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für Mathematik
in den Naturwissenschaften
Leipzig

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Preprint no.: 50 2006

σ_k-SCALAR CURVATURE AND EIGENVALUES OF THE DIRAC OPERATOR

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Abstract. On a 4-dimensional closed spin manifold (M^4, g), the eigenvalues of the Dirac operator can be estimated from below by the total σ_2-scalar curvature of M^4 as follows

$$\lambda^4 \geq \frac{32}{3} \frac{\int_{M^4} \sigma_2(g) dvol(g)}{vol(M^4, g)}.$$

Equality implies that (M^4, g) is a round sphere and the corresponding eigenspinors are Killing spinors.

Dedicated to Professor Wang Guangyin on the occasion of his 80th birthday

1. Introduction

Let (M^n, g) be a closed Riemannian spin manifold of dimension $n \geq 2$. Let $\not{D}$ be the (classical) Dirac operator. This operator is of fundamental importance in spin geometry, especially in the Atiyah-Singer index theorem. Let R_g denote the scalar curvature of g and ∇ the Levi-Civita connection on spinors. By the Schrödinger-Lichnerowicz formula ([20])

$$\not{D}^2 = \nabla^* \nabla + \frac{R_g}{4}, \tag{1}$$

it is easy to show that all eigenvalues λ of $\not{D}$ satisfy

$$\lambda^2 \geq \frac{1}{4} \min_{x \in M} R_g(x). \tag{2}$$

It is obvious that (2) is interesting only if the scalar curvature is positive. This in fact gives a topological obstruction to the existence of metrics of positive scalar curvature. One can also see from (1) that the Lichnerowicz inequality (2) is never sharp. The first sharp estimate for the eigenvalues of the Dirac operator was given by Friedrich in 1980 [9]. He proved the following inequality

$$\lambda^2 \geq \frac{n}{4(n-1)} \min_{x \in M} R_g(x). \tag{3}$$

The Friedrich inequality (3) is interesting also only in the case that the scalar curvature is positive. Nevertheless, this inequality is sharp in the sense that there are manifolds where equality holds. In 1986 Hijazi ([17]) generalized this inequality to obtain

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu_1, \tag{4}$$

where μ_1 is the first eigenvalue of the Yamabe operator (or the conformal Laplacian) $Y_1 = 4(n-1)/(n-2)\Delta + R_g$. It is obvious that the Hijazi inequality (4) is interesting only in the case that $\mu_1 > 0$. However, the metric g itself should not to be assumed to have positive scalar curvature. The condition $\mu_1 > 0$ is conformally invariant and is equivalent to the condition that there is a conformal metric $\tilde{g} \in [g]$ of positive scalar curvature. In dimension $n = 2$, Bär ([3]) obtained the following inequality by using the Gauss-Bonnet theorem

$$\lambda^2 \geq 2\pi\chi(M^2)/area(M^2), \tag{5}$$

where $\chi(M)$ is the Euler number of M^2. It is clear that the only interesting case is $\chi(M^2) = 2$, i.e., M^2 is a topological sphere. In this case, the Bär inequality (5) reads

$$\lambda^2 \geq \frac{4\pi}{area(S^2, g)}.$$

This inequality was conjectured in [21]. In fact, Lott [21] proved in 1986 that if $\not{D}$ is invertible, then there is a constant $C(M^n, [g]) > 0$ such that for any conformal metric $g' \in [g]$, the first positive eigenvalue $\lambda(g')$ of the Dirac operator with respect to g' satisfies

$$\lambda^2(g') \geq C(M^n, [g])/vol^{2/n}(M^n, g'). \tag{6}$$

See also [1] for a related problem. In the case $\mu_1 > 0$, the Hijazi inequality (4) provides a lower bound for the optimal constant $C(M^n, [g])$, namely

$$C(M^n, [g]) \geq \frac{n}{4(n-1)}\mu(M^n, [g]), \tag{7}$$

where $\mu(M^n, [g])$ is the Yamabe constant. See also [18].

It is a very natural question whether one can generalize the beautiful inequalities above. There are many attempts to generalize these inequalities. See for instance [19], [11], [12] and [10]. In a preliminary version of [10], the following conjecture was proposed

Conjecture. $\lambda^2 \geq \frac{n}{4(n-1)} \int_{M^n} R_g dvol(g)/vol(M^n, g)$.

Unfortunately, this conjecture was wrong ([2]). The motivation of this note is to show that this conjecture is true, at least for 4-dimensional manifolds, provided that we replace the scalar curvature by a newly introduced $n/2$-scalar curvature, .

Theorem 1. *Let (M^4, g) be a closed Riemannian spin manifold with $\mu_1 > 0$. Then all eigenvalues of the Dirac operator satisfy*

$$\lambda^4 \geq \frac{32}{3}\frac{\int_{M^4} \sigma_2(g) dvol(g)}{vol(M^4, g)} = \frac{16}{3}\frac{\int_{M^4} Q_g dvol(g)}{vol(M^4, g)}, \tag{8}$$

where $\sigma_2(g)$ is the 2-scalar curvature and Q_g the Q-curvature of g. If equality holds, then (M^4, g) is a round sphere and the corresponding eigenspinors are Killing spinors.

For the definition of 2-scalar curvature $\sigma_2(g)$, see the next section. Inequality (8) has a form similar to the Conjecture. We note that $\int_M \sigma_2(g)dvol(g)$ is a conformal invariant on a 4-dimensional manifold. The condition $\mu_1 > 0$ is necessary. On a product manifold $M^4 = \Sigma_1^2 \times \Sigma_2^2$ of two surfaces Σ_1^2 and Σ_2^2 of constant negative Gaussian curvature, one can find a spin structure such that the corresponding Dirac operator admits harmonic spinors (cf. [16]). It is clear that its σ_2 is a positive constant.

The proof of Theorem 1 is a combination of a result of Hijazi [17] and many recent results on the 2-scalar curvature $\sigma_k(g)$ obtained in [6] and [8]. The notion of k-scalar curvature $\sigma_k(g)$ was introduced by Viaclovsky in [22]. See [8] for another interesting application. For a higher dimensional manifold, a Hijazi type inequality, Proposition 2, can be obtained in terms of a nonlinear eigenvalue of a fully nonlinear operator involving the σ_k-scalar curvature.

Acknowledgement. The author would like to thank Pengfei Guan for many helpful discussions in mathematics and Jürgen Jost for his constant support. The author would also like to thank B. Ammann, T. Friedrich and the anonymous referee for their helpful suggestions.

2. Basic facts of various scalar curvatures

Let (M^n, g) be a closed, oriented Riemannian manifold and let $[g] := \{e^{-2u}g \,|\, u : M^n \to \mathbb{R}\}$ be the conformal class of g. Let Ric_g and R_g be the Ricci tensor and scalar curvature of g, respectively. The Schouten tensor of the metric g is a normalized Ricci tensor defined by

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} \cdot g\right).$$

The Schouten tensor plays an important role in conformal geometry. Recall that there is an important decomposition of the Riemann curvature tensor

$$Riem = W_g + S_g \owedge g,$$

where W_g is the Weyl tensor of g. It is well-known that the Weyl tensor $g^{-1} \cdot W_g$ is invariant in a conformal class. Let σ_k be the kth elementary symmetric function. For a symmetric $n \times n$ matrix A, set $\sigma_k(A) = \sigma_k(\Lambda)$, where $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is the set of eigenvalues of A. Following Viaclovsky [22], a *σ_k-scalar curvature* or *k-scalar curvature* of g is defined by

$$\sigma_k(g) := \sigma_k(g^{-1} \cdot S_g),$$

where $g^{-1} \cdot S_g$ is locally defined by $(g^{-1} \cdot S_g)^i_j = \sum_k g^{ik}(S_g)_{kj}$. Note that $\sigma_1(g) = \frac{1}{2(n-1)}R_g$.

Let

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \,|\, \sigma_j(\Lambda) > 0, \forall j \le k\}$$

be Garding's cone. A metric g is said to be *k-positive*, symbolically $g \in \Gamma_k^+$, if $g^{-1} \cdot S_g \in \Gamma_k^+$ for every point $x \in M$.

From the Newton-MacLaurin inequality, we have a simple lemma for k-scalar curvatures.

Lemma 1. *Let $n = 2m$ be an even number. We have*

(i) $\sigma_1^2(g) \geq \frac{2n}{n-1}\sigma_2(g)$.

(ii) *if g is k-positive, i.e., $g \in \Gamma_k^+$, then* $\frac{n!}{(n-k)!k!}\sigma_1^k(g) \geq n^k \sigma_k(g)$.

Let us introduce the integral

$$\mathcal{F}_k(g) = \int_{M^n} \sigma_k(g) dvol(g).$$

In dimension four, $\mathcal{F}_2$ is a conformal invariant. In fact, by the Chern-Gauss-Bonnet formula for 4-dimensional manifolds, we have

$$8\pi^2\chi(M^4) = \int_{M^4} |W_g|^2 dvol(g) + 4\int_{M^4} \sigma_2(g) dvol(g). \tag{9}$$

Therefore, inequality (8) can also be written as

$$\lambda^4 \geq \frac{32}{3} \frac{2\pi^2\chi(M^4) - \frac{1}{4}\int |W_g|^2 dvol(g)}{vol(M^4, g)}. \tag{10}$$

The Q-curvature introduced by Branson is another interesting quantity in conformal geometry. We have a relationship between the σ_2-scalar curvature and the Q-curvature

$$\int_{M^4} Q_g dvol(g) = 2\mathcal{F}_2(g).$$

See [6] and [15]. Note that when $n = 2m > 4$ and the underlying manifold (M^{2m}, g) is locally conformally flat, it was proved in [22] that

$$\chi(M^{2m}) = c_m \int_{M^{2m}} \sigma_m(g) dvol(g), \tag{11}$$

for some dimensional constant c_m.

Lemma 2. *Let (M^4, g) be a closed manifold. There exists a conformal metric $\tilde{g} \in [g]$ with $\tilde{g} \in \Gamma_m^+$, provided that $\mu_1 > 0$ and $\int_{M^4} \sigma_2(g) > 0$*

Proof. This is an important result, which was proved in [6]. See also [15]. ∎

3. Proof of Main Theorems

The proof of Theorem 1 follows [17] closely, after we can show the existence of solutions of equation (16) below.

Proof of Theorem 1. From [17], we have

$$\lambda^2 \geq \frac{n}{4(n-1)} \sup_u \inf\{R_{e^{-2u}g} e^{-2u}\} = \frac{n}{2} \sup_u \inf\{\sigma_1(e^{-2u}g)e^{-2u}\}, \tag{12}$$

for an n-dimensional manifold. Now we consider $n = 4$. From above, we have

$$\lambda^4 \geq 4 \sup_u \inf\{\sigma_1^2(e^{-2u}g)e^{-4u}\}. \tag{13}$$

From Lemma 2, we have

$$\sigma_1^2(e^{-2u}g)e^{-4u} \geq \frac{8}{3}\sigma_2(e^{-2u}g)e^{-4u}. \tag{14}$$

Set $\bar{g} = e^{-2u}g$. Now we claim that

$$\sup_u \inf_M \sigma_2(e^{-2u}g)e^{-4u} = \frac{1}{vol(g)}\int_M \sigma_2(g)dvol(g). \tag{15}$$

First, we have

$$\begin{aligned}\inf_M \sigma_2(\bar{g})e^{-4u} &\leq \frac{1}{vol(g)}\int_M \sigma_2(\bar{g})e^{-4u}dvol(g)\\ &= \frac{1}{vol(g)}\int_M \sigma_2(\bar{g})dvol(\bar{g}) = \frac{1}{vol(g)}\int_M \sigma_2(g)dvol(g).\end{aligned}$$

Then we consider the following nonlinear eigenvalue problem:

$$\sigma_2(e^{-2u}g)e^{-4u} = \mu_2. \tag{16}$$

Note that the only case we need to consider is $\int_{M^4}\sigma_2(g)dvol(g) > 0$. In this case, with the assumption that $\mu_1 > 0$ we can apply Lemma 2 to obtain a conformal metric $g_0 \in [g]$ with $g_0 \in \Gamma_2^+$. Then we can show that there exists a unique pair (u_0, μ_2) satisfying (16). It is important to note that $\mu_2 > 0$ and $e^{-2u_0}g_0 \in \Gamma_2^+$. A proof of this can be found in [8] or in [14]. It is easy to show that $\mu_2 = \mathcal{F}_2(g)/vol(g)$ by integrating (16) over M^4

$$\begin{aligned}\mu_2 &= \frac{1}{vol(g)}\int_M \sigma_2(e^{-2u}g)e^{-4u}dvol(g)\\ &= \frac{1}{vol(g)}\int_M \sigma_2(e^{-2u}g)dvol(e^{-2u}g) = \frac{1}{vol(g)}\int_M \sigma_2(g)dvol(g).\end{aligned}$$

Since u_0 is a solution of (16), we have

$$\inf \sigma_2(e^{-2u_0}g)e^{-4u_0} = \mu_2 = \frac{1}{vol(g)}\int_M \sigma_2(g)dvol(g).$$

Therefore, we have the claim. From (12)–(15), we have (8).

In the case of equality in (8) we have equality in (12) and (14) as well. Hence we know that the corresponding eigenspinor is a Killing spinor ([17]) and (M^4, g) is a standard sphere by a result of Friedrich ([9]). ■

4. Some remarks

In the proof of Theorem 1, it is crucial that (16) has a solution (u_0, μ_2). On a compact Riemanniann manifold, one can consider a nonlinear eigenvalue problem for a given $k > 1$:

$$\sigma_k(e^{-2u}g)e^{-2ku} = \mu_k. \tag{17}$$

When $k = 1$, μ_1 is the first eigenvalue of the Yamabe operator mentioned above. When $g \in \Gamma_k^+$, it was proved in [14] that there is a unique pair (u_0, μ_k) satisfying (17) with $\mu_k > 0$ and $e^{-2u_0}g \in \Gamma_k^+$. Hence, as above, by using a similar method of [17], we have

Proposition 1. *Let (M^n, g) be a closed spin manifold. If there is a conformal metric $\tilde g \in [g]$ with $\tilde g \in \Gamma_k^+$ $(1 \le k \le n)$, then the eigenvalues of the Dirac operator $\not{D}_g$ satisfy*

$$\lambda^{2k} \ge \left(\frac{n}{2}\right)^k \frac{(n-k)!k!n^k}{n!}\mu_k.$$

When $k = 1$, this is the Hijazi inequality (4).

By using the methods given in [2] and [13], one can show that Theorem 1 is not true, if we use σ_k-scalar curvature with $k < n/2$ to replace $\sigma_{n/2}$-scalar curvature. Namely, we have

Proposition 2. *Let (M^n, g) be a closed spin manifold with $g \in \Gamma_k$ and $k < n/2$. Then there is a constant $C > 0$ and a sequence of metrics $g_i \in [g] \cap \Gamma_k^+$ with bounded first (non-negative) eigenvalue $\lambda_1(g_i) \le C$ $(\forall i)$ and*

$$\frac{\int_M \sigma_k(g_i) dvol(g_i)}{vol(g_i)} \to \infty,$$

as $i \to \infty$.

In their proof [2], Ammann and Bär constructed metrics which they called Pinocchio metrics. Those metrics can be chosen in a conformal class $[g] \cap \Gamma_k^+$. (cf. [13]) The key part is the domain A_2 in [2] isometric to $\mathbb{S}^{n-1}(r) \times [0, L]$. A simple observation is that its σ_k-scalar curvature is positive for $k < n/2$ and $\sigma_{n/2} = 0$. Therefore, when $L \to \infty$, $\int_{A_2} \sigma_k \to \infty$ for $k < n/2$, while $\int_{A_2} \sigma_{n/2} = 0$ is fixed. This simple observation motivates this note.

For a general higher dimensional manifold, one might hope that a result similar to Theorem 1 holds. However, the proof given here relies on the conformal invariance of $\mathcal{F}_{n/2}$, which was only proved to be true in the case that $n = 4$ or (M, g) is locally conformally flat ([22]). The average Q-curvature might be a good quantity to bound the eigenvalues under certain positivity assumptions on the Paneitz-Branson operator on a higher dimensional manifold.

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