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## Dynamic stability of martensite twins under regular kinetics

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#### Abstract

This paper considers phase boundaries governed by regular kinetic relations as first proposed by Abeyaratne and Knowles [1, 2]. It shows that static configurations of hyperelastic materials, in which two different martensitic (monoclinic) states meet along a planar interface, are dynamically stable towards fully three dimensional perturbations.


Keywords: Martensite twin, dynamical stability, Lopatinski function.

## 1. Introduction

At appropriate temperatures, certain crystalline materials are characterized by a multiplicity of symmetry-related, preferred martensitic states of deformation. Nonlinear elasticity theory (Ball and James [3, 4]) models such materials by a stored-energy density

$$
W: \mathbb{R}_{+}^{3 \times 3} \longrightarrow[0,+\infty)
$$

which, as a frame-indifferent function of the deformation gradient $U \in \mathbb{R}_{+}^{3 \times 3}$, has a nonconvex, multiple well structure with several global minima. Being any of these minima a martensitic state, each corresponding well is called a martensitic phase. Many such materials allow for pairs $\left(\underline{U}^{-}, \underline{U}^{+}\right)$of energy minimizing martensitic states which satisfy the Hadamard condition

$$
\begin{equation*}
\underline{U}^{+}-\underline{U}^{-}=a \otimes v \tag{1}
\end{equation*}
$$

for some $a, v \in \mathbb{R}^{3}, a \neq 0,|v|=1$. In this case, $\underline{U}^{-}$and $\underline{U}^{+}$are called rank-one connected, and a configuration of the form

$$
U_{0}(x)= \begin{cases}U^{-}, & x \cdot v<0,  \tag{2}\\ \underline{U}^{+}, & x \cdot v>0,\end{cases}
$$

is called a martensite twin. Mathematically, $U_{0}(x)$ is a weak solution to the system

$$
\begin{align*}
\operatorname{div}_{x} \sigma(U) & =0,  \tag{3}\\
\operatorname{curl}_{x} U & =0,
\end{align*}
$$

where $\sigma(U)$ denotes the (first) Piola-Kirchhoff stress tensor

$$
\sigma=\frac{\partial W}{\partial U}
$$

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Clearly, the function $\left(U_{*}(x, t), V_{*}(x, t)\right):=\left(U_{0}(x), 0\right)$ is a (time-independent) weak solution of the equations of elastodynamics in the absence of external forces,

$$
\begin{array}{r}
U_{t}-\nabla_{x} V=0, \\
V_{t}-\operatorname{div}_{x} \sigma(U)=0, \tag{4}
\end{array}
$$

together with the physical constraint

$$
\begin{equation*}
\operatorname{curl}_{x} U=0, \tag{5}
\end{equation*}
$$

in which the spatial and temporal derivatives of the local deformation $X(x, t)$,

$$
U(x, t)=\nabla_{x} X(x, t), \quad \text { and } \quad V(x, t)=X_{t}(x, t)
$$

denote the now possibly time-dependent deformation gradient and velocity, respectively, of the elastic material.

This paper addresses the stability of martensite twins (2) under the viewpoint of elastodynamics. More precisely, we are concerned with the following question: What happens to the solution $\left(U_{*}, V_{*}\right)$ if its value at some time, say $t=0$, is modified (perturbed) slightly? In other words, will solutions $(U, V)$ of (4), whose initial data $(U(x, 0), V(x, 0))$ are only near to -but not identical with- $\left(U_{*}(x), 0\right)$, be close and similar to -or far and qualitatively different from- $\left(U_{*}, V_{*}\right)$ ?

Among the things that might happen upon perturbation of data are moving planar phase boundaries, i.e. solutions of the form

$$
(U, V)(x, t)= \begin{cases}\left(U^{-}, V^{-}\right), & x \cdot N-s t<0  \tag{6}\\ \left(U^{+}, V^{+}\right), & x \cdot N-s t>0\end{cases}
$$

that travel at a small speed $s$ and satisfy canonical jump conditions

$$
\begin{array}{r}
-s[U]-[V] \otimes N=0 \\
-s[V]-[\sigma(U)] N=0  \tag{7}\\
{[U] \times N=0 .}
\end{array}
$$

Unlike classical shock waves, the dynamics of slowly (more precisely: subsonically) moving phase boundaries for given initial data, cannot be determined solely by (4) and (7), but they must satisfy one further condition. Any additional law introduced for this purpose is called a kinetic rule. In the simplest case, a kinetic rule is given by an equation of the form

$$
\begin{equation*}
g\left(\left(U^{-}, V^{-}\right),\left(U^{+}, V^{+}\right), s, N\right)=0 \tag{8}
\end{equation*}
$$

which, just like the canonical jump conditions (7), interrelates the two states on either side (at some point) of the moving boundary with its space-time normal $(N,-s)$ (at that point). The precise circumstances under which the motion of phase boundaries in real materials can be captured by a kinetic rule of this simple kind, seem currently not clear from the literature. In the present paper we simply assume such circumstances and consider rules that, with some regular (at least once differentiable) real valued function

$$
h=h\left(\left(U^{-}, V^{-}\right),\left(U^{+}, V^{+}\right), s, N\right),
$$

are of form (8) with

$$
\begin{equation*}
g=\mathcal{F}+h, \quad \mathcal{F}\left(U^{+}, U^{-}, N\right):=[W(U)]-N^{\top}[U]^{\top}\langle\sigma(U)\rangle N \tag{9}
\end{equation*}
$$

and compatible with some martensite twin (2) in the sense that

$$
\begin{equation*}
h\left(\left(\underline{U}^{-}, 0\right),\left(\underline{U}^{+}, 0\right), 0, v\right)=0 . \tag{10}
\end{equation*}
$$

Such kinetic rules (9) have been introduced by Abeyaratne and Knowles [1, 2] under the perspective of irreversible thermodynamics. The quantity $\mathcal{F}\left(U^{+}, U^{-}, N\right)$ is often called the
driving traction (or driving force) across the boundary [1]. In particular, the present study considers both the Maxwell (or Hugoniot) rule, corresponding to

$$
\begin{equation*}
h \equiv 0 \quad \text { identically, } \tag{11}
\end{equation*}
$$

and regular Abeyaratne-Knowles rules, corresponding to $h$ satisfying

$$
\begin{equation*}
h((\cdot, \cdot),(\cdot, \cdot), 0, \cdot)=0, \quad \text { identically } \tag{12a}
\end{equation*}
$$

and

$$
\begin{equation*}
D_{s} h\left(\left(\underline{U}^{-}, 0\right),\left(\underline{U}^{+}, 0\right), 0, v\right)<0 \tag{12b}
\end{equation*}
$$

Recently, Freistühler and Plaza [8] have shown that the nonlinear dynamic stability of subsonic phase boundaries under regular kinetic rules is governed by a so-called reduced Lopatinski or stability function

$$
\Delta: \mathcal{S} \longrightarrow \mathbb{C}
$$

(defined on an appropriate compact set $\mathcal{S}$ of space-time frequencies), in the following sense: The phase boundary is dynamically [weakly] stable whenever $\Delta$ does not vanish in [the interior off $\mathcal{S}$.

The goal of this paper is to show that at least one widely considered example of a martensite twin is weakly stable under the Maxwell rule (11), and strongly stable under regular Abeyaratne-Knowles rules (12). Following Ball and James [3], the example we have chosen twins two monoclinic deformations

$$
\underline{U}^{+}=\left(\begin{array}{lll}
1 & 0 & 0  \tag{13}\\
\epsilon & 1 & 0 \\
0 & 0 & 1
\end{array}\right) \quad \text { and } \quad \underline{U}^{-}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
-\epsilon & 1 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

with $\epsilon>0$, which are clearly rank-one connected,

$$
\underline{U}^{+}-\underline{U}^{-}=2 \epsilon e_{2} \otimes e_{1}
$$

and satisfy $\operatorname{det} \underline{U}^{+}=\operatorname{det} \underline{U}^{-}=1$. The rotationally invariant energy wells are defined by $\mathcal{U}^{+}=\mathrm{SO}(3) \underline{U}^{+}$and $\mathcal{U}^{-}=\mathrm{SO}(3) \underline{U^{-}}$, where $\mathrm{SO}(3)=\left\{R \in \mathbb{R}^{3 \times 3}: R^{\top} R=I\right\}$ is the group of proper rotations. As the stored energy function, we concretely take

$$
\begin{equation*}
W(U)=\frac{1}{32}\left|C-C^{+}\right|^{2}\left|C-C^{-}\right|^{2} \tag{14}
\end{equation*}
$$

where $C(U)=U^{\top} U$ is the right Cauchy-Green strain, $C^{ \pm}=C\left(\underline{U}^{ \pm}\right)$, and $|\cdot|$ denotes the Frobenius norm on $\mathbb{R}^{3 \times 3}$,

$$
|M|^{2}=\operatorname{Tr}\left(M^{\top} M\right)
$$

Clearly, (14) is a frame-indifferent function, as $W(R U)=W(U)$ for all $R \in \mathrm{SO}$ (3), and attains its minimum only on the set $\mathcal{U}=\mathcal{U}^{-} \cup \mathcal{U}^{+}$, that is, $W(U)=0$ if and only if $U \in \mathcal{U}$. This energy density, a choice following Kružík and Luskin [11], is an example of the simplest multiple well structure where the number of different variants is two, namely, a double-well energy modelling orthorhombic-to-monoclinic transformations (see Ball and James [4]).

For this particular material, we evaluate the Lopatinski function $\Delta$ numerically and find that, (i) with Maxwell kinetics, $\Delta$ has zeroes on the boundary of $\mathcal{S}$ and no zeroes in the interior of $\mathcal{S}$ (weak stability), while, (ii) with various choices of regular Abeyaratne-Knowles kinetics, $\Delta$ has no zeroes on all of $\mathcal{S}$ (strong stability).

## Notation

In this paper we adopt the following conventions. $\mathbb{R}_{+}^{3 \times 3}$ denotes the set of all $3 \times 3$ matrices of positive determinant. With respect to the canonical (column) basis $\left\{e_{j}\right\}_{j=1}^{3}$ of $\mathbb{R}^{3}$, the PiolaKirchhoff stress tensor $\sigma(U)$ has $(i, j)$-component $\sigma_{i j}(U)=\partial W / \partial U_{i j}$, for $1 \leq i, j \leq 3 . U_{j}$ and $\sigma_{j}$ denote the $j$-th column of $U$ and $\sigma$, respectively. To express the second derivatives of the energy, for each pair $(i, j)$, define the $3 \times 3$ matrix fields $B_{i}^{j}(U)=\partial \sigma_{j} / \partial U_{i}$, whose $(l, k)$-entry is

$$
\left(B_{i}^{j}(U)\right)_{l k}=\frac{\partial^{2} W}{\partial U_{l j} \partial U_{k i}}
$$

The $B_{i}^{j}$ are mutual transposes, $\left(B_{i}^{j}\right)^{\top}=B_{j}^{i}, 1 \leq i, j \leq 3$, and define the acoustic tensor as

$$
\begin{equation*}
\mathcal{N}(\xi, U)=\sum_{i, j=1}^{3} \xi_{i} \xi_{j} B_{i}^{j}(U) \tag{15}
\end{equation*}
$$

for all $U \in \mathbb{R}^{3 \times 3}$ and $\xi \in \mathbb{R}^{3}$. When evaluating at the wells (13), we write $B_{i}^{j \pm}:=B_{i}^{j}\left(\underline{U}^{ \pm}\right)$. Equations (4) constitute a system of conservation laws with state variables and fluxes,

$$
u=\left(\begin{array}{c}
U_{1} \\
U_{2} \\
U_{3} \\
V
\end{array}\right) \in \mathbb{R}^{12 \times 1}, \text { and } F(u)=-\left(\begin{array}{ccc}
V & 0 & 0 \\
0 & V & 0 \\
0 & 0 & V \\
\sigma(U)_{1} & \sigma(U)_{2} & \sigma(U)_{3}
\end{array}\right) \in \mathbb{R}^{12 \times 3},
$$

respectively. Derivatives are written according to custom as $D_{y} g$, with $\left(D_{y} g\right)_{i j}=\partial g_{i} / \partial y_{j}$. For short, $D_{(U, V)} g$ denotes $D_{\left(U_{1}, U_{2}, U_{3}, V\right)} g$, for each function $g$. Given any function $f$ of the state variables, we denote the jump across the interface as $[f]=f^{+}-f^{-}$, and the mean value as $\langle f\rangle=\frac{1}{2}\left(f^{+}+f^{-}\right)$.

## 2. The reduced Lopatinski function

For simplicity, and in view of our choice of the wells (13), let us assume that in (2), $v$ points in the positive direction along the $x_{1}$-axis (i.e. $v=e_{1}$ ), and denote $\tilde{\xi}=\left(\xi_{2}, \xi_{3}\right)$ as the variables transversal to the direction of propagation. The set $\mathcal{S}$ of space-time frequencies is defined by

$$
\begin{equation*}
\mathcal{S}:=\left\{(\lambda, \tilde{\xi}) \in \mathbb{C} \times \mathbb{R}^{2}:|\lambda|^{2}+|\tilde{\xi}|^{2}=1, \operatorname{Re} \lambda \geq 0\right\} \tag{16}
\end{equation*}
$$

with interior,

$$
\mathcal{S}^{+}:=\mathcal{S} \cap\{\operatorname{Re} \lambda>0\} .
$$

Consider the two continuous matrix fields $\mathbb{M}_{ \pm}: \mathcal{S} \rightarrow \mathbb{R}^{6 \times 6}$ defined by

$$
\mathbb{M}_{ \pm}(\lambda, \tilde{\xi})=\mathbb{M}_{ \pm}(\lambda, \tilde{\xi}, s)_{\mid s=0}=\left(\begin{array}{ll}
M_{11}^{ \pm} & M_{12}^{ \pm}  \tag{17}\\
M_{21}^{ \pm} & M_{22}^{ \pm}
\end{array}\right)
$$

with

$$
\begin{align*}
M_{11}^{ \pm} & =i\left(B_{1}^{1 \pm}\right)^{-1}\left(\xi_{2} B_{2}^{1 \pm}+\xi_{3} B_{3}^{1 \pm}\right), \\
M_{12}^{ \pm} & =-\left(B_{1}^{1 \pm}\right)^{-1}, \\
M_{21}^{ \pm}= & \left(\xi_{2} B_{1}^{2 \pm}+\xi_{3} B_{1}^{3 \pm}\right)\left(B_{1}^{1 \pm}\right)^{-1}\left(\xi_{2} B_{2}^{1 \pm}+\xi_{3} B_{3}^{1 \pm}\right)-\lambda^{2} I+  \tag{18}\\
& \quad-\left(\xi_{2}^{2} B_{2}^{2 \pm}+\xi_{3}^{2} B_{3}^{33}+\xi_{2} \xi_{3}\left(B_{3}^{2 \pm}+B_{2}^{3 \pm}\right)\right), \\
M_{22}^{ \pm}= & i\left(\xi_{2} B_{1}^{2 \pm}+\xi_{3} B_{1}^{3 \pm}\right)\left(B_{1}^{1 \pm}\right)^{-1} .
\end{align*}
$$

The main observation in [8] is that the curl-free constraint (5) reduces the normal modes analysis for the stability of such elastic fronts to a subspace of amplitudes, whose dynamics
in the frequency (or Fourier-Laplace) space is captured by the matrix fields $\mathbb{M}_{ \pm}$. In view of the analysis of [8], the stability function $\Delta: \mathcal{S} \rightarrow \mathbb{C}$ for such a static planar interface is given by

$$
\Delta(\lambda, \tilde{\xi})=\operatorname{det}\left(\begin{array}{lll}
\hat{R}_{-}^{s} & \hat{Q} & \hat{R}_{+}^{u}  \tag{19}\\
\hat{p}^{-} & \hat{q} & \hat{p}^{+}
\end{array}\right)
$$

where

$$
\begin{align*}
& \hat{Q}:=\binom{\left[U_{1}\right]}{i[\sigma(U)](0, \tilde{\xi})^{\top}} \in \mathbb{C}^{6 \times 1},  \tag{20}\\
& \hat{q}:=-\lambda\left(D_{s} g\right)+i\left((0, \tilde{\xi})^{\top} D_{N} g\right) \in \mathbb{C}^{1 \times 1}  \tag{21}\\
& \hat{p}^{+}:=\left(D_{\left(U^{+}, V^{+}\right)} g\right) \mathcal{K}_{+}(\lambda, \tilde{\xi}) \hat{R}_{+}^{u} \in \mathbb{C}^{1 \times 3} \\
& \hat{p}^{-}:=-\left(D_{\left(U^{-}, V^{-}\right)} g\right) \mathcal{K}_{-}(\lambda, \tilde{\xi}) \hat{R}_{-}^{s} \in \mathbb{C}^{1 \times 3}  \tag{22}\\
& \mathcal{K}_{ \pm}(\lambda, \tilde{\xi}):=\left(\begin{array}{cc}
i\left(B_{1}^{1 \pm}\right)^{-1}\left(\xi_{2} B_{2}^{1 \pm}+\xi_{3} B_{3}^{1 \pm}\right) & -\left(B_{1}^{1 \pm}\right)^{-1} \\
-i \xi_{2} I & 0 \\
-i \xi_{3} I & 0 \\
-\lambda I & 0
\end{array}\right) \in \mathbb{C}^{12 \times 6}, \tag{23}
\end{align*}
$$

and $\hat{R}_{+}^{u}=\hat{R}_{+}^{u}(\lambda, \tilde{\xi})$ (resp. $\hat{R}_{-}^{s}$ ) denotes the unstable (resp. stable) space of the matrix fields $\mathbb{M}_{+}$(resp. $\mathbb{M}_{-}$). In (19), $\hat{Q}$ is the "jump vector" associated to Rankine-Hugoniot conditions [12], and $\hat{q}$ denotes its kinetic counterpart [7]. All these elements are evaluated at the wells, for which $W$ attains its minima, and at $s=0$. In addition, $\Delta$ is analytic in $(\lambda, \tilde{\xi}) \in \mathcal{S}^{+}$and continuous in $(\lambda, \tilde{\xi}) \in \mathcal{S}$. For details on the definition of $\Delta$ see [8].

### 2.1. Stable and unstable modes

Denote $\mathcal{N}_{ \pm}(\mu, \tilde{\xi}):=\mathcal{N}\left(\mu, \tilde{\xi}_{2}, \tilde{\xi}_{3}, \underline{U^{ \pm}}\right)$. It has been shown [8] that the eigenvalues $\beta=-i \mu$ of $\mathbb{M}_{ \pm}$, with $\mu \in \mathbb{C}$, satisfy the characteristic equation

$$
\begin{equation*}
\pi_{ \pm}(\mu):=\operatorname{det}\left(N_{ \pm}(\mu, \tilde{\xi})+\lambda^{2} I\right)=0 \tag{24}
\end{equation*}
$$

for all $(\lambda, \tilde{\xi}) \in \mathcal{S}$. Moreover, restricting to $\mathcal{S}^{+}$, and due to hyperbolicity at the end states, the eigenvalues split into stable (with $\operatorname{Im} \mu<0$ ) and unstable (with $\operatorname{Im} \mu>0$ ) modes [8, 9]. Under the current assumptions, there are exactly three unstable and three stable eigenmodes (counting multiplicities) for each $(\lambda, \tilde{\xi}) \in \mathcal{S}^{+}$, implying that the stable and unstable spaces of both $\mathbb{M}_{ \pm}$have constant dimension in all $\mathcal{S}^{+}$. These invariant right spaces are represented by mappings (or bundles) $\hat{R}_{ \pm}^{s, u}: \mathcal{S} \rightarrow \mathbb{C}^{6 \times 3}$ arranged in suitable column bases which can be chosen analytic on $\mathcal{S}^{+}$with continuous extensions to all of $\mathcal{S}$ (see [8] and the references therein).

From Lemma 6 in [8], we reckon that the eigenvector of $\mathbb{M}_{ \pm}$associated to an eigenvalue $\beta=-i \mu$ has the form

$$
\begin{equation*}
R=\binom{Y}{i\left(\mu B_{1}^{1 \pm}+\xi_{2} B_{2}^{1 \pm}+\xi_{3} B_{3}^{1 \pm}\right) Y} \tag{25}
\end{equation*}
$$

where

$$
\begin{equation*}
Y \in \operatorname{ker}\left(\mathcal{N}_{ \pm}(\mu, \tilde{\xi})+\lambda^{2} I\right) \tag{26}
\end{equation*}
$$

### 2.2. Kinetic-rule and jump conditions blocks

In this section we compute the elements in $\Delta$ associated to the classical jump conditions and the kinetic rule. First, since $\sigma=0$ at the wells, we have

$$
\hat{Q}=\binom{\underline{U}_{1}^{+}-\underline{U}_{1}^{-}}{0}=2 \epsilon\binom{e_{2}}{0} \in \mathbb{R}^{6 \times 1}
$$

Anticipating expressions (37) and (38), specializing them to the case where $N=e_{1}$, evaluating at the wells, and using (33), we arrive at

$$
\begin{aligned}
\left(D_{s} g\right)_{\mid s=0} & =D_{s}(\mathcal{F}+h)_{s=0}=\left(D_{s} h\right)_{\mid s=0}, \\
\left(D_{N} g\right)_{\mid s=0} & =D_{N}(\mathcal{F}+h)_{s=0}=0, \\
\left(D_{\left(U^{ \pm}, V^{ \pm}\right)} g\right)_{\mid s=0} & =D_{\left(U^{ \pm}, V^{ \pm}\right)}(\mathcal{F}+h)_{\mid s=0} \\
& =\left(-\frac{1}{2}\left[U_{1}\right]^{\top} B_{1}^{1 \pm},-\frac{1}{2}\left[\underline{U}_{1}\right]^{\top} B_{2}^{1 \pm},-\frac{1}{2}\left[\underline{U}_{1}\right]^{\top} B_{3}^{1 \pm}, 0\right) \\
& =-\epsilon^{3}\left(\left( \pm 2 \epsilon, 1+2 \epsilon^{2}, 0\right), \pm \epsilon e_{2}^{\top}, \pm \epsilon e_{3}^{\top}, 0\right) \in \mathbb{R}^{1 \times 12} .
\end{aligned}
$$

Anticipating (33) and (34), we also compute

$$
\mathcal{K}_{ \pm}(\lambda, \tilde{\xi})=\left(\begin{array}{c}
\frac{i}{2}\left(\begin{array}{ccc}
0 & \xi_{2} & \xi_{3} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) \\
-i \xi_{2} I \\
-i \xi_{3} I \\
-\lambda I
\end{array} \begin{array}{cccc}
-2 & \begin{array}{cc}
-\frac{1}{2}\left(1+2 \epsilon^{2}\right) & \pm \epsilon \\
0 \\
0 & -1
\end{array} & 0 \\
0 & 0 & -1
\end{array}\right) .
$$

Hence,

$$
\begin{aligned}
\left(D_{\left(U^{+}, V^{+}\right)} g\right)_{\mid s=0} \mathcal{K}_{+}(\lambda, \tilde{\xi}) & =-\epsilon^{3}\left(\left(2 \epsilon, 1+2 \epsilon^{2}, 0\right), \epsilon e_{2}^{\top}, \epsilon e_{3}^{\top}, 0\right) \mathcal{K}_{+}(\lambda, \tilde{\xi}) \\
& =\epsilon\left(0, e_{2}^{\top}\right) \in \mathbb{R}^{1 \times 6}, \\
\left(D_{\left(U^{-}, V^{-}\right)} g\right)_{\mid s=0} \mathcal{K}_{-}(\lambda, \tilde{\xi}) & =-\epsilon^{3}\left(\left(-2 \epsilon, 1+2 \epsilon^{2}, 0\right),-\epsilon e_{2}^{\top},-\epsilon e_{3}^{\top}, 0\right) \mathcal{K}_{-}(\lambda, \tilde{\xi}) \\
& =\epsilon\left(0, e_{2}^{\top}\right) \in \mathbb{R}^{1 \times 6},
\end{aligned}
$$

### 2.3. Using the symmetry of the twin

Thanks to the symmetries associated to the martensitic wells, we can simplify the expression for $\Delta$. For instance, let

$$
\Pi=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{array}\right), \quad \quad \Pi^{2}=I .
$$

A direct computation yields

$$
\begin{array}{ll}
\Pi B_{i}^{i+} \Pi=B_{i}^{i-}, & \text { for all } i, \\
\Pi B_{i}^{1+} \Pi=-B_{i}^{1-}, & \text { for all } i \neq 1, \\
\Pi B_{3}^{2+} \Pi=B_{3}^{2-} . &
\end{array}
$$

This implies that

$$
\Pi \mathcal{N}_{+}(\mu, \tilde{\xi}) \Pi=\mathcal{N}_{-}(-\mu, \tilde{\xi})
$$

for all $\mu \in \mathbb{C}$ and all $\tilde{\xi} \in \mathbb{R}^{2}$. Therefore if $Y \in \operatorname{ker}\left(\mathcal{N}_{+}(\mu, \tilde{\xi})+\lambda^{2} I\right)$ for some $\mu \in \mathbb{C}$, then $\left.\Pi Y \in \operatorname{ker}\left(\mathcal{N}_{-}(-\mu, \tilde{\xi})\right)+\lambda^{2} I\right)$, and the stable modes $\mu_{-}^{s}$ at $U=\underline{U^{-}}$can be computed from the unstable modes $\mu_{+}^{u}$ at the other well, by simply taking $\mu_{-}^{s}=-\mu_{+}^{u}$. Hence, it suffices to compute
the unstable modes at $U=\underline{U}^{+}$. This symmetry provides a relation between the bundles $\hat{R}$ as well. By direct computation one finds

$$
\begin{array}{ll}
\Pi M_{11}^{+} \Pi=-M_{11}^{-}, & \Pi M_{12}^{+} \Pi=M_{11}^{-} \\
\Pi M_{22}^{+} \Pi=-M_{22}^{-}, & \Pi M_{21}^{+} \Pi=M_{21}^{-}
\end{array}
$$

and letting

$$
\Lambda:=\left(\begin{array}{cc}
\Pi & 0 \\
0 & -\Pi
\end{array}\right), \quad \Lambda^{2}=I
$$

we arrive at the similarity condition

$$
\begin{equation*}
\Lambda \mathbb{M}_{+}(\lambda, \tilde{\xi}) \Lambda=-\mathbb{M}_{-}(\lambda, \tilde{\xi}) \tag{27}
\end{equation*}
$$

for all $(\lambda, \tilde{\xi}) \in \mathcal{S}$, which implies that if $\hat{R}_{+}^{u}$ is a continuous (analytic) representation on $\mathcal{S}^{+}$of the unstable space of $\mathbb{M}_{+}(\lambda, \tilde{\xi})$, then the columns of

$$
\begin{equation*}
\hat{R}_{-}^{s}:=\Lambda \hat{R}_{+}^{u}, \tag{28}
\end{equation*}
$$

span the stable space of $\mathbb{M}_{-}(\lambda, \tilde{\xi})$ and constitute an analytic representation on $\mathcal{S}^{+}$of this space (this follows by a simple dynamical systems argument). These bundles have continuous, fullrank extensions on the whole set $\mathcal{S}$. Consequently, it suffices to compute the bundle $\hat{R}_{+}^{u}$. Notice also that $\left(0, e_{2}^{\top}\right) \Lambda=\left(0, e_{2}^{\top}\right)$, and that by (28), we have

$$
\hat{p}^{+}=-\hat{p}^{-}=\epsilon\left(0, e_{2}^{\top}\right) \hat{R}_{+}^{u} .
$$

Summarizing, the stability function takes the form

$$
\Delta(\lambda, \tilde{\xi})=\operatorname{det}\left(\begin{array}{ccc}
\Lambda \hat{R}_{+}^{u} & \hat{Q} & \hat{R}_{+}^{u}  \tag{29}\\
-\hat{p} & \hat{q} & \hat{p}
\end{array}\right)
$$

with

$$
\begin{align*}
& \hat{Q}=2 \epsilon\binom{e_{2}}{0},  \tag{30}\\
& \hat{q}=-\lambda\left(D_{s} h\right)_{\mid s=0}, \\
& \hat{p}=\epsilon\left(0, e_{2}^{T}\right) \hat{R}_{+}^{u}
\end{align*}
$$

### 2.4. Using the homogeneity of $\Delta$

From the definition of $\Delta$ and the analysis of [8], it is not hard to see that for all $\rho>0$ and all $(\lambda, \tilde{\xi})$, there holds the homogeneity-like relation

$$
\Delta(\lambda, \tilde{\xi})=\Theta(\rho) \Delta(\rho \lambda, \rho \tilde{\xi})
$$

with $\Theta$ continuous, non-vanishing factor, such that $|\Theta( \pm 1)|=1, \Theta(\rho) \neq 0$. This property follows from the existence of a lifting matrix field $\mathcal{J}(\lambda, \tilde{\xi})$ with constant rank and linear in $\lambda$ and $\tilde{\xi}$, which translates between original and reduced coordinates. (For details see [8].) Therefore we can restrict to the case $|\tilde{\xi}|=1$ and parametrize in polar coordinates by

$$
\tilde{\xi}=(\cos \varphi, \sin \varphi), \quad \varphi \in[0,2 \pi) .
$$

Let us denote $\tilde{\mathcal{N}}_{+}(\mu, \varphi):=\mathcal{N}_{+}(\mu, \tilde{\xi})$. Since $\mathcal{N}_{+}(-\mu,-\tilde{\xi})=\mathcal{N}_{+}(\mu, \tilde{\xi})$ for all $\mu$ and $\tilde{\xi}$, then it suffices to take $\varphi \in[0, \pi]$. Finally, as $\tilde{\mathcal{N}}_{+}(-\mu,-\varphi)=\tilde{\mathcal{N}}_{+}(\mu, \varphi)$, we can restrict the computations to the quarter circle $\varphi \in\left[0, \frac{\pi}{2}\right]$, meaning no loss of generality. From this point on, we write indistinctively $\Delta(\lambda, \tilde{\xi})$ or $\Delta(\lambda, \varphi)$ to indicate $\Delta\left(\lambda, e^{i \varphi}\right)$.

## 3. Energy density function and kinetic rules

This section gathers needed properties of the energy density function (14) and the kinetic rules (11) and (12).

### 3.1. Derivatives of the stress

First, we compute the matrices $B_{i}^{j}$ containing the second derivatives of the energy. Denoting $\Phi_{ \pm}(U):=\left|C-C_{ \pm}\right|^{2}$, then clearly $W(U)=\frac{1}{32} \Phi_{+}(U) \Phi_{-}(U)$, with $\Phi_{ \pm}\left(\underline{U}^{ \pm}\right)=0$ and $\Phi_{ \pm}\left(\underline{U}^{\mp}\right)=$ $8 \epsilon^{2}>0$. By direct computation,

$$
\begin{align*}
& \left(D_{U_{1}} \Phi_{ \pm}\right)(U)=4\left(\left(\left|U_{1}\right|^{2}-\left(1+\epsilon^{2}\right)\right) U_{1}+\left(U_{2}^{\top} U_{1} \mp \epsilon\right) U_{2}+\left(U_{1}^{\top} U_{3}\right) U_{3}\right), \\
& \left(D_{U_{2}} \Phi_{ \pm}\right)(U)=4\left(\left(\left|U_{2}\right|^{2}-1\right) U_{2}+\left(U_{2}^{\top} U_{1} \mp \epsilon\right) U_{1}+\left(U_{2}^{\top} U_{3}\right) U_{3}\right),  \tag{31}\\
& \left(D_{U_{3}} \Phi_{ \pm}\right)(U)=4\left(\left(\left|U_{3}\right|^{2}-1\right) U_{2}+\left(U_{3}^{\top} U_{1} \mp \epsilon\right) U_{1}+\left(U_{2}^{\top} U_{3}\right) U_{2}\right) .
\end{align*}
$$

Since the stress $\sigma$ vanishes at the wells, it is easy to see that for all $i, j$,

$$
B_{i}^{j \pm}=\frac{1}{32} \Phi_{\mp}\left(\underline{U}^{ \pm}\right)\left(D_{U_{j} U_{i}}^{2} \Phi_{ \pm}\right)\left(\underline{U}^{ \pm}\right)=\frac{1}{4} \epsilon^{2}\left(D_{U_{j} U_{i}}^{2} \Phi_{ \pm}\right)\left(\underline{U}^{ \pm}\right)
$$

From (31) we obtain

$$
\begin{aligned}
& \left(D_{U_{1} U_{1}}^{2} \Phi_{ \pm}\right)(U)=4\left(2 U_{1} \otimes U_{1}+\left(\left|U_{1}\right|^{2}-\left(1+\epsilon^{2}\right)\right) I+U_{2} \otimes U_{2}+U_{3} \otimes U_{3}\right), \\
& \left(D_{U_{2} U_{2}}^{2} \Phi_{ \pm}\right)(U)=4\left(2 U_{2} \otimes U_{2}+\left(\left|U_{2}\right|^{2}-1\right) I+U_{1} \otimes U_{1}+U_{3} \otimes U_{3}\right), \\
& \left(D_{U_{3} U_{3}}^{2} \Phi_{ \pm}\right)(U)=4\left(2 U_{3} \otimes U_{3}+\left(\left|U_{3}\right|^{2}-1\right) I+U_{1} \otimes U_{1}+U_{2} \otimes U_{2}\right), \\
& \left(D_{U_{1} U_{2}}^{2} \Phi_{ \pm}\right)(U)=\left(D_{U_{2} U_{1}}^{2} \Phi_{ \pm}\right)^{\top}=4\left(U_{1} \otimes U_{2}+\left(U_{2}^{\top} U_{1} \mp \epsilon\right) I\right), \\
& \left(D_{U_{1} U_{3}}^{2} \Phi_{ \pm}\right)(U)=\left(D_{U_{3} U_{1}}^{2} \Phi_{ \pm}\right)^{\top}=4\left(U_{1} \otimes U_{3}+\left(U_{1}^{\top} U_{3}\right) I\right), \\
& \left(D_{U_{2} U_{3}}^{2} \Phi_{ \pm}\right)(U)=\left(D_{U_{2} U_{1}}^{2} \Phi_{ \pm}\right)^{\top}=4\left(U_{2} \otimes U_{3}+\left(U_{2}^{\top} U_{3}\right) I\right),
\end{aligned}
$$

and evaluating at the wells we readily get,

$$
\begin{array}{ll}
B_{1}^{1 \pm}=\epsilon^{2}\left(\begin{array}{ccc}
2 & \pm 2 \epsilon & 0 \\
\pm 2 \epsilon & 1+2 \epsilon^{2} & 0 \\
0 & 0 & 1
\end{array}\right), & B_{2}^{1 \pm}=\left(B_{1}^{2 \pm}\right)^{\top}=\epsilon^{2}\left(\begin{array}{ccc}
0 & 1 & 0 \\
0 & \pm \epsilon & 0 \\
0 & 0 & 0
\end{array}\right), \\
B_{2}^{2 \pm}=\epsilon^{2}\left(\begin{array}{ccc}
1 & \pm \epsilon & 0 \\
\pm \epsilon & 2+\epsilon^{2} & 0 \\
0 & 0 & 1
\end{array}\right), & B_{3}^{1 \pm}=\left(B_{1}^{3 \pm}\right)^{\top}=\epsilon^{2}\left(\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & \pm \epsilon \\
0 & 0 & 0
\end{array}\right),  \tag{33}\\
B_{3}^{3 \pm}=\epsilon^{2}\left(\begin{array}{ccc}
1 & \pm \epsilon & 0 \\
\pm \epsilon & 1+\epsilon^{2} & 0 \\
0 & 0 & 2
\end{array}\right), & B_{3}^{2 \pm}=\left(B_{2}^{3 \pm}\right)^{\top}=\epsilon^{2}\left(\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right) .
\end{array}
$$

From (33) one finds that

$$
\left(B_{1}^{1 \pm}\right)^{-1}=\epsilon^{-2}\left(\begin{array}{ccc}
\frac{1}{2}\left(1+2 \epsilon^{2}\right) & \mp \epsilon & 0  \tag{34}\\
\mp \epsilon & 1 & 0 \\
0 & 0 & 1
\end{array}\right) .
$$

Note that the eigenvalues of $B_{1}^{1 \pm}$ are

$$
\begin{align*}
& k_{1}=\frac{1}{2} \epsilon^{2}\left(3+2 \epsilon^{2}-\sqrt{\left(3+2 \epsilon^{2}\right)^{2}-8}\right), \\
& k_{2}=\epsilon^{2},  \tag{35}\\
& k_{3}=\frac{1}{2} \epsilon^{2}\left(3+2 \epsilon^{2}+\sqrt{\left(3+2 \epsilon^{2}\right)^{2}-8}\right),
\end{align*}
$$

with $0<k_{1}<k_{2}<k_{3}$ and constant multiplicity 1 . Therefore the six characteristic speeds associated to the front are $\alpha_{j}= \pm \sqrt{k_{j}}, j=1,2,3$. The analysis in [8] applies to all subsonic propagating fronts with speed $s^{2}<k_{j}$ for all $j$.

### 3.2. Rank-one convexity and constant multiplicity assumption

In this subsection, we verify two hypotheses made in [8], so as to guarantee that the results of our computations really allow for the said conclusions. To assure hyperbolicity at the end states, the first hypothesis is the rank-one convexity of $W$ at the wells, or equivalently, that Legendre-Hadamard condition,

$$
\begin{equation*}
\zeta^{\top} \mathcal{N}\left(\xi, \underline{U}^{ \pm}\right) \zeta>0, \quad \text { for all } \zeta, \xi \in \mathbb{R}^{3} \backslash\{0\} \tag{36}
\end{equation*}
$$

holds [5]. In order to verify (36) we express $W$ in terms of the Cauchy-Green strain as

$$
W(U)=: \hat{W}\left(C_{11}, C_{22}, C_{33}, C_{12}, C_{13}, C_{23}\right),
$$

with Hessian

$$
\left(D_{C}^{2} \hat{W}\right)_{\mid C=C^{ \pm}}=\epsilon^{2}\left(\begin{array}{cc}
\frac{1}{2} I_{3 \times 3} & 0 \\
0 & I_{3 \times 3}
\end{array}\right)>0
$$

for all $\epsilon>0$ and evaluated, of course, at the minima $C=C^{ \pm}$. Therefore, it is easy to see that the quadratic form in (36) can be expressed as

$$
b^{ \pm}(\xi, \zeta):=\zeta^{\top} \mathcal{N}\left(\xi, \underline{U}^{ \pm}\right) \zeta=\xi^{\top} H\left(\underline{U}^{ \pm}, \zeta\right)^{\top}\left(D_{C}^{2} \hat{W}\right)_{\mid C=C^{ \pm}} H\left(\underline{U}^{ \pm}, \zeta\right) \xi,
$$

where the $6 \times 3$ matrix $H\left(\underline{U}^{ \pm}, \zeta\right)$ is defined by

$$
H\left(\underline{U}^{ \pm}, \zeta\right)=\left(\begin{array}{ccc}
2\left(\underline{U}_{1}^{ \pm}\right)^{\top} \zeta & 0 & 0 \\
0 & 2\left(\underline{U}_{2}^{ \pm}\right)^{\top} \zeta & 0 \\
0 & 0 & 2\left(\underline{U}_{3}^{ \pm}\right)^{\top} \zeta \\
\left(\underline{U}_{2}^{ \pm}\right)^{\top} \zeta & \left(\underline{U}_{1}^{ \pm}\right)^{\top} \zeta & 0 \\
\left(\underline{U}_{3}^{ \pm}\right)^{\top} \zeta & 0 & \left(\underline{U}_{1}^{ \pm}\right)^{\top} \zeta \\
0 & \left(\underline{U}_{3}^{ \pm}\right)^{\top} \zeta & \left(\underline{U}_{2}^{ \pm}\right)^{\top} \zeta
\end{array}\right)
$$

Since for each $\zeta \neq 0, H\left(\underline{U}^{ \pm}, \zeta\right)$ has full rank (as $\left.\operatorname{det} \underline{U}^{ \pm}>0\right)$ and in view of $\left(D_{C}^{2} \hat{W}\right)_{\mid C=C^{ \pm}}>0$, then clearly $b^{ \pm}(\xi, \zeta)>0$ for all $\zeta, \xi \in \mathbb{R}^{3} \backslash\{0\}$, that is, $W$ is rank-one convex at $U=\underline{U}^{ \pm}$and the system is hyperbolic at the wells.

Another (though actually not essential) requirement for the analysis of [8] is the constant multiplicity condition of Métivier [13], namely, that the eigenvalues of $\mathcal{N}(\xi, U)$ are all semisimple for all $U$ near $\underline{U}^{+}$or near $\underline{U}^{-}$, and all $\xi \neq 0$. By continuity of the eigenvalues, it suffices to check this condition at the wells, as the property is preserved in open neighborhoods of $U=\underline{U}^{ \pm}$. Notice that $\mathcal{N}\left(\rho \xi, \underline{U}^{ \pm}\right)=\rho^{2} \mathcal{N}\left(\xi, \underline{U}^{ \pm}\right)$for all $\rho>0$. Hence, it suffices to consider $|\xi|=1$ which we parametrize in spherical coordinates by $\xi_{1}=\sin \psi \cos \theta, \xi_{2}=\sin \psi \sin \theta$, and $\xi_{3}=\cos \psi$, with $\theta \in[0,2 \pi], \psi \in[0, \pi]$. One can easily check Métivier's condition numerically. Figure 1 shows the computed eigenvalues $\kappa_{j}$ of $\mathcal{N}\left(\xi, \underline{U^{ \pm}}\right)$for different values of $\epsilon>0$ in a mesh of angles $(\psi, \theta)$, interpolated smoothly as surfaces. It turns out that the eigenvalues are all simple for all values of $\xi$ and the selected values of the material parameter $\epsilon$.

### 3.3. Derivatives of kinetic rules

We gather the derivatives of $h$ and $\mathcal{F}$ from (10), (11) and (12). Noticing that for each $i, j$, we have

$$
\begin{aligned}
\partial_{U_{i j}^{ \pm}}\left(N^{\top}[U]^{\top}\langle\sigma(U)\rangle N\right) & = \pm N_{j} \sum_{k} N_{k}\left\langle\sigma(U)_{i k}\right\rangle+\frac{1}{2} N^{\top}[U]^{\top} \sum_{l} N_{l} \partial_{U_{i j}^{ \pm}}\left(\sigma\left(U^{ \pm}\right)_{l}\right) \\
& = \pm N_{j} \sum_{k} N_{k}\left\langle\sigma(U)_{i k}\right\rangle+\frac{1}{2} \sum_{l, k} N_{l}([U] N)_{k}\left(B_{j}^{l \pm}\right)_{k i},
\end{aligned}
$$



Figure 1. Plot of the eigenvalues of $\mathcal{N}\left(\xi, \underline{U}^{ \pm}\right), \xi=(\sin \psi \cos \theta, \sin \psi \sin \theta, \cos \psi)$ on a mesh $(\psi, \theta) \in[0, \pi] \times[0,2 \pi]$, for values of $\epsilon=1$ (top), $\epsilon=0.75$ (center), $\epsilon=0.5$ (bottom). The three surfaces correspond to the continuous values of the three real eigenvalues $\kappa_{4,2,3}$. The azimuthal view (right) shows that the eigenvalues are simple and never coalesce.
one gets,

$$
\begin{align*}
D_{\left(U^{ \pm}, V^{ \pm}\right)} \mathcal{F}= & \left( \pm \sigma\left(U^{ \pm}\right)_{1}^{\top} \mp N_{1} N^{\top}\langle\sigma(U)\rangle^{\top}-\frac{1}{2} N^{\top}[U]^{\top} \sum_{j=1}^{3} N_{j} B_{1}^{j \pm},\right. \\
& \pm \sigma\left(U^{ \pm}\right)_{2}^{\top} \mp N_{2} N^{\top}\langle\sigma(U)\rangle^{\top}-\frac{1}{2} N^{\top}[U]^{\top} \sum_{j=1}^{3} N_{j} B_{2}^{j \pm},  \tag{37}\\
& \pm \sigma\left(U^{ \pm}\right)_{3}^{\top} \mp N_{3} N^{\top}\langle\sigma(U)\rangle^{\top}-\frac{1}{2} N^{\top}[U]^{\top} \sum_{j=1}^{3} N_{j} B_{3}^{j \pm}, \\
& 0) \quad \in \mathbb{R}^{1 \times 12},
\end{align*}
$$

(where each element is a $1 \times 3$ block), and

$$
\begin{align*}
& D_{N} \mathcal{F}=N^{\top}\left([U]^{\top}\langle\sigma(U)\rangle+\langle\sigma(U)\rangle^{\top}[U]\right) \in \mathbb{R}^{1 \times 3}  \tag{38}\\
& D_{s} \mathcal{F} \equiv 0 \tag{39}
\end{align*}
$$

Furthermore, we obviously have

$$
\begin{align*}
D_{\left(U^{ \pm}, V^{ \pm}\right)} h & =0, \quad \text { at } \quad\left(\left(\underline{U}^{+}, 0\right),\left(\underline{U}^{-}, 0\right), 0, e_{1}\right),  \tag{40}\\
D_{N} h & =0, \quad \text {, }
\end{align*}
$$

both for Maxwell and Abeyaratne-Knowles rules.

## 4. Evaluating $\Delta$

In this section we detail how to numerically evaluate the reduced Lopatinski function $\Delta$ associated to the martensite twin in such a way that the numerical output provides secure information about the dynamical stability behaviour of the twin.

### 4.1. Key ideas

4.1.1. The winding number argument We will look at the family of mappings

$$
\lambda \mapsto \bar{\Delta}(\lambda, \varphi) \quad \varphi \in[0, \pi / 2]
$$

for a suitable normalization of the stability function $\bar{\Delta}$ along the closed contours

$$
\begin{aligned}
\lambda \in C_{\rho} & :=C_{\rho}^{+} \cup C_{\rho}^{0} \\
C_{\rho}^{+} & :=\{\lambda \in \mathbb{C}:|\lambda|=\rho, \operatorname{Re} \lambda>0\} \\
C_{\rho}^{0} & :=\{\lambda \in \mathbb{C}: \lambda=i \tau, \tau \in[-\rho,+\rho]\},
\end{aligned}
$$

for some $\rho>0$ sufficiently large. Given that for $|\lambda| \gg 1$ large, $\Delta(\lambda, \tilde{\xi}) \sim \Delta(1,0) \neq 0$, the possible zeroes are bounded in $\lambda$ and it is enough to consider contours $C_{\rho}$ with finite $\varrho>0$. By the product formula of mapping degrees [6], the winding number of the curves $\lambda \mapsto \Delta(\lambda, \varphi)$ with respect to zero determines the stability of the configuration.
4.1.2. The normalization There is a major ambiguity in the definition (29), consisting in the freedom to choose specific representations $\hat{R}_{-}^{s}$ of the stable space of $\mathbb{M}_{-}$, and $\hat{R}_{+}^{u}$ of the unstable space of $\mathbb{M}_{+}$. This ambiguity can largely obscure the picture when one tries to apply the above computational evaluation of the winding number to the unmodified reduced Lopatinski function $\Delta$. In fact, in order to proceed correctly in that direction, one would have to convince oneself that the concrete representations, chosen through formulation and computing methodology, are not only continuous along the curves $C_{\rho}$, but can be extended, without modification, to continuous matrix functions of full rank at least on the whole interior of these curves. However, given the fact that they are three-dimensional sub-bundles of $\mathbb{C}^{6}$, there does not seem to exist any easy way of doing such verification. We completely circumvent this apparent difficulty by considering a normalized version of $\Delta$,

$$
\begin{equation*}
\bar{\Delta}:=\frac{\Delta}{-|R|} \tag{41}
\end{equation*}
$$

with

$$
\begin{equation*}
|R|:=\operatorname{det}\left(\hat{R}_{-}^{s} \quad \hat{R}_{+}^{u}\right)=\operatorname{det}\left(\Lambda \hat{R}_{+}^{u} \quad \hat{R}_{+}^{u}\right) . \tag{42}
\end{equation*}
$$

The simple idea of this normalization is that it automatically undoes all possible extraneous winding. In other words: By whichever concrete factor the original $\Delta$ deviates from what it would be with respect to an admissible choice $R_{-}^{s}, R_{+}^{u}$ of the representation, the scaling denominator $|R|$ deviates from the determinant $\operatorname{det}\left(R_{-}^{s} R_{+}^{u}\right)$ in exactly that way, and these two effects cancel each other out!
4.1.3. $\bar{\Delta}$ for Maxwell vs. Abeyaratne-Knowles kinetics A simple consideration (see Section 4.4 below) on $\Delta$ and $|R|$ shows that

$$
\begin{equation*}
\bar{\Delta}=\bar{\Delta}_{0}-\lambda\left(D_{s} h\right) \tag{43}
\end{equation*}
$$

where $\bar{\Delta}_{0}$ is the normalized Lopatinski function associated to Maxwell kinetics. This not only simplifies the algorithms considerably, but also gives a direct explanation of how the images of the $C_{\rho}$ change upon going from vanishing to non-vanishing driving traction. We invite the reader to play with the geometric associations that formula (43) immediately prompts in that direction.

### 4.2. The Lopatinski modes

The essential piece in reliably determining the Lopatinski spaces is the identification of the frequencies. Once this is done in a secure way, standard algorithms allow to construct the bundle representations $\hat{R}_{-}^{s}$ and $\hat{R}_{+}^{u}$ safely. We now discuss how to properly find the unstable $\mu$-roots of

$$
\pi_{+}(\mu)=\operatorname{det}\left(\mathcal{N}_{+}(\mu, \tilde{\xi})+\lambda^{2} I\right)=\sum_{j=0}^{6} a_{j} \mu^{j}
$$

for each $(\lambda, \tilde{\xi}) \in \mathbb{C} \times \mathbb{R}^{2}$, with $|\tilde{\xi}|=1$. After cumbersome but straightforward computations, one finds that the coefficients $a_{j}=a_{j}(\lambda, \tilde{\xi}, \epsilon)$ are given by

$$
\begin{align*}
& a_{6}=2 \epsilon^{6}, \\
& a_{5}=0, \\
& a_{4}=\lambda^{2} \epsilon^{4}\left(2 \epsilon^{2}+5\right)+6 \epsilon^{6}|\tilde{\xi}|^{2}, \\
& a_{3}=2 \epsilon^{5} \xi \xi_{2} \lambda,  \tag{44}\\
& a_{2}=2 \lambda^{4} \epsilon^{3}\left(2+\epsilon^{2}\right)+\lambda^{2} \epsilon^{4}\left(\left(10+3 \epsilon^{2}\right)|\tilde{\xi}|^{2}+\epsilon^{2} \xi_{3}^{2}\right)+6 \epsilon^{6}|\tilde{\xi}|^{4}, \\
& a_{1}=2 \lambda^{2} \epsilon^{3} \xi_{2}\left(\lambda^{2}+\epsilon^{2}|\tilde{\xi}|^{2}\right) \\
& a_{0}=\lambda^{6}+\lambda^{4} \epsilon^{2}|\tilde{\xi}|^{2}\left(4+\epsilon^{2}\right)+\lambda^{2} \epsilon^{4}|\tilde{\xi}|^{2}\left(\left(5+\epsilon^{2}\right)|\tilde{\xi}|^{2}+\epsilon^{2} \xi_{3}^{2}\right)+2 \epsilon^{6}|\tilde{\xi}|^{6} .
\end{align*}
$$

Since $a_{6}=2 \epsilon^{6}>0$, we implemented a standard algorithm to compute the roots of the polynomial $\pi_{+}(\mu)=0$. More precisely, we calculated the eigenvalues of

$$
C(\lambda, \tilde{\xi}, \epsilon)=\left(\begin{array}{cccccc}
-a_{5} / a_{6} & -a_{4} / a_{6} & -a_{3} / a_{6} & -a_{2} / a_{6} & -a_{1} / a_{6} & -a_{0} / a_{6} \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right)
$$

via a standard Schur factorization for complex non-Hermitian matrices [14].
For values of $\lambda$ with $\operatorname{Re} \lambda>0$, the six $\mu$-modes of $\pi_{+}(\mu)=0$ are either stable or unstable. But along the imaginary axis neutral modes can occur. These neutral modes, however, are the continuous limits of stable or unstable modes as $\operatorname{Re} \lambda \rightarrow 0^{+}$. When a neutral mode $\mu$ is the limit of a stable (resp. unstable) mode we call it neutrally stable (resp. neutrally unstable).

Therefore the roots of $\pi_{+}$come always in trios of stable/neutrally stable or unstable/neutrally unstable modes.

We designed a direct algorithm to determine the stability nature of the neutral modes along the imaginary axis, using a simple perturbation method. For each $\lambda=i \tau \in i \mathbb{R}$ we make $\tilde{\lambda}=i \tau+\delta$ with $\delta>0$ and compute the perturbed eigenmodes. Estimating the minimum distance between the latter and the original neutral mode, we select the associated stable or unstable mode which, by continuity, tends to the original neutral $\mu$ as $\delta \rightarrow 0^{+}$. In the case of coalescence, that is, when there is a multiple neutral mode, the algorithm safely keeps track of the perturbed modes already selected, in such a way that it produces exactly three stable/neutrally-stable and three unstable/neutrally-unstable values, respecting the hyperbolic dichotomy (it can happen that a multiple neutral mode is at the same time the limit of stable and unstable modes). In our computations we take $\delta$ of order $O\left(10^{-6}\right)$.

Figure 2 shows the computed values of stable and unstable modes for $\lambda \in \mathcal{C}_{\rho}^{+}$, that is, along half circles, for different values of $\rho>0$. They behave as unstable/stable in trios, with neutral limits as $\operatorname{Re} \lambda \rightarrow 0^{+}$.


Figure 2. Computed values of $\mu$ for $(\lambda, \tilde{\xi})=\left(\rho e^{i \theta}, e^{i \varphi}\right), \theta \in[0, \pi]$, for different values of $(\rho, \varphi)=(1,0)$ (left), $(2, \pi / 4)$ (center), $(3, \pi / 2)$ (right). The eigenmodes come in trios of stable $(\operatorname{Im} \mu<0)$ and unstable $(\operatorname{Im} \mu>0)$ modes along the curves.

For values of $\lambda$ along the imaginary axis we expect the presence of branch points, for example, values of $\tau$ for which the eigenmodes $\mu$ change from neutral to stable/unstable, or points where two or more modes coalesce. Figure 3 shows computed values of the unstable modes along an interesting portion of the imaginary axis, namely, near zero (for $\tau$ large the modes are always neutral), and for different values of $\varphi$. We observe that the modes coalesce, for instance, at $\lambda=0$. To illustrate the phenomenon of neutral to stable/unstable branching, figure 4 depicts the imaginary parts of the unstable modes $\mu^{u}$ against $\tau \in[-1,1]$, again, for different values of $\varphi$. It is clear the existence of six branch points of real-to-complex type.

### 4.3. Evaluation of the bundle $\hat{R}_{+}^{u}$

Once the Lopatinski frequencies have been computed, we procceed to assemble the unstable Lopatinski bundle $\hat{R}_{+}^{u}$ according to formula (25). The main step is the computation of the kernel of $\tilde{\mathcal{N}}_{+}(\mu, \varphi)+\lambda^{2} I$, where $\mu$ is the unstable mode under consideration. For that purpose, we use a simple cross-product algorithm that calculates $Y$ and numerically verify the result for each unstable/neutrally-unstable mode $\mu$. At most points, the constructed bundle (following (25)) is a valid, full-rank representation of the unstable Lopatinski space, with the exception of a discrete set of branch points associated to multiple modes where non-trivial Jordan blocks may occur. But this factor, however, appears at the same order in both the numerator and


Figure 3. Computed values of $\mu$ for $(\lambda, \tilde{\xi})=\left(i \tau, e^{i \varphi}\right), \tau \in[-1,1]$, for different values of $\varphi=0$ (left), $\varphi=\pi / 4$ (center), $\varphi=\pi / 2$ (right). The modes are neutral for $|\tau|$ large.


Figure 4. Plots of the imaginary part of the unstable modes $\mu^{\mu}$ versus $\tau$, along $\lambda=i \tau$, $\tau \in[-1,1]$ for different values of $\tilde{\xi}=e^{i \varphi}$, with $\varphi=0, \pi / 4,4 \pi / 9$, and $\pi / 2$ (upper left, upper right, lower left and lower right corners, respectively). Note the presence of six branch points of neutral-to-unstable type along this segment of the imaginary axis.
the denominator of (41), cancelling each other out, and yielding the desired effect of the normalization.

### 4.4. Reduction of determinants

Finally, one can make further reductions on the expression for $\bar{\Delta}$ in order to arrive at formula (43). Suppose $\hat{R}_{+}^{u}$ is a continuous representation of the unstable space of $\mathbb{M}_{+}$. Let us define

$$
\hat{R}_{+}^{u}=:\left(\begin{array}{c}
y_{1}  \tag{45}\\
y_{2} \\
y_{3} \\
z_{1} \\
z_{2} \\
z_{3}
\end{array}\right)
$$

so that each $y_{j}$ and each $z_{j}$ are three-dimensional row vector-valued functions of $(\lambda, \tilde{\xi})$. From (29) and (30), the reduced Lopatinski function takes the form

$$
\Delta=\left|\begin{array}{ccc}
y_{1} & 0 & y_{1} \\
-y_{2} & 2 \epsilon & y_{2} \\
-y_{3} & 0 & y_{3} \\
-z_{1} & 0 & z_{1} \\
z_{2} & 0 & z_{2} \\
z_{3} & 0 & z_{3} \\
-\epsilon z_{2} & -\lambda\left(D_{s} h\right) & \epsilon z_{2}
\end{array}\right| .
$$

Performing elementary column-row transformations and pivoting on the middle column, we obtain

$$
\begin{align*}
\Delta & =8\left|\begin{array}{ccc}
y_{1} & 0 & y_{1} \\
z_{2} & 0 & z_{2} \\
z_{3} & 0 & z_{3} \\
0 & 0 & y_{3} \\
0 & 0 & z_{1} \\
0 & 2 \epsilon & y_{2} \\
0 & -\lambda\left(D_{s} h\right) & \epsilon z_{2}
\end{array}\right| \\
& =16 \epsilon\left|\begin{array}{cc}
y_{1} & y_{1} \\
z_{2} & z_{2} \\
z_{3} & z_{3} \\
0 & y_{3} \\
0 & z_{1} \\
0 & \epsilon z_{2}
\end{array}\right|+8 \lambda\left(D_{s} h\right)\left|\begin{array}{cc}
y_{1} & y_{1} \\
z_{2} & z_{2} \\
z_{3} & z_{3} \\
0 & y_{3} \\
0 & z_{1} \\
0 & y_{2}
\end{array}\right|  \tag{46}\\
& \left.=16 \epsilon^{2}\left|\begin{array}{c|c|c|}
y_{1} \\
z_{2} \\
z_{3}
\end{array}\right|\left|\begin{array}{c}
y_{3} \\
z_{1} \\
z_{2}
\end{array}\right|+8 \lambda\left(D_{s} h\right)\left|\begin{array}{c}
y_{1} \\
z_{2} \\
z_{3}
\end{array}\right| \begin{array}{c}
y_{3} \\
z_{1} \\
y_{2}
\end{array} \right\rvert\, .
\end{align*}
$$

Similarly, from the definition (42) of $|R|$, we have

$$
|R|=\left|\begin{array}{cc}
y_{1} & y_{1} \\
-y_{2} & y_{2} \\
-y_{3} & y_{3} \\
-z_{1} & z_{1} \\
z_{2} & z_{2} \\
z_{3} & z_{3}
\end{array}\right|=8\left|\begin{array}{cc}
y_{1} & y_{1} \\
z_{2} & z_{2} \\
z_{3} & z_{3} \\
0 & z_{1} \\
0 & y_{2} \\
0 & y_{3}
\end{array}\right|=8\left|\begin{array}{l}
y_{1} \\
z_{2} \\
z_{3}
\end{array}\right|\left|\begin{array}{c}
z_{1} \\
y_{2} \\
y_{3}
\end{array}\right| .
$$

In view of the normalization (41), this finally implies that

$$
\begin{equation*}
\bar{\Delta}=\bar{\Delta}_{0}-\lambda\left(D_{s} h\right), \tag{47}
\end{equation*}
$$

where

$$
\bar{\Delta}_{0}=-2 \epsilon^{2} \frac{\left|\begin{array}{l}
y_{3}  \tag{48}\\
z_{1} \\
z_{2}
\end{array}\right|}{\left|\begin{array}{l}
z_{1} \\
y_{2} \\
y_{3}
\end{array}\right|}
$$

is the normalized Lopatinski function associated to Maxwellian kinetics.
Thus, all we need to do is to safely compute the the bundle $\hat{R}_{+}^{u}$, rearrange its entries in the two $3 \times 3$ determinants according to (45), and evaluate their quotient (48).

## 5. Output and stability results

This section displays the results of our computations, in which we fixed the value of the material parameter as $\epsilon=0.5$. In what follows we reckon kinetic relations of linear type, as proposed originally by Abeyaratne and Knowles [1] for irreversible processes close to thermodynamic equilibrium. These have the form

$$
\begin{equation*}
\mathcal{F}=\frac{s}{M}, \tag{49}
\end{equation*}
$$

where $M>0$ is a mobility coefficient; or, in other words, $h=-s / M$ and $D_{s} g=D_{s}(\mathcal{F}+h)=$ $-1 / M$. Clearly, when $M \rightarrow+\infty$ we recover the Maxwell rule.

We computed the parametrized curves $\lambda \mapsto \bar{\Delta}(\cdot, \varphi), \lambda \in C_{\rho}$, with $\rho=2$ and for different parameter values of $\varphi$, under both the Maxwell rule and linear kinetic rules with $M=10$ and $M=1$. Since $\Delta$ is continuous in $\varphi \in[0, \pi / 2]$, we chose $\varphi=0, \pi / 8, \pi / 4,3 \pi / 8$ and $\pi / 2$ in order to illustrate the behaviour along the whole parameter set. In each of the figures presented in the following pages, the blue dots correspond to values of $\bar{\Delta}$ for $\lambda=i \tau$ with $\tau>0$, whereas the red circles represent values of $\bar{\Delta}$ for $\lambda=i \tau$ with $\tau>0$; the green dots correspond to the values of $\Delta$ for $\lambda \in C_{\rho}^{+}$. Each figure shows a detailed view of an interesting part of the imaginary axis (left), say $i \tau \in[-1,1]$, and of the whole contour $\mathcal{C}_{\rho}$ (right).

### 5.1. Maxwell kinetic rule: $M=+\infty$

Our first calculation studies Maxwellian kinetics. Figures 5 and 6 show the computed curves $\lambda \rightarrow \Delta_{0}(\cdot, \varphi)$, corresponding to vanishing driving traction. For instance, in Figure 5, in thr graph corresponding to the value $\varphi=0$ (top) we notice the presence of two zeroes of $\bar{\Delta}_{0}$ at $\lambda= \pm i \tau_{0}$ for some $\tau_{0}>0$. By continuity of the Lopatinski function in $\varphi$, this behaviour persists if we move up to a critical value $\varphi_{c}<\pi / 8$. This suggests the presence of a "cone" of zeroes (cone in the variables $\tilde{\xi}$ ) along the imaginary axis for Maxwell kinetic rules. Observe that for $\varphi=\pi / 8$ (center) and $\varphi=\pi / 4$ (bottom) there are no longer zeroes along the imaginary axis. Figure 5 depicts the curves for $\varphi=3 \pi / 8$ (top) and $\varphi=\pi / 2$ (bottom), exhibiting the same behaviour. Note, however, the presence of an isolated zero at $\lambda= \pm i \tau_{*}$ for some $\tau_{*}>0$. Also note that by a winding number argument, there are no zeroes inside the contour for all values of $\varphi$ under consideration. These observations imply weak stability of the twin for Maxwellian kinetics.

### 5.2. Case $M=10$

If we perturb by linear kinetic relations as described above with positive values of $M$, we observe an stabilizing effect. Take, for instance, what happens when $M=10$. In this case we compute $\Delta$ substituting $D_{s} h=-1 / 10$ in formula (43). Figures 7 and 8 show the computed curves $\lambda \rightarrow \bar{\Delta}$ with $\lambda \in \mathcal{C}_{\rho}$, for different values of $\varphi$. The significant behaviour of the curve is such that zero remains outside the contour, showing the evanescence of the winding number of the curve with respect to zero and suggesting, in turn, strong stability. Figure 7 presents the curves for $\varphi=0$ (top), $\pi / 8$ (center) and $\pi / 4$ (bottom). Figure 8 shows the curves for $\varphi=3 \pi / 8$ (top) and $\pi / 2$ (bottom).

### 5.3. Case $M=1$

When we perform the computation of $\bar{\Delta}$, but now taking the material parameter as $M=1$, the same stabilizing effect of the linear kinetic rule is observed. Figures 9 and 10 present the numerical computation of the curves for $M=1$. Figure 9 shows the curves for $\varphi=0$ (top), $\pi / 8$ (center) and $\pi / 4$ (bottom). Figure 10 shows the curves for $\varphi=3 \pi / 8$ (top) and $\pi / 2$ (bottom). It turns out that the symmetric "opening" of the curve along the imaginary axis, which avoids zero, is even more accentuated. These pictures indicate strong stability of the twin with respect to linear kinetic rules as well.

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Figure 5. Maxwell kinetic rule (a): Images of the curve $\lambda \mapsto \bar{\Delta}_{0}$ (with $M=+\infty$ ), for a portion of the imaginary axis (left), and the whole contour $\mathcal{C}_{\rho}$ (right), with $\rho=2$. These correspond to values of $\varphi=0$ (top), $\varphi=\pi / 8$ (center) and $\varphi=\pi / 4$ (bottom).


Figure 6. Maxwell kinetic rule (b): Images of the curve $\lambda \mapsto \bar{\Delta}_{0}$ (with $M=+\infty$ ), for a portion of the imaginary axis (left), and the whole contour $\mathcal{C}_{\rho}$ (right), with $\rho=2$. These correspond to values of $\varphi=3 \pi / 8$ (top) and $\varphi=\pi / 2$ (bottom).


Figure 7. Perturbed kinetic rule with $M=10$ (a): Images of the curve $\lambda \mapsto \bar{\Delta}$ with $M=10$, for a portion of the imaginary axis (left), and the whole contour $C_{p}$ (right), with $\rho=2$. These correspond to values of $\varphi=0$ (top), $\varphi=\pi / 8$ (center), and $\varphi=\pi / 4$ (bottom).


Figure 8. Perturbed kinetic rule with $M=10(b)$ : Images of the curve $\lambda \mapsto \bar{\Delta}$ with $M=10$, for a portion of the imaginary axis (left), and the whole contour $C_{\rho}$ (right), with $\rho=2$. These correspond to values of $\varphi=3 \pi / 8$ (top) and $\varphi=\pi / 2$ (bottom).


Figure 9. Perturbed kinetic rule with $M=1$ (a): Images of the curve $\lambda \mapsto \bar{\Delta}$ with $M=1$, for a portion of the imaginary axis (left), and the whole contour $C_{p}$ (right), with $\rho=2$. These correspond to values of $\varphi=0$ (top), $\varphi=\pi / 8$ (center), and $\varphi=\pi / 4$ (bottom).


Figure 10. Perturbed kinetic rule with $M=1$ (b): Images of the curve $\lambda \mapsto \bar{\Delta}$ with $M=1$, for a portion of the imaginary axis (left), and the whole contour $C_{\rho}$ (right), with $\rho=2$. These correspond to values of $\varphi=3 \pi / 8$ (top) and $\varphi=\pi / 2$ (bottom).

