

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Delayed feedback control near Hopf bifurcation

(revised version: October 2008)

by

Fatihcan M. Atay

Preprint no.: 32

2007



Delayed Feedback Control near Hopf Bifurcation

Fatihcan M. Atay

Max Planck Institute for Mathematics in the Sciences
Inselstr. 22, Leipzig 04103, Germany
atay@member.ams.org

Preprint. Final version in:

Discrete and Continuous Dynamical Systems-S 1:197–205, (2008)

Abstract

The stability of functional differential equations under delayed feedback is investigated near a Hopf bifurcation. Necessary and sufficient conditions are derived for the stability of the equilibrium solution using averaging theory. The results are used to compare delayed versus undelayed feedback, as well as discrete versus distributed delays. Conditions are obtained for which delayed feedback with partial state information can yield stability where undelayed feedback is ineffective. Furthermore, it is shown that if the feedback is stabilizing (respectively, destabilizing), then a discrete delay is locally the most stabilizing (resp., destabilizing) one among delay distributions having the same mean. The result also holds globally if one considers delays that are symmetrically distributed about their mean.

1 Introduction

We study the effect of the feedback function f on the stability of the zero solution of the functional differential equation

$$\dot{x}(t) = Lx_t + \varepsilon g(x_t; \varepsilon) + \varepsilon \kappa f(x_t; \varepsilon), \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $x_t \in \mathcal{C} \triangleq C([- \tau, 0], \mathbb{R}^n)$, $x_t(\theta) = x(t + \theta) \in \mathbb{R}^n$, $\theta \in [- \tau, 0]$, $L : \mathcal{C} \rightarrow \mathbb{R}^n$ is linear, ε is a small real parameter, $f, g \in \mathcal{C} \times \mathbb{R} \rightarrow \mathbb{R}^n$ have continuous second derivatives with respect to each of their arguments and satisfy $f(0; \varepsilon) = g(0; \varepsilon) = 0$ for all ε , and $\kappa \in \mathbb{R}$ denotes the feedback gain. It is assumed that the linear problem obtained by setting $\varepsilon = 0$ has a pair of complex conjugate characteristic values $\pm i\omega \neq 0$, and all other characteristic values have negative real parts. Equation (1) arises in the study of a Hopf bifurcation of an equilibrium solution, after rescaling the space variable $x \rightarrow \varepsilon x$ and the bifurcation parameter $\alpha \rightarrow \varepsilon \alpha$; see e.g. [1]. In applications, the problem is related to the feedback control of oscillations, or conversely, to the oscillatory instabilities arising from delayed feedback (e.g. [2, 3]).

The aim of the present paper is to obtain precise conditions under which a delayed feedback action can stabilize or destabilize an equilibrium solution near a Hopf bifurcation, in particular when not all the system variables are available for feedback. Moreover, we are interested in the difference between discrete and distributed delays in the feedback. Taking advantage of being near a Hopf bifurcation, we use averaging theory in Section 2 to derive necessary and sufficient conditions for stability. The implications for delayed versus instantaneous feedback are investigated in Section 3. Section 4 is devoted to a discussion of discrete versus distributed delays.

2 Stability of the zero solution

We introduce some notation. For $\varepsilon = 0$, we write (1) as

$$\dot{x}(t) = Lx_t = \int_{-\tau}^0 d\eta(\theta)x(t+\theta), \quad (2)$$

where η is an $n \times n$ matrix whose components are of bounded variation on $[-\tau, 0]$. By assumption, (2) has a pair of characteristic values $\pm i\omega \neq 0$. By rescaling time it can be assumed that $\omega = 1$ without loss of generality. Assume all other characteristic values have negative real parts. Let Φ be an $n \times 2$ matrix whose columns span the eigenspace of (2) corresponding to the characteristic value $\pm i$. In particular, Φ can be chosen such that

$$\Phi(\theta) = \Phi(0)e^{J\theta}, \quad \theta \in [-\tau, 0] \quad (3)$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (4)$$

Similarly, let Ψ denote an $n \times 2$ matrix whose columns span the eigenspace corresponding to $\pm i$ for the adjoint equation

$$\dot{z}(t) = - \int_{-\tau}^0 d\eta^\top(\theta)z(t-\theta) \quad (5)$$

on the space $\mathcal{C}^* = C([0, \tau], \mathbb{R}^n)$. Let F and G be $n \times n$ matrices, with elements of bounded variation on $[-\tau, 0]$, such that

$$[D_1 f(0; 0)]\phi = \int_{-\tau}^0 dF(\theta)\phi(\theta) \quad (6)$$

$$[D_1 g(0; 0)]\phi = \int_{-\tau}^0 dG(\theta)\phi(\theta). \quad (7)$$

We define the scalar functions

$$\hat{f}_1(\theta) = \text{tr} \left(\Psi^\top(0)F(\theta)\Phi(0) \right), \quad \hat{f}_2(\theta) = \text{tr} \left(\Psi^\top(0)F(\theta)\Phi(0)J \right), \quad (8)$$

$$\hat{g}_1(\theta) = \text{tr} \left(\Psi^\top(0)G(\theta)\Phi(0) \right), \quad \hat{g}_2(\theta) = \text{tr} \left(\Psi^\top(0)G(\theta)\Phi(0)J \right), \quad (9)$$

where “tr” denotes the matrix trace, and define the real numbers q, p by

$$q = \int_{-\tau}^0 \cos \theta d\hat{g}_1(\theta) + \int_{-\tau}^0 \sin \theta d\hat{g}_2(\theta) \quad (10)$$

$$p = \int_{-\tau}^0 \cos \theta d\hat{f}_1(\theta) + \int_{-\tau}^0 \sin \theta d\hat{f}_2(\theta) \quad (11)$$

Then for sufficiently small ε , the stability of the zero solution of (1) is given by the following result.

Theorem 1. *Let $\kappa \in \mathbb{R}$. There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ the origin is asymptotically stable (unstable) if*

$$q + \kappa p < 0 \quad (> 0).$$

Proof. By assumption, the linear system (2) and the adjoint system (5) each have a 2-dimensional center subspace, which are spanned by the columns of the matrices Φ and Ψ , respectively. The bilinear form

$$(\psi, \varphi) := \psi^\top(0)\varphi(0) - \int_{-\tau}^0 \int_0^\theta \psi^\top(\zeta - \theta) d\eta(\theta) \varphi(\zeta) d\zeta, \quad (12)$$

where $\psi \in \mathcal{C}^*$ and $\varphi \in \mathcal{C}$, allows a decomposition of the space $\mathcal{C} = C([- \tau, 0], \mathbb{R}^n)$ [4]. Accordingly, the solution x_t of the perturbed equation (1) can be written as

$$x_t = \Phi y(t) + \chi_t, \quad y(t) = (\Psi, x_t)$$

for some $\chi_t \in \mathcal{C}$, where y satisfies

$$\dot{y}(t) = Jy(t) + \varepsilon \Psi^\top(0) [g(\Phi y(t) + \chi_t; \varepsilon) + \kappa f(\Phi y(t) + \chi_t; \varepsilon)],$$

with J is as defined in (4). The change of variables $y = \exp(Jt)u$, gives

$$\dot{u}(t) = \varepsilon e^{-Jt} \Psi^\top(0) (g(\Phi e^{Jt}u(t) + \chi_t; \varepsilon) + \kappa f(\Phi e^{Jt}u(t) + \chi_t; \varepsilon)). \quad (13)$$

By averaging, one obtains the equation

$$\dot{u} = \varepsilon \bar{g}(u) + \varepsilon \kappa \bar{f}(u), \quad (14)$$

where

$$\bar{f}(u) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{-Jt} \Psi^\top(0) f(\Phi e^{Jt}u; 0) dt \quad (15)$$

$$\bar{g}(u) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{-Jt} \Psi^\top(0) g(\Phi e^{Jt}u; 0) dt. \quad (16)$$

It follows by the assumptions on f and g that $\bar{f}(0) = \bar{g}(0) = 0$; thus the origin is an equilibrium point of the system of equations (14). The linear variational equation about the origin is

$$\dot{u} = \varepsilon(\bar{G} + \kappa\bar{F})u, \quad (17)$$

where the averaged matrices $\bar{F}, \bar{G} \in \mathbb{R}^{2 \times 2}$ are defined by

$$\begin{aligned} \bar{F} &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{-Jt} \Psi^\top(0) \int_{-\tau}^0 dF(\theta) \Phi(\theta) e^{Jt} dt \\ \bar{G} &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{-Jt} \Psi^\top(0) \int_{-\tau}^0 dG(\theta) \Phi(\theta) e^{Jt} dt, \end{aligned}$$

with F and G given in (6)–(7). Applying Lemma 1 in [5] to \bar{F} and \bar{G} , we obtain

$$\bar{F} = \frac{1}{2} \operatorname{tr} \left(\Psi^\top(0) \int_{-\tau}^0 dF(\theta) \Phi(\theta) \right) \cdot I - \frac{1}{2} \operatorname{tr} \left(J \Psi^\top(0) \int_{-\tau}^0 dF(\theta) \Phi(\theta) \right) \cdot J \quad (18)$$

$$\bar{G} = \frac{1}{2} \operatorname{tr} \left(\Psi^\top(0) \int_{-\tau}^0 dG(\theta) \Phi(\theta) \right) \cdot I - \frac{1}{2} \operatorname{tr} \left(J \Psi^\top(0) \int_{-\tau}^0 dG(\theta) \Phi(\theta) \right) \cdot J. \quad (19)$$

From (3), (8), and (18), the real parts of the eigenvalues of \bar{F} are both equal to

$$\begin{aligned} & \frac{1}{2} \operatorname{tr} \left(\Psi^\top(0) \int_{-\tau}^0 dF(\theta) \Phi(\theta) \right) \\ &= \frac{1}{2} \operatorname{tr} \left(\Psi^\top(0) \int_{-\tau}^0 dF(\theta) \Phi(0) e^{J\theta} \right) \\ &= \frac{1}{2} \operatorname{tr} \left(\Psi^\top(0) \int_{-\tau}^0 dF(\theta) \Phi(0) (I \cos \theta + J \sin \theta) \right) \\ &= \frac{p}{2} \end{aligned}$$

Similarly, the real parts of the eigenvalues of \bar{G} are equal to $q/2$, so that the real parts of the eigenvalues of the matrix $\bar{G} + \kappa\bar{F}$ in (17) are given by $\frac{1}{2}(q + \kappa p)$. If $(q + \kappa p) \neq 0$, the averaging theorem implies that there exists $\varepsilon_0 > 0$ and an almost periodic solution $x^*(\varepsilon)$ of the original equation (1) for each $\varepsilon \in [0, \varepsilon_0]$, which has the same stability type as the zero solution of (17) [4]. Furthermore, $x^*(0) = 0$, and x^* is unique in a neighborhood of $0 \in \mathcal{C}$ and $\varepsilon = 0$. It follows that $x^*(\varepsilon) \equiv 0$ for $0 \leq \varepsilon \leq \varepsilon_0$, since zero is an almost periodic solution of the averaged equation (14) for all ε . The theorem is then proved since the stability of the zero solution of (17) is determined by the sign of $q + \kappa p$. \square

On the basis of the above theorem, we say that the feedback is *stabilizing* or *destabilizing* depending on whether κp is negative or positive, respectively. In this sense, the quantity p quantifies and compares the (de)stabilizing effect of the various feedback schemes given by f .

3 Delayed versus instantaneous feedback

We now consider the role of delays in feedback with partial state information. For this purpose, we assume that the linearized feedback F given in (6) has the form

$$F(\theta) = Ch(\theta) \quad (20)$$

where $h : [-\tau, 0] \rightarrow \mathbb{R}$ is a function of bounded variation representing a scalar distribution of delays, and $C \in \mathbb{R}^{n \times n}$ is a structure matrix. The feedback gain κ is set to 1, or alternatively subsumed into the matrix C . The interesting case is when C does not have full rank, for instance when some of the system variables are not available for feedback. In the following we fix the number q given in (10) by fixing L and g , and investigate the effect of the structure matrix C and the delay distribution h on stability.

Let $\hat{C} = \Psi^\top(0)C\Phi(0)$. From (11) and (20), it is seen that

$$p = \alpha \operatorname{tr}(\hat{C}) + \beta \operatorname{tr}(\hat{C}J), \quad (21)$$

where

$$\alpha = \int_{-\tau}^0 \cos \theta dh(\theta), \quad \beta = \int_{-\tau}^0 \sin \theta dh(\theta). \quad (22)$$

In the absence of delays, i.e., when $h(\theta)$ is a Heaviside step function at zero, one has $\alpha = 1$ and $\beta = 0$, yielding $p = \operatorname{tr}(\hat{C})$ for undelayed feedback. Hence, delayed feedback is more stabilizing than undelayed feedback if

$$\alpha \operatorname{tr}(\hat{C}) + \beta \operatorname{tr}(\hat{C}J) < \operatorname{tr}(\hat{C})$$

or equivalently, if

$$(1 - \alpha) \operatorname{tr}(\hat{C}) > \beta \operatorname{tr}(J\hat{C}) \quad (23)$$

Similarly, delayed feedback is more destabilizing if (23) holds with the inequality reversed. Although in applications the delays are often viewed as destabilizing factors, the condition (23) shows the role of delays in inducing stability. In particular, if $\operatorname{tr}(\hat{C}) = 0$ then instantaneous feedback has no effect on the stability of the zero solution. This case occurs, for instance, if only some of the system variables are used in the feedback. However, if $\beta \operatorname{tr}(\hat{C}J) \neq 0$, then by (23) delayed feedback of the same variables can stabilize or destabilize the zero solution, depending on the values of α and β .

To illustrate with an example, consider the classical van der Pol oscillator with linear feedback control

$$\ddot{y} + \varepsilon(y^2 - 1)\dot{y} + y = \varepsilon \int_{-\tau}^0 [c_1 y(t + \theta) + c_2 \dot{y}(t + \theta)] dh(\theta), \quad 0 < \varepsilon \ll 1. \quad (24)$$

With $x = (y, \dot{y})$, the linear equation around the origin is

$$\dot{x}(t) = -Jx(t) + \varepsilon \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x(t) + \varepsilon \begin{pmatrix} 0 & 0 \\ c_1 & c_2 \end{pmatrix} \int_{-\tau}^0 x(t + \theta) dh(\theta).$$

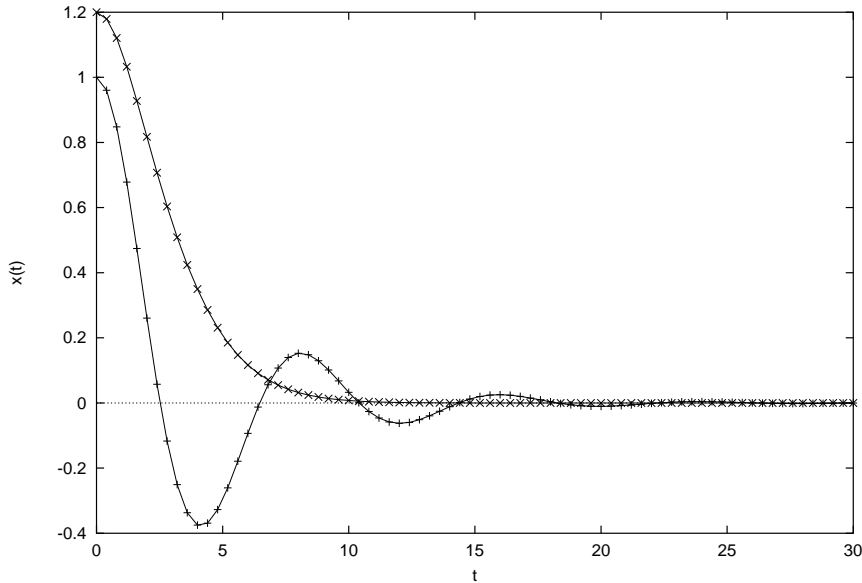


Figure 1: Stabilization of the zero solution of the van der Pol oscillator using position feedback, for feedback gain c_1 equal to 5(+) and 7.8(x), and a discrete feedback delay at $\tau = 1$. Other parameters are $c_2 = 0$, and $\varepsilon = 0.1$.

We have

$$\hat{C} = C = \begin{pmatrix} 0 & 0 \\ c_1 & c_2 \end{pmatrix},$$

giving $p = \alpha c_2 - \beta c_1$. If the feedback is instantaneous (i.e. without delays), then $\alpha = 1$ and $\beta = 0$, so that p depends only on the velocity feedback c_2 . In this case, the origin cannot be stabilized if velocity information is not available for feedback. By contrast, if the feedback is delayed, then using only position information can yield stability provided $\beta c_1 > 1$, by Theorem 1. Figure 1 shows the quenching of oscillations and the stabilization of the origin when $c_2 = 0$ and h represents a discrete delay at $\tau = 1$.

In closing this section we note that we have confined our discussion to linear feedback. If nonlinear terms are added to the feedback, then it is possible to further shape the system's behavior in addition to changing its stability. For example, it has been shown that a limit cycle of desired amplitude can be created [2], or the period can be modified within certain limitations [3].

4 Distributed versus discrete delays

An interesting question in stability investigations is how various distributions of delays about a given mean value affects stability. In particular, one is interested in the difference between distributed delays having the same mean delay $\bar{\tau} = \int_{-\tau}^0 \theta dh(\theta)$ and a

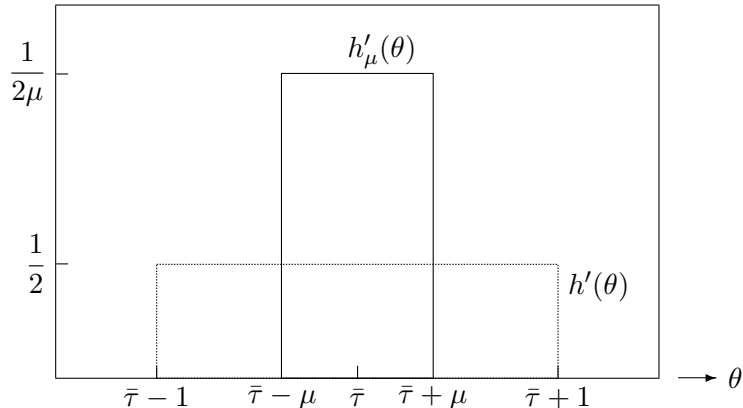


Figure 2: The densities h'_μ corresponding to uniformly distributed delays about the mean value $\bar{\tau}$ and parametrized by μ .

discrete delay at $\bar{\tau}$. For example, Ref. [6] studied the stability of the Cushing equation with discrete and gamma-distributed delays. In the context of a first-order system, it has been discussed that the stability tends to improve with increasing variance of the delay distribution [7], and it was conjectured that a discrete delay at $\bar{\tau}$ is more destabilizing than distributed delays having mean $\bar{\tau}$ [8]. A further example involving coupled oscillators indeed showed that increasing the variance of the delay distribution can enlarge the stability region in the parameter domain [9]. We shall show that these observations are true in a certain sense for Hopf instabilities of more general systems. More precisely, when the delays act towards destabilizing the system, the discrete delay is locally the most destabilizing one among delay distributions having the same mean value. On the other hand, delays can also stabilize an unstable equilibrium point, as seen in the previous section. In this case the discrete delay is locally the most stabilizing delay distribution.

To give a systematic study on the effect of delay distributions, we let $\bar{\tau}$ be fixed and consider a family of distributions having mean value $\bar{\tau}$. For this purpose, let h be some reference distribution with compact support satisfying¹

$$\int_{-\infty}^{\infty} dh(\theta) = 1, \quad \int_{-\infty}^{\infty} \theta dh(\theta) = \bar{\tau}, \quad \int_{-\infty}^{\infty} (\theta - \bar{\tau})^2 dh(\theta) = \sigma^2, \quad (25)$$

and define a family of distributions parametrized by $\mu > 0$,

$$h_\mu(\theta) = h(\bar{\tau} + (\theta - \bar{\tau})/\mu). \quad (26)$$

(Figure 2 depicts the rescaling (26) for the case of uniformly distributed delays.) It is

¹For simplicity, here we view h as a probability distribution, that is, a monotone function satisfying (25). The particular choice of h will not be important for the following discussion.

easy to check that

$$\int_{-\infty}^{\infty} dh_{\mu}(\theta) = 1, \quad \int_{-\infty}^{\infty} \theta dh_{\mu}(\theta) = \bar{\tau}, \quad \int_{-\infty}^{\infty} (\theta - \bar{\tau})^2 dh_{\mu}(\theta) = \mu^2 \sigma^2.$$

Hence, the family h_{μ} provides a natural way to change the variance $\mu^2 \sigma^2$ of the distribution while keeping the mean value fixed. We denote the corresponding values in (22) as

$$\alpha_{\mu} = \int_{-\infty}^{\infty} \cos \theta dh_{\mu}(\theta), \quad \beta_{\mu} = \int_{-\infty}^{\infty} \sin \theta dh_{\mu}(\theta). \quad (27)$$

We also define

$$\alpha_0 = \cos \bar{\tau}, \quad \beta_0 = \sin \bar{\tau}, \quad (28)$$

i.e., the values of α, β for a discrete delay at $\bar{\tau}$, which is equivalent to letting h_0 be a Heaviside step function at $\bar{\tau}$. The question is, for a fixed structure matrix \hat{C} , how the quantity

$$p_{\mu} = \alpha_{\mu} \operatorname{tr}(\hat{C}) + \beta_{\mu} \operatorname{tr}(J\hat{C}) \quad (29)$$

changes as μ is varied.

Before considering the general case, it is instructive to first look at the specific example of uniformly distributed delays shown in Figure 2. Here one has

$$\alpha_{\mu} = \frac{1}{\mu} \sin \mu \cos \bar{\tau}, \quad \beta_{\mu} = \frac{1}{\mu} \sin \mu \sin \bar{\tau},$$

which yields

$$\begin{aligned} p_{\mu} &= \frac{\sin \mu}{\mu} \left(\operatorname{tr}(\hat{C}) \cos \bar{\tau} + \operatorname{tr}(J\hat{C}) \sin \bar{\tau} \right) \\ &= \frac{\sin \mu}{\mu} p_0. \end{aligned} \quad (30)$$

It is thus seen that the dependence of p_{μ} on the parameter μ is not monotone. In fact, by Theorem 1, the sign changes of p_{μ} for varying values of μ indicates that stability switches can occur as the variance of the delay distribution is changed. Nevertheless, since $|\sin \mu| < |\mu|$ for all $\mu > 0$, one has $|p_0| > |p_{\mu}|$, which shows that a discrete delay has a stronger effect on stability than all uniformly distributed delays having the same mean value. Moreover, there are certain values of the distribution variance (given by $\sin \mu = 0$, $\mu > 0$) for which the feedback has no effect on stability. This last property is purely an effect of the delay variance and is independent of the choice of the structure matrix C .

The extremal property of discrete delays observed above can be extended to more general delay distributions. We first give a local characterization.

Proposition 2.

$$\left. \frac{\partial p_{\mu}}{\partial \mu} \right|_{\mu=0} = 0, \quad \text{and} \quad \left. \frac{\partial^2 p_{\mu}}{\partial \mu^2} \right|_{\mu=0} = -\sigma^2 p_0$$

Proof. By a change of variables in (27) and using (26), we have

$$\alpha_\mu = \int_{-\infty}^{\infty} \cos \theta dh_\mu(\theta) = \int_{-\infty}^{\infty} \cos(\bar{\tau} + \mu(s - \bar{\tau})) dh(s). \quad (31)$$

Using (28) it is seen that α_μ and β_μ are smooth functions of μ on \mathbb{R} . Differentiating under the integral gives

$$\frac{\partial \alpha_\mu}{\partial \mu} = - \int_{-\infty}^{\infty} \sin(\bar{\tau} + \mu(s - \bar{\tau}))(s - \bar{\tau}) dh(s).$$

Thus $\partial \alpha_\mu / \partial \mu|_{\mu=0} = 0$. Similarly, $\partial \beta_\mu / \partial \mu|_{\mu=0} = 0$. On the other hand,

$$\begin{aligned} \left. \frac{\partial^2 \alpha_\mu}{\partial \mu^2} \right|_{\mu=0} &= -\sigma^2 \cos \bar{\tau} = -\sigma^2 \alpha_0, \\ \left. \frac{\partial^2 \beta_\mu}{\partial \mu^2} \right|_{\mu=0} &= -\sigma^2 \sin \bar{\tau} = -\sigma^2 \beta_0. \end{aligned}$$

Using in (29), we obtain the conclusion. \square

By the above result, if p_0 is nonzero, then it is a local extremum for p_μ . This shows that discrete delays are indeed special in a certain sense. Thus, if the delayed feedback is a destabilizing one ($p_\mu > 0$), then a discrete delay is locally the most destabilizing delay distribution, and increasing the variance of the distribution reduces p_μ . This confirms the observations of [7–9] and shows that it is generally true near a Hopf instability. However, as noted above, delays can also have a stabilizing effect ($p_\mu < 0$), in which case a discrete delay is locally the most stabilizing one, and increasing the variance of the distribution can yield instability. In both cases, increasing the variance of the distribution locally about a discrete delay reduces the effect of delays in the feedback.

For symmetrically distributed delays, we can also give a global characterization of the extremal property of discrete delays.

Proposition 3. *For delay distributions that are symmetrically distributed about their mean value², $|p_\mu| \leq |p_0|$ for all $\mu > 0$.*

Proof. Expanding the cosine term in (31),

$$\alpha_\mu = \cos \bar{\tau} \int_{-\infty}^{\infty} \cos(\mu(s - \bar{\tau})) dh(s) - \sin \bar{\tau} \int_{-\infty}^{\infty} \sin(\mu(s - \bar{\tau})) dh(s).$$

The second integral vanishes because the distribution is symmetric about $\bar{\tau}$ and sine is an odd function. Thus by (28), $\alpha_\mu = \alpha_0 \int_{-\infty}^{\infty} \cos(\mu(s - \bar{\tau})) dh(s)$, and similarly $\beta_\mu = \beta_0 \int_{-\infty}^{\infty} \cos(\mu(s - \bar{\tau})) dh(s)$. Hence, from (29)),

$$p_\mu = p_0 \int_{-\infty}^{\infty} \cos(\mu(s - \bar{\tau})) dh(s).$$

²That is, $h'(\bar{\tau} + \theta) = h'(\bar{\tau} - \theta)$, where the derivative exists a. e. by assumption.

Then the estimate

$$|p_\mu| \leq |p_0| \int_{-\infty}^{\infty} |\cos(\mu(s - \bar{\tau}))| dh(s) \leq |p_0| \int_{-\infty}^{\infty} dh(s) = |p_0|$$

follows. □

Finally we note that p_0 depends only on the mean delay and not on the distribution h . Hence, the extremal properties of discrete delays given in Propositions 2 and 3 are independent of the particular choice of the reference distribution h .

References

- [1] S.-N. Chow and J. Mallet-Paret, *Integral averaging and bifurcation*, J. Differential Equations, **26** (1977), 112–159.
- [2] F. M. Atay, *Delayed-feedback control of oscillations in non-linear planar systems*, Internat. J. Control, **75** (2002), 297–304.
- [3] F. M. Atay, *Oscillation control in delayed feedback systems*, Lecture Notes in Control and Information Sciences, **273** (2002), 103–116.
- [4] J. K. Hale, *Averaging methods for differential equations with retarded arguments and a small parameter*, J. Differential Equations, **2** (1966), 57–73.
- [5] F. M. Atay, *Total and partial amplitude death in networks of diffusively coupled oscillators*, Physica D, **183** (2003), 1–18.
- [6] F. G. Boese, *The stability chart for the linearized Cushing equation with a discrete delay and with gamma-distributed delays*, J. Math. Anal. Appl., **140** (1989), 510–536.
- [7] R. F. V. Anderson, *Geometric and probabilistic stability criteria for delay systems*, Math. Biosci, **105** (1991), 81–96.
- [8] S. Bernard, J. Beláir and M. C. Mackey, *Sufficient conditions for stability of linear differential equations with distributed delay*, Discrete and Continuous Dynamical Systems B, **1** (2001), 233–256.
- [9] F. M. Atay, *Distributed delays facilitate amplitude death of coupled oscillators*, Phys. Rev. Lett., **91** (2003).