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Decompositions of the tensor products of
irreducible $\mathfrak{sl}(2)$ -modules in characteristic 3

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DECOMPOSITIONS OF THE TENSOR PRODUCTS OF IRREDUCIBLE $\mathfrak{sl}(2)$ -MODULES IN CHARACTERISTIC 3

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ABSTRACT. We completely describe the decompositions (into indecomposable submodules) of the tensor products of irreducible $\mathfrak{sl}(2)$ -modules in characteristic 3. The answer resembles analogous decompositions for the Lie superalgebras $\mathfrak{sl}(1|1)$.

1. Introduction

Texts devoted to representations of Lie algebras in characteristic $p > 0$ are often prefaced by the disclaimer that the spaces considered are of dimension less than p . To the best of the author's knowledge, this restriction is always imposed on tensor products of irreducible modules when studying the analog of Klebsch-Gordon decompositions. That is, if V and W are two irreducible modules over a simple Lie algebra \mathfrak{g} and one wishes to decompose $V \otimes W$ into indecomposable submodules, $\dim V \otimes W$ is always restricted to be less than p . In this paper, I remove this restriction and present a complete investigation of the decomposition of $V \otimes W$ for the case $\mathfrak{g} = \mathfrak{sl}(2)$ and $p = 3$ for any irreducible $\mathfrak{sl}(2)$ -modules V and W .

In $\mathfrak{sl}(2)$, we consider the natural basis

$$X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Hence, the structure constants are derived from the relations

$$[X_+, X_-] = H, \quad [H, X_{\pm}] = \pm 2X_{\pm}.$$

Let k be an algebraically closed field of characteristic 3. Irreducible $\mathfrak{sl}(2)$ -modules in characteristic $p > 2$ were completely described by Rudakov and Shafarevich in [5]. These modules are all of dimension $D \leq p$, and in the cases $D < p$, there is no difference from the case of characteristic zero (cf. [1]). For $p = 3$ and $D < 3$, these are only the modules denoted by $\underline{1}$,

$$0 \xleftarrow{X_+} V_0 \xrightarrow{X_-} 0, \tag{1}$$

and $\underline{2}$,

$$0 \xleftarrow{X_+} V_1 \xrightleftharpoons[X_+]{X_-} V_{-1} \xrightarrow{X_-} 0. \tag{2}$$

In these diagrams, V_{ρ} denotes the 1-dimensional *weight space* of eigenvectors of H with eigenvalue ρ . The arrows indicate the action of the operators in the sub- or superscript.

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1.1. Remark. More generally, over a field of prime characteristic p , we always have the irreducible $\mathfrak{sl}(2)$ modules \underline{N} for $N \in \{1, \dots, p\}$, given diagrammatically by

$$0 \xleftarrow[X_+]{X_-} V_{N-1} \xleftrightarrow[X_+]{X_-} V_{N-3} \xleftrightarrow[X_+]{X_-} \cdots \xleftrightarrow[X_+]{X_-} V_{-N+3} \xleftrightarrow[X_+]{X_-} V_{-N+1} \xrightarrow[X_-]{X_+} 0. \quad (3)$$

Let us return to our case of characteristic $p = 3$. For $D = 3$, i.e., 3-dimensional irreducible representations, we have more than in the case of characteristic 0, where there is only the module $\underline{3}$,

$$0 \xleftarrow[X_+]{X_-} V_{-1} \xleftrightarrow[X_+]{X_-} V_0 \xleftrightarrow[X_+]{X_-} V_1 \xrightarrow[X_-]{X_+} 0. \quad (4)$$

There is in fact an entire family of irreducible representations, parametrized by a 3-dimensional variety, of which $\underline{3}$ is a special case. Writing the images of the generators of $\mathfrak{sl}(2)$ as matrices acting on a 3-dimensional vector space, these representations are given as follows. First, we have the irreducible modules that we denote by $T(b, c, d)$:

$$X_- = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} d-1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d+1 \end{pmatrix} \quad X_+ = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & a_2 \\ b & 0 & 0 \end{pmatrix}, \quad (5)$$

where

$$\begin{aligned} a_1 &= bc + d - 1, \\ a_2 &= a_1 + d = bc - d - 1. \end{aligned} \quad (6)$$

We also have the family of “opposite” irreducible modules, where the forms of X_+ and X_- are exchanged, which we denote by $\tilde{T}(b, c, d)$:

$$X_- = \begin{pmatrix} 0 & 0 & b \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \end{pmatrix} \quad H = \begin{pmatrix} d-1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d+1 \end{pmatrix} \quad X_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c & 0 & 0 \end{pmatrix}. \quad (7)$$

In both of these cases, b , c , and d are arbitrary elements of the ground field k , *however* we don't allow the cases

$$T(0, 0, 1) \quad \text{or} \quad T(0, 0, -1), \quad (8)$$

since in these cases the representation is not irreducible. Once the other parameters are chosen, a_1 and a_2 are necessarily given in terms of b , c , and d by (6) if the matrices (5) and (7) are to be representations of $\mathfrak{sl}(2)$. (To see this, one can explicitly solve, for example, the equation $[X_+, X_-] = H$ for a_1 and a_2 using the above matrix representations, then check that the relations $[H, X_\pm] = \pm 2X_\pm$ are satisfied).

In addition to these two kinds of irreducible modules, we have each of their duals, which we will denote by $T^*(b, c, d)$ and $\tilde{T}^*(b, c, d)$.

1.2. Remark. Note that $\underline{3} \simeq T(0, 0, 0)$ as $\mathfrak{sl}(2)$ -modules.

All of these irreducible modules can be glued into the following indecomposables.

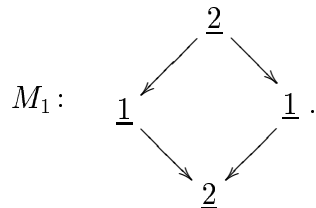
We let $M \oplus \widetilde{M}$ denote the *semidirect sum* of the subspaces M and \widetilde{M} . By this, we mean that \widetilde{M} is a submodule.

A diagram of subspaces (e.g. $M \rightarrow \widetilde{M}$) indicates something similar, but gives more information. A subspace that is the source of no arrows (in our example, \widetilde{M}) is a submodule. A subspace that is the source of some arrows becomes a submodule upon taking the quotient modulo the targets of those arrows (in our example, M/\widetilde{M}). Note that a subspace that is the source of some arrows cannot be selected uniquely, since it is a quotient space. Instead,

we should think of it as the span of a collection of vectors, the representatives of which form a basis for the quotient space.

The direction of an arrow in a diagram also carries information. An arrow pointing to the right indicates that we get an element of the target by acting on an element of the source with X_- . An arrow pointing to the left indicates the same for X_+ . Note that the “direction” of the arrow refers only to the left/right direction. That is, an arrow that points up/down and right is still thought of as an “arrow pointing to the right”, and similarly for up/down and left. We also say, taking the example from the previous paragraph, that “ M is glued in to \widetilde{M} via X_- ”.

For later, we let M_1 denote the submodule (cf. [4]):



To make the statements of the last two paragraphs concrete, we dissect this particular case. The symbol $\underline{2}$ at the bottom indicates an irreducible submodule. The symbol $\underline{1}$ on the left is represented by the span of a single vector v with $X_+v = 0$ and X_-v a vector of the irreducible submodule $\underline{2}$ at the bottom. Similarly, the symbol $\underline{1}$ on the right is represented by the span of a single vector w with $X_-w = 0$ and X_+w a vector of the irreducible submodule $\underline{2}$ at the bottom. Finally, the symbol $\underline{2}$ at the top stands for the span of two vectors, v' and w' , with

$$\begin{aligned} X_-v' &= w', & X_-w' &= \mu w, \\ X_+v' &= \lambda v, & X_+w' &= v', \end{aligned} \tag{9}$$

for some $\lambda, \mu \in k$, i.e. X_+v' is contained in the left “ $\underline{1}$ ” and X_-w' is contained in the right “ $\underline{1}$ ”.

The main result of the paper is the following theorem.

1.3. Theorem. *The decompositions (into indecomposable submodules) of tensor products of all irreducible $\mathfrak{sl}(2)$ -modules ($\underline{1}$, $\underline{2}$, $T(b, c, d)$, $\widetilde{T}(b, c, d)$, and their duals) are completely described by*

- (1) $\underline{1} \otimes V \simeq V$ for any $\mathfrak{sl}(2)$ -module V ,
- (2) $\underline{2} \otimes \underline{2} = \underline{1} \oplus \underline{3}$,
- (3) $\underline{2} \otimes \widetilde{T}(b, \frac{1}{b}, 0) = \widetilde{T}(b, \frac{1}{b}, 0) \oplus T(\frac{1}{b}, 0, 0)$,

together with Tables 3–7, found at the end of the paper.

1.4. Remark. Statement (1) of the theorem is obvious.

1.5. Remark. In §2, we will explain how we arrived at the list of modules examined in Tables 3–7, as well as why we are allowed to seemingly ignore certain modules.

The paper is organized as follows. In §2 we will closely examine the families of modules and reduce the number of different cases we must consider separately by demonstrating certain correspondences. Then, we will study the structure of the modules we are concerned with and, in particular, semidirect sums. In §3, we will briefly present the notation we use for the calculations. Finally, in §4, we break down the various tensor products we can form case by case and compute their decompositions.

All computations for this paper were made with the assistance of **SuperLie** ([2], [3]).

2. Preliminaries

Before we get into the meat of the paper, let us carefully describe the $\mathfrak{sl}(2)$ -modules in characteristic 3, first the irreducible ones, then certain indecomposables (to describe all indecomposables is an open problem).

We begin by proving a couple of lemmas which will help us to reduce the amount of work we have to do. In particular, we can show (via a change of basis) that some seemingly different modules are actually isomorphic. In the end, we will only have to consider the cases given in the following lemma (in addition, of course, to $\underline{2}$):

2.1. Lemma. *We can represent every 3-dimensional irreducible $\mathfrak{sl}(2)$ -module by a member of one of the following two families:*

- (1) $T(b, c, d)$, where $b, c, d \in k$ are arbitrary subject to (8);
- (2) $\tilde{T}(b, \frac{1}{b}, 0)$.

The proof of Lemma 2.1 is immediately implied by Lemmas 2.2, 2.3, and 2.4 below.

2.2. Lemma. *For the dual modules of $T(b, c, d)$ and $\tilde{T}(b, c, d)$, given any $b, c, d \in k$, we have:*

- (1) $T^*(b, c, d) \simeq T(b', c', d')$ for some $b', c', d' \in k$,
- (2) $\tilde{T}^*(b, c, d) \simeq \tilde{T}(b', c', d')$ for some $b', c', d' \in k$.

Proof. 1) Recall that, given a matrix representation X of the action of an element of a Lie algebra on a module, the action of X on the dual module is given by $-X^t$, i.e., the negative transpose of the original matrix. So here, the action of $\mathfrak{sl}(2)$ on the dual module $T^*(b, c, d)$ is as follows:

$$X_- = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -c & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} -d+1 & 0 & 0 \\ 0 & -d & 0 \\ 0 & 0 & -d-1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 0 & -b \\ -a_1 & 0 & 0 \\ 0 & -a_2 & 0 \end{pmatrix}. \quad (10)$$

We apply a similarity transformation given by the matrix

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (11)$$

We then rename the resulting matrices to $X'_\pm := SX_\pm S^{-1}$ and $H' := SHS^{-1}$:

$$X'_- = \begin{pmatrix} 0 & 0 & c' \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad H' = \begin{pmatrix} d'-1 & 0 & 0 \\ 0 & d' & 0 \\ 0 & 0 & d'+1 \end{pmatrix}, \quad X'_+ = \begin{pmatrix} 0 & a'_1 & 0 \\ 0 & 0 & a'_2 \\ b' & 0 & 0 \end{pmatrix}. \quad (12)$$

Here one easily checks that $a'_1 = a_2$, $a'_2 = a_1$, $b' = -b$, $c' = -c$, and $d' = -d$. We have transformed our original representation $T^*(b, c, d)$ into one of the form $T(b', c', d')$.

2) The action on the dual module $\tilde{T}^*(b, c, d)$ is given by:

$$X_- = \begin{pmatrix} 0 & -a_1 & 0 \\ 0 & 0 & -a_2 \\ -b & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} -d+1 & 0 & 0 \\ 0 & -d & 0 \\ 0 & 0 & -d-1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 0 & -c \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}. \quad (13)$$

Since this is in complete analogy with the previous case, we skip the detailed calculation. (We do note that one can, in fact, even use the same similarity transformation as in (11) above.) \square

2.3. Lemma. *For an appropriate choice of $b', c', d' \in k$ we have*

$$\tilde{T}(b, c, d) \simeq T(b', c', d'), \quad (14)$$

if and only if at most one of a_1, a_2 , and b are equal 0.

Proof. First, we note that if two of a_1, a_2 , and b are zero, then we will clearly get no isomorphism, since for modules of the form $T(b, c, d)$, the matrix of X_- can have at most one eigenvector with eigenvalue 0.

The proof of the converse statement is broken up into different cases.

1) $a_1, a_2 \neq 0$. We apply the similarity transformation with the matrix

$$S = \begin{pmatrix} \frac{1}{a_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a_1 a_2} \end{pmatrix}. \quad (15)$$

This yields matrices H', X'_\pm of the same form as in (12) above. In this case, using the same notation as above, we have $a'_1 = a_1$, $a'_2 = a_2$, $b' = \frac{c}{a_1 a_2}$, $c' = a_1 a_2 b$, and $d' = d$.

2) $a_1 = 0, a_2, b \neq 0$. We apply the similarity transformation with the matrix

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a_2} \\ \frac{1}{ba_2} & 0 & 0 \end{pmatrix}. \quad (16)$$

We get matrices as in the form of (12), where this time $a'_1 = a_2$, $a'_2 = bc$, $b' = \frac{1}{ba_2}$, $c' = 0$, and $d' = d + 1$.

3) $a_2 = 0, a_1, b \neq 0$. We apply the similarity transformation with the matrix

$$S = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{b} & 0 & 0 \\ 0 & \frac{1}{ba_1} & 0 \end{pmatrix}. \quad (17)$$

We again get matrices in the form of (12), where $a'_1 = bc$, $a'_2 = a_1$, $b' = \frac{1}{ba_1}$, $c' = 0$, and $d' = d - 1$. \square

The following lemma will tell us more about $\tilde{T}(b, c, d)$ when two of a_1, a_2 , and b are zero.

2.4. Lemma. *If two of a_1, a_2 , and b are zero, then $\tilde{T}(b, c, d) \simeq \tilde{T}(b, \frac{1}{b}, 0)$.*

Proof. There are three possibilities: $a_1, a_2 = 0$, $a_1, b = 0$, and $a_2, b = 0$. Let us examine each one in turn.

1) $a_1, a_2 = 0$. In this case, we have

$$\begin{aligned} bc + d - 1 &= 0, \\ bc - d - 1 &= 0. \end{aligned} \quad (18)$$

Subtracting the second equation from the first implies $d = 0$, which we can plug back into the first equation to get $bc - 1 = 0$, or $c = \frac{1}{b}$. Hence, the module must be of the form $\tilde{T}(b, \frac{1}{b}, 0)$.

2) $a_1, b = 0$. Here we have $0 = a_1 = bc + d - 1 = d - 1$. Hence, $d = 1$. Furthermore, by Lemma 2.5 below, we can transform $T(0, c, 1)$ into a representation with $d = 0$ provided that $c \neq 0$. Since X_- will still have two eigenvectors with eigenvalue 0, the transformed representation will necessarily be as in case 1). (When $a_2 = 0$ and $b = 0$, d is necessarily -1 ;

see case 3) below.) Therefore, the only truly new case is $\tilde{T}(0, 0, 1)$. However, $\tilde{T}(0, 0, 1)$ is not irreducible (see the Introduction) and, since we assumed our module to be irreducible, is disallowed. So we have now completely reduced to case 1).

3) $a_2, b = 0$. In this case, we get $0 = a_2 = bc - d - 1 = -d - 1$, or $d = -1$. As above, we can reduce this to case 1) if and only if $c \neq 0$. Therefore, the only new case is $\tilde{T}(0, 0, -1)$. However, from the Introduction we know that $\tilde{T}(0, 0, -1)$ is not irreducible, and as in case 2) we have now completely reduced to case 1). \square

We also want to examine the cases where $d = 0$ or ± 1 in more detail, since they will turn out to be special once we start tensoring. It turns out that all three of these cases correspond to a module where $d = 0$ unless $c = 0$:

2.5. Lemma. *Let $d = \pm 1$ and $c \neq 0$. Then*

$$T(b, c, d) \simeq T(b', c', 0) \quad (19)$$

for an appropriate choice of b' and c' .

The statement remains true if we replace T by \tilde{T} everywhere above.

Proof. We will prove the statement for $T(b, c, 1)$ (i.e. for $d = 1$). The other cases are completely analogous.

In this case, our representation is given by the following matrices:

$$X_- = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & a_2 \\ b & 0 & 0 \end{pmatrix}. \quad (20)$$

We apply the similarity transformation with matrix

$$S = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{c} & 0 & 0 \\ 0 & \frac{1}{c} & 0 \end{pmatrix}, \quad (21)$$

renaming the resulting matrices to $X'_\pm := SX_\pm S^{-1}$ and $H' := SHS^{-1}$ to get

$$X'_- = \begin{pmatrix} 0 & 0 & c' \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad H' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X'_+ = \begin{pmatrix} 0 & a'_1 & 0 \\ 0 & 0 & a'_2 \\ b' & 0 & 0 \end{pmatrix}. \quad (22)$$

Explicitly, we have $a'_1 = bc$, $a'_2 = a_1$, $b' = \frac{a_2}{c}$, and $c' = c$. So the change of basis gives us a representation of the form $T(b', c', 0)$. \square

Now that we have proved Lemma 2.1, we move on to studying semidirect sums and the structure of our $\mathfrak{sl}(2)$ -modules. Let S and T be two 3-dimensional irreducible $\mathfrak{sl}(2)$ -modules, and let us consider the tensor product $V = S \otimes T$ as an $\mathfrak{sl}(2)$ -module. This is a 9-dimensional space, but we may divide it into three distinguished 3-dimensional subspaces, the *weight spaces*, i.e., the eigenspaces of H . (It is simple to check that H has three distinct eigenvalues for any such S and T .)

2.6. Remark. For the remainder of this section, all vectors are assumed to be weight vectors, i.e., eigenvectors of H .

Thus, instead of considering the full 9-dimensional space V , we restrict attention to one of the 3-dimensional weight spaces. We denote them by

$$V_\rho = \{v \in V \mid Hv = \rho v\}. \quad (23)$$

The element $X_+X_- \in U(\mathfrak{sl}(2))$ acts on each weight space, since if $Hv = \rho v$ for some $\rho \in k$, then

$$\begin{aligned} H(X_+X_-v) &= X_+HX_-v + [H, X_+]X_-v \\ &= X_+X_-Hv + X_+[H, X_-]v + 2X_+X_-v \\ &= \rho X_+X_-v. \end{aligned} \tag{24}$$

Therefore, we may consider X_+X_- as a linear transformation of one such space and look for its eigenvalues and eigenvectors. We relate this to semidirect sums in the following lemma:

2.7. Lemma. *Let X_+X_- have two distinct eigenvalues, λ_1 and λ_2 , on the subspace V_ρ for some eigenvalue ρ of H . Suppose that V contains a semidirect sum $M \ltimes \widetilde{M}$. Further, let $\widetilde{M} \cap V_\rho = \text{span}(v_1, v_2)$, where $X_+X_-v_i = \lambda_i v_i$. Consider the action of $\mathfrak{sl}(2)$ on the quotient space and a vector $m \in M/\widetilde{M}$ with $Hm = \rho m$. Then $X_+X_-m = \lambda_i m$ for some i .*

Proof. Suppose, on the contrary, that $X_+X_-m = \mu m$ (equality being in the quotient space) for some $\mu \neq \lambda_i$ for all i . We will show that there is a vector $v \in V$ with $Hv = \rho v$ and $X_+X_-v = \mu v$, a contradiction.

For the remainder of the proof, the action of $\mathfrak{sl}(2)$ will be on the full space V , not the quotient space.

We know that $X_+X_-m = \mu m + \widetilde{m}$ for some $\widetilde{m} \in \widetilde{M}$. By the assumptions of the lemma, we can write $\widetilde{m} = \widetilde{m}_1 + \widetilde{m}_2$, where $X_+X_- \widetilde{m}_i = \lambda_i \widetilde{m}_i$. We set

$$v = m + \frac{1}{\mu - \lambda_1} \widetilde{m}_1 + \frac{1}{\mu - \lambda_2} \widetilde{m}_2. \tag{25}$$

Since $\mu \neq \lambda_1$ or λ_2 , v is well-defined, and it is easily seen that $X_+X_-v = \mu v$. \square

We can consider other eigenvalue equations on the weight spaces of a tensor product, in particular for X_+^3 and X_-^3 . The proof of the following lemma is straightforward and is left to the reader.

2.8. Lemma. *The action of X_+^3 and X_-^3 on any weight vector v in the module $V = S \otimes T$, where S and T are of the form $\widetilde{T}(b, \frac{1}{b}, 0)$ or $T(b, c, d)$, is given by:*

- (1) $\widetilde{T}(b, \frac{1}{b}, 0) \otimes \widetilde{T}(\beta, \frac{1}{\beta}, 0)$: $X_+^3 v = \frac{b+\beta}{b\beta} v$, $X_-^3 v = 0$
- (2) $\widetilde{T}(b, \frac{1}{b}, 0) \otimes T(\beta, \gamma, \delta)$: $X_+^3 v = (\frac{1}{b} + \beta\alpha_1\alpha_2)v$, $X_-^3 v = \gamma v$
- (3) $T(b, c, d) \otimes T(\beta, \gamma, \delta)$: $X_+^3 v = (ba_1a_1 + \beta\alpha_1\alpha_2)v$, $X_-^3 v = (c + \gamma)v$

Furthermore, assuming v is some weight vector, V contains highest weight vectors if and only if $X_+^3 v = 0$, and lowest weight vectors if and only if $X_-^3 v = 0$.

Combining Lemmas 2.7 and 2.8, we get:

2.9. Lemma. *Given a module $T(b, c, d)$ with no lowest weight vectors (i.e. $c \neq 0$), or a module $\widetilde{T}(b, \frac{1}{b}, 0)$, we can determine b , as well as c and d (where applicable), from the actions of X_+X_- and X_-^3 on the weight spaces. If $c = 0$ in $T(b, c, d)$, we may still determine b from the action of X_+^3 if we know d .*

Proof. Let us fix a basis $\{v_1, v_2, v_3\}$ such that the $\mathfrak{sl}(2)$ -action is given by the matrices of (5) or (7).

For $\widetilde{T}(b, \frac{1}{b}, 0)$ we must simply note that $X_+^3 v_1 = \frac{1}{b} v_1$.

For $T(b, c, d)$, where $c \neq 0$ we note that the equations

$$\begin{aligned} X_+X_-v_1 &= (bc - 1 + d)v_1, \\ X_+X_-v_3 &= bcv_3, \\ X_-^3v_1 &= cv_1 \end{aligned} \tag{26}$$

allow us to determine b , c , and d , since $c \neq 0$.

In the case of $T(b, 0, d)$ where d is known, we note that

$$X_+^3v_1 = ba_1a_2v_1 = b(1 - d^2)v_1, \tag{27}$$

allowing us to determine b . □

The number of linearly independent eigenvectors of X_+X_- will be important to us when determining the structure of a decomposition, as the following lemma shows.

2.10. Lemma. *Let there be either no highest or no lowest weight vectors in V .*

1) *If X_+X_- has three linearly independent eigenvectors v_1 , v_2 , and v_3 in V_ρ for an eigenvalue ρ of H , then there are 3-dimensional irreducible submodules $M, M', M'' \subset V$ such that $V = M \oplus M' \oplus M''$.*

2) *If X_+X_- has only two linearly independent eigenvectors v_1 and v_2 in V_ρ , then there are 3-dimensional submodules $M, M', M'' \subset V$ such that $V = M \oplus (M' \oplus M'')$. In this case, M' and M'' are irreducible.*

2.11. Remark. In the cases we consider, X_+X_- will always have at least two distinct eigenvectors on any V_ρ .

Proof. We will prove this for the case where $X_+v \neq 0$ for all $v \in V$. The other case is analogous.

Clearly, in an irreducible submodule of $\mathfrak{sl}(2)$, for any weight vector v , we have $X_+X_-v = \lambda v$ for some $\lambda \in k$. Further, a one-dimensional subspace belongs to at most one irreducible submodule. So we can have at most as many irreducible submodules as we have eigenvectors of X_+X_- in some V_ρ .

Furthermore, if $X_+X_-v = \lambda v$ for some $\lambda \in k$ and v a weight vector, then v , X_+v , and X_+^2v form an irreducible submodule. From Lemma 2.8, it is clear that X_+^3v is a nonzero multiple of v . From the relations of $\mathfrak{sl}(2)$, it is easy to check that $X_-X_+^n v$ is a multiple of $X_+^{n-1}v$ for any nonnegative integer n . So these three vectors do form a submodule, and it is irreducible because X_+^n sends any subspace to any other for some n .

This completes the proof of heading 1), since in that case we can build three 3-dimensional irreducible submodules by the procedure in the last paragraph. For heading 2), we can only build two such irreducible submodules M' and M'' this way. To complete the proof, we consider $M = V/(M' \oplus M'')$ and the action of $\mathfrak{sl}(2)$ on this quotient. We take any nonzero weight vector $m \in M$ and note again that m , X_+m , and X_+^2m form a basis of this quotient module, and that it is irreducible. They are nonzero because X_+^3m is a nonzero multiple of m . They are linearly independent because they are of different weights, and they form an irreducible submodule because there is no invariant subspace (X_+^n sends any subspace to any other for some n). □

3. Notation

For the sake of convenience, we fix the notation for our modules for the rest of the paper. We denote vectors in our modules according to the following scheme:

$$\begin{aligned} q_i &\in \underline{2} \text{ for } i = 1, 2; \\ t_i &\in \tilde{T}(b, \frac{1}{b}, 0), \quad u_i \in \tilde{T}(\beta, \frac{1}{\beta}, 0) \text{ for } i = 1, 2, 3; \\ v_i &\in T(b, c, d), \quad w_i \in T(\beta, \gamma, \delta) \text{ for } i = 1, 2, 3. \end{aligned} \tag{28}$$

The index in the subscript refers to the weight of the vector. For $\underline{2}$, the vector q_1 is of weight 1 and q_2 is of weight -1 . For t_i , u_i , v_i , and w_i , we refer to the matrix representations (5) and (7) and set $t_i = e_i$, $u_i = e_i$, etc., where $\{e_i \mid i = 1, 2, 3\}$ is the usual basis of k^3 .

In addition, since we will need them so often and the notation becomes awkward, we will omit the “ \otimes ” when writing tensor products of vectors. For example, $q_1 \otimes v_2 \in \underline{2} \otimes T(b, c, d)$ will be expressed simply as $q_1 v_2$.

4. Case-by-case calculations

All calculations were done with the assistance of the *Mathematica*-based package *SuperLie* ([2], [3]).

Due to space considerations, and for the flow of arguments, calculations will not be repeated here in detail. Instead, we refer to the *Mathematica* notebooks, which have been made available online at:

<http://personal-homepages.mis.mpg.de/clarke/Tensor-Calculations.tar.gz>

We have focused on producing explicit decompositions where possible. Many decompositions could have also been deduced using more general arguments based on the lemmas of §2, as was done in 4.5.10 or 4.6.19.

4.1. The case $V = \underline{2} \otimes \underline{2}$. This is the simplest case of all; in fact, there is no difference from the decomposition in characteristic 0. We start from the two highest weight vectors, $r_1 r_1$ and $r_2 r_1 - r_1 r_2$. Applying X_- gives an irreducible submodule of the form $\underline{3}$ from the first, and $\underline{1}$ from the second. Hence, we have a direct sum of two irreducible modules,

$$\underline{2} \otimes \underline{2} = \underline{3} \oplus \underline{1}. \tag{29}$$

This could have been deduced from more general considerations as well. Working over fields of characteristic p , if we have two modules of the form \underline{N} and \underline{M} , where $N, M \in \{1, \dots, p\}$ (defined in (3)), they have highest weights $N - 1$ and $M - 1$, respectively. Taking their tensor product $\underline{N} \otimes \underline{M}$ will give us vectors with weights at most $N + M - 2$. If $N + M - 2 < p$, then we will see behavior that is no different from the case of characteristic 0. Things start getting interesting when $N + M - 2 \geq p$, which we will have in all following cases.

4.2. The case $V = \underline{2} \otimes \tilde{T}(b, \frac{1}{b}, 0)$. In this case, we find two highest weight vectors, $q_2 t_1$, and $q_2 t_2$. Applying X_- to these, we construct two 3-dimensional irreducible submodules that supply a complete decomposition. From $q_2 t_1$, we get the module $T(\frac{1}{b}, 0, 0)$, and from $q_2 t_2$, we get the module $\tilde{T}(b, \frac{1}{b}, 0)$. So

$$\underline{2} \otimes \tilde{T}(b, \frac{1}{b}, 0) = T(\frac{1}{b}, 0, 0) \oplus \tilde{T}(b, \frac{1}{b}, 0). \tag{30}$$

4.3. The case $V = \underline{2} \otimes T(b, c, d)$. In this case, we have the lowest weight vectors q_2v_3 and $q_1v_3 - q_2v_2$ if and only if $c = 0$. We have highest weight vectors if and only if we are in one of the following situations:

- (1) q_1v_1 and $q_1v_2 + (1 - d)q_2v_1 \iff b = 0$,
- (2) q_1v_3 and $q_1v_1 - bq_2v_3 \iff 1 - bc + d = 0$,
- (3) q_1v_2 and $q_1v_3 + (1 - bc + d)q_2v_2 \iff 1 - bc - d = 0$.

The “only if” part of these statements is provided by Lemma 2.8.

Furthermore, the eigenvalues of H are d and $d \pm 1$. Hence, whenever $d = 0, 1$, or 2 , we will have different behavior—e.g., the appearance of submodules like $\underline{3}$ and $\underline{1} \rightarrow \underline{2}$.

Therefore, we divide our computations into subcases. For the precise breakdown of these subcases, we note that for $c = 0$, whether we have the highest weight vectors described in (2) and (3) above depends only on d —we have (2) if and only if $d = -1$ and (3) if and only if $d = 1$. Furthermore, if $c \neq 0$, then we need not consider $d = \pm 1$ separately from $d = 0$, since in both of these cases $T(b, c, d)$ is isomorphic to a module with $T(b', c', 0)$ for some b' and c' by Lemma 2.5.

4.3.1. The subcase $c = 0; d = 0; b = 0$. In this case, by acting on the two lowest and highest weight vectors listed above, we compute a submodule of the form $\underline{1} \rightarrow \underline{2} \leftarrow \underline{1}$. We quickly verify that this contains all highest and lowest weight vectors.

The full module has dimension six, so we do not yet have a complete decomposition. Since any weight vectors not contained in $\underline{1} \rightarrow \underline{2} \leftarrow \underline{1}$ have weights 1 and -1 , we know that they will form a module of the form $\underline{2}$ after quotienting by $\underline{1} \rightarrow \underline{2} \leftarrow \underline{1}$ (to form two modules $\underline{1}$, they would have to have weight 0). A direct computation shows that all together, the module decomposes as M_1 .

4.3.2. The subcase $c = 0; d = 0; b \neq 0$. Here, we only have lowest weight vectors to work with. Acting on them by X_+ gives rise to one irreducible submodule, $M := \tilde{T}(\frac{1}{b}, b, 0)$, which contains both lowest weight vectors.

Since there are no more highest or lowest weight vectors, there are now two possibilities. Either the remainder of the full module forms a submodule of the form $T(b', c', d')$ with no highest or lowest weight vectors, or the remainder forms irreducible submodules only upon taking some quotients.

The first possibility can hold if and only if X_+X_- has a basis of eigenvectors for each weight space. A quick check shows that the minimal polynomial of X_+X_- is $(\lambda - 1)^2$ when acting on the space of weight 1 vectors. Hence, the module contains no more irreducible submodules.

A direct computation shows us now that the quotient module V/M is again of the form $\tilde{T}(\frac{1}{b}, b, 0)$, so we get the complete decomposition

$$\tilde{T}\left(\frac{1}{b}, b, 0\right) \oplus \tilde{T}\left(\frac{1}{b}, b, 0\right). \quad (31)$$

4.3.3. The subcase $c = 0; d = 1; b = 0$. We begin by acting on the lowest weight vectors by X_+ , which gives us two irreducible submodules, $\underline{3}$ and $\underline{1}$. This exhausts the lowest weight vectors, yet V is not yet completely decomposed—we are missing a 2-dimensional subspace.

Since $1 - bc - d = 1 - d = 0$, there is one remaining highest weight vector. Acting on it by X_- gives us the final submodule $\underline{2}$, which is glued into $\underline{1}$ via X_- . Hence, we get

$$\underline{3} \oplus (\underline{2} \rightarrow \underline{1}). \quad (32)$$

4.3.4. The subcase $c = 0; d = 1; b \neq 0$. As in the last case, we immediately get two irreducible submodules, $\underline{3}$ and $\underline{1}$. Here, there are no remaining highest weight vectors, but

we note that X_+X_- has a basis of eigenvectors for the space of weight 1 vectors. We use this to compute that the remaining 2-dimensional submodule is irreducible after quotienting by $\underline{1}$, so we get

$$\underline{3} \oplus (\underline{2} \oplus \underline{1}). \quad (33)$$

4.3.5. *The subcase $c = 0; d = 2; b = 0$.* This case proceeds exactly analogously to Section 4.3.3. However, since we have different highest weight vectors (here $1 - bc + d = 1 + d = 0$), we get the slightly different decomposition

$$\underline{3} \oplus (\underline{1} \rightarrow \underline{2}). \quad (34)$$

4.3.6. *The subcase $c = 0; d = 2; b \neq 0$.* As above, this is completely analogous to Section 4.3.4, but we get the decomposition

$$\underline{3} \oplus (\underline{1} \oplus \underline{2}). \quad (35)$$

4.3.7. *The subcase $c = 0; d \neq 0, 1, 2$.* In this case, we may have highest weight vectors if $b = 0$, but it turns out that in either case, acting on the lowest weight vectors by X_+ immediately gives us a complete decomposition,

$$T\left(\frac{b(d-1)}{d}, 0, d-1\right) \oplus T\left(\frac{b(d+1)}{d}, 0, d+1\right). \quad (36)$$

4.3.8. *The subcase $c \neq 0; d = 0; b = 0$.* By acting on the highest weight vectors with X_- , we get a module of the form $M := T(0, c, 1)$, which exhausts the highest weight vectors. Since there are no more highest or lowest weight vectors, we have the same situation as in Section 4.3.2. As there, we can check that X_+X_- does not have a basis of eigenvectors for the space of weight 1 vectors. We then directly compute that V/M is again of the form $T(0, c, 1)$. In all, we get

$$T(0, c, 1) \oplus T(0, c, 1) \quad (37)$$

4.3.9. *The subcase $c \neq 0; d = 0; b = \frac{1}{c}$.* In this case there are three highest weight vectors, since $1 - bc + d = 1 - bc - d = 0$. By acting on these with X_- , we get a complete decomposition,

$$T(0, c, 1) \oplus T(0, c, 1). \quad (38)$$

4.3.10. *The subcase $c \neq 0; d = 0; b \neq 0$ or $\frac{1}{c}$.* Here there are no highest or lowest weight vectors. However, the minimal polynomial for X_+X_- acting on the space of weight 0 vectors is $\lambda^2 + bc\lambda + bc(bc - 1)$, which has the two distinct roots $bc \pm \sqrt{bc}$. Hence, X_+X_- has a basis of eigenvectors for this space. Explicitly solving for these eigenvectors and acting on them by X_{\pm} gives the decomposition

$$T\left(b + \sqrt{\frac{b}{c}}, c, 1\right) \oplus T\left(b - \sqrt{\frac{b}{c}}, c, 1\right). \quad (39)$$

4.3.11. *The subcase $c \neq 0; d \neq 0, 1$ or $2; b = 0$.* In this case, acting by X_- on the two highest weight vectors gives us a complete decomposition,

$$T(0, c, d+1) \oplus T(0, c, d-1). \quad (40)$$

4.3.12. *The subcase $c \neq 0; d \neq 0, 1$ or $2; 1 - bc + d = 0$.* Here again, acting by X_- on the two highest weight vectors gives us a complete decomposition,

$$T(0, c, d) \oplus T(0, c, d+1). \quad (41)$$

4.3.13. *The subcase $c \neq 0; d \neq 0, 1$ or $2; 1 - bc - d = 0$.* Here again, acting by X_- on the two highest weight vectors gives us a complete decomposition,

$$T(0, c, d-1) \oplus T(0, c, d). \quad (42)$$

4.3.14. *The subcase $c \neq 0$; $d \neq 0, 1$ or 2 ; $b \neq 0$; $1 - bc \pm d \neq 0$.* As in Section 4.3.10, there are no highest or lowest weight vectors. Investigating the action of X_+X_- on the space of weight d vectors shows that it has eigenvalues $bc - d \pm \sqrt{bc + d^2}$, and that X_+X_- has a basis of eigenvectors for this space if and only if the eigenvalues are distinct, i.e., if $bc + d^2 \neq 0$.

If $bc + d^2 \neq 0$, we solve for these eigenvectors and act on them by X_\pm to get a complete decomposition,

$$T\left(b + \frac{d + \sqrt{bc + d^2}}{c}, c, d + 1\right) \oplus T\left(b + \frac{d - \sqrt{bc + d^2}}{c}, c, d + 1\right). \quad (43)$$

If $bc + d^2 = 0$, the two eigenvectors from above degenerate to one, and acting on it we get a submodule $M := T(b + \frac{d}{c}, c, d + 1)$. We can then explicitly compute that $V/M = T(b + \frac{d}{c}, c, d + 1)$ as well. So in this case, the complete decomposition is

$$T\left(b + \frac{d}{c}, c, d + 1\right) \oplus T\left(b + \frac{d}{c}, c, d + 1\right). \quad (44)$$

4.4. The case $V = \tilde{T}(b, \frac{1}{b}, 0) \otimes \tilde{T}(\beta, \frac{1}{\beta}, 0)$. In this case, by Lemma 2.8, there are always lowest weight vectors, namely t_1u_1 , t_1u_2 , t_2u_1 , $t_1u_3 + t_3u_1$, and t_2u_2 . We have the highest weight vectors $bt_1u_1 + t_2u_3 - t_3u_2$, $bt_1u_2 - bt_2u_1 - t_3u_3$, and $t_1u_3 - t_2u_2 + t_3u_1$ if and only if $\frac{b+\beta}{b\beta} = 0$, or equivalently $\beta = -b$. Therefore, we need to split this case into two subcases.

4.4.1. *The subcase $\beta = -b$.* Here, we start by acting on the highest weight vectors by X_- . This immediately generates three irreducible modules, $\underline{3}$, $\underline{1}$, and $\underline{2}$. This exhausts all of the highest weight vectors.

Since the full module has two lowest weight vectors not contained in the above submodules, there are two possibilities. Either these lowest weight vectors make up a module of the form $\tilde{T}(b', \frac{1}{b'}, 0)$, or they form irreducible modules only upon quotienting. The first possibility is ruled out since X_+^3 of any weight vector is zero, which is not the case for any $\tilde{T}(b', \frac{1}{b'}, 0)$. Hence, we have the second possibility.

By acting on the remaining lowest weight vectors by X_+ , we obtain a module $\underline{2}$ glued into the module $\underline{1}$ via X_+ , and a module $\underline{1}$ glued into the module $\underline{2}$ via X_+ . Hence the full decomposition is

$$\underline{3} \oplus (\underline{2} \leftarrow \underline{1}) \oplus (\underline{1} \leftarrow \underline{2}). \quad (45)$$

4.4.2. *The subcase $\beta \neq -b$.* In this case, acting on the lowest weight vectors by X_+ immediately gives us a complete decomposition,

$$T\left(\frac{b+\beta}{b\beta}, 0, 0\right) \oplus \tilde{T}\left(\frac{b\beta}{b+\beta}, \frac{b+\beta}{b\beta}, 0\right) \oplus \tilde{T}\left(\frac{b\beta}{b+\beta}, \frac{b+\beta}{b\beta}, 0\right). \quad (46)$$

4.5. The case $V = \tilde{T}(b, \frac{1}{b}, 0) \otimes T(\beta, \gamma, \delta)$. Again referring to Lemma 2.8, we determine that V contains the lowest weight vectors t_2w_3 , $bt_1w_2 - t_3w_3$, and t_1w_3 if and only if $\gamma = 0$. Furthermore, V contains the highest weight vectors

$$\begin{aligned} & t_1w_1 - \beta t_2w_3 - \beta(1 - \beta\gamma + \delta)t_3w_2, \\ & t_1w_2 + (1 - \gamma - \delta)t_2w_1 - \beta(1 - \beta\gamma - \delta)t_3w_3, \\ & t_1w_3 + (1 - \beta\gamma + \delta)t_2w_1 + (1 - \beta\gamma + \delta)(1 - \beta\gamma - \delta)t_3w_1 \end{aligned} \quad (47)$$

if and only if

$$1 + b\beta(1 - \beta\gamma + \delta)(1 - \beta\gamma - \delta) = 0. \quad (48)$$

Furthermore, similarly to the case of $\underline{2} \otimes T(b, c, d)$, the eigenvalues of H are δ and $\delta \pm 1$.

Based on this, we split this case into subcases. Note that when we have both that $\gamma = 0$ and $\delta = \pm 1$, it is impossible to have highest weight vectors. In addition, if $\gamma \neq 0$, then we

need not consider the cases $\delta = \pm 1$ separately from $\delta = 0$, since by Lemma 2.5, $T(\beta, \gamma, \pm 1)$ is isomorphic to a module $T(\beta', \gamma', 0)$ for some β', γ' .

4.5.1. *The subcase $\gamma = 0; \delta = 0; \beta = -\frac{1}{b}$.* In this case, we begin by acting on the three available lowest weight vectors by X_+ , followed by acting on the highest weight vectors by X_- . This immediately allows us to compute two indecomposable submodules of the forms $\underline{3}$ and $\underline{1} \rightarrow \underline{2} \leftarrow \underline{1}$.

The above submodules have dimension seven, while V has dimension nine. The weight vectors linearly independent from the above submodules have weights 1 and -1 . Since there are no more highest or lowest weight vectors, these vectors form an irreducible module of the form $\underline{2}$ only after passage to the quotient. A quick check of the minimal equation of X_+X_- acting on the space of vectors of weight 1 shows that X_+X_- has only two eigenvectors on this space. So we start with an arbitrary vector that is linearly independent from the submodules of the last paragraph and find that the complete decomposition is of the form

$$\underline{3} \oplus (\underline{2} \oplus (\underline{1} \rightarrow \underline{2} \leftarrow \underline{1})). \quad (49)$$

4.5.2. *The subcase $\gamma = 0; \delta = 0; \beta \neq -\frac{1}{b}$.* Here, we have three lowest weight vectors and no highest weight vectors. Acting on the lowest weight vectors yields two submodules, $\tilde{T}(\frac{b}{1+b\beta}, \frac{1+b\beta}{b}, 0)$ and $T(\frac{1+b\beta}{b}, 0, 0)$, which contain all of the lowest weight vectors.

One quickly checks that X_+X_- does not have a basis of eigenvectors for the space of weight 1 vectors. Therefore, the subspace of remaining vectors will be irreducible only upon quotienting. A direct computation shows that the quotient is of the form $\tilde{T}(\frac{b}{1+b\beta}, \frac{1+b\beta}{b}, 0)$, giving the complete decomposition

$$\tilde{T}\left(\frac{b}{1+b\beta}, \frac{1+b\beta}{b}, 0\right) \oplus \left(\tilde{T}\left(\frac{b}{1+b\beta}, \frac{1+b\beta}{b}, 0\right) \oplus T\left(\frac{1+b\beta}{b}, 0, 0\right)\right). \quad (50)$$

4.5.3. *The subcase $\gamma = 0; \delta = 1$.* In this case there are three lowest weight vectors but, as noted above, no highest weight vectors. By acting on the lowest weight vectors with X_+ , we immediately get two submodules, $\tilde{T}(b, \frac{1}{b}, 0)$ and $T(\frac{1}{b}, 0, 0)$.

The above submodules contain all of the highest and lowest weight vectors, and we can check that X_+X_- does not have a basis of eigenvectors for any weight space (actually, checking one particular weight space suffices). Hence, starting with an arbitrary weight vector that is linearly independent from the two submodules above, we compute the final submodule, which is irreducible upon quotienting and gives the decomposition

$$\tilde{T}\left(b, \frac{1}{b}, 0\right) \oplus \left(\tilde{T}\left(b, \frac{1}{b}, 0\right) \oplus T\left(\frac{1}{b}, 0, 0\right)\right). \quad (51)$$

4.5.4. *The subcase $\gamma = 0; \delta = 2$.* This case is completely analogous to that of Section 4.5.3. The end result is also identical.

4.5.5. *The subcase $\gamma = 0; \delta \neq 0, 1$ or $2; b\beta(1 - \delta^2) = -1$.* For this case, please refer to Section 4.5.9 below. The calculation there is also completely valid for the case $\gamma = 0$.

4.5.6. *The subcase $\gamma = 0; \delta \neq 0, 1$ or $2; b\beta(1 - \delta^2) \neq -1$.* Here, there are no highest weight vectors, but acting on the three lowest weight vectors by X_+ gives the entire decomposition,

$$T(1 + b\beta(1 - \delta^2), 0, \delta - 1) \oplus T(1 + b\beta(1 - \delta^2), 0, \delta) \oplus T(1 + b\beta(1 - \delta^2), 0, \delta + 1). \quad (52)$$

4.5.7. *The subcase $\gamma \neq 0; \delta = 0; b\beta(1 - \beta\gamma)^2 = -1$.* Here there are no lowest weight vectors, but three highest weight vectors. Acting on these by X_- yields two 3-dimensional submodules, $T(0, \gamma, 0)$ and $T(0, \gamma, -1)$.

A quick check then shows that X_+X_- does not have a basis of eigenvectors for any weight space. However, by selecting a weight vector linearly independent from the two submodules

above and applying X_- to it, we find that the quotient of V by the two submodules above is of the form $T(0, \gamma, -1)$. In all, we get

$$T(0, \gamma, -1) \oplus (T(0, \gamma, 0) \otimes T(0, \gamma, -1)) \quad (53)$$

4.5.8. *The subcase $\gamma \neq 0$; $\delta = 0$; $b\beta(1 - \beta\gamma)^2 \neq -1$.* This case is covered by the calculation of Section 4.5.10 below. Note that in the current case, the condition (58) cannot hold. The assumption $\delta = 0$ implies $(\delta(\delta + 1)(\delta - 1))^2 = 0$ and we have assumed that

$$1 + b\beta(1 - \beta\gamma + \delta)(1 - \beta\gamma - \delta) \neq 0. \quad (54)$$

4.5.9. *The subcase $\gamma \neq 0$; $\delta \neq 0, 1$ or 2 ; $b\beta(1 - \beta\gamma + \delta)(1 - \beta\gamma - \delta) = -1$.* In this case, there are no lowest weight vectors, but three highest weight ones. Acting on them by X_- yields a full decomposition,

$$T(0, \gamma, \delta - 1) \oplus T(0, \gamma, \delta) \oplus T(0, \gamma, \delta + 1). \quad (55)$$

4.5.10. *The subcase $\gamma \neq 0$; $\delta \neq 0, 1$ or 2 ; $b\beta(1 - \beta\gamma + \delta)(1 - \beta\gamma - \delta) \neq -1$.* Here, we have neither highest nor lowest weight vectors to exploit. Hence, we must rely on Lemmas 2.8 and 2.10 and examine the eigenspaces of X_+X_- .

We begin with the space $V_{\delta+1}$. A straightforward computation shows that the characteristic polynomial of X_+X_- acting on $V_{\delta+1}$ is

$$\lambda^3 + (1 - \delta^2)\lambda^2 + \lambda - \frac{\gamma}{b}(1 + b\beta(1 - \beta\gamma + \delta)(1 - \beta\gamma - \delta)). \quad (56)$$

To apply Lemma 2.10, we must know how many linearly independent eigenvectors X_+X_- has in $V_{\delta+1}$; that is, we must know what the *minimal* polynomial of X_+X_- is.

Noting that, in a field of characteristic 3, $(\lambda - \mu)^3 = \lambda^3 - \mu^3$, it is easily seen that (56) cannot be written as a perfect cube. So it must have at least two solutions.

We then note that

$$(\lambda - \mu)^2(\lambda - \rho) = \lambda^3 + (\mu - \rho)\lambda^2 + (\mu^2 - \mu\rho)\lambda - \mu^2\rho. \quad (57)$$

By equating coefficients of λ , we deduce that (56) and (57) can be equal if and only if $\mu = 1 - \delta^2$ and $\rho = -\delta^2$, as well as the following relation between the parameters of $\tilde{T}(b, \frac{1}{b}, 0)$ and $T(\beta, \gamma, \delta)$ is satisfied:

$$\frac{\gamma}{b}(1 + b\beta(1 - \beta\gamma + \delta)(1 - \beta\gamma - \delta)) = \mu^2\rho = -\delta^2(1 - \delta)^2(1 + \delta)^2. \quad (58)$$

Of course, even if the characteristic polynomial has a root of multiplicity two, X_+X_- may still have a basis of eigenvectors if its minimal polynomial factors into distinct linear factors, that is if X_+X_- satisfies the equation

$$(X_+X_- - (1 - \delta^2)I)(X_+X_- + \delta^2I) = 0, \quad (59)$$

where I is the identity operator on $V_{\delta+1}$. However, a direct computation shows that this is never the case.

This argumentation shows that we have a direct sum decomposition,

$$T(b_1, c_1, d_1) \oplus T(b_2, c_2, d_2) \oplus T(b_3, c_3, d_3), \quad (60)$$

if (58) does not hold. If (58) holds, we have a decomposition involving a semidirect sum,

$$T(b_3, c_3, d_3) \oplus (T(b_1, c_1, d_1) \oplus T(b_2, c_2, d_2)). \quad (61)$$

Now we are interested in determining the possible values of the parameters b_i, c_i, d_i . Let ρ_1, ρ_2 , and ρ_3 be the (not necessarily distinct) roots of the polynomial (56), and assume that at least $\rho_1 \neq \rho_2$. Now, let $v_{3,i}$ be the distinct eigenvectors of X_+X_- in $V_{\delta+1}$, so that i ranges

from one to the number of distinct eigenvectors. Finally, set $v_{2,i} = X_+ v_{3,i}$ and $v_{1,i} = X_+^2 v_{3,i}$. We take $\{v_{1,i}, v_{2,i}, v_{3,i}\}$ as our basis for $T(b_i, c_i, d_i)$ in the matrix representation (5).

With this notation, and recalling Lemma 2.9, we can determine the parameters. We already know that $X_-^3 v_{j,i} = \gamma v_{j,i}$ for all i and j , so $c_i = \gamma$ for all i . Now, on the one hand, $X_+ X_- v_{3,i} = b_i c_i v_{3,i}$, and the other hand, we know that $X_+ X_- v_{3,i} = \rho_i v_{3,i}$. Therefore, $b_i = \frac{\rho_i}{\gamma}$. Finally, since each $v_{3,i}$ belongs to $V_{\delta+1}$, we know from the algebra equations for $\mathfrak{sl}(2)$ that $v_{2,i}$ is of weight $(\delta + 1) + 2 = \delta$. So $d_i = \delta$. This determines the parameters completely for the case that $X_+ X_-$ has a basis of eigenvectors for $V_{\delta+1}$.

If $X_+ X_-$ has only two distinct eigenvectors for $V_{\delta+1}$, b_3 is yet to be determined. To do this, we note the following general fact. Let $A : W \rightarrow W$ be a linear endomorphism of a finite-dimensional vector space W , and let U be an A -invariant subspace of W . Furthermore, let the minimal polynomial of $A : W \rightarrow W$ be m , that of $A : U \rightarrow U$ be m_1 , and that of the induced map $A : W/U \rightarrow W/U$ be m_2 . Then $m = m_1 \cdot m_2$.

With this in mind, assume we have nondistinct eigenvalues $\rho_1 = \rho_3 = 1 - \delta^2$ and $\rho_2 = -\delta^2$. Then the minimal polynomial of $X_+ X_-$ on $V_{\delta+1}$ is

$$m = (\lambda - (1 - \delta^2))^2 (\lambda + \delta^2). \quad (62)$$

Using the notation of the previous paragraph with $A = X_+ X_-$, $W = V_{\delta+1}$, and $U = \text{span}(v_{3,1}, v_{3,2})$, we then have

$$m_1 = (\lambda - (1 - \delta^2))(\lambda + \delta^2), \quad (63)$$

implying $m_2 = (\lambda - (1 - \delta^2))$. Hence, letting $v_{3,3}$ be a representative for a nonzero vector in the quotient space W/U , we get $X_+ X_-$ is $X_+ X_- v_{3,3} = (1 - \delta^2) v_{3,3}$. Therefore,

$$b_3 = \frac{1 - \delta^2}{c}. \quad (64)$$

Thus, we have determined all parameters for the decomposition.

4.6. The case $T(b, c, d) \otimes T(\beta, \gamma, \delta)$. By Lemma 2.8, we have lowest weight vectors if and only if $c + \gamma = 0$, and highest weight vectors if and only if $ba_1 a_2 + \beta \alpha_1 \alpha_2 = 0$, where

$$\begin{aligned} a_1 &= bc + d - 1, & \alpha_1 &= \beta \gamma + \delta - 1, \\ a_2 &= bc - d - 1, & \alpha_2 &= \beta \gamma - \delta - 1. \end{aligned} \quad (65)$$

The lowest weight vectors are then

$$\begin{aligned} &cv_1 w_1 + v_2 w_3 - v_3 w_2, \\ &cv_1 w_2 - cv_2 w_1 - v_3 w_3, \\ &v_1 w_3 - v_2 w_2 + v_3 w_1, \end{aligned} \quad (66)$$

and the highest weight vectors are

$$\begin{aligned} &a_1 a_2 v_1 w_1 - \beta a_2 v_2 w_3 + \beta \alpha_2 v_3 w_2, \\ &a_1 a_2 v_1 w_2 - a_2 \alpha_1 v_2 w_1 + \beta \alpha_1 v_3 w_3, \\ &a_1 a_2 v_1 w_3 - a_2 \alpha_2 v_2 w_2 + \alpha_1 \alpha_2 v_3 w_1. \end{aligned} \quad (67)$$

The highest weight vectors were computed using the following method (here, as an example, for the weight space $\text{span}(v_1 w_1, v_2 w_3, v_3 w_2)$). We have, for the action of X_+ on this weight space,

$$\begin{aligned} X_+(v_1 w_1) &= \beta v_1 w_3 + b v_3 w_1, \\ X_+(v_2 w_3) &= a_1 v_1 w_3 + \alpha_2 v_2 w_2, \\ X_+(v_3 w_2) &= a_2 v_2 w_2 + \alpha_1 v_3 w_1. \end{aligned} \quad (68)$$

b	d	β	δ
0	1	0	1
0	-1	0	-1
0	1	$\frac{1}{\gamma}$	0
0	-1	0	1
0	1	0	-1
0	-1	$\frac{1}{\gamma}$	0
$\frac{1}{c}$	0	0	1
$\frac{1}{c}$	0	0	-1
$\frac{1}{c}$	0	$\frac{1}{\gamma}$	0

TABLE 1. Parameter values for which all “representations” of some highest weight vector are zero

We first try to cancel the factors of v_1w_3 , noting that

$$X_+(a_1v_1w_1 - \beta v_1w_3) = -\beta\alpha_2v_2w_2 + ba_1v_3w_1. \quad (69)$$

From here, we cancel the factors of v_2w_2 :

$$X_+(a_2(a_1v_1w_1 - \beta v_2w_3) + \beta\alpha_2v_3w_2) = (ba_1a_2 + \beta\alpha_1\alpha_2)v_3w_1 \quad (70)$$

Since we have assumed that $ba_1a_2 + \beta\alpha_1\alpha_2 = 0$, this tells us that $a_1a_2v_1w_1 - \beta a_2v_2w_3 + \beta\alpha_2v_3w_2$ is a highest weight vector.

The problem with this method of computing the highest weight vector is that the vector we end up with might be the zero vector. We can try to rectify this by changing the order of the factors that we cancel. Doing so gives us two additional “representations” (by an abuse of language) for the highest weight vectors:

$$\begin{aligned} & a_1\alpha_1v_1w_1 - \beta\alpha_1v_2w_3 - ba_1v_3w_2, \\ & \beta\alpha_2v_1w_2 + ba_2v_2w_1 - b\beta v_3w_3, \\ & \beta\alpha_1v_1w_3 + ba_2v_2w_2 + b\alpha_1v_3w_1 \end{aligned} \quad (71)$$

and

$$\begin{aligned} & \alpha_1\alpha_2v_1w_1 + ba_2v_2w_3 - b\alpha_2v_3w_2, \\ & a_1\alpha_2v_1w_2 - \alpha_1\alpha_2v_2w_1 - ba_1v_3w_3, \\ & \beta a_1v_1w_3 - \beta\alpha_2v_2w_2 + ba_1v_3w_1. \end{aligned} \quad (72)$$

We then hope that one of these “representations” is nonzero. However, for certain choices of the parameters, all three “representations” of some highest weight vector are zero. A lengthy but straightforward analysis shows that the only such choices are given by Table 1.

We do not have to consider all of these cases separately, however. Note that in all of these cases, $c \neq 0$. Whenever $b = \frac{1}{c}$, it is assumed, and whenever $b = 0$, we have $d = \pm 1$, and so we must have $c \neq 0$ if $T(b, c, d)$ is to be irreducible (see the Introduction). Likewise, $\gamma \neq 0$ in all of these cases. Therefore, we may use Lemma 2.5 (and the explicit calculations of its proof) to see that $T(0, c, \pm 1) \simeq T(\frac{1}{c}, c, 0)$ and $T(0, \gamma, \pm 1) \simeq T(\frac{1}{\gamma}, \gamma, 0)$. Hence, all nine cases are equivalent to the case where $b = \frac{1}{c}$, $d = 0$, $\beta = \frac{1}{\gamma}$, and $\delta = 0$. So this means we must consider that particular case separately from the other cases where highest weight vectors are present.

One may ask the related question of whether there is only one highest weight vector in each weight space. For the lowest weight vectors this is clear by inspection, but in the case of highest weight vectors, it is not so easily seen directly whether this is true. We may, however, consider the characteristic polynomials of X_+X_- on the weight spaces. These are

$$\lambda^3 + \lambda^2 + ((d + \delta)^2 - 1)\lambda + (c + \gamma)(ba_1a_2\beta\alpha_1\alpha_2) \text{ on } V_{d+\delta+1}, \quad (73)$$

$$\lambda^3 + \lambda^2 + (d + \delta)((d + \delta) - 1)\lambda + (c + \gamma)(ba_1a_2\beta\alpha_1\alpha_2) \text{ on } V_{d+\delta-1}, \quad (74)$$

$$\lambda^3 + \lambda^2 + (d + \delta)((d + \delta) + 1)\lambda + (c + \gamma)(ba_1a_2\beta\alpha_1\alpha_2) \text{ on } V_{d+\delta}. \quad (75)$$

From this, we clearly see that X_+X_- can have at most two eigenvectors with eigenvalue 0, since the geometric multiplicity of an eigenvalue is at most its algebraic multiplicity in the characteristic equation. Furthermore, we note that if $c + \gamma = 0$, one of these zero eigenvectors will come from a vector v with $X_-v = 0$. Therefore, we focus on the case where $c + \gamma \neq 0$. Another lengthy but straightforward calculation shows us that the only cases where X_+X_- has two eigenvectors with eigenvalue 0 in some weight space are again exactly given by Table 1. Hence, we have no extra cases here to consider specially.

For the action of H , we note that the possible weights for vectors in V are $d + \delta$ and $d + \delta \pm 1$. As in the case of $\widetilde{T}(b, \frac{1}{b}, 0) \otimes T(\beta, \gamma, \delta)$, we will be concerned with when these weights can be 0, 1, or -1 , since in these cases we will see phenomena that are not possible otherwise. Hence, we consider $d + \delta = 0$ or ± 1 separately.

However, for $\gamma \neq -c$ (i.e., when there are no lowest weight vectors), we may use Lemma 2.5 to reduce the cases $d + \delta = \pm 1$ to the case $d + \delta = 0$. This is because we can assume that either c or γ is nonzero. Without loss of generality, we assume $c \neq 0$. We can then use the isomorphisms

$$T(b, c, 0) \simeq T(b', c', 1) \simeq T(b'', c'', -1) \quad (76)$$

(for appropriate b' , b'' , c' and c'') to modify the value of $d + \delta$.

If $\gamma = -c$, on the other hand, we could have $\gamma = 0 = c$, in which case Lemma 2.5 is inapplicable. For the calculations where $d + \delta = \pm 1$, we therefore assume $\gamma = 0 = c$. This simplifies equations and sometimes allows sharper decompositions.

Finally, we note the following factorizations for $K = ba_1a_2 + \beta\alpha_1\alpha_2$ for special values of $d + \delta$ when $\gamma = -c$:

$$\begin{aligned} d + \delta = 0 &\implies K = (b + \beta)(1 + bc + b^2c^2 - d^2 - c\beta - bc^2\beta + c^2\beta), \\ d + \delta = 1 &\implies K = (1 + d)((1 - d)b - d\beta), \\ d + \delta = -1 &\implies K = (1 - d)((1 + d)b + d\beta). \end{aligned} \quad (77)$$

With all of this in mind, we are now ready to break down the necessary subcases. Except in the two subcases where we explicitly state this to be the case, we assume that we do *not* have all of the conditions $b = \frac{1}{c}$, $d = 0$, $\beta = \frac{1}{\gamma}$, and $\delta = 0$. (The same assumption goes for Tables 6 and 7.)

4.6.1. *The subcase $\gamma = -c$; $b = \frac{1}{c}$; $d = 0$; $\beta = \frac{1}{\gamma}$; $\delta = 0$.* This computation is implied by that in 4.6.16 below. We note that $T(0, 0, 0) \simeq \underline{3}$, $T(0, 0, 1) \simeq \underline{1} \rightarrow \underline{2}$, and $T(0, 0, -1) \simeq \underline{2} \rightarrow \underline{1}$.

4.6.2. *The subcase $\gamma = -c$; $d + \delta = 0$; $\beta = -b$; $1 + bc + b^2c^2 - d^2 - c\beta - bc^2\beta + c^2\beta^2 = 0$.* In this case, we have three lowest weight vectors. Acting on them by X_+ , we obtain three submodules, of the forms $\underline{1}$, $\underline{2}$, and $\underline{3}$. This exhausts both the highest and lowest weight vectors.

Examining the action of X_+X_- on V_1 , we notice by calculating the characteristic and minimal polynomials that X_+X_- has two eigenvectors with eigenvalue 1. We have only

exploited one of these; acting on the other by X_+ and X_- gives a module that is of the form $\underline{2}$ after quotienting with the $\underline{1}$ from the last paragraph. Together, these submodules have dimension eight, while V has dimension nine, so we deduce the complete decomposition

$$\underline{1} \oplus ((\underline{2} \oplus \underline{1}) \oplus \underline{2} \oplus \underline{3}). \quad (78)$$

4.6.3. *The subcase $\gamma = -c$; $d + \delta = 0$; $\beta = -b$; $1 + bc + b^2c^2 - d^2 - c\beta - bc^2\beta + c^2\beta^2 \neq 0$.* As in the last case, we begin with acting by X_+ on the lowest weight vectors, which gives two submodules of the forms $\underline{3}$ and $\underline{1} \leftarrow \underline{2}$. These submodules contain all but one highest weight vector, which has weight 1. Acting on it by X_- gives a submodule of the form $\underline{2}$. Since there are no more highest or lowest weight vectors, the complete decomposition is of the form

$$\underline{1} \oplus (\underline{2} \oplus (\underline{3} \oplus (\underline{1} \leftarrow \underline{2}))). \quad (79)$$

4.6.4. *The subcase $\gamma = -c$; $d + \delta = 0$; $\beta \neq -b$; $1 + bc + b^2c^2 - d^2 - c\beta - bc^2\beta + c^2\beta^2 = 0$.* This case proceeds completely analogously to above, only after exploiting the lowest weight vectors, we have two submodules of the forms $\underline{3}$ and $\underline{2} \leftarrow \underline{1}$. The one remaining highest weight vector is of weight 0. Acting on it by X_- yields a vector in $\underline{3} \oplus (\underline{2} \leftarrow \underline{1})$. With no remaining highest or lowest weight vectors, the decomposition is of the form

$$\underline{2} \oplus (\underline{1} \oplus (\underline{3} \oplus (\underline{2} \leftarrow \underline{1}))). \quad (80)$$

4.6.5. *The subcase $\gamma = -c$; $d + \delta = 0$; $\beta \neq -b$; $1 + bc + b^2c^2 - d^2 - c\beta - bc^2\beta + c^2\beta^2 \neq 0$.* In this case there are no highest weight vectors. Acting on the lowest weight vectors by X_+ gives two irreducible submodules, $T(K, 0, 0)$ and $\tilde{T}(\frac{1}{K}, K, 0)$. This exhausts all of the lowest weight vectors. By Lemma 2.8 there can be no module of the form $T(b, c, d)$ without highest or lowest weight vectors. Therefore, the remainder of the module can be irreducible only upon quotienting. Selecting an arbitrary vector from $V/(T(K, 0, 0) \oplus \tilde{T}(\frac{1}{K}, K, 0))$ and acting on it by X_+ shows that the quotient is of the form $\tilde{T}(\frac{1}{K}, K, 0)$. In all, we get

$$\tilde{T}(\frac{1}{K}, K, 0) \oplus (T(K, 0, 0) \oplus \tilde{T}(\frac{1}{K}, K, 0)) \quad (81)$$

4.6.6. *The subcase $\gamma = -c$; $d + \delta = 1$; $d = -1$; $(1 - d)b = d\beta$.* This case is done completely analogously to Section 4.6.2 and gives the same result.

4.6.7. *The subcase $\gamma = -c$; $d + \delta = 1$; $d = -1$; $(1 - d)b \neq d\beta$.* This case proceeds as in Section 4.6.3. However, here the equations are a bit simpler, and we can achieve the somewhat sharper decomposition

$$\underline{1} \oplus (\underline{3} \oplus (\underline{2} \rightarrow \underline{1} \leftarrow \underline{2})). \quad (82)$$

4.6.8. *The subcase $\gamma = -c$; $d + \delta = 1$; $d \neq -1$; $(1 - d)b = d\beta$.* This case proceeds as in Section 4.6.4. As in the previous case, we can achieve a somewhat sharper decomposition,

$$\underline{2} \oplus (\underline{3} \oplus (\underline{1} \rightarrow \underline{2} \leftarrow \underline{1})). \quad (83)$$

4.6.9. *The subcase $\gamma = -c$; $d + \delta = 1$; $d \neq -1$; $(1 - d)b \neq d\beta$.* This case is completely analogous to that of Section 4.6.5, and gives an identical result.

4.6.10. *The subcase $\gamma = -c$; $d + \delta = 2$; $d = 1$; $(1 + d)b = -d\beta$.* This case is handled just like Section 4.6.6 and gives the same result.

4.6.11. *The subcase $\gamma = -c$; $d + \delta = 2$; $d = 1$; $(1 + d)b \neq -d\beta$.* Here, we proceed as in Section 4.6.7 and get the same result.

4.6.12. *The subcase $\gamma = -c$; $d + \delta = 2$; $d \neq 1$; $(1 + d)b = -d\beta$.* This case is analogous to Section 4.6.8 and again gives the same result.

4.6.13. *The subcase $\gamma = -c$; $d + \delta = 2$; $d \neq 1$; $(1 + d)b \neq -d\beta$.* Here, we compute the same result as in Section 4.6.9 in exactly the same manner.

4.6.14. *The subcase $\gamma = -c$; $d + \delta \neq 0, 1$ or 2 ; $ba_1a_2 = -\beta\alpha_1\alpha_2$.* The decomposition for this case is implied by that in Section 4.6.15 simply by substituting $K = 0$. The computations are all still valid.

4.6.15. *The subcase $\gamma = -c$; $d + \delta \neq 0, 1$ or 2 ; $ba_1a_2 \neq -\beta\alpha_1\alpha_2$.* Here, we have no highest weight vectors, but there are three lowest weight vectors. Acting on them by X_+ immediately gives us the complete decomposition,

$$T(K, 0, d + \delta - 1) \oplus T(K, 0, d + \delta) \oplus T(K, 0, d + \delta + 1). \quad (84)$$

4.6.16. *The subcase $\gamma \neq -c$; $b = \frac{1}{c}$; $d = 0$; $\beta = \frac{1}{\gamma}$; $\delta = 0$.* In this case, there are no lowest weight vectors, but we have five highest weight vectors to exploit. Acting on them by X_- gives the complete decomposition

$$T\left(0, \frac{b + \beta}{b\beta}, -1\right) \oplus T\left(0, \frac{b + \beta}{b\beta}, 0\right) \oplus T\left(0, \frac{b + \beta}{b\beta}, 1\right). \quad (85)$$

4.6.17. *The subcase $\gamma \neq -c$; $ba_1a_2 = -\beta\alpha_1\alpha_2$; $d + \delta = 0$.* In this case, there are three highest weight vectors. Acting on them by X_- , we obtain two submodules, $T(0, c + \gamma, -1)$ and $T(0, c + \gamma, 0)$. This exhausts all highest weight vectors.

The two remaining possibilities are that we have some module of the form $T(b', c', d')$ without highest or lowest weight vectors, or that the remainder of the module forms an irreducible module only after quotienting. Lemma 2.8 rules out the first possibility, so we must have the second. Furthermore, again by Lemma 2.8, we know that the quotient module must have the form $T(b', c', d')$ after quotienting, since it can have no lowest weight vectors. Furthermore, it must have $c' = c + \gamma$.

Let us examine the action of X_+X_- on V_0 , and argue analogously to Section 4.5.10. The minimal polynomial of X_+X_- on this space is $\lambda^2(\lambda + 1)$. There is one eigenvector of eigenvalue 0 and one of eigenvalue -1 . The quotient of V_0 by the span of these eigenvectors is a 1-dimensional space, and the minimal polynomial of X_+X_- on this space is λ . Therefore, choosing a basis vector v'_3 for the quotient, we have $X_+X_-v'_3 = 0$ for the action on the quotient. Setting $v'_2 = X_+v'_3$ and $v'_1 = X_+^2v'_3$ and taking $\{v'_1, v'_2, v'_3\}$ as a basis for the quotient space, we determine that $b' = 0$ and $d' = 0 + 2 = -1$.

4.6.18. *The subcase $\gamma \neq -c$; $ba_1a_2 = -\beta\alpha_1\alpha_2$; $d + \delta \neq 0, 1, 2$.* Here, we have three highest weight vectors. Acting on them by X_- yields, after a lengthy but straightforward calculation, three irreducible submodules which form a complete decomposition,

$$T(0, c + \gamma, d + \delta - 1) \oplus T(0, c + \gamma, d + \delta) \oplus T(0, c + \gamma, d + \delta + 1). \quad (86)$$

4.6.19. *The subcase $\gamma \neq -c$; $ba_1a_2 \neq -\beta\alpha_1\alpha_2$.* Here, there are no highest or lowest weight vectors. Therefore, we want to proceed upon the lines of 4.5.10, trying to apply Lemma 2.10 to determine the structure of the decomposition.

The characteristic polynomial of X_+X_- acting on $V_{d+\delta+1}$ is given by

$$\lambda^3 + \lambda^2 + (1 - (d + \delta)^2)\lambda - (c + \gamma)(ba_1a_2 + \beta\alpha_1\alpha_2). \quad (87)$$

We are now interested in using this to determine how many linearly independent eigenvectors X_+X_- has on $V_{d+\delta+1}$.

We note, as in 4.5.10, that (87) must have at least two distinct roots, since $(\lambda - \mu)^3 = \lambda^3 - \mu^3$ in characteristic 3.

It is, however, possible that (87) has only two distinct roots. Equating coefficients of (87) and (57) gives that (87) is of the form $(\lambda - \mu)^2(\lambda - \rho)$ if and only if $\mu = -(d + \delta)(1 + d + \delta)$

and $\rho = -1 - (d + \delta)(1 + d + \delta)$, *as well as* the following condition on the parameters b , c , d , etc. is satisfied:

$$(d + \delta)^2(1 - (d + \delta)^2)^2 = -(c + \gamma)(ba_1a_2 + \beta\alpha_1\alpha_2). \quad (88)$$

We will still have a direct sum decomposition even if (88) is satisfied, provided the minimal polynomial of X_+X_- on $V_{d+\delta+1}$ factors into distinct linear factors, i.e.,

$$(X_+X_- - \mu I)(X_+X_- - \rho I) = 0, \quad (89)$$

This matrix equation can be viewed as a system of nine equations. Taking sums and differences of these equations, and using a good deal of brute force, we can reduce these nine equations to the following three conditions:

$$\begin{aligned} bd(c + \gamma) &= (d + \delta)(1 - (d + \delta)^2), \\ \beta\delta &= bd, \\ \gamma(d - d^3) &= c(\delta - \delta^3). \end{aligned} \quad (90)$$

Therefore, we have the decomposition

$$T(b_3, c_3, d_3) \oplus (T(b_1, c_1, d_1) \oplus T(b_2, c_2, d_2)). \quad (91)$$

if and only if (88) is satisfied but (90) is not. Otherwise, we have the direct sum decomposition

$$T(b_1, c_1, d_1) \oplus T(b_2, c_2, d_2) \oplus T(b_3, c_3, d_3) \quad (92)$$

To determine the b_i , c_i , and d_i , let us repeat our considerations from Section 4.5.10 in this case. Let μ_1 , μ_2 , and μ_3 be the (not necessarily distinct) roots of the polynomial (87), and assume that at least $\mu_1 \neq \mu_2$. As before, let $v_{3,i}$ be the distinct eigenvectors of X_+X_- in $V_{d+\delta+1}$, so that i ranges from one to the number of distinct eigenvectors. Then set $v_{2,i} = X_+v_{3,i}$ and $v_{1,i} = X_+^2v_{3,i}$. We take $\{v_{1,i}, v_{2,i}, v_{3,i}\}$ as our basis for $T(b_i, c_i, d_i)$ in the matrix representation (5).

Since $X_-^3v_{j,i} = (c + \gamma)v_{j,i}$ for all i and j , it follows that $c_i = c + \gamma$ for all i . We then have the two equations $X_+X_-v_{3,i} = b_ic_iv_{3,i}$ and $X_+X_-v_{3,i} = \mu_iv_{3,i}$. These imply that

$$b_i = \frac{\mu_i}{c + \gamma}. \quad (93)$$

To determine d_i , remember that each $v_{3,i}$ belongs to $V_{d+\delta+1}$, so $v_{2,i}$ is of weight $(d + \delta + 1) + 2 = d + \delta$. Hence, $d_i = d + \delta$.

As in Section 4.5.10, we must finally determine b_3 in the case that there are only two distinct eigenvectors, that is, when the minimal polynomial does not factor into linear factors with multiplicity one. In this case we have $\mu_1 = \mu_3 = -(d + \delta)(1 + d + \delta)$ and $\mu_2 = -1 - (d + \delta)(1 + d + \delta)$. Here also, The minimal polynomial of X_+X_- on $V_{d+\delta+1}$ is

$$(\lambda + (d + \delta)(1 + d + \delta))^2(\lambda + (1 + (d + \delta)(1 + d + \delta))). \quad (94)$$

By the same arguments as in Section 4.5.10, we deduce that

$$b_3 = \frac{-(d + \delta)(1 + d + \delta)}{c + \gamma}. \quad (95)$$

This completes our determination of the parameters for this decomposition, and thus our computations for Theorem 1.3.

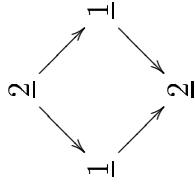
Symbol	Meaning
a_1	$bc + d - 1$
a_2	$bc - d - 1$
α_1	$\beta\gamma + \delta - 1$
α_2	$\beta\gamma - \delta - 1$
J	$1 + \alpha_1\alpha_2b\beta$
K	$a_1a_2b + \alpha_1\alpha_2\beta$
D	$(d + \delta)^2(1 - (d + \delta)^2)^2$
Δ	$(d + \delta)(1 + d + \delta)$
ρ_1, ρ_2, ρ_3	The roots of the polynomial $\lambda^3 + (1 - \delta^2)\lambda^2 + \lambda - \frac{\gamma}{b}(1 + b\beta(1 - \beta\gamma + \delta)(1 - \beta\gamma - \delta))$.
μ_1, μ_2, μ_3	The roots of the polynomial $\lambda^3 + \lambda^2 + (1 - (d + \delta)^2)\lambda - (c + \gamma)(ba_1a_2 + \beta\alpha_1\alpha_2)$.
M_1	

Table 2: Symbols used in Tables 3–7

Relations			Decomposition
$c = 0$	$d = 0$	$b = 0$	M_1
		$b \neq 0$	$\tilde{T}(\frac{1}{b}, b, 0) \oplus \tilde{T}(\frac{1}{b}, b, 0)$
	$d = 1$	$b = 0$	$\underline{3} \oplus (\underline{2} \rightarrow \underline{1})$
		$b \neq 0$	$\underline{3} \oplus (\underline{2} \oplus \underline{1})$
	$d = 2$	$b = 0$	$\underline{3} \oplus (\underline{1} \rightarrow \underline{2})$
		$b \neq 0$	$\underline{3} \oplus (\underline{1} \oplus \underline{2})$
	$d \neq 0, 1, 2$		$T(\frac{b(d-1)}{d}, 0, d-1) \oplus T(\frac{b(d+1)}{d}, 0, d+1)$
$c \neq 0$	$d = 0$	$b = 0$	$T(0, c, 1) \oplus T(0, c, 1)$
		$b = \frac{1}{c}$	$T(0, c, 1) \oplus T(0, c, 1)$
		$b \neq 0, \frac{1}{c}$	$T(b + \sqrt{\frac{b}{c}}, c, 1) \oplus T(b - \sqrt{\frac{b}{c}}, c, 1)$
	$d \neq 0, 1, 2$	$b = 0$	$T(0, c, d+1) \oplus T(0, c, d-1)$
		$1 - bc + d = 0$	$T(0, c, d) \oplus T(0, c, d+1)$
		$1 - bc - d = 0$	$T(0, c, d-1) \oplus T(0, c, d)$
		$b \neq 0,$ $1 - bc \pm d \neq 0$	$T(b + \frac{d}{c}, c, d+1) \oplus T(b + \frac{d}{c}, c, d+1)$
			$T(b + \frac{d+\sqrt{bc+d^2}}{c}, c, d+1) \oplus T(b + \frac{d-\sqrt{bc+d^2}}{c}, c, d+1)$

Table 3: $2 \otimes T(b, c, d)$

Relations	Decomposition
$b = -\beta$	$\underline{3} \oplus (\underline{2} \leftarrow \underline{1}) \oplus (\underline{1} \leftarrow \underline{2})$
$b \neq -\beta$	$T(\frac{b+\beta}{b\beta}, 0, 0) \oplus \tilde{T}(\frac{b\beta}{b+\beta}, \frac{b+\beta}{b\beta}, 0) \oplus \tilde{T}(\frac{b\beta}{b+\beta}, \frac{b+\beta}{b\beta}, 0)$

 Table 4: $\tilde{T}(b, \frac{1}{b}, 0) \otimes \tilde{T}(\beta, \frac{1}{\beta}, 0)$

Relations		Decomposition
$\gamma = 0$	$\delta = 0$	$\beta = -\frac{1}{b}$
		$\beta \neq -\frac{1}{b}$
	$\delta = 1$	$\tilde{T}(\frac{b}{1+b\beta}, \frac{1+b\beta}{b}, 0) \oplus (\tilde{T}(\frac{b}{1+b\beta}, \frac{1+b\beta}{b}, 0) \oplus T(\frac{1+b\beta}{b}, 0, 0))$
	$\delta = 2$	$\tilde{T}(b, \frac{1}{b}, 0) \oplus (\tilde{T}(b, \frac{1}{b}, 0) \oplus T(\frac{1}{b}, 0, 0))$
	$\delta \neq 0, 1, 2$	$\tilde{T}(b, \frac{1}{b}, 0) \oplus (\tilde{T}(b, \frac{1}{b}, 0) \oplus T(\frac{1}{b}, 0, 0))$
	$\delta \neq 0, 1, 2$	$T(0, 0, \delta - 1) \oplus T(0, 0, \delta) \oplus T(0, 0, \delta + 1)$
$\gamma \neq 0$	$\delta = 0$	$T(J, 0, \delta - 1) \oplus T(J, 0, \delta) \oplus T(J, 0, \delta + 1)$
	$\delta = 0$	$T(0, \gamma, -1) \oplus (T(0, \gamma, 0) \oplus T(0, \gamma, -1))$
	$\delta = 0$	$T(\rho_1, c, 0) \oplus T(\rho_2, c, 0) \oplus T(\rho_3, c, 0)$
	$\delta \neq 0, 1, 2$	$T(0, \gamma, \delta - 1) \oplus T(0, \gamma, \delta) \oplus T(0, \gamma, \delta + 1)$
	$J \neq 0$	$T(\rho_1, c, \delta) \oplus T(\rho_2, c, \delta) \oplus T(\rho_3, c, \delta)$
	$J \neq 0$	$T(\frac{1-\delta^2}{c}, c, \delta) \oplus (T(\frac{1-\delta^2}{c}, c, \delta) \oplus T(-\frac{\delta^2}{c}, c, \delta))$

 Table 5: $\tilde{T}(b, \frac{1}{b}, 0) \otimes T(\beta, \gamma, \delta)$

Relations		Decomposition
$d + \delta = 0$	$b = \frac{1}{c}, d = 0, \beta = \frac{1}{\gamma}, \delta = 0$	$\underline{3} \oplus (\underline{1} \rightarrow \underline{2}) \oplus (\underline{2} \rightarrow \underline{1})$
	$\beta = -b$	$\underline{1} \oplus ((\underline{2} \oplus \underline{1}) \oplus \underline{2} \oplus \underline{3})$
	$1 + bc + b^2c^2 - d^2 - c\beta - bc^2\beta + c^2\beta^2 = 0$	$\underline{1} \oplus (\underline{2} \oplus (\underline{3} \oplus (\underline{1} \leftarrow \underline{2})))$
	$\beta \neq -b$	$\underline{2} \oplus (\underline{1} \oplus (\underline{3} \oplus (\underline{2} \leftarrow \underline{1})))$
	$1 + bc + b^2c^2 - d^2 - c\beta - bc^2\beta + c^2\beta^2 \neq 0$	$\tilde{T}(\frac{1}{K}, K, 0) \oplus (T(K, 0, 0) \oplus \tilde{T}(\frac{1}{K}, K, 0))$
$d + \delta = 1$	$\beta = b$	$\underline{1} \oplus ((\underline{2} \oplus \underline{1}) \oplus \underline{2} \oplus \underline{3})$
	$\beta \neq b$	$\underline{1} \oplus (\underline{3} \oplus (\underline{2} \rightarrow \underline{1} \leftarrow \underline{2}))$
	$d \neq -1$	$\underline{2} \oplus (\underline{3} \oplus (\underline{1} \rightarrow \underline{2} \leftarrow \underline{1}))$
$d + \delta = 2$	$(1 - d)b \neq d\beta$	$\tilde{T}(\frac{1}{K}, K, 0) \oplus (T(K, 0, 0) \oplus \tilde{T}(\frac{1}{K}, K, 0))$
	$\beta = b$	$\underline{3} \oplus (\underline{2} \oplus \underline{1}) \oplus (\underline{1} \oplus \underline{2})$
	$\beta \neq b$	$\underline{1} \oplus (\underline{3} \oplus (\underline{2} \rightarrow \underline{1} \leftarrow \underline{2}))$
	$d \neq 1$	$\underline{2} \oplus (\underline{1} \oplus ((\underline{1} \leftarrow \underline{2}) \oplus \underline{3}))$
	$(1 + d)b \neq -d\beta$	$\tilde{T}(\frac{1}{K}, K, 0) \oplus (T(K, 0, 0) \oplus \tilde{T}(\frac{1}{K}, K, 0))$
$d + \delta \neq 0, 1, 2$	$b(1 - bc + d)(1 - bc - d) = -\beta(1 + c\beta + \delta)(1 + c\beta - \delta)$	$T(0, 0, d + \delta - 1) \oplus T(0, 0, d + \delta) \oplus T(0, 0, d + \delta + 1)$
	$b(1 - bc + d)(1 - bc - d) \neq -\beta(1 + c\beta + \delta)(1 + c\beta - \delta)$	$T(K, 0, d + \delta - 1) \oplus T(K, 0, d + \delta) \oplus T(K, 0, d + \delta + 1)$
<i>Note:</i> Except where explicitly state this to be the case, we assume that we do <i>not</i> have all of the conditions $b = \frac{1}{c}, d = 0, \beta = \frac{1}{\gamma}$, and $\delta = 0$.		

Table 6: $T(b, c, d) \otimes T(\beta, \gamma, \delta)$ ($\gamma = -c$)

Relations		Decomposition
$b = \frac{1}{c}, d = 0, \beta = \frac{1}{\gamma}, \delta = 0$		$T(0, \frac{b+\beta}{b\beta}, -1) \oplus T(0, \frac{b+\beta}{b\beta}, 0) \oplus T(0, \frac{b+\beta}{b\beta}, 1)$
$K = 0$	$d + \delta = 0$	$T(0, c + \gamma, -1) \oplus (T(0, c + \gamma, 0) \oplus T(0, c + \gamma, -1))$
	$d + \delta \neq 0, 1, 2$	$T(0, c + \gamma, d + \delta - 1) \oplus T(0, c + \gamma, d + \delta) \oplus T(0, c + \gamma, d + \delta + 1)$
$K \neq 0$	$D = -K(c + \gamma)$	$T(\frac{\mu_1}{c+\gamma}, c + \gamma, d + \delta) \oplus T(\frac{\mu_2}{c+\gamma}, c + \gamma, d + \delta) \oplus T(\frac{\mu_3}{c+\gamma}, c + \gamma, d + \delta)$
		$bd(c + \gamma) = \sqrt{D},$ $\beta\delta = bd,$ $\gamma(d - d^3) = c(\delta - \delta^3)$
		<i>otherwise</i>
	$D \neq -K(c + \gamma)$	$T(-\frac{\Delta}{c+\gamma}, c + \gamma, d + \delta) \oplus (T(-\frac{\Delta}{c+\gamma}, c + \gamma, d + \delta) \oplus T(\frac{1-\Delta}{c+\gamma}, c + \gamma, d + \delta))$ $T(\frac{\mu_1}{c+\gamma}, c + \gamma, d + \delta) \oplus T(\frac{\mu_2}{c+\gamma}, c + \gamma, d + \delta) \oplus T(\frac{\mu_3}{c+\gamma}, c + \gamma, d + \delta)$
<i>Note:</i> Except where explicitly state this to be the case, we assume that we do <i>not</i> have all of the conditions		
$b = \frac{1}{c}, d = 0, \beta = \frac{1}{\gamma},$ and $\delta = 0.$		

 Table 7: $T(b, c, d) \otimes T(\beta, \gamma, \delta)$ ($\gamma \neq c$)

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