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# Wavelet radiosity

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Abstract

The present paper considers the fast solution of boundary integral equations on unstructured meshes by the Galerkin scheme and its application to the special situation of the radiosity equation. It is known that the system matrix of the scheme in a wavelet basis which provides vanishing moments with respect to traces of polynomials in the space can be compressed to $\mathcal{O}(N \log N)$ relevant matrix entries, where N denotes the number of degrees of freedom. However, the presence of the visibility function in the kernel of the radiosity equation provides discontinuities for non-convex geometries, which cause trouble for most of the fast methods. Nevertheless, we have purchased a wavelet Galerkin method which is able to produce a system matrix with $\mathcal{O}(N \log^3 N)$ relevant matrix coefficients for the radiosity equation on a reasonable geometry.

*Keywords: BEM, wavelets, radiosity*

## 1. The radiosity equation - Introduction

In the present paper we consider the radiosity equation on the closed surface $\Gamma$ of a 3-dimensional domain $\Omega \in \mathbb{R}^3$ with $\Gamma = \bigcup_l \Gamma_l$ and $\Gamma_l \in C^1$. The inner product in $L^2(\Gamma)$ on the boundary shall be given by

$$\langle u, v \rangle := \int_\Gamma v(x)u(x)d\Gamma. \tag{1}$$

Furthermore, we will denote the Sobolev spaces by $H^s(\Gamma)$ as well as the associated norm by $\|\cdot\|_s$ for further considerations.

The radiosity equation represents a model for the brightness of a surface when its reflectivity and emissivity is given. It is formulated by

$$B(x) = E(x)+$$

$$r_d(x) \int_\Gamma \frac{\langle \check{n}_x, x-y\rangle \langle \check{n}_y, y-x\rangle}{\pi \|x-y\|^4} B(y) Vis(x,y) \partial\Gamma_y, \quad x \in \Gamma,$$

with the radiosity $B(x)$, the emitted radiation $E(x)$, the diffuse reflection coefficient $r_d(x)$, the inner normal vectors $\check{n}_x$, $\check{n}_y$ and the visibility function

$$Vis(x,y) := \begin{cases} 1 & \text{if} \quad \forall t \in (0,1) : (x + t(y-x)) \in \Omega \\ 0 & \text{else} \end{cases}.$$

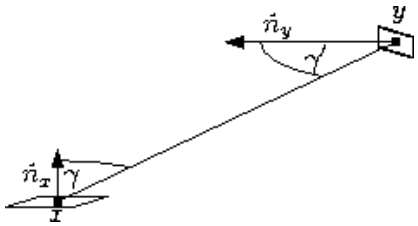


Figure 1: Geometrical relation between two areas

Alternatively, we can write the radiosity equation in the equivalent formulation

$$E(x) = \mathcal{R}(B)(x) = B(x)-$$

$$r_d(x) \int_\Gamma \frac{\cos\gamma \cos\gamma`}{\pi \|x-y\|^4} B(y) Vis(x,y) \partial\Gamma_y, \quad x \in \Gamma,$$

in which $\gamma$ and $\gamma'$ are the angles between the inner normal $\check{n}_x$ and $\overrightarrow{xy}$ or $\check{n}_y$ and $\overrightarrow{yx}$, resp., cp. figure 1. With it, we have a strictly coercive operator $\mathcal{R} : L^2(\Gamma) \to L^2(\Gamma)$, whose kernel function

$$k_r(x,y) = \frac{\langle \check{n}_x, x-y\rangle \langle \check{n}_y, y-x\rangle}{\pi \|x-y\|^4} Vis(x,y)$$

fulfills the decay property

$$|\partial_x^\alpha \partial_y^\beta k(x,y)| \lesssim \frac{(\alpha+\beta)!}{(s\|x-y\|)^{n+2q+|\alpha|+|\beta|}}, \qquad s > 0, \tag{2}$$

uniformly in the 3-dimensional multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ for the unoccluded case, i.e. $Vis(x,y) = 1, \forall x, y \in \Gamma$.

Let be remarked that the numerical treatment of the radiosity equation is completely similar to the treatment of the heat radiation equation,

$$q_0(x) = \epsilon(x)\sigma T^4(x)+$$

$$(1-\epsilon(x)) \int_\Gamma \frac{\langle \check{n}_x, x-y\rangle \langle \check{n}_y, y-x\rangle}{\|x-y\|^4} q_0(y) Vis(x,y) d\Gamma_y,$$

with the rate of outgoing radiant energy $q_0(x)$, the temperature $T(x)$, the emissivity coefficient $\epsilon$, the Stefan-Boltzmann constant $\sigma$ and the visibility function $Vis(x,y)$.

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## 2. Galerkin scheme - discretization

Due to the following Galerkin discretization with piecewise constant ansatz functions we assume that the boundary $\Gamma$ is only provided by its polygonial approximation $\Gamma_N$, i.e. by a set of 2-dimensional plane simplices in $\mathbb{R}^3$ such that $\Gamma_N = \bigcup_{i=1}^N \pi_i$ and $\pi_i \cap \pi_j$ $(i \neq j)$ is either empty or a lower dimensional face. Furthermore, this triangulation should be quasi uniform, i.e.

$$\min_{i=1}^{N} \rho_i \sim h_N \sim N^{-\frac{1}{2}}$$

is valid with $\rho_i$ as the inscribed circle of the triangles $\pi_i$ and the step width $h_N := \max_{i=1}^N \operatorname{diam}(\pi_i)$.

On this triangulation we define the spaces of piecewise constant ansatzfunctions

$$V_N := \operatorname{span}\{\phi_i : i = 1, 2, \ldots, N\},$$

in which the support of every single piecewise constant ansatzfunction is restricted to a simplex $\pi_i$, i.e.

$$\phi_i = \begin{cases} \frac{1}{\sqrt{|\pi_i|}}, & x \in \pi_i, \\ 0, & \text{else.} \end{cases}$$

With the notations $f(x) = E(x)$, $u(x) = B(x)$ as well as $\mathcal{R}(x) = A(x)$ we can easily develop the Galerkin scheme for the resulting operator equation $Au = f$ on $\Gamma_N$, which reads as:

seek $u_N \in V_N$ such that $\langle Au_N, v_N \rangle = \langle f, v_N \rangle \; \forall \; v_N \in V_N$.

Making the ansatz $u_N = \sum_{i=1}^N u_i \phi_i$, we have to solve the linear system of equations

$$\mathbf{A}^\phi \mathbf{u}^\phi = \mathbf{f}^\phi$$

with the solution vector $\mathbf{u}^\phi = [u_i]_{i=1}^N$, the right hand side vector $\mathbf{f}^\phi = [\langle f, \phi_i \rangle]_{i=1}^N$ and the system matrix $\mathbf{A}^\phi = [\langle A\phi_{i'}, \phi_i \rangle]_{i,i'=1}^N$.

The last one represents the core of our problem, since it is usually densely populated and would produce unacceptable $\mathcal{O}(N^2)$ computing and memory costs. For the unoccluded case there exist several fast methods to avoid these costs, such as methods based on hierarchical matrices or multipole, e.g. in [1]. However, in the occluded case, i.e. $\exists x, y_1, y_2 : Vis(x, y_1) = 0$, $Vis(x, y_2) = 1$, the jump in the visibility function provides discontinuities in the kernel or in the derivatives of the kernel, which prevent a reduction to linear-logarithmic memory and computing time costs by standard approaches of these fast methods. We will show that an approach by a multiscale basis, namely wavelets, instead of the traditional singlescale basis $\{\phi_i\}$ will solve parts of this problem. While the idea to use wavelets for the radiosity equation is not essentially new, cf. [5], this paper presents in a short overview the necessary compression and cut-off parameters for the occluded case as well as it gives some ideas for the computation of the system matrix. For a deeper consideration we would refer to the upcoming papers [10] and [11].

## 3. The wavelet bases

Since the representation of our geometry automatically limits the finest level of any finite consideration to a single simplex, we cannot use refinement strategies. Instead, we apply the idea of coarsing procedure to the wavelet construction following [8, 14] to obtain the multiscale hierarchy

$$V_0 \subset V_1 \subset \cdots \subset V_J = V_N. \tag{3}$$

For that, we require a uniform hierarchical subdivision of $\Gamma_N$, called cluster tree.

A cluster $\nu$ is defined as the non-empty union $\nu = \pi_1 \cup \cdots \cup \pi_i$ of a certain set of simplices $\pi_i$. For a set of clusters we define the relation $\prec$ by $\nu' \prec \nu \Longleftrightarrow \nu' \subsetneq \nu$ and $\nexists \nu'' : \nu' \subsetneq \nu'' \subsetneq \nu$, in which $\nu'$ shall be a son cluster of $\nu$ and $\nu$ shall be the father cluster of $\nu'$. If we order a set of clusters hierarchical concerning this father-son relation, the resulting structure is called cluster tree $T$, provided that the following properties are valid.

- The cluster which is equivalent to $\Gamma_N$ is the only cluster which has no father cluster and is denoted as *root* of $T$.
- The pairwise intersection $\nu' \cap \nu''$ of son clusters of a cluster $\nu$ is either empty or a lower dimensional face.
- The union of all son clusters $\nu_i^{son}$ of a cluster $\nu$ holds $\bigcup_i \nu_i^{son} = \nu$.

A cluster $\nu$ of $T$ should belong to the *level* $j$, if there exist $j$ clusters $\{\nu_i\}_{i=0}^{j-1}$ with

$$\nu \prec \nu_{j-1} \prec \ldots \prec \nu_0 = \Gamma,$$

but not $j+1$ clusters with the same property. Consequently, the root $\Gamma_N$ of $T$ is on level 0 and there exists a maximum level $J$, which is also denoted as *depth* of the cluster tree. Furthermore, clusters which have no son clusters are called *leaves*.

Moreover, we call the cluster tree a balanced quad tree, if it additionally fulfills the following properties.

- A cluster has either 0 or 4 sons.
- The diameter $\operatorname{diam}(\nu)$ of a cluster $\nu$ on the level $j$ scales approximately like $2^{-j}$.
- The number $\#\nu$ of simplices of a cluster $\nu$ on level $j$, which we denote as *cardinality* of the cluster, behaves like $2^{2(J-j)}$.

The balanced quad tree with the mentioned properties should be given for our further consideration. There are several methods to obtain such a cluster tree or asymptotically similar strutures. For more details we refer to [7] for example.

With the hierarchical struture of the quad-tree at hand we are in the position to construct the wavelet basis. It should realize a hierarchical structure, which means that the support of a wavelet $\psi_{j,k}^\mu$ on level $j$ should be restricted to a single cluster $\mu$ on the level $j$ to ensure the locality of the wavelets. Moreover, for matrix compression the wavelets should provide *vanishing moments* of order $\widetilde{d}$, that is

$$\int_\Gamma x^\alpha \psi_{j,k}^\mu(x) d\Gamma = 0, \qquad |\alpha| < \widetilde{d}. \tag{4}$$

Let be remarked that $x^\alpha$ is considered as a spatial polynomial in $\mathbb{R}^3$ and only their traces on $\Gamma$ enter in definition (4).

We consider a cluster $\mu$ on the level $j$. The scaling functions $\Phi_j^\mu = \{\varphi_{j,k}^\mu\}$ and wavelets $\Psi_j^\mu = \{\psi_{j,k}^\mu\}$ supported in this cluster shall be defined as a linear combinations from the scaling functions $\Phi_{j+1}^\mu = \{\Phi_{j+1}^{\mu_1^{son}}, ..., \Phi_{j+1}^{\mu_4^{son}}\}$ of $\mu$'s son clusters $\mu_i^{son}$ on the finer level $j+1$

$$\left[\mathbf{\Phi}_j^\mu, \mathbf{\Psi}_j^\mu\right] = \mathbf{\Phi}_{j+1}^\mu \left[Q_{j,\Phi}^\mu, Q_{j,\Psi}^\mu\right], \tag{5}$$

in which the matrix $[Q_{j,\Phi}^\mu, Q_{j,\Psi}^\mu]$ is supposed to be orthogonal and the matrices $\mathbf{\Phi}_j^\mu$, $\mathbf{\Psi}_j^\mu$ and $\mathbf{\Phi}_{j+1}^\mu$ are an ordered representation of the function sets $\Phi_j^\mu$, $\Psi_j^\mu$ and $\Phi_{j+1}^\mu$. The beginning of the

recursion on the finest level $J$ are initiated by the piecewise constant ansatz functions supported in a cluster $\mu$, $\Phi^\mu_{J+1} := \{\phi_i : \mathrm{supp}(\phi_i) \subset \mu\}$. That way, we obtain the wanted multiscale hierarchy (3) via

$$V_j := \mathrm{span}\{\Phi^\mu_j : \mu \text{ is a cluster from the level } j\}. \tag{6}$$

Besides, the spaces

$$W_j := \mathrm{span}\{\Psi^\mu_j : \mu \text{ is a cluster from the level } j\}$$

satisfy

$$V_{j+1} = V_j \overset{\perp}{\oplus} W_j$$

due to the orthogonality of the matrices $[Q^\mu_{j,\Phi}, Q^\mu_{j,\Psi}]$.

In order to realize (4) we determine the moment matrix

$$M^\mu_j := \left[\int_\Gamma x^\alpha \Phi^\mu_{j+1}(x) d\Gamma\right]_{|\alpha|<\widetilde{d}} \tag{7}$$

of the cluster $\mu$ on the level $j$, employ the singular value decomposition

$$M^\mu_j = U\Sigma V^\top = U\,[S,0]\left[Q^\mu_{j,\Phi}, Q^\mu_{j,\Psi}\right]^\top \tag{8}$$

and obtain the coefficient matrices $Q^\mu_{j,\Phi}$ and $Q^\mu_{j,\Psi}$, cf. [14].

This algorithm provides recursively defined wavelet functions. For a complete basis we have to add the scaling functions on the coarsest level to the wavelet functions. Hence, the wavelet basis is defined by

$$\Psi_N := \Phi^\Gamma_0 \cup \{\Psi^\nu_j : \nu \in T\}.$$

The following theorem provides the most important properties of the wavelet basis, cp. [8, 14].

**Theorem 3.1** *The wavelets $\{\Psi_N\}$ define an orthonormal basis with respect to the inner product* (1)*. The amount of wavelets on the level $j$ is approximately $2^{jn}$ while the diameter of their support scales like $2^{-j}$. The wavelets provide vanishing moments in terms of* (4) *of order $\widetilde{d}$.*

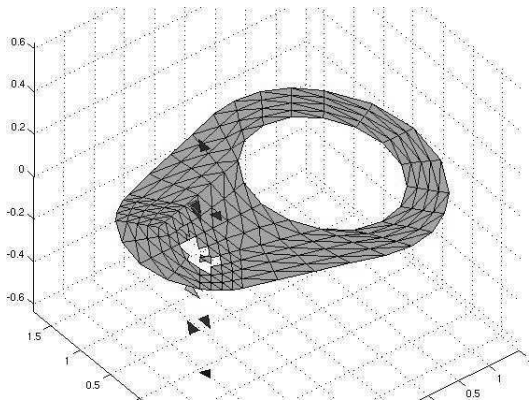

Figure 2: Wavelet with three vanishing moments

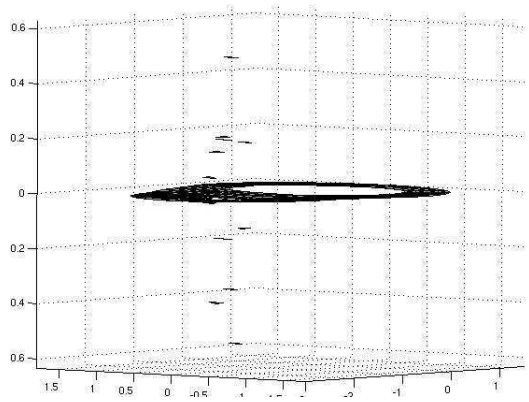

Figure 3: Wavelet with three vanishing moments

The wavelet basis can be recursively computed within $\mathcal{O}(N)$ operations. One wavelet with three vanishing moments is shown in figure 2 and 3 from two different angles.

For the application of the wavelets to the Galerkin scheme it will be necessary to switch between the wavelet and the singlescale representations

$$\sum_{j,k}\langle f, \psi_{j,k}\rangle\psi_{j,k} = \sum_i \langle f, \phi_i\rangle\phi_i$$

for the determination of the right hand side vector as well as of the solution vector. Similar to the wavelet construction, we can determine the coefficients recursively via the fast wavelet transform

$$\left[\langle f, \mathbf{\Phi}^\nu_j\rangle, \langle f, \mathbf{\Psi}^\nu_j\rangle\right] = \langle f, \mathbf{\Phi}^\nu_{j+1}\rangle\left[Q^\nu_{j,\Phi}, Q^\nu_{j,\Psi}\right],$$

j = J, J-1,...,0, with linear efforts. Since $[Q^\nu_{j,\Phi}, Q^\nu_{j,\Psi}]$ is orthogonal, we can pass backwards through this algorithm to gain the inverse wavelet transform. For further details we refer to [12, 14, 2].

## 4. Wavelet compression

The idea of the wavelet method is the replacing of the traditional singlescale basis $\{\phi_i\}$ by the wavelet basis $\Psi_N$ within the Galerkin scheme. That means, that we make the ansatz $u_N = \sum_{j,k}\overline{u}_{j,k}\psi_{j,k}$ and set up the Galerkin scheme

$$\mathbf{A}^\Psi\mathbf{u}^\Psi = \mathbf{f}^\Psi. \tag{9}$$

within the wavelet basis with $\mathbf{A}^\Psi = [\langle A\psi_{j',k'}, \psi_{j,k}\rangle]_{(j,k),(j',k')}$, $\mathbf{u}^\Psi = [\overline{u}_{j,k}]_{(j,k)}$ and $\mathbf{f}^\Psi = [\langle f, \psi_{j,k}\rangle]_{(j,k)}$. In general, for coercive operators whose kernel functions provides the decay porperty (2) the matrix $\mathbf{A}^\psi$ is a quasi sparse matrix. That means, it can be compressed to a sparse matrix without compromising the stability and accuracy of the underlying Galerkin scheme. However, the standard compression requires a certain smoothness of the kernel function, which is partly prevented by the visibility function in the case of the radiosity equation.

Let us start with the unoccluded case, in which the visibility function is not apparent due to $Vis(x,y) = 1$. In this case we can use the standard compression. Following [6, 8, 13] we receive the following important proposition for the unoccluded case.

**Proposition 4.1** *Let the cut-off parameter $B_{j,j'}$ be defined by*

$$\mathcal{B}_{j,j'} := a \max \left\{ 2^{-\min\{j,j'\}}, 2^{\frac{2J(d')-(j+j')(d'+\tilde{d})}{2(\tilde{d})}} \right\}. \qquad (10)$$

*with $a > 1$, $1 < d' < \widetilde{d}$. Furthermore, let $\Theta_{j,k} = \mu$ be the support of the wavelet function $\psi^{\mu}_{j,k}$. Then, the system matrix $\mathbf{A}^{\Psi}$ of the radiosity equation in the unoccluded case can be compressed in accordance with*

$$\mathbf{A}^{\Psi}_{c\ (j,k),(j',k')}$$

$$:= \begin{cases} 0, & \text{if } \operatorname{dist}(\Theta_{j,k}, \Theta_{j',k'}) > \mathcal{B}_{j,j'}, \\ \langle A\psi_{j',k'}, \psi_{j,k} \rangle & \text{otherwise}, \end{cases} \qquad (11)$$

*to only $\mathcal{O}(N \log N)$ nonzero matrix coefficients without compromising the stability and accuracy of the underlying Galerkin scheme.*

In the occluded case the visibility function takes effect and we have to distinct between three different cases. Let $\mu$ and $\nu$ be two clusters. We say that these clusters are

- *completely visible* or $\mu \Leftrightarrow \nu$, if $Vis(x,y) = 1 \quad \forall x \in \mu, y \in \nu$ is valid,
- *partly visible* or $\mu \leftrightarrow \nu$, if the pair holds $\exists x \in \mu, y_1, y_2 \in \nu : Vis(x,y_1) = 0, Vis(x,y_1) = 1$ is valid, and
- *non-visible* or $\mu \not\leftrightarrow \nu$, if $Vis(x,y) = 0 \quad \forall x \in \mu, y \in \nu$ is fulfilled.

If $\mu$ and $\nu$ are are completely visible, the visibility function takes no effect, again. So, we can apply the standard wavelet compression (11) to these cluster pairs. For a non-visible cluster pair $(\mu, \nu)$ we can formulate the simple compression

$$\mathbf{A}^{\Psi}_{c\ (j,k),(j',k')} = 0$$

due to the resulting zero kernel.

The difficulties appear in the consideration of partly visible clusters. For that, it is necessary to make certain assumptions and specifications to the geometry. A partly visible cluster pair $(\mu, \nu)$ appears, if we have a jump in the visibility function $Vis(x,y)$ on $\mu \times \nu$. The jumps can be identified by the shadowlines depending on the view point $x_0$. This is the part of $\Gamma_N$, which contains the jumps in $Vis(x_0, y)$. Due to the idea that this shadowline is a manifold of lower dimension, which means, that it can be one-dimensional at most, we formulate the following assumption.

**Assumption 4.2 (Size of shadowlines)** *Let $\Gamma_N$ be the given polygonial approximation of the surface. Then, it holds*

$\forall x \in \Gamma_N :$ *#*$\{\pi_i : \exists y_1, y_2 \in \pi_i$ *with* $Vis(x,y_1) = 1$ *and*

$Vis(x,y_2) = 0\} \sim \mathcal{O}(\sqrt{(N)})$.

Assumption 4.2 is kept in a very general way. So, we want to specify the cases of partial visibility in the following. All the cases are connected to the asymptotical considerations, since the special properties of the macro structure of the geometry vanish for a sufficient large $N$.

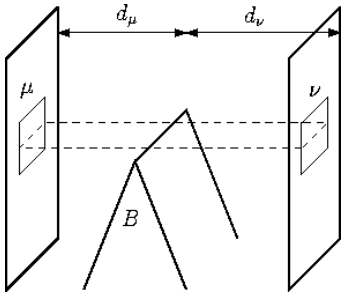


Figure 4: Partly visible case A

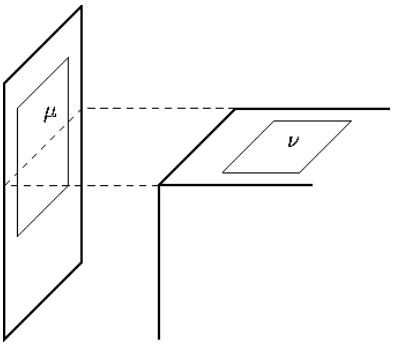


Figure 5: Partly visible case B

**Case A:** Let $\mu$ and $\nu$ be two clusters on different planes. They should be partly visible due to a barrier $B$ between the two planes. In this context we demand, that the distance between the clusters and the barrier $d_\mu$ and $d_\nu$ holds $d_\mu, d_\nu > 0$, cp. figure 4, and the angles fulfill $\gamma, \gamma' < \frac{\Pi}{2}$.

**Case B:** Let $\mu$ and $\nu$ be two clusters on different planes in such a kind that the hyperplane spanned by one cluster and the other cluster intersect in a one dimensional manifold, cp. figure 5. This case includes the degenerations of case A.

**Case C:** Let $\mu$ be a cluster in a plane and $\nu$ be a cluster on a curved plane such that surface part $\Gamma_\nu \subset \Gamma$, which is approximated by $\nu$, holds $\Gamma_\nu \in C^1$ and the intersection of $\mu$ and a tangential plane on $\nu$ is not empty. The partial visibility appears due to the curve of $\nu$, cp. figure 6.

**Case D:** Let $\mu$ be a cluster in a plane and $\nu$ be a cluster on an edge such that parts of $\nu$ are completely visible by $\mu$ and parts are non-visible by $\mu$, cp. figure 7.

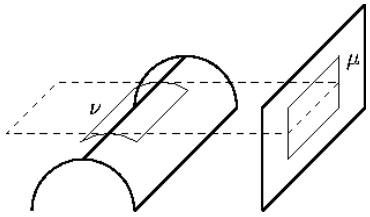


Figure 6: Partly visible case C

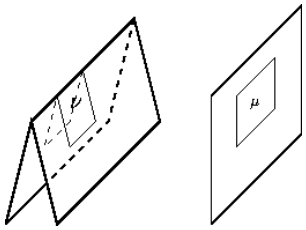


Figure 7: Partly visible case D

To find a suitable compression for the presented cases of partial visibility we use the next proposition, following [13], which is based on the multiscale structure of the wavelet basis.

**Proposition 4.3** *Let $\mathbf{A}^{\Psi}$ be the exact and $\widetilde{\mathbf{A}}^{\Psi}$ be the disturbed system matrix. Furthermore, let $\mathbf{A}^{\Psi}_{\mathbb{I}_j,\mathbb{I}_{j'}}$ be the submatrix of all scalar products $\langle A\psi^{\mu}_{j,k}, \psi^{\nu}_{j',k'}\rangle$ with fixed levels $j$ and $j'$. Similarly, we define the submatrix $\widetilde{\mathbf{A}}^{\Psi}_{\mathbb{I}_j,\mathbb{I}_{j'}}$ of the disturbed system matrix. If the estimate*

$$\|\mathbf{A}^{\Psi}_{\mathbb{I}_j,\mathbb{I}_{j'}} - \widetilde{\mathbf{A}}^{\Psi}_{\mathbb{I}_j,\mathbb{I}_{j'}}\|_2 \lesssim 2^{-2Jq} 2^{-2(J-\frac{j+j'}{2})} J^{-1}$$

*is valid for wavelets based on piecewise constant ansatz functions, the wavelet Galerkin scheme is stable and the error estimate holds*

$$\|u - u_N\|_s \lesssim h^{(t-s)} \|u\|_t$$

*with $-1 \leq s < \frac{1}{2}$ and $0 \leq t \leq 1$, in which $u \in H^1$ denotes the exact solution of the boundary integral equation $Au = f$ and $u_N$ is the solution derived from the disturbed system matrix $\widetilde{\mathbf{A}}^{\Psi}$.*

This allows us to formulate the new cut-off parameter

$$B^{Rad}_{j,j'} = a_2 2^{J-\frac{j}{2}-\frac{j'}{2}-\frac{\max(j,j')}{2}} J^{\frac{1}{2}} \tag{12}$$

with the compression

$$\mathbf{A}^{\Psi}_{(j,k),(j',k')} = 0 \quad \text{if } \operatorname{dist}(\Theta_{j,k}, \Theta_{j',k'}) > \mathcal{B}^{Rad}_{j,j'}$$

for the cases $A$, $B$ and $C$ of partial visibility. The compression by the cut-off parameter $B^{Rad}_{j,j'}$ does not cover the case $D$. However, our geometry is a union of a fixed number of surface parts $\Gamma_l \in C^1$. This allows us to adapt the part of the cluster tree for coarser levels in such a way that from a certain level the edges in the case $D$ merge with the boundaries of the clusters. With it, the case $D$ of partial visibility takes only effect up to fixed level $j_0$. This has influence on constants in certain estimates but no influence on the general assymptical behaviour.

With this considerations we can finally formulate the following theorem for the occluded case of the radiosity equation.

**Theorem 4.4** *Let the cut-off parameter $B_{j,j'}$ be from* (10) *and $B^{Rad}_{j,j'}$ from* (12)*. Furthermore, the assumption 4.2 shall be valid and the cases of partial visibility shall be restricted to the mentioned ones. Then, the system matrix $\mathbf{A}^{\Psi}$ of the radiosity equation in the occluded case can be compressed in accordance with*

$$\mathbf{A}^{\Psi}_{c\ (j,k),(j',k')}$$

$$:= \begin{cases} 0, & \text{if } \operatorname{dist}(\Theta_{j,k}, \Theta_{j',k'}) > \mathcal{B}_{j,j'} \\ & \text{and } \Theta_{j,k} \Leftrightarrow \Theta_{j',k'}, \\ 0, & \text{if } \operatorname{dist}(\Theta_{j,k}, \Theta_{j',k'}) > \mathcal{B}^{Rad}_{j,j'} \\ & \text{and } \Theta_{j,k} \leftrightarrow \Theta_{j',k'}, \\ 0, & \text{if } \Theta_{j,k} \not\leftrightarrow \Theta_{j',k'}, \\ \langle A\psi_{j',k'}, \psi_{j,k}\rangle & \text{otherwise}, \end{cases}$$

*to only $\mathcal{O}(N \log^3 N)$ nonzero matrix coefficients without compromising the stability and accuracy of the underlying Galerkin scheme.*

The final compression unites the three different compression techniques for the system matrix of the radiosity equation. For its illustration let us consider the system matrix of the radiosity equation on the $L$-block, cp. figure 8. In the following illustrations of the system matrix dark points show non-zero entries, while light points describe zero entries.

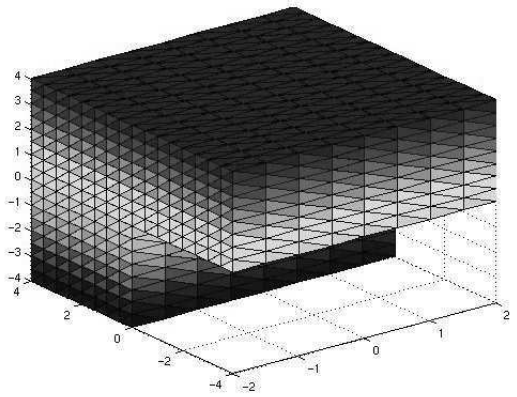

Figure 8: L-Block with 1792 surface triangles

In figure 9 we have the quasi sparse system matrix in the wavelet basis. The matrix has a $3 \times 3$ block structure, which corresponds to the lower left part, the upper right and the upper left part of the $L$-block. The only compression which has been performed was made for entries concerning wavelets whose supports are non-visible.

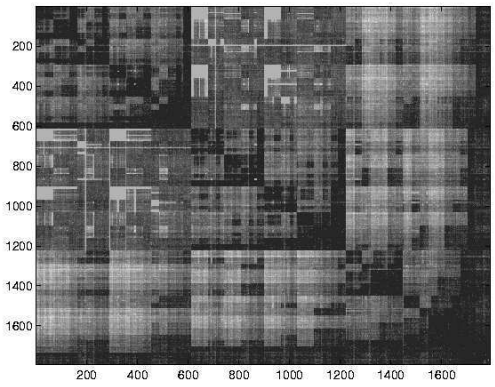


Figure 9: Quasi sparse system matrix of the radiosity equation in the wavelet basis on the $L$-block for $N = 1792$

Figure 10 shows the system matrix, in which the compression for non-visible cluster pairs are performed as well as the standard compression for completely visible cluster pairs. We can see, that the matrix blocks concerning the interactions between the lower left part and the upper right part of the $L$-block, which represent the cases of partial visibility, are hardly compressed.

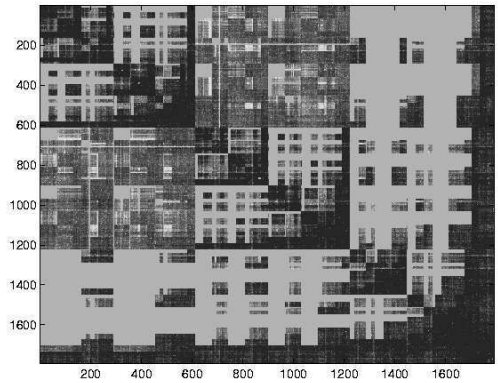


Figure 10: Partly compressed system matrix of the radiosity equation in the wavelet basis on the $L$-block for $N = 1792$

Finally, in figure 11 we see the result of the complete compression and the necessity of the compression for the case of partial visibility.

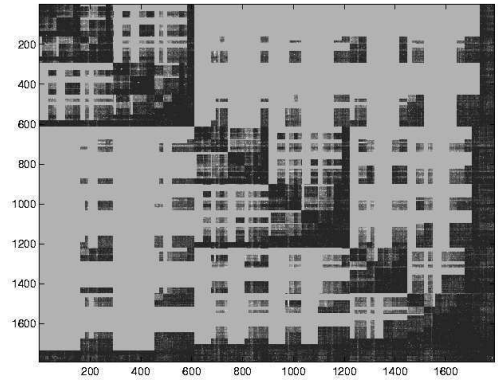


Figure 11: Compressed system matrix of the radiosity equation in the wavelet basis on the $L$-block for $N = 1792$

**Remark 4.5** *Let us remark that it would be possible to intensify the cut-off parameter $B^{Rad}_{j,j'}$ to*

$$B^{Rad}_{j,j'} = a_2 2^{J-\frac{j}{2}-\frac{j'}{2}-\frac{\max(j,j')}{2}},$$

*which would result in the reduction of the sparse matrix to $\mathcal{O}(N \log^2 N)$ entries. However, the error estimate in the proposition 4.3 would increase to*

$$\|u - u_N\|_s \lesssim \log(\frac{1}{h}) h^{(t-s)} \|\rho\|_t$$

*with $-1 \leq s < \frac{1}{2}$ and $0 \leq t \leq 1$.*

For further details as well as for the proofs of the presented results we would refer to [10].

## 5. Determination of the system matrix and numerical results

In the previous section we saw the possibility to obtain a sparse system matrix. In this section we will shortly indicate how we achieve the sparse system matrix.

To establish the system matrix we use the a priori information which are provided by the cut-off parameters. We only compute such entries which are indicated by the cut-off parameter to be relevant.

For the computation of a single entry we use the following two ideas. Let $\mathbf{A}^{\Psi}_{\mu,\nu} := [\langle A\psi^{\nu}_{j',k'}, \psi^{\mu}_{j,k}\rangle]_{k,k'}$, $\mathbf{A}^{\Phi}_{\mu,\nu} := [\langle A\varphi^{\nu}_{j',k'}, \varphi^{\mu}_{j,k}\rangle]_{k,k'}$, $\mathbf{A}^{\Phi,\Psi}_{\mu,\nu} := [\langle A\psi^{\nu}_{j',k'}, \varphi^{\mu}_{j,k}\rangle]_{k,k'}$ and $\mathbf{A}^{\Psi,\Phi}_{\mu,\nu} := [\langle A\varphi^{\nu}_{j',k'}, \psi^{\mu}_{j,k}\rangle]_{k,k'}$ be the submatrices of scalar products between wavelets $\psi^{\mu}_{j,k}$ or scaling functions $\varphi^{\mu}_{j,k}$ from clusters $\mu$ and $\nu$. We can apply the idea of the recursive definition of the wavelets (5) to the determination of submatrices by

$$\begin{bmatrix} \mathbf{A}^{\Phi}_{\mu,\nu} & \mathbf{A}^{\Phi,\Psi}_{\mu,\nu} \\ \mathbf{A}^{\Psi,\Phi}_{\mu,\nu} & \mathbf{A}^{\Psi}_{\mu,\nu} \end{bmatrix}$$

$$= \begin{bmatrix} Q^{\mu\,\top}_{\Phi} \\ Q^{\mu\,\top}_{\Psi} \end{bmatrix} \begin{bmatrix} \mathbf{A}^{\Phi}_{\mu_1^{son},\nu_1^{son}} & \cdots & \mathbf{A}^{\Phi}_{\mu_1^{son},\nu_4^{son}} \\ \vdots & \ddots & \vdots \\ \mathbf{A}^{\Phi}_{\mu_4^{son},\nu_1^{son}} & \cdots & \mathbf{A}^{\Phi}_{\mu_4^{son},\nu_4^{son}} \end{bmatrix} [Q^{\nu}_{\Phi}, Q^{\nu}_{\Psi}].$$

This recursion allows us to compute from the submatrices concerning the scaling functions of the son clusters $\mathbf{A}^{\Phi}_{\mu_i^{son},\nu_{i'}^{son}}$ the wanted entries of the system matrix $\mathbf{A}^{\Psi}_{\mu,\nu}$ as well as the submatrices concerning the scaling functions of the present cluster for further recursions. This approach is very useful, if the entries of the system matrix concerning the son clusters are relevant, too.

In other cases we apply the idea of hierarchical matrices, in particular the idea of the $\mathcal{H}^2$-matrices [3, 4, 9], to the wavelet method. For two clusters $\mu$ and $\nu$ which are completely visible and fulfill the admissibility condition

$$\max\{\operatorname{diam}(\nu), \operatorname{diam}(\mu)\} < \eta \operatorname{dist}(\nu, \mu)$$

with the admissibility constant $\eta$ an entry $\mathbf{A}^{\Psi}_{k,k'} = \langle A\psi^{\nu}_{j',k'}, \psi^{\mu}_{j,k}\rangle$ can be approximated by interpolation by

$$\langle A\psi^{\nu}_{k'}, \psi^{\mu}_{k}\rangle_{L^2} \approx \sum_{s\in K}\sum_{t\in K} \underbrace{k(x^{\mu}_s, y^{\nu}_t)}_{S^{\mu,\mu}_{s,t}}$$

$$\underbrace{\left(\int_{\Gamma_N} \psi^{\mu}_k(x)\mathcal{L}^{\mu}_s(x)\partial\Gamma_x\right)}_{V^{\Psi}_{\mu\,k,s}} \underbrace{\left(\int_{\Gamma_N} \psi^{\nu}_{k'}(y)\mathcal{L}^{\nu}_t(y)\partial\Gamma_y\right)}_{V^{\Psi}_{\nu\,k',s}}$$

with the Lagrange polynomials $\mathcal{L}^{\mu}_s$ and the interpolation points $x^{\mu}_s$. This approximation can be applied to whole submatrices, which reads

$$\mathbf{A}^{\Psi}_{\mu,\nu} \approx V^{\mu}_{\Psi} S^{\mu,\nu} V^{\nu\,\top}_{\Psi},$$

Table 1: Overall computing time, $L_2$- as well as $L_\infty$-error and storage requirements for the system matrix $\mathbf{A}^\Psi$

| $N$ | computing time | $\|u-u_N\|_{L_2}$ | $\|u-u_N\|_{L_\infty}$ | Kbytes/N |
|---|---|---|---|---|
| 112 | 2.1 | 0.82e-02 | 1.72e-02 | 0.66 |
| 448 | 15.7 | 0.42e-02 | 1.04e-02 | 2.29 |
| 1792 | 119.5 | 0.19e-02 | 0.71e-02 | 4.42 |
| 7168 | 838.8 | 0.83e-03 | 0.49e-02 | 6.50 |
| 28672 | 7058 | 0.38e-03 | 0.35e-02 | 8.54 |

as well as

$$\mathbf{A}^\Phi_{\mu,\nu} \approx V^\mu_\Phi S^{\mu,\nu} V^{\nu\top}_\Phi,$$

with the corresponding matrix entries

$$V^\Phi_{\mu\ k,s} := \int_{\Gamma_N} \varphi^\mu_k(x) \mathcal{L}^\mu_s(x) \partial\Gamma_x.$$

The combination of both ideas provides the next theorem.

**Theorem 5.1** *The system matrix $\mathbf{A}^\Psi$ of the radiosity equation can be established within $\mathcal{O}(N^{\frac{3}{2}}\#K)$ operations, in which $\#K$ describes the amount of interpolation points of the Lagrange interpolation and depends on the necessaray accuracy. Furthermore, the determination of the system matrix only requires temporary memory in the size of $\mathcal{O}(N\log^3 N)$.*

For the details of the computation method as well as for the proof of the theorem we refer to [11].

The theorem 5.1 shows that we require almost $\mathcal{O}(N^{\frac{3}{2}})$ operations to set up the system matrix. However, once we have the sparse system matrix all matrix-vector multiplications, which are necessary for iterative solvers, have only linear-logarithmic costs in computing time. This provides a great advantage for the solution of the radiosity equation for different right hand sides, which can appear in form of different light sources for a fixed geometry.

Finally, we want to present in table 1 some numerical results for the $L$-block. For $r_d(x) = 0.5$ and $f(x) = E(x) = 0.5$ $\forall x \in \Gamma$ we know the exact solution $u(x) = B(x) = 1$. So, we can consider the errors in the $L_2$- as well as in the $L_\infty$-norm. The computing times in the table 1 shows the predicted $\mathcal{O}(N^{\frac{3}{2}})$ behaviour. Besides, the quotient of used memory per degree of freedom shows that we have achieved a sparse system matrix.

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