m(x))} \qquad 1 \le j \le m \tag{3.4}$$

$$c_j^m(x) = \frac{|x_j^m(x) - x_{j-1}^m(x)|}{|x_{j-1}^m(x) - x_{j-2}^m(x)|} \qquad 2 \le j \le m \tag{3.5}$$

and $\kappa(z)$ denotes the curvature at the point z.

Remark 3.2. Note that the definition of the sequence $\{R_j^m(x)\}_{1 \le j \le m}$ coincides precisely with that of $\{S_{j-1}\}_{1 \le j \le m}$ in equations (A.9)-(A.10) of Appendix A and it is therefore related to the second derivatives of the phase through the corresponding version of (A.8).

As a simple consequence of Theorem 3.1, we have the following corollary which will constitute the starting point of our derivation of a rate of convergence formula on periodic orbits in §4.

Corollary 3.3. *For any $m \geq 1$, the iterated current η_m satisfies*

$$\eta_m(x) = (1 + \mathcal{O}(k^{-1}))\, 2ik\, \eta_m^A(x) \tag{3.6}$$

on any compact subset S_m of the m-th illuminated region ∂K_m^{IL} as $k \to \infty$. Here, η_m^A is defined over the entire boundary ∂K_m *by*

$$\eta_m^A(x) = (-1)^m\, e^{ik\varphi_m(x)} \beta_m(x)\, \gamma_m(x) \tag{3.7}$$

where

$$\beta_m(x) = \prod_{j=1}^{m} \frac{1}{\sqrt{R_j^m(x)}} \quad \text{and} \quad \gamma_m(x) = \frac{x - x_{m-1}^m(x)}{|x - x_{m-1}^m(x)|} \cdot \nu(x) \tag{3.8}$$

and where $R_j^m(x)$ are as given in Theorem 3.1.

Proof. Given $x \in \partial K_m$, let $(x_0, \ldots, x_m)$ be the broken $(m+1)$-ray terminating at x. As $\eta_m(x) = \exp(ik\varphi_m(x))\eta_m^{\text{slow}}(x)$, and

$$\alpha \cdot v_0 = -\frac{x_1 - x_0}{|x_1 - x_0|} \cdot \nu_0 \quad \text{and} \quad \frac{x_{j+1} - x_j}{|x_{j+1} - x_j|} \cdot \nu_j = -\frac{x_j - x_{j-1}}{|x_j - x_{j-1}|} \cdot \nu_j$$

$(1 \leq j \leq m-1)$, repeated application of Theorem 3.1 yields

$$\eta_m(x) = (1 + \mathcal{O}(k^{-1}))\, 2ik(-1)^m e^{ik\varphi_m(x)}\, \gamma_m(x) \prod_{j=1}^{m} \frac{1}{\sqrt{R_j^j(x_j)}} .$$

Since, for $1 \leq j \leq m$, $R_j^m(x) = R_j^j(x_j)$, the result follows. □

The proof of Theorem 3.1 is based on an asymptotic analysis of the integrals in (2.32)-(2.33). The first result below determines the asymptotic value of the right-hand side of (2.33), which correspond to the (normal derivative of) the field

$$u_{m-1}^{\text{scat}}(x) \equiv \int_{\partial K_{m-1}} G(x,y)\ \eta_{m-1}^{\text{slow}}(y) e^{ik\varphi_{m-1}(y)}\, ds(y) \quad x \in \mathbb{R}^2 \backslash \overline{K}_{m-1} \tag{3.9}$$

scattered by a current generated on the $m-1$st obstacle in the path and evaluated on the mth obstacle.

Lemma 3.4 (Asymptotic Expansions of Right-hand Sides)**.** *For $m \geq 1$, the asymptotic expansion, as $k \to \infty$, of the right-hand side of (2.33) coincides with the right-hand side of (3.2) on any compact subset of $\partial K_m \backslash \partial K_m^{SB}$.*

Proof. Given $x = x_m(t_m) = x_m^m \in \partial K_m$, let $(x_0^m, \ldots, x_m^m) \in \partial K_0 \times \cdots \times \partial K_m$ be the broken $(m+1)$-ray terminating at x. We write the right-hand side integral in (2.33) as

$$\int_{\partial K_{m-1}} \frac{\partial G(x,y)}{\partial \nu(x)}\, \eta_{m-1}^{\text{slow}}(y) e^{ik\,\varphi_{m-1}(y)}\, ds(y) = \sum_{j=1}^{2} \int_{\partial K_{m-1}} \Lambda_j(y)\, \frac{\partial G(x,y)}{\partial \nu(x)}\, \eta_{m-1}^{\text{slow}}(y) e^{ik\,\varphi_{m-1}(y)}\, ds(y)$$

where $\{\Lambda_j\}_{j=1,2}$ is a smooth partition of unity defined on the curve ∂K_{m-1}, and the support of Λ_1 is chosen to be a small neighborhood of x_{m-1}^m whose size is independent of k. Then, the convexity, visibility and no-occlusion conditions combined with Lemma A.2 in Appendix A, imply that [24]

$$\int_{\partial K_{m-1}} \Lambda_2(y)\, \frac{\partial G(x,y)}{\partial \nu(x)}\, \eta_{m-1}^{\text{slow}}(y) e^{ik\,\varphi_{m-1}(y)}\, ds(y) = \mathcal{O}(k^{-\infty})$$

since the integrand is regular and its phase is never stationary on the support of Λ_2. Concerning the integral on the support of Λ_1, since $dH_0^{(1)}(z)/dz = -H_1^{(1)}(z)$ [15], we have

$$\frac{\partial G(x,y)}{\partial \nu(x)} = -\frac{i}{2} \frac{\partial H_0^{(1)}(k|x-y|)}{\partial \nu(x)} = \frac{ik}{2} H_1^{(1)}(k|x-y|) \frac{x-y}{|x-y|} \cdot \nu(x)$$

so that

$$\frac{\partial G(x,y)}{\partial \nu(x)} = \sqrt{\frac{k}{2\pi}}\,\frac{e^{i(k|x-y|-\pi/4)}}{\sqrt{|x-y|}}\,\frac{x-y}{|x-y|}\cdot\nu(x)\left(1+\mathcal{O}\left(\frac{1}{k|x-y|}\right)\right);$$

accordingly

$$\int_{\partial K_{m-1}} \Lambda_1(y)\,\frac{\partial G(x,y)}{\partial \nu(x)}\,\eta^{\mathrm{slow}}_{m-1}(y)\,e^{ik\,\varphi_{m-1}(y)}\,ds(y) = \mathcal{I} + \mathcal{O}(\mathcal{I}/k)$$

where

$$\mathcal{I} = e^{-i\pi/4}\sqrt{\frac{k}{2\pi}}\int_{\partial K_{m-1}} e^{ik\{|x-y|+\varphi_{m-1}(y)\}}\,h(y)\,\eta^{\mathrm{slow}}_{m-1}(y)\,ds(y) \tag{3.10}$$

and

$$h(y) = \frac{\Lambda_1(y)}{\sqrt{|x-y|}}\,\frac{x-y}{|x-y|}\cdot\nu(x)\,.$$

To estimate $\mathcal{I}$, first let

$$\begin{aligned}\varphi_{t_m}(t_{m-1}) &\equiv |x - x_{m-1}(t_{m-1})| + \varphi_{m-1}(x_{m-1}(t_{m-1}))\\ &= |x - x_{m-1}(t_{m-1})| + \alpha\cdot x_0^{m-1} + \sum_{j=0}^{m-2} |x_{j+1}^{m-1} - x_j^{m-1}|\end{aligned} \tag{3.11}$$

where $x^{m-1}_{m-1} = x_{m-1}(t_{m-1})$ and $(x_0^{m-1},\cdots,x^{m-1}_{m-1})$ is the broken m-ray terminating at $x_{m-1}(t_{m-1})$, denote the phase of the integrand in (3.10), so that the integral in (3.10) has the parametric representation

$$\int_0^{2\pi} e^{ik\,\varphi_{t_m}(t_{m-1})}\,h(x_{m-1}(t_{m-1}))\,|\dot{x}_{m-1}(t_{m-1})|\,dt_{m-1}. \tag{3.12}$$

Lemma A.2 states that the phase in (3.11) is stationary at t_{m-1} only when

$$x_{m-1}(t_{m-1}) = x^m_{m-1}\,. \tag{3.13}$$

Second, recall that on any compact set $S \subset \partial K^{IL}_{m-1}$, we have [23, 24]

$$\eta^{\mathrm{slow}}_{m-1}(x) \sim \sum_{j=0}^{\infty} k^{1-j} a_j(x)$$

in the sense that

$$\left| D_x^r\left(\eta^{\mathrm{slow}}_{m-1}(x) - \sum_{j=0}^{M} k^{1-j}a_j(x)\right)\right| \le C(M,r,S)\,(1+k)^{-M}$$

for all $M \ge 0$ and $r \ge 0$. Therefore

$$\begin{aligned}\left|\int_{\partial K_{m-1}} e^{ik\{|x-y|+\varphi_{m-1}(y)\}} h(y)\left[\eta^{\mathrm{slow}}_{m-1}(y) - \eta^{\mathrm{slow}}_{m-1}(x^m_{m-1})\right] ds(y)\right| &\le 2C(M,S)(1+k)^{-M}\int_{\partial K_{m-1}} |h(y)|\,ds(y)\\ &+ \sum_{j=0}^{M} k^{1-j}\left|\int_{\partial K_{m-1}} e^{ik\{|x-y|+\varphi_{m-1}(y)\}} h(y)\left[a_j(y) - a_j(x^m_{m-1})\right] ds(y)\right|\end{aligned} \tag{3.14}$$

for all $M \ge 0$. Since $\phi'_{t_m}(t_{m-1}) = 0$ only when (3.13) is satisfied and, in this case, $\phi''_{t_m}(t_{m-1}) > 0$ (cf. equation (A.8) and Theorem A.3 in Appendix A), an appeal to stationary phase lemma [18] yields

$$\int_{\partial K_{m-1}} e^{ik\{|x-y|+\varphi_{m-1}(y)\}}\,h(y)\,\left[a_j(y) - a_j(x^m_{m-1})\right] ds(y) = \mathcal{O}(k^{-3/2})$$

where we have used that $a_j(y) - a_j(x^m_{m-1})$ vanishes at $y = x^m_{m-1}$. By (3.14), we therefore get that

$$\left|\int_{\partial K_{m-1}} e^{ik\{|x-y|+\varphi_{m-1}(y)\}} h(y)\left[\eta^{\mathrm{slow}}_{m-1}(y) - \eta^{\mathrm{slow}}_{m-1}(x^m_{m-1})\right] ds(y)\right|$$

is bounded by $\mathcal{O}(k^{-M}) + \mathcal{O}(k^{-1/2})$ *for all* $M \geq 0$. This implies, in particular, that

$$\left| \mathcal{I} - e^{-i\pi/4} \sqrt{\frac{k}{2\pi}}\, \eta^{\text{slow}}_{m-1}(x^m_{m-1})\, \mathcal{J} \right| \leq \mathcal{O}(1) \tag{3.15}$$

where we have set

$$\mathcal{J} = \int_{\partial K_{m-1}} e^{ik\{|x-y|+\varphi_{m-1}(y)\}} h(y)\, ds(y)\,.$$

Applying the stationary phase lemma once more, we obtain

$$\mathcal{J} = e^{i\pi/4} \sqrt{\frac{2\pi}{k\phi''_{t_m}(t_{m-1})}} h(x^m_{m-1}) |\dot{x}^m_{m-1}| e^{ik\varphi_m(x)} + \mathcal{O}(k^{-3/2}) \tag{3.16}$$

where we have used that, under the condition (3.13),

$$\varphi_{t_m}(t_{m-1}) = |x - x^m_{m-1}| + \alpha \cdot x^m_0 + \sum_{j=0}^{m-2} |x^m_{j+1} - x^m_j| = \varphi_m(x).$$

It therefore follows from (3.15) and (3.16) that

$$\mathcal{I} = \left(\frac{h(x^m_{m-1})|\dot{x}^m_{m-1}|}{\sqrt{\phi''_{t_m}(t_{m-1})}} e^{ik\varphi_m(x)} + \mathcal{O}(k^{-1}) \right) \eta^{\text{slow}}_{m-1}(x^m_{m-1}) + \mathcal{O}(1) = \frac{h(x^m_{m-1})|\dot{x}^m_{m-1}|}{\sqrt{\phi''_{t_m}(t_{m-1})}} e^{ik\varphi_m(x)} \eta^{\text{slow}}_{m-1}(x^m_{m-1}) + \mathcal{O}(1)\,.$$

Accordingly the right-hand side of (2.33) satisfies

$$\begin{aligned}
e^{-ik\,\varphi_m(x)} \int_{\partial K_{m-1}} \frac{\partial G(x,y)}{\partial \nu(x)}\, \eta^{\text{slow}}_{m-1}(y) e^{ik\varphi_{m-1}(y)}\, ds(y) &= \frac{h(x^m_{m-1})|\dot{x}^m_{m-1}|}{\sqrt{\varphi''_{t_m}(t_{m-1})}}\, \eta^{\text{slow}}_{m-1}(x^m_{m-1}) + \mathcal{O}(1) \\
&= \frac{|\dot{x}^m_{m-1}|}{\sqrt{|x - x^m_{m-1}| \varphi''_{t_m}(t_{m-1})}}\, \frac{x - x^m_{m-1}}{|x - x^m_{m-1}|} \cdot \nu(x)\, \eta_{m-1}(x^m_{m-1}) + \mathcal{O}(1) \\
&= \frac{|\dot{x}^m_{m-1}|\, Q_m(x)}{\sqrt{|x - x^m_{m-1}| \varphi''_{t_m}(t_{m-1})}}\, \frac{x - x^m_{m-1}}{|x - x^m_{m-1}|} \cdot \nu(x^m_{m-1})\, \eta_{m-1}(x^m_{m-1}) + \mathcal{O}(1) \\
&= [R^m_m(x)]^{-1/2}\, Q_m(x)\, \eta_{m-1}(x^m_{m-1}) + \mathcal{O}(1)
\end{aligned}$$

where the last equality follows from Remark 3.2 and the definition (A.8). □

To complete the proof of Theorem 3.1 we need only show that, for a target point in the m-th illuminated region, the integrals on the left-hand sides of (2.32)–(2.33) are negligible. To this end, we will make use of the following identities.

Lemma 3.5. *Let* $m \geq 0$ *and* $x_m = x_m(t_m) \in \partial K_m$ *be fixed. Then, letting*

$$f(\tau_m) = \varphi_m(x_m(\tau_m)) - \varphi_m(x_m(t_m)) \qquad \text{and} \qquad g(\tau_m) = |x_m(\tau_m) - x_m(t_m)|$$

we have

$$\frac{df(\tau_m)}{d\tau_m} = \begin{cases} \alpha \cdot \dot{x}_m(\tau_m) & \text{if } m = 0 \\[2ex] \dfrac{x_m(\tau_m) - x_{m-1}(\tau_m)}{|x_m(\tau_m) - x_{m-1}(\tau_m)|} \cdot \dot{x}_m(\tau_m) & \text{if } m \geq 1 \end{cases} \tag{3.17}$$

and

$$\frac{dg(\tau_m)}{d\tau_m} = \begin{cases} \dfrac{x_m(\tau_m) - x_m(t_m)}{|x_m(\tau_m) - x_m(t_m)|} \cdot \dot{x}_m(\tau_m) & \text{if } \tau_m \neq t_m, \\[2ex] \dfrac{\dot{x}_m(t_m)}{|\dot{x}_m(t_m)|} \cdot \dot{x}_m(t_m) & \text{if } \tau_m = t_m\ . \end{cases} \tag{3.18}$$

where, for $m \geq 1$ and $x_m(\tau_m) \in \partial K_m$, we denote by $(x_0, \ldots, x_m) \in \partial K_0 \times \cdots \times \partial K_m$ the broken $(m+1)$-ray terminating at $x_m(\tau_m)$.

Proof. A direct calculation yields

$$\frac{df(\tau_m)}{d\tau_m} = \left(\alpha - \frac{x_1 - x_0}{|x_1 - x_0|}\right) \cdot \dot{x}_0 \frac{d\tau_0}{d\tau_m} + \sum_{i=1}^{m-1} \left(\frac{x_i - x_{i-1}}{|x_i - x_{i-1}|} - \frac{x_{i+1} - x_i}{|x_{i+1} - x_i|}\right) \cdot \dot{x}_i \frac{d\tau_i}{d\tau_m} + \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \dot{x}_m$$

which reduces to (3.17) since the tuple $(x_0, \ldots, x_m)$ is the broken $(m+1)$-ray terminating at $x_m(\tau_m)$. The proof of (3.18) is straighforward. □

With the identities (3.17) and (3.18), we are now in a position to prove the following lemma.

Lemma 3.6 (Asymptotic Expansions of Left-hand Sides)**.** *For any $m \geq 0$, the left-hand side integral in (2.32)–(2.33) satisfies*

$$\int_{\partial K_m} \frac{\partial G(x,y)}{\partial \nu(x)} e^{ik\{\varphi_m(y) - \varphi_m(x)\}} \eta_m^{\mathrm{slow}}(y) ds(y) = \mathcal{O}(k^{-1}) \eta_m^{\mathrm{slow}}(x) + \mathcal{O}(k^{-1})$$

on any compact subset of ∂K_m^{IL} as $k \to \infty$.

Proof. Let S be a compact subset of ∂K_m^{IL}, and $x \in S$. Denoting the parametrization of the curve ∂K_m by $y(\tau) = (y^1(\tau), y^2(\tau))$ and assuming that $x = y(t)$, we have

$$\int_{\partial K_m} \frac{\partial G(x,y)}{\partial \nu(x)} e^{ik\{\varphi_m(y) - \varphi_m(x)\}} \eta_m^{\mathrm{slow}}(y) ds(y) = \int_0^{2\pi} F(\tau) \eta_m^{\mathrm{slow}}(y(\tau)) d\tau$$

where we have set

$$F(\tau) = \frac{ik}{2} e^{ik(\varphi_m(y(\tau)) - \varphi_m(y(t)) + |y(\tau) - y(t)|)} \times H(k|y(\tau) - y(t)|) \frac{y(\tau) - y(t)}{|y(\tau) - y(t)|} \cdot \nu(y(t)) \, |\dot{y}(\tau)| \tag{3.19}$$

with $H(z) = e^{-iz} H_1^{(1)}(z)$. Parallel with the treatment of the right-hand side integrals, we write

$$\int_0^{2\pi} F(\tau) \, \eta_m^{\mathrm{slow}}(y(\tau)) \, d\tau = \sum_{j=1}^{2} \int_0^{2\pi} \Lambda_j(\tau) \, F(\tau) \, \eta_m^{\mathrm{slow}}(y(\tau)) \, d\tau \tag{3.20}$$

where Λ_1 and Λ_2 form a smooth partition of unity of the curve ∂K_m but, in this case, the support of Λ_1 is a small neighborhood of x whose size will be chosen below, independently of k. As with the right-hand-side integrals [24], the integral involving Λ_2 is of order $\mathcal{O}(k^{-\infty})$ so that

$$\int_0^{2\pi} F(\tau) \eta_m^{\mathrm{slow}}(y(\tau)) d\tau = \int_0^{2\pi} \Lambda_1(\tau) F(\tau) \eta_m^{\mathrm{slow}}(y(\tau)) d\tau + \mathcal{O}(k^{-\infty}) \tag{3.21}$$

Since the estimation of the integral on the right-hand side of (3.21) is the same on the intervals $[t, t+\pi]$ and $[t-\pi, t]$, we shall only consider the one on $[t, t+\pi]$. To this end, we introduce the change of variables

$$u = h(\tau) := \varphi_m(y(\tau)) - \varphi_m(y(t)) + |y(\tau) - y(t)| = f(\tau) + g(\tau)$$

where the functions f and g are as given in Lemma 3.5. From this lemma it follows that we can choose ε so that the functions f, g and h are invertible in $[t, t+\varepsilon]$ and we let the support of Λ_1 be contained in $[t-\varepsilon, t+\varepsilon]$. Note that, as we anticipated, ε is independent of k and, moreover, it can be chosen to depend only on S. Therefore, setting $\epsilon = h^{-1}(t+\varepsilon)$, we have

$$\int_t^{t+\pi} \Lambda_1(\tau) F(\tau) \eta_m^{\mathrm{slow}}(y(\tau)) d\tau = \frac{ik}{2} \int_0^{\epsilon} e^{iku} F_1(k,u) \rho(k,u) du \tag{3.22}$$

where

$$F_1(k,u) = \Lambda_1(h^{-1}(u)) H(kg(h^{-1}(u))) |\dot{y}(h^{-1}(u))| [h^{-1}(u)]' \, \frac{y(h^{-1}(u)) - y(t)}{|y(h^{-1}(u)) - y(t)|} \cdot \nu(y(t)) \tag{3.23}$$

and

$$\rho(k,u) = \eta_m^{\mathrm{slow}}(y(h^{-1}(u))) \,. \tag{3.24}$$

To estimate the integral in the right-hand-side of (3.22) we first recall that on any compact set $S \subset \partial K_m^{IL}$, we have [23, 24]

$$\eta_m^{\text{slow}}(x) \sim \sum_{j=0}^{\infty} k^{1-j} a_j(x) \tag{3.25}$$

in the sense that

$$\left| D_x^r \left(\eta_m^{\text{slow}}(x) - \sum_{j=0}^{M} k^{1-j} a_j(x) \right) \right| \leq C(M,r,S)\,(1+k)^{-M} \tag{3.26}$$

for all $M \geq 0$ and $r \geq 0$. Then, setting $b_j(u) = a_j(y(h^{-1}(u)))$, we obtain

$$\begin{aligned} \left| \int_0^{\epsilon} e^{iku} F_1(k,u)\,[\rho(k,u) - \rho(k,0)]\,du \right| \leq C(M,S)\,(1+k)^{-M} \int_0^{\epsilon} |F_1(k,u)|\,du \\ + \sum_{j=0}^{M} k^{1-j} \left| \int_0^{\epsilon} e^{iku}\,[b_j(u) - b_j(0)]\,F_1(k,u)du \right| \end{aligned}$$

for all $M \geq 0$. Since $\rho(k,0) = \eta_m^{\text{slow}}(x)$ the result will follow provided that

$$\int_0^{\epsilon} e^{iku} F_1(k,u)du = \mathcal{O}(k^{-2}) \tag{3.27}$$

and

$$\int_0^{\epsilon} e^{iku}\,[b_j(u) - b_j(0)]\,F_1(k,u)du = \mathcal{O}(k^{-3})\,. \tag{3.28}$$

These equalities, on the other hand, follow from the first order vanishing of $F_1(k,u)$ at $u = 0$. Indeed, if $b(u)$ vanishes at 0, an integration by parts yields

$$k^3 \int_0^{\epsilon} e^{iku} b(u) F_1(k,u)du = ik^2 \int_0^{\epsilon} e^{iku}\,[b(u)F_1(k,u)]'\,du \tag{3.29}$$

and the expression on the right-hand side of (3.29) can be treated similar to (3.27) by expanding b into a Taylor series. We shall therefore verify only (3.27). To this end, expanding F_1 in a Taylor series around 0, we see that the left-hand side of (3.27) has the same asymptotic order in k (as $k \to \infty$) with that of

$$k \int_0^{\epsilon} e^{iku}\, H\!\left(\frac{ku}{\beta}\right) \frac{ku}{\beta}\, \Lambda_1\!\left(t + \frac{u}{\beta|\dot{x}|}\right) du$$

for some non-zero constant β. To complete the proof, it therefore suffices to show that, as $k \to \infty$, we have

$$k \int_0^{\epsilon_1} e^{ik\beta v}\, H(kv)\, kv\, \Lambda(v)\, dv = \mathcal{O}(1)$$

where $\epsilon_1 = \epsilon/\beta$ and $\Lambda(v) = \Lambda_1(t + v/|\dot{x}|)$. Equivalently, with $\beta_1 = \beta - 1$, it suffices to show that

$$k \int_0^{a+b} e^{i\beta_1 kv}\, H_1^{(1)}(kv)\, kv\, \Gamma_{a,b}(v)\, dv = \mathcal{O}(1) \tag{3.30}$$

where $\Gamma_{a,b} = \Lambda$, $a + b = \epsilon_1$, $\Gamma_{a,b} \equiv 1$ on $[-a, a]$ and reduces smoothly to zero on $[-\epsilon_1, \epsilon_1]\backslash[-a, a]$.

Now, let $\Gamma_{\epsilon_2,\epsilon_2}$ be a smooth cut-off function that is identically 1 on $[-\epsilon_2, \epsilon_2]$ and that reduces smoothly to zero on $[-2\epsilon_2, 2\epsilon_2] - [-\epsilon_2, \epsilon_2]$; here we have chosen ϵ_2 so that $\epsilon_2 \gg k^{-1}$ but $\epsilon_2 k = \mathcal{O}(1)$ is independent of k. Therefore

$$\left| k\int_0^{2\epsilon_2} e^{i\beta_1 kv} H_1^{(1)}(kv) kv \Gamma_{\epsilon_2,\epsilon_2}(v) dv \right| = \left| \int_0^{2\epsilon_2 k} e^{i\beta_1 v} \Gamma_{\epsilon_2,\epsilon_2}(v/k) H_1^{(1)}(v) v dv \right| \leq \int_0^{2\epsilon_2 k} \left| H_1^{(1)}(v) \right| v\, dv = \mathcal{O}(1)$$

since $\epsilon_2 k = \mathcal{O}(1)$. On the other hand, on account of the asymptotic expansion (2.15) and our choice that $\epsilon_2 k \gg 1$, the integral

$$k \int_{\epsilon_2}^{a+b} e^{i\beta_1 kv} \, H_1^{(1)}(kv) \, kv \, (\Gamma_{a,b}(v) - \Gamma_{\epsilon_2,\epsilon_2}(v)) \, dv$$

have the same order in k (as $k \to \infty$) with that of

$$k^{3/2} \int_{\epsilon_2}^{a+b} e^{i\beta kv} \, v^{1/2} \, (\Gamma_{a,b}(v) - \Gamma_{\epsilon_2,\epsilon_2}(v)) \, dv \, ; \tag{3.31}$$

and integrating by parts, we see that (3.31) have the same order in k as

$$\begin{aligned} k^{1/2} \int_{\epsilon_2}^{a+b} & e^{i\beta kv} \left[(\Gamma_{a,b}(v) - \Gamma_{\epsilon_2,\epsilon_2}(v)) \, v^{1/2} \right]' dv \\ &= k^{1/2} \left\{ \int_{a}^{a+b} e^{i\beta kv} \, v^{1/2} \, \Gamma'_{a,b}(v) \, dv - \int_{\epsilon_2}^{2\epsilon_2} e^{i\beta kv} \, v^{1/2} \, \Gamma'_{\epsilon_2,\epsilon_2}(v) \, dv + \frac{1}{2} \int_{2\epsilon_2}^{a} e^{i\beta kv} \, v^{-1/2} dv \right. \\ & \qquad \left. + \frac{1}{2} \int_{\epsilon_2}^{2\epsilon_2} e^{i\beta kv} \, v^{-1/2} \, \left(1 - \Gamma_{\epsilon_2,\epsilon_2}(v)\right) dv + \frac{1}{2} \int_{a}^{a+b} e^{i\beta kv} \, \Gamma_{a,b}(v) \, v^{-1/2} dv \right\} . \end{aligned} \tag{3.32}$$

Now, by repeated integration by parts, we see that the first integral on the right-hand side of (3.32) is of order $\mathcal{O}(k^{-n})$ for all $n \geq 1$; using the relation $\epsilon_2 k = \mathcal{O}(1)$, the second and fourth integrals are easily shown to be of order $\mathcal{O}(k^{-1/2})$; and an integration by parts shows that the last integral is of order $\mathcal{O}(k^{-1/2})$. Finally, concerning the second integral on the right-hand side of (3.32), we recall that [15]

$$\int_0^1 e^{i\gamma} w^{-1/2} dw = 2 \, {}_1F_1(1/2, 3/2, i\gamma) \, , \qquad \gamma > 0$$

where ${}_1F_1$ is a hypergeometric function. Therefore, a straightforward computation yields

$$\int_{2\epsilon_2}^{a} e^{i\beta kv} v^{-1/2} dv = 2a^{1/2} {}_1F_1(1/2, 3/2, i\beta ak) - 2(2\epsilon_2)^{1/2} {}_1F_1(1/2, 3/2, 2i\beta\epsilon_2 k) \, . \tag{3.33}$$

Since a is independent of k, the asymptotic expansion [15]

$${}_1F_1(1/2, 3/2, i\beta ak) = \mathcal{O}(1/\sqrt{ak})$$

implies that the first term on the right-hand side of (3.33) is of $\mathcal{O}(k^{-1/2})$. On the other hand, since $\epsilon_2 k = \mathcal{O}(1)$, the second term on the right-hand side of (3.33) is of order $\epsilon_2^{1/2} = \mathcal{O}(k^{-1/2})$. This completes the proof. □

4. Rate of convergence on periodic orbits

In this section we analyze the asymptotic expansions in Corollary 3.3 to derive high-frequency rate-of-convergence formulas for periodic orbits. Throughout this section, we shall assume that $\{\sigma_m\}_{m\geq 0} \in \mathcal{I}^\infty$ is a *fixed* $n-$periodic multiple-scattering sequence (i.e. $\sigma_{m+1} \neq \sigma_m$ for all $m \geq 0$, and $\sigma_{r+qn} = \sigma_r$ for $0 \leq r \leq n-1$ and $q \geq 0$); as before, we will write K_m, η_m, ... instead of K_{σ_m}, η_{σ_m}, etc.

As is apparent from Corollary 3.3, the analysis of the currents η_m on an $n-$periodic orbit requires the analysis of the ratios η^A_{m+n}/η^A_m and of the *jointly* illuminated regions $\partial K^{IL}_{m+n} \cap \partial K^{IL}_m$. Our main results in this direction are summarized in the next two theorems. The first result identifies a rate-of-convergence $\mathcal{R}_{n,k}$ for the "approximate currents" η^A_m defined in (3.7).

Theorem 4.1. *There exist uniquely determined constants*

$$\Phi_n \in \mathbb{R}_+ \qquad \text{and} \qquad \{\mathcal{L}_r : r = 0, \ldots, n-1\} \subset (1, \infty) \tag{4.1}$$

with the property that, for any $m > 2n$ and $x \in \partial K_m$,

$$\left| \eta^A_{m+n}(x) - \mathcal{R}_{n,k} \eta^A_m(x) \right| \leq \left| \frac{\eta^A_m(x)}{\gamma_m(x)} \right| \mathcal{F} \leq \delta^{m/2} \mathcal{F}$$

where

$$\mathcal{R}_{n,k} = (-1)^n e^{ik\Phi_n} \prod_{r=0}^{n-1} \frac{1}{\sqrt{\mathcal{L}_r}},$$

$$\mathcal{F} = \mathcal{F}(C,k,\delta,m,n) = \min\left\{2, e^{Ck\delta^{m/2}} - 1\right\} \delta^{n/2} + C\delta^{(m-n)/2}$$

and the constants $C = C(K,\alpha)$ and $\delta = \delta(K,\alpha) \in (0,1)$ are independent of the given periodic orbit.

The next result, in turn, guarantees the existence of jointly illuminated regions for large values of the index m in a periodic orbit $\{\sigma_m\}_{m\geq 0}$.

Theorem 4.2. *For $0 \leq r \leq n-1$, there exist compact connected subsets S_r and T_r of ∂K_r with the property that*

$$\exists m_0 \geq 1 : \forall m \geq m_0 \quad [m \equiv r \mod n] \Rightarrow J_{m+1}(\partial K_{m+1}) \subset T_r \subset \operatorname{int}(S_r) \subset S_r \subset \partial K_m^{IL}$$

where, for $m \geq 1$, $J_m : \partial K_m \to \partial K_{m-1} : x \mapsto x_{m-1}^m(x)$.

We have separated the proof of Theorem 4.1 into four parts. First, we begin in §4.1 by recalling some classical estimates from the "theory of dispersing billiard flows" [19, 27] that we use throughout the rest of this section. In §4.2, we characterize the constant Φ_n appearing in Theorem 4.1, and explain its relationship with the phase differences $\varphi_{m+n} - \varphi_m$. The connection between the ratios β_{m+n}/β_m (cf. (3.8)) and the constants $\mathcal{L}_r$ as well as an algorithm for the efficient determination of these latter quantities are discussed in §4.3. Finally, in §4.4, we study the ratios γ_{m+n}/γ_m and complete the proof of Theorem 4.1.

The proof of Theorem 4.2, on the other hand, is presented in §4.5, where we further characterize the limiting behavior of the jointly illuminated regions $\partial K_{m+n}^{IL} \cap \partial K_m^{IL}$. Finally, in §4.6, we combine Theorems 4.1 and 4.2 to obtain an asymptotic rate of convergence formula for the actual currents η_m (cf. (3.6)) that are valid throughout the *entire* boundary ∂K_m.

4.1. **Properties of broken rays.** In this section, we recall two classical results from the theory of dispersing billiard flows [19, 27]. The first one depends only on the convexity and the visibility conditions and is given in the next lemma.

Lemma 4.3. *There exist constants $C_1 = C_1(K)$ and $\delta_1 = \delta_1(K) < 1$ with the property that, given any sequence $\{K_{\sigma_j}\}_{j=0,\dots,m}$ of obstacles with $\sigma_{j-1} \neq \sigma_j$ $(j = 1,\dots,m)$, and any two sequences $\{\xi_j\}_{j=0,\dots,m}$ and $\{\zeta_j\}_{j=0,\dots,m}$ in $\partial K_{\sigma_0} \times \dots \times \partial K_{\sigma_m}$ satisfying the conditions*

(a) *the segments $[\xi_{j-1},\xi_j]$ and $[\xi_j,\xi_{j+1}]$ (resp. $[\zeta_{j-1},\zeta_j]$ and $[\zeta_j,\zeta_{j+1}]$) satisfy the law of reflection at ξ_j (resp. ζ_j) $(j = 1,\dots,m-1)$, and*
(b) *neither of the segments $[\xi_{j-1},\xi_j]$ nor $[\zeta_{j-1},\zeta_j]$ have a point in common with the interior of K $(j = 2,\dots,m-1)$*

we have

$$|\xi_j - \zeta_j| \leq C_1(\delta_1^j + \delta_1^{m-j}) \qquad (0 \leq j \leq m).$$

In addition, we have

$$\xi_0 = \zeta_0 \quad \Rightarrow \quad |\xi_j - \zeta_j| \leq C_1\delta_1^{m-j} \qquad (0 \leq j \leq m),$$

and

$$\xi_m = \zeta_m \quad \Rightarrow \quad |\xi_j - \zeta_j| \leq C_1\delta_1^{j} \qquad (0 \leq j \leq m).$$

The second result, given in the next lemma, makes use of the no-occlusion condition in addition to convexity and visibility.

Lemma 4.4. *If $\alpha \in S^1 = \{\alpha \in \mathbb{R}^2 : |\alpha| = 1\}$ is such that the no-occlusion condition is satisfied, then there exist constants $C_2 = C_2(K,\alpha)$ and $\delta_2 = \delta_2(K,\alpha) < 1$ with the property that, for any two sequences $\{\xi_j\}_{j=0,\dots,m}$ and $\{\zeta_j\}_{j=0,\dots,m}$ satisfying the conditions in Lemma 4.3, the additional condition that these sequences correspond to broken rays with initial direction α implies*

$$|\xi_j - \zeta_j| \leq C_2\delta_2^{m-j} \qquad (0 \leq j \leq m)\ .$$

4.2. **Analysis of phase the differences $\varphi_{m+n} - \varphi_m$ on $n-$periodic orbits.** To characterize the phase constant Φ_n appearing in Theorem 4.1, we consider the *"n-periodic distance function"*

$$\Phi_n(x_0, \dots, x_{n-1}) = |x_{n-1} - x_0| + \sum_{r=0}^{n-2} |x_{r+1} - x_r| \tag{4.2}$$

defined on $\partial K_0 \times \ldots \times \partial K_{n-1}$. As the next lemma shows, the minimum of Φ_n has a simple geometric characterization. Its proof is immediate from the convexity and visibility conditions.

Lemma 4.5. *Φ_n attains its minimum at a* uniquely *determined point $(a_0, \dots, a_{n-1}) \in \partial K_0 \times \ldots \times \partial K_{n-1}$. Moreover, with the extended definition*

$$a_{r+qn} := a_r \qquad \textit{for} \qquad [0 \le r \le n-1 \quad \textit{and} \quad q \in \mathbb{Z}]\,,$$

the points $\{a_j\}_{j \in \mathbb{Z}}$ satisy

$$\frac{a_{j+1} - a_j}{|a_{j+1} - a_j|} = \frac{a_j - a_{j-1}}{|a_j - a_{j-1}|} - 2\left(\frac{a_j - a_{j-1}}{|a_j - a_{j-1}|} \cdot \nu(a_j)\right)\nu(a_j)\ .$$

That is, a ray starting from a_j and arriving at a_{j+1} transverses the path formed by the points $\{a_j\}_{j \in \mathbb{Z}}$ indefinitely.

The next result provides a relationship between $\Phi_n(a_0, \dots, a_{n-1})$ and the phase differences $\varphi_{m+n} - \varphi_m$.

Lemma 4.6. *For any $m > 2n$ and any $x \in \partial K_m$, we have*

$$|\varphi_{m+n}(x) - \varphi_m(x) - \Phi_n(a_0, \dots, a_{n-1})| \le C\,\delta^{m/2} \tag{4.3}$$

where the constants $C = C(K, \alpha)$ and $\delta = \delta(K, \alpha) \in (0,1)$ are independent of the given periodic orbit.

Proof. Repeated application of triangle inequality yields

$$\begin{aligned}
&|\varphi_{m+n}(x) - \varphi_m(x) - \Phi_n(a_0, \dots, a_{n-1})| \\
&\quad \le |\alpha|\,|x_0^{m+n}(x) - x_0^m(x)| + \sum_{j=0}^{p-1}\left[|x_{j+1}^{m+n}(x) - x_{j+1}^m(x)| + |x_j^{m+n}(x) - x_j^m(x)|\right] \\
&\quad + \sum_{j=p}^{m-1}\left[|x_{j+n+1}^{m+n}(x) - x_{j+1}^m(x)| + |x_{j+n}^{m+n}(x) - x_j^m(x)|\right] + \sum_{r=0}^{n-1}\left[|x_{p+r+1}^{m+n}(x) - a_{p+r+1}| + |x_{p+r}^{m+n}(x) - a_{p+r}|\right]
\end{aligned}$$

where we have set $p = [m/2]$. Letting C and δ be the maxima of the corresponding constants from Lemmas 4.3 and 4.4, we obtain

$$\begin{aligned}
&C^{-1}|\varphi_{m+n}(x) - \varphi_m(x) - \Phi_n(a_0, \dots, a_{n-1})| \\
&\quad \le \delta^m + \sum_{j=0}^{p-1}\left[\delta^{m-(j+1)} + \delta^{m-j}\right] + \sum_{j=p}^{m-1}\left[\delta^{j+1} + \delta^j\right] + \sum_{r=0}^{n-1}\left[\delta^{p+r+1} + \delta^{(m+n)-(p+r+1)} + \delta^{p+r} + \delta^{(m+n)-(p+r)}\right]
\end{aligned}$$

so that

$$\begin{aligned}
&C^{-1}|\varphi_{m+n}(x) - \varphi_m(x) - \Phi_n(a_0, \dots, a_{n-1})| \\
&\qquad \le \delta^m + \frac{1+\delta}{1-\delta}\left[(1-\delta^p)\delta^{m-p} + (1-\delta^{m-p})\delta^p + (1-\delta^n)\left(\delta^p + \delta^{m-p}\right)\right] \le 2\frac{1+\delta}{1-\delta}\left(\delta^p + \delta^{m-p}\right)
\end{aligned}$$

completing the proof. ◻

4.3. **Analysis of the ratios $\beta_{m+n}(x)/\beta_m(x)$ on n-periodic orbits.** In this section we analyze the ratios of the quantities $\beta_m(x)$ defined in (3.8) as the product of the inverse of the square roots of the "continued fractions" $R_j^m(x)$ in (3.3).

Theorem 4.7. *There exist uniquely determined constants*

$$\{\mathcal{L}_r : r = 0, \dots, n-1\} \subset (1, \infty) \tag{4.4}$$

with the property that, for any $m > 2n$ and $x \in \partial K_m$, we have

$$\left| \frac{\beta_{m+n}(x)}{\beta_m(x)} - \prod_{r=0}^{n-1} \frac{1}{\sqrt{\mathcal{L}_r}} \right| \le C\delta^{(m-n)/2}$$

where the constants $C = C(K, \alpha)$ and $\delta = \delta(K, \alpha) \in (0, 1)$ are independent of the given periodic orbit.

We need two technical lemmas for the proof of Theorem 4.7. First we introduce a sequence $\{L_j\}$ defined by the geometry of the periodic orbit $\partial K_0 \times \ldots \times \partial K_{n-1}$ at the points $(a_0, \dots, a_{n-1})$.

Definition 4.8. The sequence $\{L_j\}_{j\ge1}$ is defined by

$$L_j = \begin{cases} b_1 & \text{if } j = 1 \\ b_j + c_j \left(1 - \dfrac{1}{L_{j-1}}\right) & \text{if } j \ge 2 \end{cases} \tag{4.5}$$

where

$$b_j = 1 + \frac{2\kappa(a_{j-1})\,|a_j - a_{j-1}|}{\dfrac{a_j - a_{j-1}}{|a_j - a_{j-1}|} \cdot \nu(a_{j-1})} \qquad \text{for} \qquad j \in \mathbb{Z} \tag{4.6}$$

$$c_j = \frac{|a_j - a_{j-1}|}{|a_{j-1} - a_{j-2}|} \qquad \text{for} \qquad j \in \mathbb{Z}\,. \tag{4.7}$$

The connection between the ratios $\beta_{m+n}(x)/\beta_m(x)$, and the sequence $\{L_j\}$ is given in the next lemma.

Lemma 4.9. *For any $m > 2n$ and $x \in \partial K_m$, we have*

$$\left| \frac{\beta_{m+n}(x)}{\beta_m(x)} - \prod_{j=p-n}^{p-1} \frac{1}{\sqrt{L_j}} \right| \le C\delta^{(m-n)/2} \tag{4.8}$$

where $p = [m/2]$ and the constants $C = C(K, \alpha)$ and $\delta = \delta(K, \alpha) \in (0, 1)$ are independent of the given periodic orbit.

Remark 4.10. As we show in Lemma B.10 in Appendix B, each one of the first two products on the right-hand side of the equation

$$\frac{\beta_{m+n}(x)}{\beta_m(x)} = \left(\prod_{j=1}^{p-1} \frac{R_j^m(x)}{R_j^{m+n}(x)} \prod_{j=p}^{m} \frac{R_j^m(x)}{R_{j+n}^{m+n}(x)} \prod_{j=p}^{p+n-1} \frac{1}{R_j^{m+n}(x)} \right)^{1/2}$$

can be approximated by 1, and the last one behaves like $\prod_{j=p-n}^{p-1} 1/L_j$.

Proof of Lemma 4.9 Since for any $A, B, C \in \mathbb{R}$ the identity

$$ABC - 1 = [(A-1)(B-1) + (A-1) + (B-1) + 1]\,(C-1) + [(A-1)(B-1) + (A-1) + (B-1)]$$

holds, an appeal to triangle inequality yields

$$|ABC - 1| \le [|A-1||B-1| + |A-1| + |B-1| + 1]\,|C-1| + [|A-1||B-1| + |A-1| + |B-1|]\ .$$

Therefore, a simple manipulation using Lemma B.10 shows that, with the choice of the constant $C = 2(C_6+1)^2 \max\{C_6, 1\}$, we have

$$\left| \prod_{j=1}^{p-1} \frac{R_j^m(x)}{R_j^{m+n}(x)} \prod_{j=p}^{m} \frac{R_j^m(x)}{R_{j+n}^{m+n}(x)} \prod_{j=p}^{p+n-1} \frac{L_{j-n}}{R_j^{m+n}(x)} - 1 \right| \le C\delta^{m/2-n}$$

Therefore, it follows from Remark B.11 that

$$\left|\frac{\beta_{m+n}(x)}{\beta_m(x)} - \prod_{j=p-n}^{p-1} \frac{1}{\sqrt{L_j}}\right| = \left|\left(\prod_{j=1}^{p-1} \frac{R_j^m(x)}{R_j^{m+n}(x)} \prod_{j=p}^{m} \frac{R_j^m(x)}{R_{j+n}^{m+n}(x)} \prod_{j=p}^{p+n-1} \frac{L_{j-n}}{R_j^{m+n}(x)}\right)^{1/2} - 1\right| \prod_{j=p-n}^{p-1} \frac{1}{\sqrt{L_j}}$$

$$\leq C\delta^{m/2-n} \prod_{j=p-n}^{p-1} \frac{1}{\sqrt{L_j}}$$

and accordingly, (4.8) follows from Lemma B.2 and Remark B.6. □

Combined with the next one, Lemma 4.9 completes the proof of Theorem 4.7.

Lemma 4.11. *The limits*

$$\mathcal{L}_r := \lim_{q\to\infty} L_{r+qn} \qquad \textit{for} \qquad 0 \leq r \leq n-1 \tag{4.9}$$

exist and are all in $(1, \infty)$*. Moreover, for any* $m > 2n$

$$\left|\prod_{j=p-n}^{p-1} \frac{1}{\sqrt{L_j}} - \prod_{r=0}^{n-1} \frac{1}{\sqrt{\mathcal{L}_r}}\right| \leq C\delta^{(m-n)/2} \tag{4.10}$$

where $p = [m/2]$ *and the constants* $C = C(K)$ *and* $\delta = \delta(K) \in (0, 1)$ *are independent of the given periodic orbit.*

Proof. First note that (B.15) holds with $R_1^m(x)$ and $R_{1+n}^{m+n}(x)$ replaced respectively by L_1 and L_{1+n}; on the other hand

$$\frac{L_j}{L_{j+n}} - 1 = \frac{c_j}{L_{j+n}L_{j-1}}\left(\frac{L_{j-1}}{L_{j+n-1}} - 1\right) \qquad \text{for} \qquad j \geq 2$$

(since, in this case, $b_{j+n} - b_j = c_{j+n} - c_j = 0$) so that by Lemma B.2

$$\left|\frac{L_j}{L_{j+n}} - 1\right| \leq \frac{c_j}{\theta^2}\left|\frac{L_{j-1}}{L_{j+n-1}} - 1\right| \qquad \text{for} \qquad j \geq 2\ .$$

Therefore, by Remark B.4,

$$\left|\frac{L_j}{L_{j+n}} - 1\right| \leq \frac{2\vartheta\theta d_{\max}}{d_{\min}} \frac{1}{\theta^{2j}} := \frac{C}{\theta^{2j}} \qquad \text{for} \qquad j \geq 1\ . \tag{4.11}$$

For $j \geq 1$ and $p \geq 1$, writing

$$\frac{L_j}{L_{j+pn}} = \prod_{s=0}^{p-1} \frac{L_{j+sn}}{L_{j+(s+1)n}}$$

and applying (B.19) yields on account of (4.11)

$$\left|\frac{L_j}{L_{j+pn}} - 1\right| \leq C \exp\left(C\sum_{s=0}^{p-1} \frac{1}{\theta^{2(j+sn)}}\right) \sum_{s=0}^{p-1} \frac{1}{\theta^{2(j+sn)}}$$

$$\leq \frac{C\theta^2}{\theta^2 - 1} \exp\left(\frac{C\theta^2}{\theta^2 - 1}\right) \frac{1}{\theta^{2j}} := \frac{C'}{\theta^{2j}}\ . \tag{4.12}$$

The first implication of (4.12) is that, by Lemma B.2, it gives

$$|L_{j+pn} - L_j| \leq \frac{C' L_{j+pn}}{\theta^{2j}} \leq \frac{C'\vartheta}{\theta^{2j}} := \frac{C''}{\theta^{2j}}$$

and this, in turn, implies that, for $0 \leq r \leq n-1$ and $p, q \geq 1$,

$$\left|L_{r+(p+q)n} - L_{r+qn}\right| \leq \frac{C''}{\theta^{2(r+qn)}}\ .$$

Therefore the sequences $\{L_{r+qn}\}_{q\geq 1}$ converge. That their limits $\mathcal{L}_r$ belong to $(1,\infty)$ is immediate from Lemma B.2. On the other hand, applying (B.19) yields on account of (4.12)

$$\left|\prod_{j=q}^{q+n-1}\frac{L_j}{L_{j+pn}}-1\right|\leq C^{'}\exp\left(C^{'}\sum_{j=q}^{q+n-1}\frac{1}{\theta^{2j}}\right)\sum_{j=q}^{q+n-1}\frac{1}{\theta^{2j}}\leq\frac{C^{'}\theta^2}{\theta^2-1}\exp\left(\frac{C^{'}\theta^2}{\theta^2-1}\right)\frac{1}{\theta^{2q}}:=\frac{C^{'''}}{\theta^{2q}}$$

for $q\geq 1$. Since

$$\prod_{j=q}^{q+n-1}\frac{1}{\sqrt{L_{j+pn}}}-\prod_{j=q}^{q+n-1}\frac{1}{\sqrt{L_j}}=\left(\left(\prod_{j=q}^{q+n-1}\frac{L_j}{L_{j+pn}}\right)^{1/2}-1\right)\prod_{j=q}^{q+n-1}\frac{1}{\sqrt{L_j}}$$

we therefore obtain by Remark B.11 and Lemma B.2

$$\left|\prod_{j=q}^{q+n-1}\frac{1}{\sqrt{L_{j+pn}}}-\prod_{j=q}^{q+n-1}\frac{1}{\sqrt{L_j}}\right|\leq\frac{C^{'''}}{\theta^{2q}}\prod_{j=q}^{q+n-1}\frac{1}{\sqrt{L_j}}\leq\frac{C^{'''}}{\theta^{2q+n/2}}$$

Letting $p\to\infty$, we get

$$\left|\prod_{r=0}^{n-1}\frac{1}{\sqrt{\mathcal{L}_r}}-\prod_{j=q}^{q+n-1}\frac{1}{\sqrt{L_j}}\right|\leq\frac{C^{'''}}{\theta^{2q+n/2}}$$

and this gives (4.10) upon choosing $q=[m/2]-n$. □

To derive an explicit formula for the product $\prod_{r=0}^{n-1}\sqrt{\mathcal{L}_r}$, we extend the definition of $\mathcal{L}_r$ by setting

$$\mathcal{L}_{r+pn}=\mathcal{L}_r \qquad \text{for} \qquad [0\leq r\leq n-1 \quad \text{and} \quad p\in\mathbb{Z}]$$

and define

$$d_j=b_j+c_j \qquad \text{for} \qquad j\in\mathbb{Z}\ .$$

It follows from equations (4.5) and (4.9) that

$$\mathcal{L}_r=d_r-\frac{c_r}{\mathcal{L}_{r-1}} \qquad \text{for} \qquad r\in\mathbb{Z}\ . \tag{4.13}$$

As is apparent then, $\mathcal{L}_0$ determines $\{\mathcal{L}_r\}_{r=0,\ldots,n-1}$ (and thereby their product) uniquely through (4.13), and it also satisfies the equation

$$x=d_n-\cfrac{c_n}{d_{n-1}-\cfrac{c_{n-1}}{\ddots\ \overline{\qquad\ d_1-\cfrac{c_1}{x}}}}\ ; \tag{4.14}$$

equation (4.14), in turn, can be used to obtain a quadratic equation in x whose solutions x_1,x_2 must be real since $\mathcal{L}_0\in\{x_1,x_2\}$. Once these roots are computed, Lemma 4.11 provides an efficient way to recover the product $\prod_{r=0}^{n-1}\sqrt{\mathcal{L}_r}$.

Moreover, in the special case that the period is $n=2$, it can be readily verified that, with $d=|a_1-a_0|$ and $\kappa_j=\kappa(a_j)$,

$$\mathcal{L}_0\mathcal{L}_1=(1+d\kappa_0)(1+d\kappa_1)\left(1+\sqrt{1-\frac{1}{(1+d\kappa_0)(1+d\kappa_1)}}\right)\ .$$

4.4. **Analysis of the differences $\gamma_{m+n}(x) - \gamma_m(x)$ on $n-$periodic orbits and proof of Theorem 4.1.**

Lemma 4.12. *For any $m \geq 1$ and $x \in \partial K_m$, we have*

$$|\gamma_{m+n}(x) - \gamma_m(x)| \leq C\delta^m$$

where the constants $C = C(K,\alpha)$ and $\delta = \delta(K,\alpha) \in (0,1)$ are independent of the given periodic orbit.

Proof. Using the notation $J_m(x) = x^m_{m-1}(x)$ for $m \geq 1$, we have

$$\gamma_{m+n}(x) - \gamma_m(x) = \frac{J_m(x) - J_{m+n}(x)}{|J_{m+n}(x) - x|} \cdot \nu(x) + \frac{|J_{m+n}(x) - x| - |J_m(x) - x|}{|J_{m+n}(x) - x|} \frac{J_m(x) - x}{|J_m(x) - x|} \cdot \nu(x)$$

so that by triangle inequality

$$|\gamma_{m+n}(x) - \gamma_m(x)| \leq 2\frac{|J_{m+n}(x) - J_m(x)|}{|J_{m+n}(x) - x|} \leq \frac{2}{d_{\min}} |J_{m+n}(x) - J_m(x)|\,;$$

applying Lemma 4.3 completes the proof with the choice of the constants $C = 2C_1/\delta_1 d_{\min}$ and $\delta = \delta_1$. $\square$

Proof of Theorem 4.1 Setting $\beta = \prod_{r=0}^{n-1} 1/\sqrt{\mathcal{L}_r}$, we have

$$\begin{aligned}(-1)^{m+n}\left(\eta^A_{m+n} - \mathcal{R}_{n,k}\eta^A_m\right) = e^{ik\varphi_{m+n}}\left(\beta_{m+n} - \beta\beta_m\right)\gamma_{m+n} + e^{ik\varphi_{m+n}}\beta\beta_m(\gamma_{m+n} - \gamma_m) \\ + \left(e^{ik\varphi_{m+n}} - e^{ik\{\varphi_m + \Phi_n(a_0,\ldots,a_{n-1})\}}\right)\beta\beta_m\gamma_m.\end{aligned}$$

Taking absolute values, and using triangle inequality, we obtain on account of Theorem 4.7 and Lemmas 4.6 and 4.12 (choosing C and δ to be the maximums of the corresponding constants)

$$\left|\eta^A_{m+n}(x) - \mathcal{R}_{n,k}\eta^A_m(x)\right| \leq C\left(\delta^{(m-n)/2}|\gamma_{m+n}(x)| + \delta^m\beta\right)\beta_m(x) + \min\{2, e^{Ck\delta^{m/2}} - 1\}\beta\beta_m(x)|\gamma_m(x)|$$

where we made use of the inequality

$$\left|e^{ik\varphi_{m+n}} - e^{ik\{\varphi_m + \Phi_n(a_0,\ldots,a_{n-1})\}}\right| \leq \min\{2, e^{k|\varphi_{m+n}(x) - \varphi_m(x) - \Phi_n(a_0,\ldots,a_{n-1})|} - 1\}.$$

By Lemma B.2 and Remark B.6, we have $\beta \leq \delta^{n/2}$; also γ_m and γ_{m+n} are bounded above by 1. Therefore, replacing C by $2C$, we obtain

$$\left|\eta^A_{m+n}(x) - \mathcal{R}_{n,k}\eta^A_m(x)\right| \leq \mathcal{F}\beta_m(x)\,.$$

Since $\beta_m(x) = |\eta^A_m(x)|/|\gamma_m(x)|$, an appeal to Lemma B.2 and Remark B.6 completes the proof. $\square$

4.5. **Analysis of illuminated regions ∂K^{IL}_m on $n-$periodic orbits and proof of Theorem 4.2.** Our main result in this section, summarized in the next lemma, provides a quantitative analysis of the illuminated regions ∂K^{IL}_m on $n-$periodic orbits. In particular, Theorem 4.2 is an immediate consequence of (4.19).

Lemma 4.13. *Given $0 \leq r \leq n-1$ and $x \in \partial K_r$, there exists a* unique *direction $\alpha_r(x) \in S^1 = \{\alpha \in \mathbb{R}^2 : |\alpha| = 1\}$ with the property that the broken ray with direction $\alpha_r(x)$ passes through x and transverses the* backwards infinite path $\{K_{r-m}\}_{m\geq 1}$ *where $K_{j_1} = K_{j_2}$ provided $j_1 \equiv j_2 \mod n$. The function*

$$\partial K_r \to \mathbb{R} : x \mapsto \alpha_r(x)\cdot\nu(x) \tag{4.15}$$

is continuous; the limiting illuminated and shadow regions

$$\partial K^{IL}_{r,\infty} := \{x \in \partial K_r : \alpha_r(x)\cdot\nu(x) > 0\} \tag{4.16}$$

$$\partial K^{SR}_{r,\infty} := \{x \in \partial K_r : \alpha_r(x)\cdot\nu(x) < 0\} \tag{4.17}$$

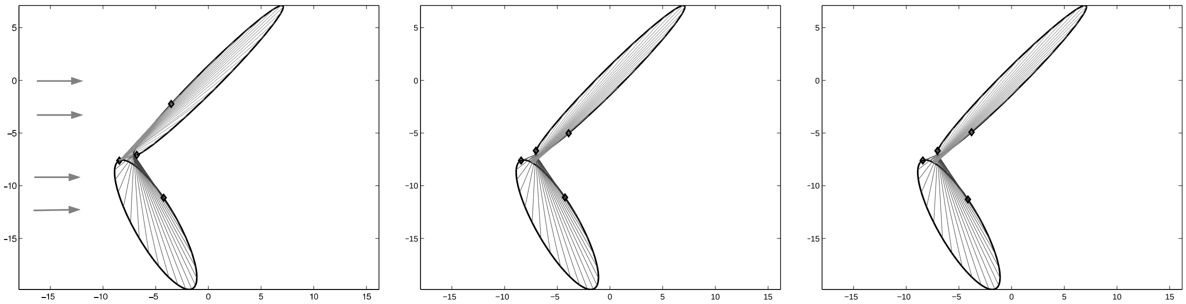
are non-empty and connected, and the limiting shadow boundary

$$\partial K^{SB}_{r,\infty} := \{x \in \partial K_r : \alpha_r(x)\cdot\nu(x) = 0\} \tag{4.18}$$

consists precisely of two distinct points. Moreover, for all $m \geq 1$,

$$\partial K^{SB}_m \subset \{x \in \partial K_r : |\alpha_r(x)\cdot\nu(x)| \leq C\delta^m\} \tag{4.19}$$

(cf. Figure 4) provided $m \equiv r \mod n$ where the constants $C = C(K,\alpha)$ and $\delta = \delta(K,\alpha) \in (0,1)$ are independent of the given periodic orbit.



(a) First reflection (b) Second reflection (c) Third reflection

FIGURE 4. Geometrical Optics solutions for a two-periodic configuration.

Proof. For a fixed $0 \le r \le n-1$ and $x \in \partial K_r$, let

$$(x_r^m(x), \dots, x_{r-m}^m(x)) \in \partial K_r \times \dots \times \partial K_{r-m} \qquad (m \ge 1)$$

be the *unique* broken ray with

$$x_r^m(x) = x \quad \text{and} \quad x_m^m(x) = a_{m-r}$$

and define the sequence of functions $\{F_m\}_{m \ge 1}$ by

$$F_m : \partial K_r \longrightarrow \partial K_{r-1} : x \longmapsto x_{r-1}^m(x) \ . \tag{4.20}$$

Since the functions F_m are continuous, and Lemma 4.3 implies that

$$|F_{m_1}(x) - F_{m_2}(x)| = \left| x_{r-1}^{m_1}(x) - x_{r-1}^{m_2}(x) \right| \le C\delta^{\min\{m_1,m_2\}-1}$$

for all $m_1, m_2 \ge 1$, we conclude that there exists a *continuous* function $\mathcal{F}_r : \partial K_r \to \partial K_{r-1}$ such that $F_m \to \mathcal{F}_r$ uniformly on ∂K_r. As is apparent from the construction, the broken ray with direction

$$\alpha_r(x) := \frac{\mathcal{F}_r(x) - x}{|\mathcal{F}_r(x) - x|} \tag{4.21}$$

passes through x and transverses the infinite path $\{\partial K_{r-m}\}_{m \ge 1}$; if $\alpha \in S^1$ is another direction possessing the same properties with $\alpha_r(x)$, then we choose $y \in \partial K_{r-1}$ with

$$\frac{y - x}{|y - x|} = \alpha$$

and note that Lemma 4.3 implies

$$|y - \mathcal{F}_r(x)| \le C\delta^{m-1}$$

for all $m \ge 1$; and this, in turn, implies that $y = \mathcal{F}_r(x)$ and thus $\alpha = \alpha_r(x)$. Since $\mathcal{F}_r$ is continuous and $\mathrm{dist}(\partial K_r, \partial K_{r-1}) > 0$, the function $\alpha_r : \partial K_r \to S^1$ is continuous, and therefore, so is the function (4.15).

Now, consider the $m-$th illuminated regions

$$\partial K_{r,m}^{IL} := \left\{ x \in \partial K_r : \frac{F_m(x) - x}{|F_m(x) - x|} \cdot \nu(x) > 0 \right\}$$

and the $m-$th shadow regions

$$\partial K_{r,m}^{SR} := \left\{ x \in \partial K_r : \frac{F_m(x) - x}{|F_m(x) - x|} \cdot \nu(x) < 0 \right\}$$

determined by the sequence $\{F_m\}_{m \ge 1}$. Clearly each one of the sets $\partial K_{r,m}^{IL}$ and $\partial K_{r,m}^{SR}$ are open and connected; writing

$$\frac{\mathcal{F}_r(x) - x}{|\mathcal{F}_r(x) - x|} \cdot \nu(x) - \frac{F_m(x) - x}{|F_m(x) - x|} \cdot \nu(x) = \left(\frac{\mathcal{F}_r(x) - x}{|\mathcal{F}_r(x) - x|} - \frac{F_m(x) - x}{|F_m(x) - x|} \right) \cdot \nu(x)$$

and letting $m \to \infty$ one obtains using the convergence $F_m \to \mathcal{F}_r$

$$\partial K^{IL}_{r,\infty} = \bigcup_{m=1}^{\infty} \bigcap_{p=m}^{\infty} \partial K^{IL}_{r,p} \quad \text{and} \quad \partial K^{SR}_{r,\infty} = \bigcup_{m=1}^{\infty} \bigcap_{p=m}^{\infty} \partial K^{SR}_{r,p} \,.$$

Accordingly, to conclude that the sets $\partial K^{IL}_{r,\infty}$ and $\partial K^{SR}_{r,\infty}$ are connected, it suffices to show that

$$\bigcap_{m=1}^{\infty} \partial K^{IL}_{r,m} \neq \varnothing \quad \text{and} \quad \bigcap_{m=1}^{\infty} \partial K^{SR}_{r,m} \neq \varnothing \,.$$

In fact, Lemma 4.5 implies that $a_r \in \partial K^{IL}_{r,m}$ for all $m \geq 1$; and, on the other hand, convexity implies that there exists a (unique) $x_r \in \partial K_r$ such that

$$\frac{a_r - x_r}{|a_r - x_r|} = \alpha_r(a_r) = \alpha_r(x_r)$$

and therefore $x_r \in \partial K^{SR}_{r,m}$ for all $m \geq 1$. Therefore, the sets $\partial K^{IL}_{r,\infty}$ and $\partial K^{SR}_{r,\infty}$ are non-empty and connected. This implies, in particular, that the set $\partial K^{SB}_{r,\infty}$ consists precisely of two closed connected components; and a simple (purely geometric) argument shows that these components must both be singletons.

Finally, to prove (4.19), suppose that $m \geq 1$ and $m \equiv r \mod n$; given $x \in \partial K_m$, let $(x^m_0(x), \ldots, x^m_m(x)) \in \partial K_0 \times \ldots \times \partial K_m$ be the unique broken ray as defined by (2.25). Defining the function

$$J_m : \partial K_m \to \partial K_{m-1} : x \mapsto x^m_{m-1}(x)\,,$$

we get exactly as in the proof of Lemma 4.12

$$\left| \frac{\mathcal{F}_r(x) - x}{|\mathcal{F}_r(x) - x|} \cdot \nu(x) - \frac{J_m(x) - x}{|J_m(x) - x|} \cdot \nu(x) \right| \leq \frac{2}{d_{\min}} |\mathcal{F}_r(x) - J_m(x)|$$

so that an appeal to Lemma 4.3 gives

$$\left| \frac{\mathcal{F}_r(x) - x}{|\mathcal{F}_r(x) - x|} \cdot \nu(x) - \frac{J_m(x) - x}{|J_m(x) - x|} \cdot \nu(x) \right| \leq C\delta^m$$

with the choice of the constants $C = 2C_1/\delta_1 d_{\min}$ and $\delta = \delta_1$. Accordingly, we have the inclusion

$$\partial K^{SB}_m \subset \left\{ x \in \partial K_r : |\alpha_r(x) \cdot \nu(x)| = \left| \frac{\mathcal{F}_r(x) - x}{|\mathcal{F}_r(x) - x|} \cdot \nu(x) \right| \leq C\delta^m \right\}$$

completing the proof. □

4.6. **Asymptotic behavior of the actual currents η_m on $n-$periodic orbits.** Appealing to Corollary 3.3, we have

$$\eta_m(x) = (1 + k^{-1} P_m(x,k))\, 2ik\, \eta^A_m(x) \quad \text{as } k \to \infty$$

on any compact subset of ∂K^{IL}_m where $P_m(k,x) = \mathcal{O}(k^0)$. Accordingly

$$\begin{aligned} \eta_{m+n}(x) - \mathcal{R}_{n,k}\eta_m(x) = {} & \frac{P_{m+n}(x,k) - P_m(x,k)}{k + P_m(x,k)} \mathcal{R}_{n,k}\, \eta_m(x) \\ & + \frac{k + P_{m+n}(x,k)}{k + P_m(x,k)} 2ik \left(1 + \frac{P_m(x,k)}{k}\right) \left(\eta^A_{m+n}(x) - \mathcal{R}_{n,k}\eta^A_m(x)\right) \end{aligned}$$

holds, as $k \to \infty$, on any compact subset of the jointly illuminated regions $\partial K^{IL}_{m+n} \cap \partial K^{IL}_m$; Theorem 4.1, in turn, yields

$$\begin{aligned} |\eta_{m+n}(x) - \mathcal{R}_{n,k}\eta_m(x)| \leq {} & \left| \frac{P_{m+n}(x,k) - P_m(x,k)}{k + P_m(x,k)} \right| |\mathcal{R}_{n,k}|\, |\eta_m(x)| \\ & + \left| \frac{k + P_{m+n}(x,k)}{k + P_m(x,k)} \right| \left| 2ik \left(1 + \frac{P_m(x,k)}{k}\right) \eta^A_m(x) \right| \frac{\mathcal{F}}{|\gamma_m(x)|} \end{aligned}$$

so that, since $|\gamma_m(x)| \leq 1$, we get

$$|\eta_{m+n}(x) - \mathcal{R}_{n,k}\eta_m(x)| \leq \left(\left| \frac{P_{m+n}(x,k) - P_m(x,k)}{k + P_m(x,k)} \right| |\mathcal{R}_{n,k}| + \left| \frac{k + P_{m+n}(x,k)}{k + P_m(x,k)} \right| \mathcal{F} \right) \left| \frac{\eta_m(x)}{\gamma_m(x)} \right|$$
$$:= (\mathcal{S}_{m,n}(x,k)\, |\mathcal{R}_{n,k}| + \mathcal{T}_{m,n}(x,k)\, \mathcal{F}) \left| \frac{\eta_m(x)}{\gamma_m(x)} \right|$$

on any compact subset of $\partial K^{IL}_{m+n} \cap \partial K^{IL}_m$ as $k \to \infty$. As $|\mathcal{R}_{n,k}| \leq \delta^{n/2}$, and $|\gamma_m|$ is bounded away from zero on any compact subset of ∂K^{IL}_m, replacing δ in Theorem 4.1 with $\delta^{1/2}$, we obtain the following result.

Corollary 4.14. *On any compact subset of the jointly illuminated regions $\partial K^{IL}_{m+n} \cap \partial K^{IL}_m$, if $\mathcal{S}_{m,n}(x,k) = \mathcal{O}(k^{-1})$ and $\mathcal{T}_{m,n}(x,k) = \mathcal{O}(k^0)$ as $k \to \infty$ independently of m, then*

$$\begin{aligned} |\eta_{m+n}(x) - \mathcal{R}_{n,k}\eta_m(x)| &= \left(\mathcal{O}(k^{-1}\delta^n) + \mathcal{O}(k^0 \mathcal{F}) \right) |\eta_m(x)| \\ &= \left(\mathcal{O}(k^{-1}\delta^n) + \mathcal{O}(k\delta^{m+n}) + \mathcal{O}(k^0\delta^{m-n}) \right) |\eta_m(x)| \end{aligned} \tag{4.22}$$

provided $m > 2n$ where all the order terms depend only on the compact subset in consideration.

To see that the relation given in Corollary 4.14 must hold *uniformly* on ∂K_r for $0 \leq r \leq n-1$ (not only on the jointly illuminated regions), we argue as follows.

Theorem 4.2 implies that the relation given in Corollary 4.14 holds for the right-hand side integrals (2.33) on any compact subset of $\partial K_m \backslash \partial K^{SB}_m$. For $k \gg 1$ and $m = m(k) \gg 1$, the factor modulating η_m on the right-hand side of (4.22) being essentially zero, this means that η_{m+n+1} and η_{m+1} are the solutions of the same linear integral equation with the right-hand sides modulated by the *constant* $\mathcal{R}_{n,k}$. Therefore, they must satisfy the relation

$$|\eta_{m+n+1}(x) - \mathcal{R}_{n,k}\eta_{m+1}(x)| \leq \left(\mathcal{O}(k^{-1}\delta^n) + \mathcal{O}(k\delta^{m+n+1}) + \mathcal{O}(k^0\delta^{m-n+1}) \right) |\eta_{m+1}(x)|$$

over the *whole boundary* ∂K_{m+1}.

5. Numerical results

5.1. **Examples of asymptotic convergence rates.** In this section, we present numerical examples testing our rate of convergence formula (4.22) on two– and three–periodic orbits.

In Figures 5−8, the left pane provides the corresponding geometrical configuration; middle pane displays

$$\log_{10} \left\| \frac{\eta_{m+n}(x)}{\eta_m(x)} - \mathcal{R}_{n,k} \right\|_{L_\infty(\partial K_m)} \tag{5.1}$$

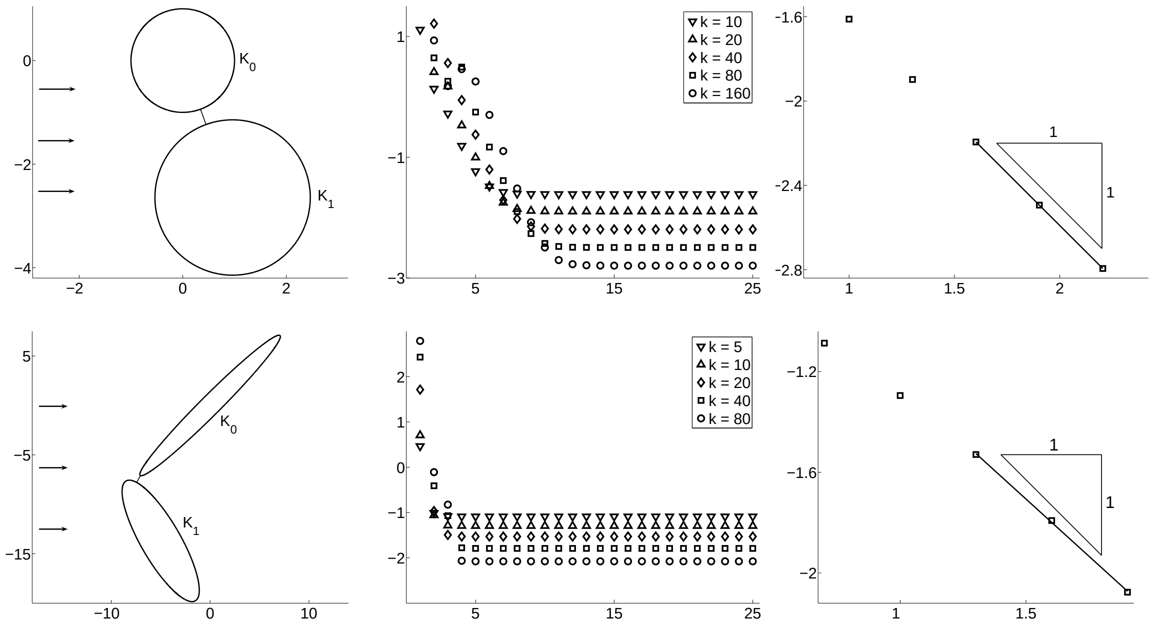
for $m = 0, n, \ldots, 25n$ on ∂K_0; the right pane, on the other hand, displays (5.1) against $\log_{10} k$ for the particular value $m = 25n$.

As is apparent in these Figures, the error involved in any finite number of reflections is, as implied by our rate of convergence formula (4.22), of order $\mathcal{O}(k^{-1})$ (right panes); and this error is *uniformly bounded* as the number of reflections m tends to infinity (middle panes).

Moreover, as is clear from these Figures, our rate of convergence formula (4.22) does not depend on the direction of incidence *even when the obstacles are occluded* (cf. Figures 6–8); it *applies to non-convex geometries* in the case that an observer walking on the boundary of any particular obstacle in the period would think that all the other obstacles are convex (cf. Figures 7–8); and it also applies without any modification when *illumination by a point source* is considered (see the second rows of Figures 7–8).

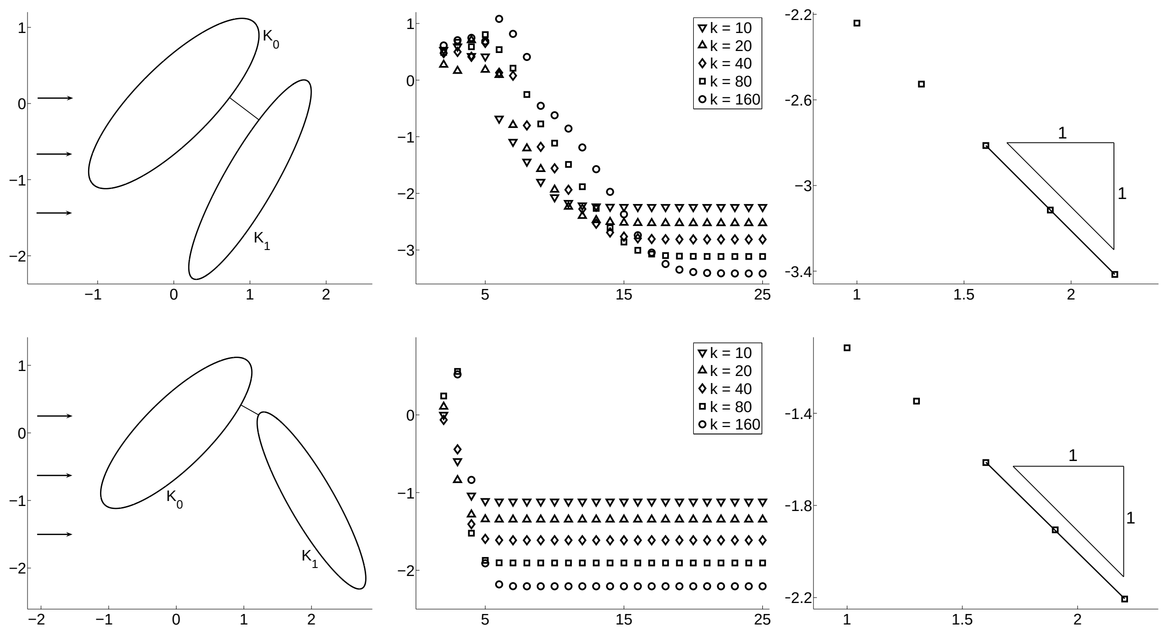
5.2. **Acceleration of convergence.** Here we present two algorithms, based on our analysis, accelerating the convergence of multiple-scattering iterates.

The first one, which we shall call "geometrical optics correction (GOC)," utilizes the rate of convergence formula (4.22) to obtain $\mathcal{O}(k^{-1})$ corrections at each reflection. The underlying idea is, as is apparent from (4.22), the tail of the multiple-scattering iterates on an $n-$periodic orbit are at a uniform distance of $\mathcal{O}(k^{-1})$ from a geometric series. The resultant sum of the iterates on this orbit must therefore lie in an $\mathcal{O}(k^{-1})$ neighborhood of the sum of the first m iterates plus the terms $\eta_{m-n+1}, \ldots, \eta_m$ multiplied by $1/(1 - \mathcal{R}_{n,k})$.



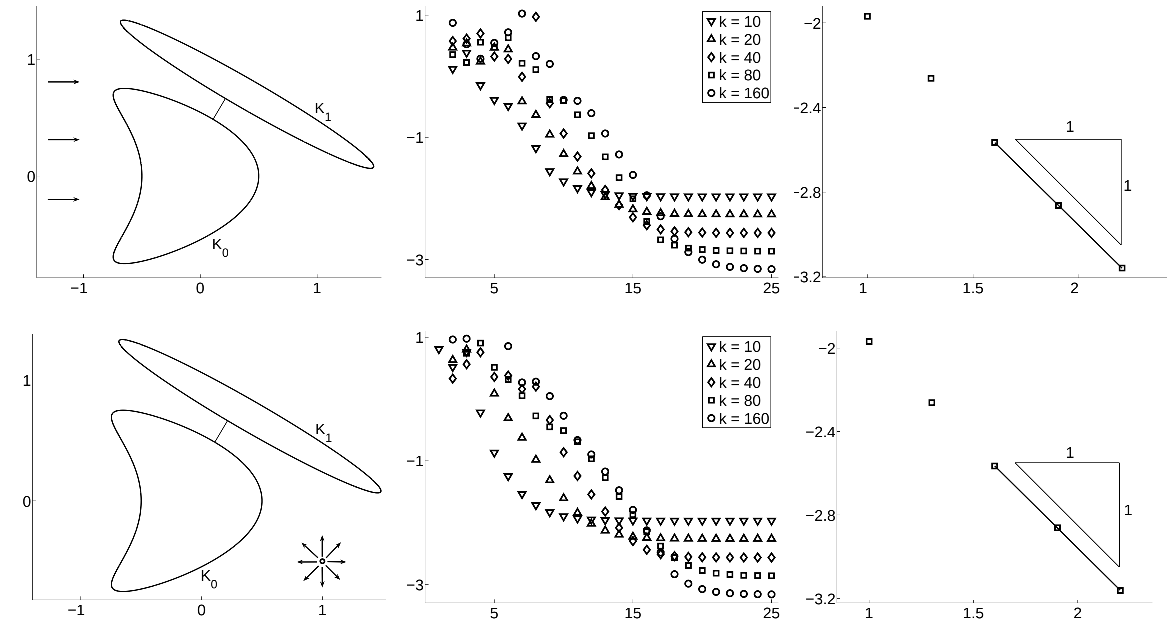
(a) Configuration (b) Stabilization (c) Convergence

Figure 5. Two-periodic configurations without occlusion; plane-wave illumination from the left.



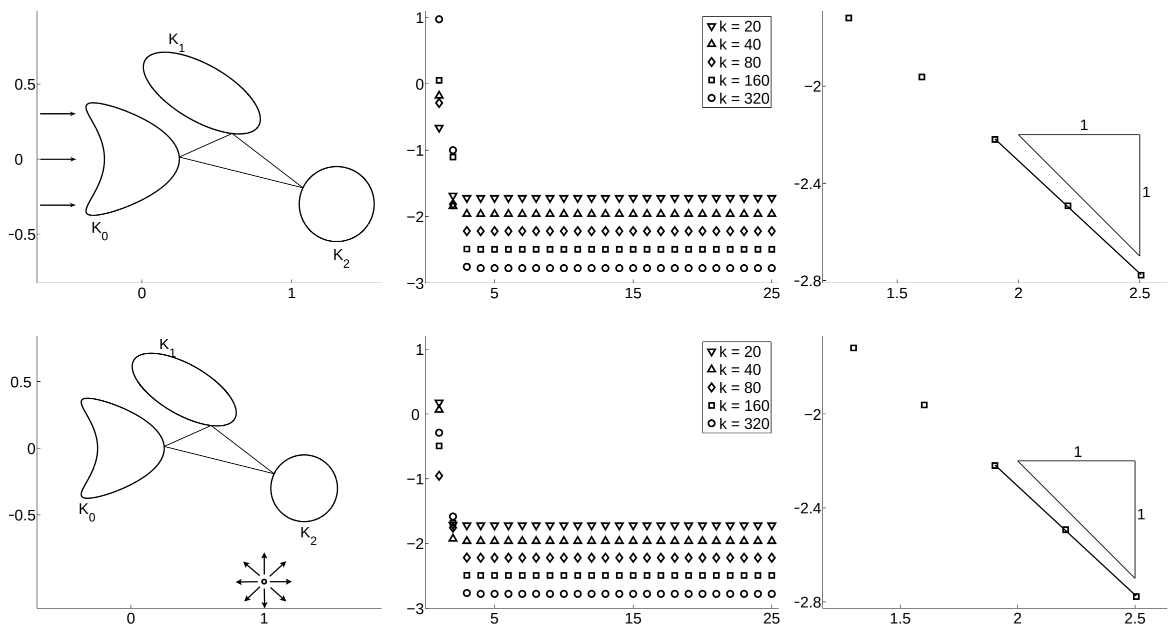
(a) Configuration (b) Stabilization (c) Convergence

Figure 6. Two-periodic configurations *with* occlusion; plane-wave illumination from the left.



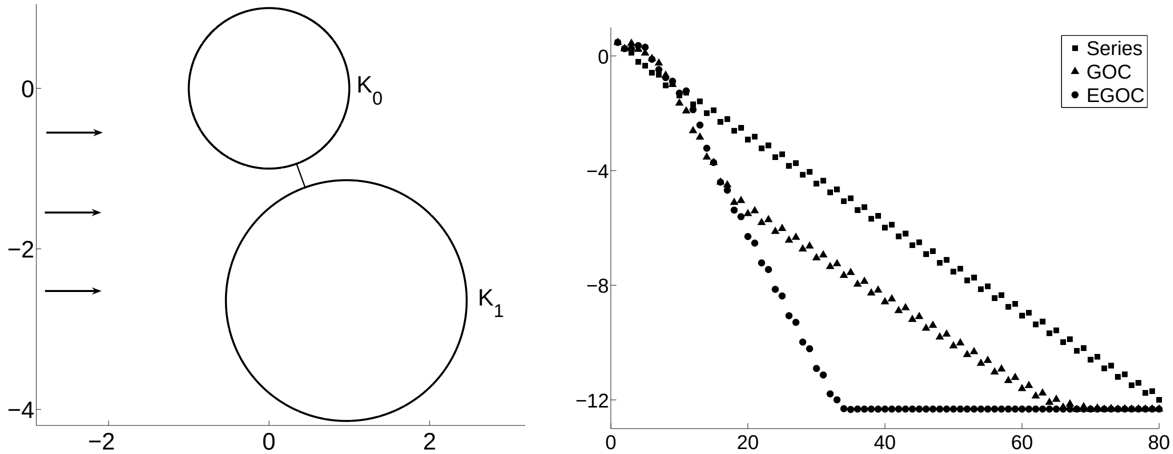
(a) Configuration (b) Stabilization (c) Convergence

Figure 7. Two-periodic configurations *with occlusion* and *non-convex* scatterers; top: plane-wave illumination from left; bottom: point-source illumination (at $[1, -0.5]$).



(a) Configuration (b) Stabilization (c) Convergence

Figure 8. Three-periodic configurations *with occlusion* and *non-convex* scatterers; top: plane-wave illumination from the left; bottom: point-source illumination (at $[1, -1]$).



(a) Configuration (b) $\log_{10}$ (Maximum error / k)

FIGURE 9. Left: two-periodic configuration illuminated by a plane-wave from the left; Right: number of reflections versus the maximum error on ∂K_0 ($k = 400$).

More precisely, assuming that the first M currents η_m on an n–periodic are computed, the GOC algorithm reads

$$\sum_{m=0}^{\infty} \eta_m \approx \sum_{m=0}^{M} \eta_m + \frac{1}{1-\mathcal{R}_{n,k}} \sum_{r=0}^{n-1} \eta_{M-r} \ . \tag{5.2}$$

Our second algorithm, which we shall call "exact geomerical optics corrections (EGOC)," takes into account of the the numerical experiments in Sect. 5.1 that display the fact that the tail of the multiple-scattering iterates on an $n-$periodic orbit are themselves (perturbations) of a geometric series. Therefore, instead of using (5.2), one computes

$$\sum_{m=0}^{\infty} \eta_m \approx \sum_{m=0}^{M} \eta_m + \sum_{r=0}^{n-1} \frac{1}{1-\gamma_r} \eta_{M-r}$$

where, for $0 \le r \le n-1$,

$$\gamma_r = \text{average} \left\{ \frac{\eta_{M-r}(x)}{\eta_{M-r-n}(x)} : x \in \partial K_{M-r} \right\} \ .$$

Figure 9 depicts the improvements provided by these two algorithms on a 2–periodic orbit where we compare the exact solution with the solutions obtained by summing the first $M-$terms of the series, and by using the GOC and EGOC algorithms. As we claimed, GOC provides an $\mathcal{O}(k^{-1})$ improvement once the series stabilizes, while EGOC, taking direct account of the geometric nature of the series, provides a significant impronment on the rate of convergence.

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Appendix A. Derivatives of phase functions

In this appendix we collect some detailed properties of the phase functions (2.24), particularly on their derivatives, that are used in Sect. 3 to derive the asymptotic expression (3.2). More precisely, these derivations necessitate expressions for the first and second derivatives of the phase function (cf. (3.11))

$$\varphi_{t_{m+1}}(t_m) = |x_{m+1} - x_m| + \varphi_m(x_m) = \alpha \cdot x_0 + \sum_{j=0}^{m} |x_{j+1} - x_j| \ , \quad x_m = x_m(t_m) \in \partial K_m, \tag{A.1}$$

where $x_{m+1} = x_{m+1}(t_{m+1})$ is an arbitrary but *fixed* point on ∂K_{m+1}, φ_m is given by (2.24), and $(x_0, \ldots, x_m) \in \partial K_0 \times \cdots \times \partial K_m$ denotes the broken $(m+1)$-ray terminating at $x_m \in \partial K_m$. The first lemma provides an explicit representation for the first derivatives.

Lemma A.1 (First Derivatives). *The derivatives of the phase functions (A.1) are given by*

$$\frac{d}{dt_0}\varphi_{t_1}(t_0) = \left(\alpha - \frac{x_1 - x_0}{|x_1 - x_0|}\right) \cdot \dot{x}_0 \ , \tag{A.2}$$

and

$$\frac{d}{dt_m}\varphi_{t_{m+1}}(t_m) = \left(\frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} - \frac{x_{m+1} - x_m}{|x_{m+1} - x_m|}\right) \cdot \dot{x}_m \ , \quad m \geq 1. \tag{A.3}$$

Proof. The proof of (A.2) is straightforward. To obtain (A.3), we differentiate (A.1) with respect to t_m to obtain

$$\begin{aligned} \frac{d}{dt_m}\varphi_{t_{m+1}}(t_m) = \left(\alpha - \frac{x_1 - x_0}{|x_1 - x_0|}\right) \cdot \dot{x}_0 \frac{dt_0}{dt_m} + \sum_{i=0}^{m-2} \left(\frac{x_{i+1} - x_i}{|x_{i+1} - x_i|} - \frac{x_{i+2} - x_{i+1}}{|x_{i+2} - x_{i+1}|}\right) \cdot \dot{x}_{i+1} \frac{dt_{i+1}}{dt_m} \\ + \left(\frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} - \frac{x_{m+1} - x_m}{|x_{m+1} - x_m|}\right) \cdot \dot{x}_m \ . \end{aligned}$$

Since the tuple $(x_0, \ldots, x_m)$ is the broken $(m+1)$-ray terminating at x_m, this reduces to (A.3). □

The next result states that $\varphi_{t_{m+1}}(t_m)$ is stationary at a point t_m if, and only if, the tuple $(x_0, \ldots, x_m, x_{m+1})$ is the broken $(m+2)$-ray terminating at x_{m+1}.

Lemma A.2 ((i) Stationary Points in First Reflections). *For $m = 0$, the phase (A.1) is stationary at a point x_0 if and only if*

$$\frac{x_1 - x_0}{|x_1 - x_0|} = \alpha + 2\frac{x_1 - x_0}{|x_1 - x_0|} \cdot \nu_0 \ \nu_0 \tag{A.4}$$

or

$$\frac{x_1 - x_0}{|x_1 - x_0|} = \alpha. \tag{A.5}$$

((ii) Stationary Points in Further Reflections). *For $m \geq 1$, the phase (A.1) is stationary at a point x_m if and only if*

$$\frac{x_{m+1} - x_m}{|x_{m+1} - x_m|} = \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} + 2\frac{x_{m+1} - x_m}{|x_{m+1} - x_m|} \cdot \nu_m \ \nu_m \tag{A.6}$$

or

$$\frac{x_{m+1} - x_m}{|x_{m+1} - x_m|} = \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|}. \tag{A.7}$$

Proof. Equation (A.2) implies

$$\frac{d}{dt_0}\varphi_{t_1}(t_0) = 0 \;\Leftrightarrow\; \alpha = \lambda_0 \nu_0 + \frac{x_1 - x_0}{|x_1 - x_0|}$$

for some λ_0. Also, since $|\alpha| = 1$, we have

$$1 = \alpha \cdot \alpha = \lambda_0^2 + 2\lambda_0 \frac{x_1 - x_0}{|x_1 - x_0|} \cdot \nu_0 + 1$$

so that

$$\lambda_0 = -2\frac{x_1 - x_0}{|x_1 - x_0|} \cdot \nu_0 \qquad \text{or} \qquad \lambda_0 = 0.$$

Similarly, for $m \geq 1$, equation (A.3) gives

$$\frac{d}{dt_m}\varphi_{t_{m+1}}(t_m) = 0 \;\Leftrightarrow\; \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} = \lambda_m \nu_m + \frac{x_{m+1} - x_m}{|x_{m+1} - x_m|}$$

for some λ_m. Since

$$1 = \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} = \lambda_m^2 + 2\lambda_m \frac{x_{m+1} - x_m}{|x_{m+1} - x_m|} \cdot \nu_m + 1$$

we get

$$\lambda_m = -2\frac{x_{m+1} - x_m}{|x_{m+1} - x_m|} \cdot \nu_m \qquad \text{or} \qquad \lambda_m = 0$$

completing the proof. □

The derivations in Sect. 3 further demand the evaluation of the second derivatives of the phase (A.1) at the stationary points as derived in Lemma A.2. We note, however, that the conditions (A.5) and (A.7) cannot hold under the no-occlusion and visibility assumptions. Our next result then provides and expression for the second derivatives at the points characterized by (A.4) and (A.6). To simplify the notation in what follows, we introduce the quantities S_m defined by the identities

$$\frac{d^2}{dt_m^2}\varphi_{t_{m+1}}(t_m) = \frac{|\dot{x}_m|^2}{|x_{m+1} - x_m|}\left(\frac{x_{m+1} - x_m}{|x_{m+1} - x_m|} \cdot \nu_m\right)^2 S_m \tag{A.8}$$

for $m \geq 0$.

Theorem A.3 ((i) Second Derivatives in First Reflections)**.** *For $m = 0$, if (A.4) holds, then*

$$S_0 = 1 + 2\kappa_0 |x_1 - x_0| \left(\frac{x_1 - x_0}{|x_1 - x_0|} \cdot \nu_0\right)^{-1}. \tag{A.9}$$

((ii) Second Derivatives in Further Reflections)**.** *For $m \geq 1$, if (A.6) holds, then*

$$S_m = 2\kappa_m |x_{m+1} - x_m| \left(\frac{x_{m+1} - x_m}{|x_{m+1} - x_m|} \cdot \nu_m\right)^{-1} + T_m \tag{A.10}$$

where

$$T_m = 1 + \frac{|x_{m+1} - x_m|}{|x_m - x_{m-1}|}\left(1 - \frac{1}{S_{m-1}}\right). \tag{A.11}$$

Proof. Differentiating (A.2) yields

$$\begin{aligned}
\frac{d^2}{dt_0^2}\varphi_{t_1}(t_0) &= \left(\alpha - \frac{x_1 - x_0}{|x_1 - x_0|}\right) \cdot \ddot{x}_0 + \frac{|\dot{x}_0|^2}{|x_1 - x_0|}\left(1 - \left(\frac{x_1 - x_0}{|x_1 - x_0|} \cdot \frac{\dot{x}_0}{|\dot{x}_0|}\right)^2\right) \\
&= \left(\alpha - \frac{x_1 - x_0}{|x_1 - x_0|}\right) \cdot \ddot{x}_0 + \frac{|\dot{x}_0|^2}{|x_1 - x_0|}\left(\frac{x_1 - x_0}{|x_1 - x_0|} \cdot \nu_0\right)^2.
\end{aligned}$$

Substituting the condition (A.4) in the first term on the right hand side, equation (A.9) readily follows. □

To prove the result for further reflections we need several additional lemmas. For future reference we first record a simple geometrical identity.

Lemma A.4. *Let u, v and w be three* unit *vectors, and let Θ be the matrix of rotation by $\pi/2$ in the counterclockwise direction. Then*

$$\Theta u \cdot \Theta v - (w \cdot \Theta u)\,(w \cdot \Theta v) = (w \cdot u)\,(w \cdot v)\ . \tag{A.12}$$

Proof. Let $\alpha_1 = \cos^{-1}(w \cdot u)$ and $\alpha_2 = \cos^{-1}(w \cdot v)$. Then, (A.12) is equivalent to the trigonometric difference formula $\cos(\alpha_1 - \alpha_2) - \sin\alpha_1 \sin\alpha_2 = \cos\alpha_1 \cos\alpha_2$. □

The next lemma provides a preliminary representation for the second derivatives in terms of the relative coordinate derivatives dt_{m-1}/dt_m. Again here, to simplify notation we define the quantities V_m as

$$\frac{dt_{m-1}}{dt_m} = \frac{|\dot{x}_m|}{|\dot{x}_{m-1}|} \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \nu_m \left(\frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \nu_{m-1} \right)^{-1} V_m. \tag{A.13}$$

Lemma A.5 (Second Derivatives in terms of Coordinate Derivatives)**.** *For $m \geq 1$, if (A.6) holds, then*

$$S_m = 2\kappa_m |x_{m+1} - x_m| \left(\frac{x_{m+1} - x_m}{|x_{m+1} - x_m|} \cdot \nu_m \right)^{-1} + U_m \tag{A.14}$$

where

$$U_m = 1 + \frac{|x_{m+1} - x_m|}{|x_m - x_{m-1}|} (1 - V_m)\ . \tag{A.15}$$

Proof. Differentiating (A.3) and using Lemma A.4 we obtain

$$\begin{aligned}
\frac{d^2}{dt_m^2} \varphi_{t_{m+1}}(t_m) = &\left(\frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} - \frac{x_{m+1} - x_m}{|x_{m+1} - x_m|} \right) \cdot \ddot{x}_m \\
&+ \frac{|\dot{x}_m|^2}{|x_m - x_{m-1}|} \left(\frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \nu_m \right)^2 + \frac{|\dot{x}_m|^2}{|x_{m+1} - x_m|} \left(\frac{x_{m+1} - x_m}{|x_{m+1} - x_m|} \cdot \nu_m \right)^2 \\
&- \frac{|\dot{x}_{m-1}||\dot{x}_m|}{|x_m - x_{m-1}|} \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \nu_{m-1} \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \nu_m \frac{dt_{m-1}}{dt_m}.
\end{aligned}$$

Now, if (A.6) holds, then

$$\begin{aligned}
\frac{d^2}{dt_m^2} \varphi_{t_{m+1}}(t_m) = 2\, \frac{x_{m+1} - x_m}{|x_{m+1} - x_m|} \cdot \nu_m\, \kappa_m\, |\dot{x}_m|^2 + |\dot{x}_m|^2 \left(\frac{1}{|x_m - x_{m-1}|} + \frac{1}{|x_{m+1} - x_m|} \right) \left(\frac{x_{m+1} - x_m}{|x_{m+1} - x_m|} \cdot \nu_m \right)^2 \\
- \frac{|\dot{x}_{m-1}||\dot{x}_m|}{|x_m - x_{m-1}|} \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \nu_{m-1} \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \nu_m \frac{dt_{m-1}}{dt_m}
\end{aligned}$$

completing the proof. □

Finally, comparing (A.10) and (A.11) with (A.14) and (A.15) we see that, to establish (A.10) and thus complete the proof of Theorem A.3, there only remains to show that

$$V_m = S_{m-1}^{-1}\ , \quad m \geq 1. \tag{A.16}$$

This next lemma establishes this identity.

Lemma A.6 ((i) Relative Coordinate Derivatives in First Reflections)**.** *For $m = 1$, if (A.4) holds, then*

$$V_1 = \left(1 + 2\kappa_0 |x_1 - x_0| \left(\frac{x_1 - x_0}{|x_1 - x_0|} \cdot \nu_0 \right)^{-1} \right)^{-1}. \tag{A.17}$$

((ii) Relative Coordinate Derivatives in Further Reflections)**.** *For $m > 1$, if (A.6) holds, then*

$$V_m = \left(2\kappa_{m-1} |x_m - x_{m-1}| \left(\frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \nu_{m-1} \right)^{-1} + U_{m-1} \right)^{-1} \tag{A.18}$$

where U_m is as given by (A.15).

Proof. Differentiating the identity

$$\left(\alpha - \frac{x_1 - x_0}{|x_1 - x_0|}\right) \cdot \dot{x}_0 = 0$$

with respect to t_1, and using Lemma A.4, we obtain

$$\frac{dt_0}{dt_1} = \frac{|\dot{x}_0||\dot{x}_1|}{|x_1 - x_0|} \frac{x_1 - x_0}{|x_1 - x_0|} \cdot \nu_0 \frac{x_1 - x_0}{|x_1 - x_0|} \cdot \nu_1 \left(\left(\alpha - \frac{x_1 - x_0}{|x_1 - x_0|}\right) \cdot \ddot{x}_0 + \frac{|\dot{x}_0|^2}{|x_1 - x_0|} \left(\frac{x_1 - x_0}{|x_1 - x_0|} \cdot \nu_0\right)^2 \right)^{-1}$$

from which (A.17) follows upon straightforward computations. To obtain (A.18), we differentiate the identity

$$\left(\frac{x_{m-1} - x_{m-2}}{|x_{m-1} - x_{m-2}|} - \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|}\right) \cdot \dot{x}_{m-1} = 0$$

with respect to t_m

$$\begin{aligned}
0 = &\left(\frac{x_{m-1} - x_{m-2}}{|x_{m-1} - x_{m-2}|} - \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|}\right) \cdot \ddot{x}_{m-1} \frac{dt_{m-1}}{dt_m} \\
&+ \frac{1}{|x_{m-1} - x_{m-2}|} \left(|\dot{x}_{m-1}|^2 - \left(\frac{x_{m-1} - x_{m-2}}{|x_{m-1} - x_{m-2}|} \cdot \dot{x}_{m-1} \right)^2 \right) \frac{dt_{m-1}}{dt_m} \\
&+ \frac{1}{|x_m - x_{m-1}|} \left(|\dot{x}_{m-1}|^2 - \left(\frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \dot{x}_{m-1} \right)^2 \right) \frac{dt_{m-1}}{dt_m} \\
&- \frac{1}{|x_{m-1} - x_{m-2}|} \frac{dt_{m-2}}{dt_{m-1}} \frac{dt_{m-1}}{dt_m} \\
&\times \left(\dot{x}_{m-2} \cdot \dot{x}_{m-1} - \frac{x_{m-1} - x_{m-2}}{|x_{m-1} - x_{m-2}|} \cdot \dot{x}_{m-2} \frac{x_{m-1} - x_{m-2}}{|x_{m-1} - x_{m-2}|} \cdot \dot{x}_{m-1} \right) \\
&- \frac{1}{|x_m - x_{m-1}|} \left(\dot{x}_{m-1} \cdot \dot{x}_m - \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \dot{x}_{m-1} \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \dot{x}_m \right),
\end{aligned}$$

so that if (A.6) holds, an appeal to Lemma A.4 yields

$$\begin{aligned}
0 = &\left(\frac{x_{m-1} - x_{m-2}}{|x_{m-1} - x_{m-2}|} - \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|}\right) \cdot \ddot{x}_{m-1} \frac{dt_{m-1}}{dt_m} \\
&+ |\dot{x}_{m-1}|^2 \left(\frac{1}{|x_{m-1} - x_{m-2}|} + \frac{1}{|x_m - x_{m-1}|} \right) \left(\frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \nu_{m-1} \right)^2 \frac{dt_{m-1}}{dt_m} \\
&- \frac{|\dot{x}_{m-2}||\dot{x}_{m-1}|}{|x_{m-1} - x_{m-2}|} \frac{x_{m-1} - x_{m-2}}{|x_{m-1} - x_{m-2}|} \cdot \nu_{m-2} \frac{x_{m-1} - x_{m-2}}{|x_{m-1} - x_{m-2}|} \cdot \nu_{m-1} \frac{dt_{m-2}}{dt_{m-1}} \frac{dt_{m-1}}{dt_m} \\
&- \frac{|\dot{x}_{m-1}||\dot{x}_m|}{|x_m - x_{m-1}|} \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \nu_{m-1} \frac{x_m - x_{m-1}}{|x_m - x_{m-1}|} \cdot \nu_m
\end{aligned}$$

from which (A.18) follows, thus completing the proof of the lemma and of Therorem A.3. □

Appendix B. Analysis of β ratios

Here we present the proof of Lemmas 4.9 and 4.11. To begin with, we note two simple geometrical facts.

Remark B.1. The visibility condition holds if and anly if there exists an angle $\phi_v \in (0, \pi/2)$ with the property that given any three points $\xi_1, \xi_2, \xi_3 \in \partial K$ such that the segments $[\xi_1, \xi_2]$ and $[\xi_2, \xi_3]$ (a) have no point in common with the interior of the connected component of K containing ξ_2, and (b) satisfy the law of reflection at ξ_2, we have

$$\frac{\xi_1 - \xi_2}{|\xi_1 - \xi_2|} \cdot \nu(\xi_2) = \frac{\xi_3 - \xi_2}{|\xi_3 - \xi_2|} \cdot \nu(\xi_2) \geq \cos \vartheta_v \ .$$

Similarly, the no-occlusion condition holds if and only if there exists an angle $\phi_{no} \in (0, \pi/2)$ with the property that given any two points $\xi_1, \xi_2 \in \partial K$ such that the segment $[\xi_1, \xi_2]$ have no point in common with the interior of the connected component of K containing ξ_1, we have

$$\alpha \cdot \nu(\xi_1) = \frac{\xi_1 - \xi_2}{|\xi_1 - \xi_2|} \cdot \nu(\xi_1) \Rightarrow \frac{\xi_2 - \xi_1}{|\xi_2 - \xi_1|} \cdot \nu(\xi_1) \geq \cos \phi_{no} \ .$$

Lemma B.2. *There exist constants $\theta = \theta(K)$, $\vartheta_v = \vartheta_v(K)$ and $\vartheta_{v,no} = \vartheta_{v,no}(K, \alpha)$ such that, for $1 \leq j \leq m$, we have*

$$1 < \theta \leq R_j^m(x) \leq \vartheta_{v,no} \tag{B.1}$$

and, for $j \geq 1$, we have

$$1 < \theta \leq L_j \leq \vartheta_v \tag{B.2}$$

In particular, the functions $\beta_m(x)$ are non-vanishing on ∂K_m.

Proof. Set $\phi_0 = \max\{\phi_{no}, \phi_v\}$ and define the quantities θ_j and ϑ_j by

$$\theta_1 = 1 + 2 d_{\min} \kappa_{\min} \qquad \vartheta_1 = 1 + \frac{2 d_{\max} \kappa_{\max}}{\cos \phi_0}$$

and, for $j \geq 2$,

$$\theta_j = \theta_1 + \frac{d_{\min}}{d_{\max}} \left(1 - \frac{1}{\theta_{j-1}}\right) \qquad \vartheta_j = \vartheta_1 + \frac{d_{\max}}{d_{\min}} \left(1 - \frac{1}{\vartheta_{j-1}}\right) \ .$$

It follows from definition (3.3) that, for $1 \leq j \leq m$, we have

$$1 < \theta_1 \leq \theta_j \leq R_j^m(x) \leq \vartheta_j \leq \vartheta_1 + \frac{d_{\max}}{d_{\min}} \ .$$

Setting $\theta = \theta_1$ and $\vartheta_{v,no} = \vartheta_1 + d_{\max}/d_{\min}$ yields (B.1) Similarly, using $\phi_0 = \phi_v$ in the definition of ϑ_1 one obtains (B.2). □

In what follows, we shall simply write ϑ instead of $\vartheta_{v,oc}$ and ϑ_v.

Lemma B.3. *There exists a constant $C_3 = C_3(K, \alpha)$ such that*

$$\left| \frac{R_1^m(x)}{R_1^{m+n}(x)} - 1 \right| \leq \frac{C_3}{\theta} \sum_{l=0}^{1} |x_l^{m+n}(x) - x_l^m(x)| \tag{B.3}$$

and, for $2 \leq j \leq m$, we have

$$\left| \frac{R_j^m(x)}{R_j^{m+n}(x)} - 1 \right| \leq \frac{C_3}{\theta} \sum_{l=0}^{2} |x_{j-l}^{m+n}(x) - x_{j-l}^m(x)| + \frac{c_j^m(x)}{\theta^2} \left| \frac{R_{j-1}^m(x)}{R_{j-1}^{m+n}(x)} - 1 \right| \tag{B.4}$$

and

$$\left| \frac{R_j^m(x)}{R_{j+n}^{m+n}(x)} - 1 \right| \leq \frac{C_3}{\theta} \sum_{l=0}^{2} |x_{j+n-l}^{m+n}(x) - x_{j-l}^m(x)| + \frac{c_j^m(x)}{\theta^2} \left| \frac{R_{j-1}^m(x)}{R_{j+n-1}^{m+n}(x)} - 1 \right| \tag{B.5}$$

and

$$\left| \frac{L_j}{R_{j+n}^{m+n}(x)} - 1 \right| \leq \frac{C_3}{\theta} \sum_{l=0}^{2} |x_{j+n-l}^{m+n}(x) - a_{j-l}| + \frac{c_j}{\theta^2} \left| \frac{L_{j-1}}{R_{j+n-1}^{m+n}(x)} - 1 \right| \tag{B.6}$$

Proof. We note that the identity

$$\begin{aligned} R_j^{m+n}(x) \left(\frac{R_j^m(x)}{R_j^{m+n}(x)} - 1 \right) &= \left(b_j^m(x) - b_j^{m+n}(x) \right) \\ &\quad + (1 - \delta_{1j}) \left(c_j^m(x) \left(1 - \frac{1}{R_{j-1}^m(x)} \right) - c_j^{m+n}(x) \left(1 - \frac{1}{R_{j-1}^{m+n}(x)} \right) \right) \end{aligned} \tag{B.7}$$

(where δ_{1j} is the *Kronecker symbol*) holds for $1 \le j \le m$. Setting

$$d_j^m(x) = |x_{j+1}^m(x) - x_j^m(x)|$$

and

$$\cos\alpha_j^m(x) = \frac{x_{j+1}^m(x) - x_j^m(x)}{|x_{j+1}^m(x) - x_j^m(x)|} \cdot \nu(x_j^m(x))$$

we see that the identities

$$\begin{aligned} b_j^m(x) - b_j^{m+n}(x) &= \frac{2d_{j-1}^{m+n}(x)}{\cos\alpha_{j-1}^{m+n}(x)} \left(\kappa(x_{j-1}^m(x)) - \kappa(x_{j-1}^{m+n}(x))\right) \\ &+ \frac{2\kappa(x_{j-1}^m(x))}{\cos\alpha_{j-1}^{m+n}(x)} \frac{d_{j-1}^m(x)}{\cos\alpha_{j-1}^m(x)} \left(\cos\alpha_{j-1}^{m+n}(x) - \cos\alpha_{j-1}^m(x)\right) + \frac{2\kappa(x_{j-1}^m(x))}{\cos\alpha_{j-1}^{m+n}(x)} \left(d_{j-1}^m(x) - d_{j-1}^{m+n}(x)\right) \end{aligned} \tag{B.8}$$

and

$$\begin{aligned} \cos\alpha_{j-1}^{m+n}(x) - \cos\alpha_{j-1}^m(x) &= \frac{[x_j^{m+n}(x) - x_{j-1}^{m+n}(x)] - [x_j^m(x) - x_{j-1}^m(x)]}{d_{j-1}^m(x)} \cdot \nu(x_{j-1}^m(x)) \\ + \frac{d_{j-1}^m(x) - d_{j-1}^{m+n}(x)}{d_{j-1}^m(x)} \frac{x_j^{m+n}(x) - x_{j-1}^{m+n}(x)}{|x_j^{m+n}(x) - x_{j-1}^{m+n}(x)|} \cdot \nu(x_{j-1}^m(x)) &+ \frac{x_j^{m+n}(x) - x_{j-1}^{m+n}(x)}{|x_j^{m+n}(x) - x_{j-1}^{m+n}(x)|} \cdot \left(\nu(x_{j-1}^{m+n}(x)) - \nu(x_{j-1}^m(x))\right) \end{aligned} \tag{B.9}$$

hold for $1 \le j \le m$, and the identity

$$\begin{aligned} &c_j^m(x)\left(1 - \frac{1}{R_{j-1}^m(x)}\right) - c_j^{m+n}(x)\left(1 - \frac{1}{R_{j-1}^{m+n}(x)}\right) = \frac{1}{d_{j-2}^{m+n}(x)} \left(d_{j-1}^m(x) - d_{j-1}^{m+n}(x)\right)\left(1 - \frac{1}{R_{j-1}^{m+n}(x)}\right) \\ &\quad + \frac{d_{j-1}^m(x)}{d_{j-2}^{m+n}(x) d_{j-2}^m(x)} \left(d_{j-2}^{m+n}(x) - d_{j-2}^m(x)\right)\left(1 - \frac{1}{R_{j-1}^{m+n}(x)}\right) - \frac{c_j^m(x)}{R_{j-1}^{m+n}(x)} \left(\frac{R_{j-1}^{m+n}(x)}{R_{j-1}^m(x)} - 1\right) \end{aligned} \tag{B.10}$$

holds for $2 \le j \le m$. Now, since the curvature is a smooth function of the (compact) boundary curve, an appeal to the mean value theorem implies that

$$\left|\kappa(x_{j-1}^{m+n}(x)) - \kappa(x_{j-1}^m(x))\right| \le C_\kappa |x_{j-1}^{m+n}(x) - x_{j-1}^m(x)|$$

for some constant $C_\kappa = C_\kappa(K)$. We also note that

$$\left|d_{j-1}^{m+n}(x) - d_{j-1}^m(x)\right| \le |x_j^{m+n}(x) - x_j^m(x)| + |x_{j-1}^{m+n}(x) - x_{j-1}^m(x)|$$

and

$$\left|d_{j-2}^{m+n}(x) - d_{j-2}^m(x)\right| \le |x_{j-1}^{m+n}(x) - x_{j-1}^m(x)| + |x_{j-2}^{m+n}(x) - x_{j-2}^m(x)|$$

and

$$\left|[x_j^m(x) - x_{j-1}^m(x)] - [x_j^{m+n}(x) - x_{j-1}^{m+n}(x)]\right| \le |x_j^{m+n}(x) - x_{j-1}^m(x)| + |x_{j-1}^{m+n}(x) - x_{j-1}^m(x)| \; ;$$

and using a simple geometric argument, we obtain

$$\begin{aligned} &\left|\frac{x_j^{m+n}(x) - x_{j-1}^{m+n}(x)}{|x_j^{m+n}(x) - x_{j-1}^{m+n}(x)|} \cdot \left(\nu(x_{j-1}^m(x)) - \nu(x_{j-1}^{m+n}(x))\right)\right| \\ &\qquad \le \left|\nu(x_{j-1}^m(x)) - \nu(x_{j-1}^{m+n}(x))\right| \le \kappa_{\max}|x_{j-1}^{m+n}(x) - x_{j-1}^m(x)| \end{aligned}$$

Finally, we note that $0 < 1/R_j^m(x) < 1/\theta$ and $|1 - 1/R_j^m(x)| < 1 - 1/\vartheta < 1$. In light of these estimates using triangle inequality in equations (B.8), (B.9) and (B.10) yields the estimates (B.3) and (B.4) for the choice of the constant

$$C_3 = \frac{2\kappa_{\max}}{\cos\phi_0} + \frac{4\kappa_{\max}}{\cos^2\phi_0} + \frac{2\kappa_{\max}^2 d_{\max}}{\cos^2\phi_0} + \frac{2C_\kappa d_{\max}}{\cos\phi_0} + \frac{1}{d_{\min}} + \frac{d_{\max}}{d_{\min}^2} \; .$$

Similarly (B.5) and (B.6) hold with the same constant. □

Equation (3.5) and Definition 4.8 yield the following estimates.

Lemma B.4. *For any $2 \le j_1 \le j_2 \le m$, we have*

$$\prod_{q=j_1}^{j_2} c_q^m(x) \le \frac{d_{\max}}{d_{\min}} \qquad \text{and} \qquad \prod_{q=j_1}^{j_2} c_q \le \frac{d_{\max}}{d_{\min}} \tag{B.11}$$

Remark B.5. In what follows, we utilize the conventions that *an empty sum is* 0 and *an empty product is* 1.

Remark B.6. From now on, we set $\delta = \max\{\delta_1, \delta_2, 1/\theta\}$.

Lemma B.7. *There exists a constant $C_4 = C_4(K, \alpha)$ such that, for $1 \le j \le m$, we have*

$$\left| \frac{R_j^m(x)}{R_j^{m+n}(x)} - 1 \right| \le C_4 \delta^{m-j} \tag{B.12}$$

and, for $2 \le j \le m$, we have

$$\left| \frac{R_j^m(x)}{R_{j+n}^{m+n}(x)} - 1 \right| \le C_4 \left(\delta^j + \delta^{m-j} \right) \tag{B.13}$$

and

$$\left| \frac{L_j}{R_{j+n}^{m+n}(x)} - 1 \right| \le C_4 \left(\delta^j + \delta^{m-j} \right) \tag{B.14}$$

Proof. Using Lemma 4.4 in (B.3) and (B.4) yields with $C = C_2 C_3$

$$C^{-1} \left| \frac{R_j^m(x)}{R_j^{m+n}(x)} - 1 \right| \le \delta^{m-(j-1)+3(j-1)} (1+\delta) \prod_{q=2}^{j} c_q^m(x) + \sum_{l=0}^{j-2} \delta^{m-(j-1)+3l} \left(1 + \delta + \delta^2\right) \prod_{q=j-l+1}^{j} c_q^m(x)$$

On account of Lemma B.4, we therefore obtain

$$\left| \frac{R_j^m(x)}{R_j^{m+n}(x)} - 1 \right| \le \frac{C\delta}{1-\delta} \frac{d_{\max}}{d_{\min}} \delta^{m-j}$$

which proves (B.12). To prove (B.13), first we note that

$$\left| \frac{R_1^m(x)}{R_{1+n}^{m+n}(x)} - 1 \right| \le \frac{2\vartheta}{\theta} \le 2\vartheta\delta \ . \tag{B.15}$$

Using (B.15) and Lemma 4.3 in (B.5) yields with $C = \max\{C_1 C_3, 2\vartheta\}$

$$C^{-1} \left| \frac{R_j^m(x)}{R_{j+n}^{m+n}(x)} - 1 \right| \le \delta^{2j-1} \prod_{q=2}^{j} c_q^m(x) + \left(1 + \delta + \delta^2\right) \sum_{l=0}^{j-1} \left(\delta^{j-1+l} + \delta^{m-(j-1)+3l} \right) \prod_{q=j-l+1}^{j} c_q^m(x)$$

so that, using Lemma B.4, we obtain

$$\left| \frac{R_j^m(x)}{R_{j+n}^{m+n}(x)} - 1 \right| \le C \frac{d_{\max}}{d_{\min}} \max\left\{ \delta, \frac{1}{\delta} \right\} \frac{1+\delta+\delta^2}{1-\delta} \left(\delta^j + \delta^{m-j} \right)$$

proving (B.13). The proof of (B.14) is similar to that of (B.13); first one notes that (B.15) holds with $R_1^m(x)$ replaced by L_1, and uses this inequality together with Lemma 4.3 in (B.6) to obtain (B.14). $\square$

Lemma B.8. *There exists a constant $C_5 = C_5(K, \alpha)$ such that, for $[m/2] \le j \le m$, we have*

$$\left| \frac{R_j^m(x)}{R_{j+n}^{m+n}(x)} - 1 \right| \le C_5 \delta^j \tag{B.16}$$

Proof. It follows from (B.13) that with $p = [m/2]$

$$\left| \frac{R_p^m(x)}{R_{p+n}^{m+n}(x)} - 1 \right| \leq C_4 \left(\delta^p + \delta^{m-p}\right) \tag{B.17}$$

Using (B.17) and Lemma 4.3 in (B.5) yields, for $0 \leq j \leq m-p$,

$$C^{-1} \left| \frac{R_{p+j}^m(x)}{R_{p+j+n}^{m+n}(x)} - 1 \right| \leq \left(\delta^{p+2j} + \delta^{m-p+2j}\right) \prod_{q=1}^{j} c_{p+q}^m(x) + \left(1 + \delta + \delta^2\right) \sum_{l=0}^{j-1} \delta^{p-1+j+l} \prod_{q=j-l+1}^{j} c_{p+q}^m(x)$$

where $C = \max\{C_2 C_3, C_4\}$. Therefore, Lemma B.4 yields

$$\left| \frac{R_{p+j}^m(x)}{R_{p+j+n}^{m+n}(x)} - 1 \right| \leq \frac{4C}{\delta(1-\delta)} \frac{d_{\max}}{d_{\min}} \delta^{p+j}$$

for $0 \leq j \leq m-p$. □

Lemma B.9. *Let $\{A_j\}_{j\in\mathbb{Z}}$ and $\{B_j\}_{j\in\mathbb{Z}}$ be two sets of complex numbers. Then, for any $m_1 \leq m_2$, we have*

$$\left| \prod_{j=m_1}^{m_2} A_j - \prod_{j=m_1}^{m_2} B_j \right| \leq \sum_{j=m_1}^{m_2} |A_j - B_j| \prod_{p=m_1}^{j-1} |A_p| \prod_{q=j+1}^{m_2} |B_q| \tag{B.18}$$

and

$$\left| \prod_{j=m_1}^{m_2} A_j - 1 \right| \leq \exp\left(\sum_{j=m_1}^{m_2} |A_j - 1| \right) \sum_{j=m_1}^{m_2} |A_j - 1| \tag{B.19}$$

Proof. A straightforward induction yields (B.18). Applying (B.18) with $B_j = 1$, and noting that $|A_j| \leq 1 + |A_j - 1|$, we obtain

$$\left| \prod_{j=m_1}^{m_2} A_j - 1 \right| \leq \prod_{j=m_1}^{m_2} \left(1 + |A_j - 1|\right) \sum_{j=m_1}^{m_2} |A_j - 1| \,.$$

Since the function $f(x) = \ln(1+x) - x$ is decreasing on $[0, \infty)$ and $f(0) = 0$, (B.19) follows. □

Lemma B.10. *There exists a constant $C_6 = C_6(K, \alpha)$ such that with $p = [m/2]$*

$$\left| \prod_{j=1}^{p-1} \frac{R_j^m(x)}{R_j^{m+n}(x)} - 1 \right| \leq C_6 \delta^{m/2} \tag{B.20}$$

and

$$\left| \prod_{j=p}^{m} \frac{R_j^m(x)}{R_{j+n}^{m+n}(x)} - 1 \right| \leq C_6 \delta^{m/2} \tag{B.21}$$

and

$$\left| \prod_{j=p}^{p+n-1} \frac{L_{j-n}}{R_j^{m+n}(x)} - 1 \right| \leq C_6 \delta^{m/2-n} \tag{B.22}$$

provided $n < m/2$.

Proof. Applying (B.19) with $A_j = R_j^m(x)/R_j^{m+n}(x)$ yields via (B.12)

$$\left| \prod_{j=1}^{p-1} \frac{R_j^m(x)}{R_j^{m+n}(x)} - 1 \right| \leq C_4 \exp\left(C_4 \sum_{j=1}^{p-1} \delta^{m-j} \right) \sum_{j=1}^{p-1} \delta^{m-j}$$

so that

$$\left|\prod_{j=1}^{p-1}\frac{R_j^m(x)}{R_j^{m+n}(x)}-1\right|\le\frac{C_4}{1-\delta}\exp\left(\frac{C_4}{1-\delta}\right)\delta^{m/2}$$

which proves (B.20). The proof of (B.21) making use of (B.19) and Lemma B.8 is similar. Finally, to prove (B.22), applying (B.19) with $A_j=L_{j-n}/R_j^{m+n}(x)$ yields on account of (B.14)

$$\left|\prod_{j=p}^{p+n-1}\frac{L_{j-n}}{R_j^{m+n}(x)}-1\right|\le C_4\exp\left(C_4\sum_{j=p}^{p+n-1}(\delta^{j-n}+\delta^{m-(j-n)})\right)\sum_{j=p}^{p+n-1}(\delta^{j-n}+\delta^{m-(j-n)})$$

so that

$$\left|\prod_{j=p}^{p+n-1}\frac{L_{j-n}}{R_j^{m+n}(x)}-1\right|\le\frac{2C_4}{1-\delta}\exp\left(\frac{2C_4}{1-\delta}\delta^{p-n}\right)\delta^{p-n}$$

from which (B.22) follows at once. □

Remark B.11. For a positive real number A, we have

$$\left|\sqrt{A}-1\right|=\frac{|A-1|}{\sqrt{A}+1}\le|A-1|\ .$$

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