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On the relationship between rank-(n-1) convexity and S-quasiconvexity

by

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ON THE RELATIONSHIP BETWEEN RANK-$(n-1)$ CONVEXITY AND $\mathcal{S}$-QUASICONVEXITY

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Abstract. We prove that rank-$(n-1)$ convexity does not imply $\mathcal{S}$-quasiconvexity (i.e., quasiconvexity with respect to divergence free fields) in $\mathbb{M}^{m\times n}$ for $m > n$, by adapting the well-known Šverák's counterexample [3] to the solenoidal setting. On the other hand, we also remark that rank-$(n-1)$ convexity and $\mathcal{S}$-quasiconvexity turn out to be equivalent in the space of $n \times n$ diagonal matrices. This follows by a generalization of Müller's work [2].

Key words: $\mathcal{A}$-quasiconvexity, quasiconvexity, lower semicontinuity.

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1. Introduction

The purpose of this note is to generalize some known results about the relationship between rank-one convexity and quasiconvexity to the context of divergence free fields. This is motivated by the lower semicontinuity results provided by Fonseca and Müller ([1], Theorems 3.6-3.7). Let us recall the relevant definitions. A function $f : \mathbb{M}^{m\times n} \to \mathbb{R}$ on the $m \times n$ matrices is called rank-one convex if it is convex on each rank-one line, i.e., for every $A, Y \in \mathbb{M}^{m\times n}$ with $\text{rank}(Y) = 1$, the function $t \to f(A + tY)$ is convex. It is quasiconvex if

$$\int_{\mathbb{T}^n} f(A + \nabla\varphi)\, dx \geq f(A)\,,$$

for all $A \in \mathbb{M}^{m\times n}$ and for all $\mathbb{T}^n$-periodic functions $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$, where $\mathbb{T}^n := (0,1)^n$. Quasiconvexity implies rank-one convexity. Whether the converse is true for $m = 2$ and $n \geq 2$ is an outstanding open problem. In the higher dimensional case $m \geq 3$, Šverák's counterexample [3] shows that rank-one convexity is not the same as quasiconvexity. On the other hand, Müller [2] proved that the two notions are equivalent for 2×2 diagonal matrices.

In the spirit of $\mathcal{A}$-quasiconvexity (see, e.g., [1]), we provide in this note the counterpart of these results in the context of divergence free fields. The corresponding notion of quasiconvexity for solenoidal fields, that we call $\mathcal{S}$-quasiconvexity, is defined as follows.

Definition 1.1. *A continuous function $f : \mathbb{M}^{m\times n} \to \mathbb{R}$ with quadratic growth is said to be $\mathcal{S}$-quasiconvex if for every $\mathbb{T}^n$-periodic matrix field $B \in L^2_{\text{loc}}(\mathbb{R}^n, \mathbb{M}^{m\times n})$ such that* $\text{Div}B = 0$ *in*

$\mathcal{D}'(\mathbb{R}^n, \mathbb{R}^m)$, *the following inequality holds:*

$$\int_{\mathbb{T}^n} f(B)\, dx \geq f\left(\int_{\mathbb{T}^n} B\, dx\right). \tag{1.1}$$

The symbol Div in the Definition 1.1 denotes the operator which acts as the distributional divergence on each row of the matrix field B. While quasiconvexity implies convexity along rank-one lines, it is easily checked that $\mathcal{S}$-quasiconvexity implies convexity along rank-$(n-1)$ lines. Indeed if a function f is $\mathcal{S}$-quasiconvex, then $t \to f(A+tV)$ is convex for every $A, V \in \mathbb{M}^{m\times n}$ with $\operatorname{rank}(V) \leq n-1$. Our aim in this note is to show that rank-$(n-1)$ convexity does not imply $\mathcal{S}$-quasiconvexity in $\mathbb{M}^{m\times n}$, for $m \geq n+1 \geq 4$. More precisely we prove the following result.

Theorem 1.2. *For all $n \geq 3$ and $m \geq n+1$, there exists $F : \mathbb{M}^{m\times n} \to \mathbb{R}$ such that F is rank-$(n-1)$ convex but not $\mathcal{S}$-quasiconvex.*

The proof of Theorem 1.2 is essentially based on the Šverák's counterexample adapted to the solenoidal setting and is reminiscent of an example given by Tartar in connection with a theorem in compensated compactness (see [4], pp. 185–6).

We do not know whether $\mathcal{S}$-quasiconvexity and rank-$(n-1)$ convexity are equivalent in the case when $m = n$. We expect that the latter case carries difficulties similar to those for gradient fields in the case $m = 2$ and $n \geq 2$.

Nevertheless Müller's result on quasiconvexity on diagonal matrices extends as well to the divergence free fields. If we identify the space $D(n)$ of diagonal $n \times n$ matrices with $\mathbb{R}^n$ via $y \to \operatorname{diag}(y_1, \ldots, y_n)$, then a rank-$(n-1)$ convex function on $D(n)$ may be regarded as a function on $\mathbb{R}^n$ which is convex on each hyperplane $\{y_i = \text{const}\}$, $i = 1, \ldots, n$. Then, a straightforward generalization of Theorem 1 in [2] asserting that rank-one convexity implies quasiconvexity on diagonal matrices, leads to the following statement assuring that rank-$(n-1)$ convexity implies $\mathcal{S}$-quasiconvexity on diagonal matrices.

Theorem 1.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex on each hyperplane $\{y_i = \text{const}\}$, $i = 1, \ldots, n$ and let f satisfy $0 \leq f(y) \leq C(1+|y|^2)$. Suppose that*

$$u_h^i \rightharpoonup u_\infty^i\,, \qquad \text{in } L^2_{\text{loc}}(\mathbb{R}^n) \text{ as } h \to \infty\,, \quad i = 1, \ldots, n\,,$$
$$\partial_i u_h^i \to \partial_i u_\infty^i\,, \qquad \text{in } H^{-1}_{\text{loc}}(\mathbb{R}^n) \text{ as } h \to \infty\,, \quad i = 1, \ldots, n.$$

Then for every open set $V \subset \mathbb{R}^n$

$$\int_V f(u_\infty^1, \ldots, u_\infty^n)\, dx \leq \liminf_{h\to\infty} \int_V f(u_h^1, \ldots, u_h^n)\, dx\,.$$

We remark that, for $n = 2$, Theorem 1.3 reduces itself to Theorem 1 in [2]. Indeed, in dimension two, the notion of $\mathcal{S}$-quasiconvexity coincides with that of quasiconvexity since any divergence free field defines a gradient field upon left multiplication by the rotation $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Finally we recall that another result in the direction of Theorem 1.3 is that if F is a quadratic form and is rank-$(n-1)$ convex, then F is also $\mathcal{S}$-quasiconvex (see [5]).

2. Proof of Theorem 1.2

Theorem 1.2 is a consequence of Lemma 2.2 and Corollary 2.3 below. We will basically follow Šverák's strategy. The key idea is to find three rank-$(n-1)$ directions such that these directions are the only rank-$(n-1)$ directions in the vector space spanned by them, which we call L. Then one defines a rank-$(n-1)$ convex function on L and seeks a divergence free field that takes values only in L and for which the inequality (1.1) is violated. The desired function F is then obtained by suitably extending the rank-$(n-1)$ convex function defined on L to the whole space. We first construct an example in $\mathbb{M}^{4\times 3}$ and then we will extend it to $\mathbb{M}^{m\times n}$. Let $V_1, V_2, V_3 \in \mathbb{M}^{4\times 3}$ be given by

$$V_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}. \tag{2.1}$$

We consider the three-dimensional subspace of $\mathbb{M}^{4\times 3}$ generated by V_1, V_2, V_3:

$$L := \operatorname{span}\{V_1, V_2, V_3\} = \{\eta_1 V_1 + \eta_2 V_2 + \eta_3 V_3 \,, \eta_1, \eta_2, \eta_3 \in \mathbb{R}\}\,, \tag{2.2}$$

and we define the function $f : L \to \mathbb{R}$ in the following way

$$\forall\, \eta_1, \eta_2, \eta_3 \in \mathbb{R} \qquad f(\eta_1 V_1 + \eta_2 V_2 + \eta_3 V_3) = -\eta_1\eta_2\eta_3\,. \tag{2.3}$$

It can be checked that the only rank-two directions in L are given by V_1, V_2, V_3 and therefore the function f is convex (in fact linear) on each rank-two line contained in L.

Lemma 2.1. *Let L and f be defined by* (2.2) *and* (2.3) *respectively and let $P : \mathbb{M}^{4\times 3} \to L$ be the orthogonal projection onto L. Then for each $\varepsilon > 0$ there exists $k = k(\varepsilon) > 0$ such that the function $F : \mathbb{M}^{4\times 3} \to \mathbb{R}$ given by*

$$F(X) = f(PX) + \varepsilon|X|^2 + \varepsilon|X|^4 + k|X - PX|^2 \tag{2.4}$$

is rank-two convex on $\mathbb{M}^{4\times 3}$.

Lemma 2.1 is an obvious extension of Lemma 2 in [3] and therefore we refer the reader to [3] for its proof. We remark that an extension of the form (2.4) is always possible if V_1, V_2, V_3 are any three rank-$(n-1)$ directions in $\mathbb{M}^{m\times n}$ such that they are the only rank-$(n-1)$ directions in the subspace spanned by them and f is defined as in (2.3).

Lemma 2.2. *There exist $\varepsilon > 0$ and $k > 0$ such that the function F given by* (2.4) *is rank-two convex but not $\mathcal{S}$-quasiconvex.*

Proof. Let $B : \mathbb{T}^3 \to \mathbb{M}^{4\times 3}$ be defined by

$$B(x) = \begin{pmatrix} \cos 2\pi x_3 & \cos 2\pi x_1 & 0 \\ 0 & \cos 2\pi x_3 & 0 \\ 0 & \cos 2\pi(x_1 - x_3) & \cos 2\pi x_1 \\ \cos 2\pi(x_1 - x_3) & \cos 2\pi(x_1 - x_3) & \cos 2\pi(x_1 - x_3) \end{pmatrix}.$$

It is readily seen that the matrix field B defined above is divergence-free and it satisfies

$$\begin{cases} B \in L \quad \text{a.e}\,, \\ \displaystyle\int_{\mathbb{T}^3} B\, dx = 0\,, \\ \displaystyle\int_{\mathbb{T}^3} f(B)\, dx = -\int_{\mathbb{T}^3} (\cos 2\pi x_1)^2 (\cos 2\pi x_3)^2\, dx < 0\,. \end{cases}$$

Since B is bounded, we can choose $\varepsilon > 0$ such that

$$\int_{\mathbb{T}^3} \Big(f(B) + \varepsilon |B|^2 + \varepsilon |B|^4 \Big)\, dx < 0\,. \tag{2.5}$$

By Lemma 2.1 there exists $k = k(\varepsilon)$ such that the function

$$F(X) = f(PX) + \varepsilon |X|^2 + \varepsilon |X|^4 + k|X - PX|^2$$

is rank-two convex. Since $|B(x) - PB(x)| = 0$ a.e., we have from (2.5)

$$\int_{\mathbb{T}^3} F(B(x))\, dx < 0$$

which concludes the proof. □

Corollary 2.3. *For all $n > 3$ and $m \geq n+1$, there exists $F^{(n)} : \mathbb{M}^{m\times n} \to \mathbb{R}$ such that F is rank-$(n-1)$ convex but not $\mathcal{S}$-quasiconvex.*

Proof. We show how to adapt the counterexample constructed in Lemma 2.2 to an arbitrary dimension n. Since one can always increase the number of rows by adding some zeros while preserving the rank of the matrices, it is enough to consider the case when $m = n+1$. In this situation we will exhibit three matrices $V_1^{(n)}, V_2^{(n)}, V_3^{(n)}$ which satisfy the following properties

$$\operatorname{rank}(V_i^{(n)}) \leq n-1 \quad \forall\, i = 1,2,3\,, \tag{2.6}$$

$$\operatorname{rank}(\alpha_1 V_1^{(n)} + \alpha_2 V_2^{(n)} + \alpha_3 V_3^{(n)}) = n \quad \forall\, \alpha \in \mathbb{S}^2 \setminus \{(\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1)\}\,. \tag{2.7}$$

For each α in $\mathbb{S}^2$ we set

$$M^{(n)}(\alpha) := \alpha_1 V_1^{(n)} + \alpha_2 V_2^{(n)} + \alpha_3 V_3^{(n)}\,.$$

We first consider the case when $n = 4$. We define $V_1^{(4)}, V_2^{(4)}, V_3^{(4)}$ as follows

$$V_1^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad V_2^{(4)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad V_3^{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.8}$$

We have that $\operatorname{rank}(V_1^{(4)}) = \operatorname{rank}(V_3^{(4)}) = 3$ and $\operatorname{rank}(V_2^{(4)}) = 2$. In order to see that the condition (2.7) is satisfied it is convenient to write the explicit formula for $M^{(4)}(\alpha)$:

$$M^{(4)}(\alpha) = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & \alpha_3 & \alpha_2 & 0 \\ \alpha_3 & \alpha_3 & \alpha_3 & \alpha_1 \\ 0 & 0 & 0 & \alpha_3 \end{pmatrix}.$$

Observe that the 4×3 minor of $M^{(4)}(\alpha)$ which is obtained eliminating the fifth row and the fourth column is a linear combination of the matrices V_1, V_2, V_3 defined by (2.1). Then using the fact that V_1, V_2, V_3 satisfy (2.7) for $n = 3$, one easily checks that $\operatorname{rank}(M^4(\alpha)) = 4$. Remark that replacing the entry $M_{4,5}^{(4)}(\alpha) = \alpha_1$ by $M_{4,5}^{(4)}(\alpha) = \alpha_2$ would give another possible choice of $V_1^{(4)}, V_2^{(4)}, V_3^{(4)}$.

For $n = 5$ we choose $V_1^{(5)}, V_2^{(5)}, V_3^{(5)}$ such that $M^{(5)}(\alpha)$ is given by

$$M^{(5)}(\alpha) = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_3 & \alpha_2 & 0 & 0 \\ \alpha_3 & \alpha_3 & \alpha_3 & a_{44} & 0 \\ 0 & 0 & 0 & \alpha_3 & a_{55} \\ 0 & 0 & 0 & 0 & \alpha_3 \end{pmatrix}$$

where a_{ii} can be chosen in the set $\{\alpha_1, \alpha_2\}$. Proceeding in a similar way, for every $n \geq 4$ we define the matrix $M^{(n)}(\alpha)$ such that

$$\begin{aligned} & M^{(n)}(\alpha) \in \mathbb{M}^{(n+1)\times n}\,, \\ & M^{(n)}_{i,j}(\alpha) = M^{(n-1)}_{i,j}(\alpha) \qquad \text{for } i \leq n, j \leq n-1\,, \\ & M^{(n)}_{n,n}(\alpha) = a_{nn} \qquad \text{where } a_{nn} \in \{\alpha_1, \alpha_2\}\,, \\ & M^{(n)}_{n+1,n}(\alpha)) = \alpha_3\,, \\ & M^{(n)}_{i,j}(\alpha) = 0 \qquad \text{otherwise}\,. \end{aligned}$$

By construction we have that $\text{rank}(M^{(n)}(\alpha)) = n$ for all $\alpha \in \mathbb{S}^2 \setminus \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$. For each n we set $L^{(n)} := \text{span}\{V_1^{(n)}, V_2^{(n)}, V_3^{(n)}\}$ and we define the function $f^{(n)} : L^{(n)} \to \mathbb{R}$ as in (2.3), i.e., $f^{(n)}(\eta_1 V_1^{(n)} + \eta_2 V_2^{(n)} + \eta_3 V_3^{(n)}) := \eta_1\eta_2\eta_3$. The sought function $F^{(n)}$ is then defined as in (2.4) with $f^{(n)}$ in the place of f and with P the orthogonal projection onto $L^{(n)}$. Considering the divergence free field $B^{(n)}(x) := \cos(2\pi x_3)V_1^{(n)} + \cos(2\pi x_1)V_2^{(n)} + \cos 2\pi(x_1 - x_3)V_3^{(n)}$, we see that $F^{(n)}$ is not $\mathcal{S}$-quasiconvex. $\square$

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