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by

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AN INTEGRO-DIFFERENTIAL EQUATION MODEL FOR ALIGNMENT AND ORIENTATIONAL AGGREGATION

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Abstract

We study an integro-differential equation modeling angular alignment of interacting bundles of cells or filaments. A bifurcation analysis of the related stationary problem was done by Geigant and Stoll in [J. Math. Biol. 46 (2003), no. 6, 537–563]. Here we analyze the time dependent problem and prove that the type of alignment (one or multidirectional) depends on the initial distribution, the interaction potential, and the preferred optimal orientation of the bundles of cells or filaments. Our main technical tool is the analysis of the evolution of suitable functionals for the cell density, which allows to also specify the direction(s) where the final alignment takes place.

1 Introduction

In this paper we analyze how a population of small, stiff cells or filaments with defined orientations will align, either by being attracted towards each other or being repelled. Here repulsion can be interpreted as attraction to the back of an elongated filament or, like in myxobacteria, to the back of another bacterium. A major question in this context is if bundles of the same orientation are formed, how many there are, and how they are organized relative to each other. Our ansatz is closely related to papers by [1], [3], [4], [5], [6], [7], [8], [9], and [10].

We assume a two-dimensional geometry. To describe the orientational aggregation of the bundles of cells or filaments we consider an integro-differential equation for the evolution of an integrable function f on the unit circle $(\mathbb{R}/\mathbb{Z})$ with arc length normalized to

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one. We will chose a representation which is $\mathcal{I} = [-\frac{1}{2}, \frac{1}{2}]$. In many of the following arguments it is convenient though to think in geometrical terms, namely $u \in \mathcal{I} = [-\frac{1}{2}, \frac{1}{2}) \to (\cos(2\pi u), \sin(2\pi u)) \in \mathbb{S}^1$. . We will use this notation freely in the figures, unless confusion is to be expected. Now $f = f(u,t)$ denotes the mean density distribution over the orientation $u \in \mathcal{I}$. The temporal evolution of f is given by

$$\partial_t f(u,t) = -\int_{\mathcal{I}} T[f](u,v) f(u,t) dv + \int_{\mathcal{I}} T[f](v,u) f(v,t) dv. \tag{1}$$

The first term on the right hand side describes the bundles of cells or filaments which reorient away from u, and the second term the bundles orienting themselves into direction u. The stationary version of equation (1) was analyzed in detail in [3] and [6].

For notational convenience in the following we sometimes omit the explicit t-dependencies. The turning rate T in (1) maps a function f acting on $\mathcal{I}$ to a function $T[f]$ acting on $\mathcal{I} \times \mathcal{I}$ with

$$T[f](u,v) = \int_{\mathcal{I}} h(w-u) G_\sigma(v - M_w(u)) f(w,t) dw. \tag{2}$$

Here $G_\sigma : (-1,1) \to \mathbb{R}_+$, $\sigma \geq 0$ is an even, bounded probability density, thus $\int_{\mathcal{I}} G_\sigma = 1$, i.e. the standard periodic Gaussian: $G_\sigma(u) = (4\pi\sigma)^{-1/2} \sum_{m \in \mathbb{Z}} \exp(-(u+2m)^2/(4\sigma))$.
So the process of turning is considered to be probabilistic. The smaller σ is, the narrower is G_σ, which means that reorientation happens with higher accuracy. The extreme case is the Dirac mass $G_0(x) = \delta_0$, which describes deterministic turning.
The measurable function $M_w : \mathcal{I} \to \mathcal{I}$, is called the optimal reorientation, indicating reorientation of bundles of cells or filaments due to their interaction with w. More precisely, if the system is invariant under rotations, we assume

$$M_w(v) = v + V(w - v),$$

where $V : [-1,1] \to \mathbb{R}$ is referred to as the orientational angle, compare fig. 1. A more detailed descriptions of M_w and V will be given in Section 2. The interaction rate $h : \mathcal{I} \to \mathbb{R}_+$ is positive and bounded.

In this paper, we analyze the behavior of solutions for the Dirac mass, δ_0 only. In this case (1) is referred to as the "limiting" equation. We expect less singular alignment patterns of solutions to (1) for G_σ, in case $\sigma > 0$, sufficiently small. This is because for any given $T > 0$ uniform convergence in $[0,T]$ for the solutions corresponding to G_σ and δ_0, respectively, was established in [5] for $\sigma \to 0$.
Our main result states the development of peaks as long time dynamics, i.e. alignment of bundles of cells or filaments. This can be proved for suitable classes of initial distributions combined with various types of optimal orientation and different ranges of interaction, both attractive and repulsive (see Theorems 4.5, 4.8, 5.3, and 5.6). Our main tool is to analyze the dynamics of suitable functionals of the cell density, which describe its behavior in a subset of all possible directions. With this we can also specify the direction(s) in which the bundles finally align.

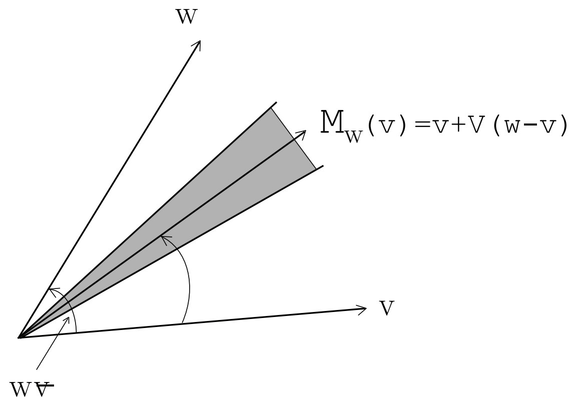

Figure 1: Geometrical interpretation of the turning rate v.

The paper is organized as follows: Section 2 introduces the class of optimal reorientations $M_w(\cdot)$ and orientational angles $V(\cdot)$, we are interested in. In Section 3 global existence of a unique solution of equation (1) is proved. Section 4 is dealing with the analysis of the limiting equation. We show that uni-directional, bi-directional, or multi-directional alignment develops for a set of prescribed initial distributions which are suitably separated, in case the optimal orientation for a certain interaction range is attractive. In Section 5 we consider the situation that bundles of cells or filaments repel each other unless they are close. First it is shown that for non-separated, continuously varying and symmetric initial distributions the solution will eventually develop two symmetric peaks. Then we consider non-symmetric initial data. We prove that if bundles of cells or filaments are attractive and repulsive they do finally align in two exactly opposite directions. Our main result is the local stability of Dirac masses.

2 Optimal reorientation and the orientational angle

In this section, we introduce various types of optimal reorientations $M_w(v)$ and orientation angles $V(\cdot)$, which may cause uni-, bi-, and multi-directional aggregation of bundles of cells or filaments, depending on their initial distribution. First we give conditions for the interaction rate h, though it does not play a crucial role for our further analysis.

Assumption 2.1 *Let $k \geq 1$ be a positive integer.*
The interaction rate $h : \mathbb{R} \to \mathbb{R}$ is a bounded, positive, and 1-periodic function.
In the interval $[-1/2, 1/2]$, h is symmetric and radially decreasing with respect to 0.
There exists $0 < \eta < 1$ such that $\eta\, h(0) \leq h(x) \leq h(0)$ in $[-1/2, 1/2)$.

Next we recall some reasonable assumptions for the optimal reorientation (compare e.g. [6]):

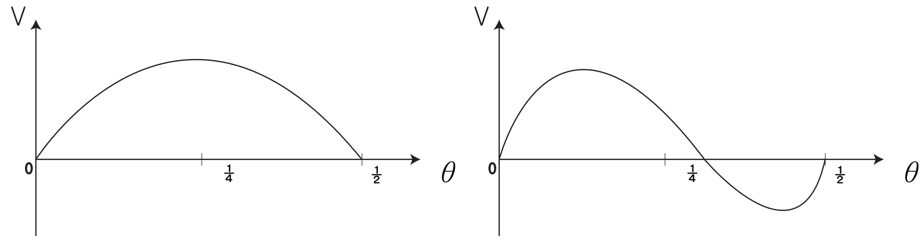

Figure 2: (i) Attracting case (ii) Attracting and repulsive case

Assumption 2.2 *The optimal orientation is of the form* $M_w(v) = v + V(w-v)$ *with* $v, w \in \mathcal{I} = [-1/2, 1/2)$*, where* $V : \mathbb{R} \to \mathbb{R}$ *is the optimal reorientation angle, with*

$$V(\theta) = -V(-\theta), \quad V(1+\theta) = V(\theta). \tag{3}$$

Let $V \in C^2$ *and* $0 < a < 1$, $0 < b < 1$ *such that*

$$|V(\theta)| \le \max\left\{a\theta, b\left(\frac{1}{2} - \theta\right)\right\}, \quad \theta \in \left[0, \frac{1}{2}\right]. \tag{4}$$

We will see more concrete examples for V later.

Remark 2.3 *Since* V *is odd and 1-periodic, it is sufficient to define the values of* V *in* $[0, 1/2]$*. Conditions* (3) *are equivalent to*

$$M_w(v) + M_v(w) = v + w, \quad M_w(1+u) = M_w(u) + 1, \quad M_{w+1}(u) = M_w(u).$$

Next we classify the properties of the optimal orientational angle $V(\theta)$, depending on the maximal range of attraction and repulsion between the bundles of cells or filaments. Instead of repulsion one may want to describe the phenomenon also as attraction to the ends of the interacting partners (e.g. like in myxobacteria), [2] .

(a) *Uni-directional alignment (attraction),* see figure 2 (i):
Here the bundles of cells or filaments attract each other towards the direction with angle smaller than π. Thus additionally to Assumption 2.2, V is required to fulfill

$$V(\theta) \ge 0 \quad \text{in } \left[0, \frac{1}{2}\right]. \tag{5}$$

(b) *Bi- and Multi- directional alignment (k-directional alignment: attraction):*
Depending on the initial distribution, two or multiple peaks of aligned cells may develop. In terms of the orientational angle V, this is achieved for short ranged attraction. E.g., for the bi-directional case, we suppose that $V > 0$ in $(0, \theta)$ where

$0 < \theta < 1/4$ and $V = 0$ in $[\theta, 1/2]$.
Generally speaking, the shorter the range of optimal turning is, the more likely is the development of many peaks. For simplicity, fix $\theta = 1/(2k)$, $k \in \mathbb{N}$. The k-directional orientational angle is then given by

$$V(\theta) > 0 \;\text{ in }\; \left(0, \frac{1}{2k}\right) \quad , \quad V(\theta) = 0 \;\text{ in }\; \left[\frac{1}{2k}, \frac{1}{2}\right]. \tag{6}$$

(c) *Bi-directional alignment (attraction and repulsion)*, see figure 2 (ii)
Here we consider the following situation: if the angle between the filaments is close, then they attract each other, if the angle between the filaments is large, then they are repulsive, respectively attracted to the ends of the interacting partners. Let $\theta_0 \in (0, 1/2)$ then the orientational angle is given as

$$V(\theta) > 0 \;\text{ in }\; (0, \theta_0) \quad , \quad V(\theta) < 0 \;\text{ in }\; \left(\theta_0, \frac{1}{2}\right). \tag{7}$$

In the following C will denote a constant which may vary from line to line.

3 Global Existence

In this section we show that for bounded interaction rate h the unique solution of (1)-(2) is globally bounded in time.

Lemma 3.1 *Let $0 < T < \infty$. Suppose that the interaction rate $h(\cdot)$ is nonnegative and bounded, i.e. $h \in L^\infty(\mathcal{I})$. Let $f_0 \in L^\infty(\mathcal{I})$ and $\int_{\mathcal{I}} f_0(u)du = m$. Then there exists a unique solution f of (1)-(2) such that $f(\cdot, t) \in L^\infty(\mathcal{I} \times [0, T))$ and $\int_{\mathcal{I}} f(u,t)du = m$ for all $t \in [0, T)$. Furthermore, if f_0, h, and $M_v(w)$ are smooth, then this is true also for f in $\mathcal{I} \times [0, T)$.*

Proof. Due to equation (1) we observe a priori that mass is preserved.
Let $(h * f)(u,t) = \int_{\mathcal{I}} h(u-v) f(v,t) dv$ denote the convolution operator. Then (1) can be rewritten as $\partial_t f(u,t) = -(h * f)(u,t) f(u,t) + R(f)(u,t)$, where

$$R(f)(u,t) = \int_{\mathcal{I}} \int_{\mathcal{I}} h(v-u) G_\sigma\left(u - M_w(v)\right) f(w,t) f(v,t) dw dv.$$

With the assumptions of Lemma 3.1 we can estimate

$$R(f)(u,t) \le \|h\|_{L^\infty(\mathcal{I})} \int_{\mathcal{I}} \int_{\mathcal{I}} G_\sigma\left(u - M_w(v)\right) f(w,t) f(v,t) dw dv$$

$$\le \|h\|_{L^\infty(\mathcal{I})} \|f(\cdot, t)\|_{L^\infty(\mathcal{I})} \int_{\mathcal{I}} \int_{\mathcal{I}} G_\sigma\left(u - M_w(v)\right) dw f(v,t) dv$$

$$\leq Km \|h\|_{L^\infty(\mathcal{I})} \|f(\cdot,t)\|_{L^\infty(\mathcal{I})} \leq C \|f(\cdot,t)\|_{L^\infty(\mathcal{I})},$$

where $K = \|G_\sigma\|_\infty$ and $C = C(K, m, ||h||_{L^\infty})$. Since our equation is of the form $\partial_t f = -a(u,t)f(u,t) + g$ with $g \geq 0$ we have that f is positive and thus $-(h * f)(u,t)f(u,t)$ is non-positive. With this and the previous estimate we obtain

$$\frac{\partial_t |f(u,t)|^2}{2} = f(u,t)\partial_t f(u,t)$$

$$= -(h * f)(u,t) |f(u,t)|^2 + f(u,t)R(f)(u,t) \leq C \|f(\cdot,t)\|^2_{L^\infty(\mathcal{I})}.$$

Using Gronwall's lemma, we get

$$\|f(\cdot,t)\|_{L^\infty(\mathcal{I})} \leq \|f(\cdot,0)\|_{L^\infty(\mathcal{I})} \exp(Ct).$$

It remains to show uniqueness of f.
Suppose that f_1 and f_2 are solutions of (1) with $f_1(u,0) = f_2(u,0)$. We will prove that

$$|\partial_t[f_1(u,t) - f_2(u,t)]| \leq C \|f_1(\cdot,t) - f_2(\cdot,t)\|_{L^\infty(\mathcal{I})},$$

where $C = C(K, m, \|h\|_{L^\infty})$. First note that

$$|(h * f_1)(u,t)f_1(u,t) - (h * f_2)(u,t)f_2(u,t)|$$

$$\leq C \left(\|f_1(.,t)\|_{L^\infty(\mathcal{I})} + \|f_2(.,t)\|_{L^\infty(\mathcal{I})} \right) \|f_1(.,t) - f_2(.,t)\|_{L^\infty(\mathcal{I})},$$

for given t. On the other hand, we have

$$|R(f_1)(u,t) - R(f_2)(u,t)|$$

$$= \left| \int_{\mathcal{I}} \int_{\mathcal{I}} h(v-u) G_\delta(u - M_w(v)) \left(f_1(w,t)f_1(v,t) - f_2(w,t)f_2(v,t) \right) dw dv \right|$$

$$\leq C \|h\|_{L^\infty(\mathcal{I})} \|f_1(.,t) - f_2(.,t)\|_{L^\infty(\mathcal{I})}.$$

Summing up, we obtain

$$\partial_t |f_1(u,t) - f_2(u,t)|^2 = 2 \left(f_1(u,t) - f_2(u,t) \right) |\partial_t \left(f_1(u,t) - f_2(u,t) \right)|$$

$$\leq \tilde{C} \|f_1(.,t) - f_2(.,t)\|^2_{L^\infty(\mathcal{I})}.$$

Uniqueness follows again from Gronwall's lemma. Smoothness of f is straightforward from the equation, if the initial distribution, interaction rate, and optimal reorientation are regular. Since the proof is similar to the one above, details are omitted. $\square$

Next we consider the "limiting" equation of (1) by substituting G_σ by δ_0.

$$\partial_t f(u,t) = -(h * f)(u,t) f(u,t) + \int_{\mathcal{I}} \int_{\mathcal{I}} \delta_0 \left(u - M_w(v)\right) f(w,t) f(v,t) dw dv. \qquad (8)$$

It is easy to see that the solution of (8) satisfies Lemma 3.1, as it is the case for the usual Gaussian $G_\sigma, \sigma > 0$. For convenience, denote by $\tilde{f}$ and f_σ the solutions of (1) for δ_0 and G_σ with $\sigma > 0$, respectively. In [5] pointwise convergence of f_σ to $\tilde{f}$ for any finite time interval was proved, more precisely

Lemma 3.2 *Let T be fixed with $0 < T < \infty$ and let the assumptions in Lemma 3.1 hold. Let f_σ and $\tilde{f}$ be solutions of (1) for G_σ and G_0 with the same initial distribution. Then in $[0,T]$ the solution f_σ converges to $\tilde{f}$ in the $L^\infty-$norm as $\sigma \to 0$.*

Proof. See [5, Theorem 2.1]. $\square$

4 Peak Solutions for the Limiting Equation

To introduce the main ideas and techniques we first start with a simple setting and study the behavior of solutions for the limiting equation, which means equations (8) respectively (1) corresponding to the Dirac mass δ_0. We start with assumptions on the classes of initial distributions, which cause uni-, bi-, and multi-directional bundles for large times.

Assumption 4.1 *Let $k \geq 1$ be a fixed integer. Suppose f_0 is smooth and $\int_{\mathcal{I}} f_0(u) du = m$. Assume that* supp $f_0 \subset \cup_{i=1}^k A_i$ *where the $A_i \subset \mathcal{I}$ are nonempty, open, connected, and mutually disjoint such that for A_i, $i = 1, 2, \cdots, k$*

1. $\int_{A_i} f_0 = m_i > 0$ *for each* $i = 1, 2, \cdots, k$, *with* $\sum_{i=1}^k m_i = m$.
2. *dist*$(A_i, A_j) \geq 1/(2k)$ *for* $i \neq j$, *thus* $\left|\mathcal{I} \setminus \cup_{i=1}^k A_i\right| \geq 1/2$.
3. $|A_i| \leq \frac{1}{2k}$ *for* $i = 1, 2, \cdots, k$.

Remark 4.2 *If $k = 1$, then there is only one $A_1 \subset \mathcal{I}$ with $|A_1| \leq 1/2$, e.g. $A_1 = (-1/4, 1/4)$. If $k = 2$, then there are disjoint open intervals, A_1 and A_2, such that $|A_1| = |A_2| \leq 1/4$. We assume $A_1 = (-3/8, -1/8)$ and $A_2 = (1/8, 3/8)$. For general $k \geq 3$, we can take $A_i = \Big(-1/2 + (2i-1)/2k, -1/2 + i/k \Big)$ where $i = 1, 2, \cdots, k$.*

In the following subsections, we derive different types of alignment with respect to the optimal reorientations relevant for assumed initial conditions, in case the deviation σ of the Gaussian is sufficiently small.

4.1 Uni-Directional Alignment

Here we take $k = 1$ in Assumption 4.1, namely supp $f_0 = A_1$. Thus the interaction range between the bundles of cells or filaments can be wide. Without loss of generality, we assume that $A_1 = (-1/4, 1/4)$. For the Dirac mass, we have

$$\int_{\mathcal{I}} u\delta_0(u-u_0)du = u_0, \qquad \int_{\mathcal{I}} |u|^2\,\delta_0(u-u_0)du = |u_0|^2\,. \tag{9}$$

The interaction rate h is as given in Assumption 2.1. For simplicity suppose $h = 1$. Thus we can rewrite (8) as follows:

$$\partial_t f(u,t) = -mf(u,t) + \int_{\mathcal{I}}\int_{\mathcal{I}} \delta_0\left(u - M_w(v)\right) f(w,t)f(v,t)dwdv. \tag{10}$$

In the next lemma, we show that the first momentum of f is preserved, provided that the orientational angle $M_w(v)$ is of uni-directional type, compare (5).

Lemma 4.3 *Let $k = 1$ and f_0 be an initial distribution satisfying Assumption 4.1. Suppose that $M_w(v)$ and V satisfy Assumption 2.2. Let the interaction rate $h = 1$. Then* supp $f(\cdot,t) \subset$ supp f_0 *for all $t \in [0,T)$, and the first momentum of f is preserved, i.e.*

$$\int_{\mathcal{I}} uf(u,t)du = \int_{\mathcal{I}} uf_0(u)du \quad \textit{for all} \;\; t \geq 0. \tag{11}$$

Proof. To show the first assertion, we consider the discrete version of (10). Let $\tau > 0$ be small, then we define $D^f_\tau(u,t)$ by

$$\frac{D^f_\tau(u,t+\tau) - D^f_\tau(u,t)}{\tau} = -mD^f_\tau(u,t)$$

$$+\int_{\mathcal{I}}\int_{\mathcal{I}} \delta_0(u - M_w(v))D^f_\tau(w,t)D^f_\tau(v,t)dwdv, \quad t \geq \tau,$$

with initial conditions

$$D^f_\tau(u,t) = f(u,0), \qquad 0 \leq t < \tau.$$

With the above identity, one can check that supp $D^f_\tau(u,t) \subset$ supp $f(u,0)$ for all $\tau > 0$. By following a procedure similar as in Lemma 3.1, we can see that D^f_τ is bounded for all t and D^f_τ converges to a function g in L^∞ as $\tau \to 0$. Since g solves equation (10), by uniqueness we conclude that g is identical to the solution f of (10). Therefore, supp $f(u,t) \subset$ supp $f(u,0)$.

It remains to prove that the first moment is preserved. Using (9), we have

$$\frac{d}{dt}\int_{\mathcal{I}} uf(u,t)du = -m\int_{\mathcal{I}} uf(u,t)du + \int_{\mathcal{I}} u\int_{\mathcal{I}}\int_{\mathcal{I}} \delta_0(u-M_w(v))f(w,t)f(v,t)dwdvdu$$

$$= -m\int_{\mathcal{I}} uf(u,t)du + \int_{\mathcal{I}}\int_{\mathcal{I}} M_w(v)f(w,t)f(v,t)dwdv$$

$$= -m\int_{\mathcal{I}} uf(u,t)du + \int_{\mathcal{I}}\int_{\mathcal{I}} \left(v+V(w-v)\right) f(w,t)f(v,t)dwdv$$

$$= \int_{\mathcal{I}}\int_{\mathcal{I}} V(w-v)f(w,t)f(v,t)dwdv.$$

The last term equals zero, since with $V(\theta) = -V(-\theta)$ we have

$$\int_{\mathcal{I}}\int_{\mathcal{I}} V(w-v)f(w,t)f(v,t)dwdv = \int_{\mathcal{I}}\int_{\mathcal{I}} V(v-w)f(v,t)f(w,t)dvdw$$

$$= -\int_{\mathcal{I}}\int_{\mathcal{I}} V(w-v)f(v,t)f(w,t)dvdw.$$

This completes the proof. □

Remark 4.4 *Lemma 4.3 is also valid when the interaction rate h is not constant, i.e. if h satisfies Assumption 2.1, the mass and first momentum are preserved and the support of f is contained in the initial distribution of f. The proof is analogous to the one before.*

Using the result of Lemma 4.3, namely that the first momentum is preserved, we can define the mean of the first moment $\xi \in \mathcal{I}$, which is constant and given as

$$\xi = \frac{\int_{\mathcal{I}} uf(u,t)du}{\int_{\mathcal{I}} f(u,t)du} = \frac{1}{m}\int_{\mathcal{I}} uf(u,t)du. \tag{12}$$

Now we show that more general types of second moments of f are decreasing in time.

Theorem 4.5 *Let $k = 1$ and f_0 be an initial distribution satisfying Assumption 4.1. Let ξ as defined in (12). Suppose that $M_w(v)$ and V satisfy Assumption 2.2 and condition (5). Assume further that $h = 1$. Then*

$$\frac{d}{dt}\int_{\mathcal{I}} (u-\xi)^2 f(u,t)du \le 0. \tag{13}$$

Equality in (13) only holds in case $f(u,t) = m\delta_{\{u=\xi\}}$, where δ is the Dirac mass.

Proof. Let $\mathcal{S}(t) := \int_{\mathcal{I}}(u-\xi)^2 f(u,t)du$. Direct calculations show that

$$\frac{d}{dt}\mathcal{S}(t) = -m\int_{\mathcal{I}}(u-\xi)^2 f(u,t)du$$

$$+\int_{\mathcal{I}}(u-\xi)^2\int_{\mathcal{I}}\int_{\mathcal{I}}\delta_0(u-M_w(v))f(w,t)f(v,t)dwdvdu$$

$$= -m\int_{\mathcal{I}}(u-\xi)^2 f(u,t)du + \int_{\mathcal{I}}\int_{\mathcal{I}}\big(M_w(v)-\xi\big)^2 f(w,t)f(v,t)dwdv$$

$$= -m\int_{\mathcal{I}}(u-\xi)^2 f(u,t)du + \frac{1}{2}\int_{\mathcal{I}}\int_{\mathcal{I}}\big(M_w(v)-\xi\big)^2 f(w,t)f(v,t)dwdv$$

$$+\frac{1}{2}\int_{\mathcal{I}}\int_{\mathcal{I}}\big(M_v(w)-\xi\big)^2 f(v,t)f(w,t)dvdw$$

$$= -m\int_{\mathcal{I}}(u-\xi)^2 f(u,t)du + \frac{1}{2}\int_{\mathcal{I}}\int_{\mathcal{I}}\big(v+V(w-v)-\xi\big)^2 f(w,t)f(v,t)dwdv$$

$$+\frac{1}{2}\int_{\mathcal{I}}\int_{\mathcal{I}}\big(w+V(v-w)-\xi\big)^2 f(v,t)f(w,t)dvdw$$

$$= (-m+m)\int_{\mathcal{I}}(u-\xi)^2 f(u,t)du$$

$$+\int_{\mathcal{I}}\int_{\mathcal{I}}\big(v-w+V(w-v)\big)V(w-v)f(w,t)f(v,t)dwdv.$$

From the assumptions on V we can deduce that $V(\theta)(V(\theta)-\theta) \leq 0$. Thus the last expression is non-positive, and $\mathcal{S}'(t)=0$ only holds if $V(v-w)=0$ for all $v,w\in\mathcal{I}$. □

Remark 4.6 *The estimate is also valid if h is a general function, satisfying Assumption 2.1. Again the computation is like the one before.*

4.2 Bi- and Multi-Directional Alignments

For completeness we shortly discuss bi- and multi-directional alignment of bundles of cells or filaments. Basically the technical arguments are similar to the ones before. This situation is extremely unstable. Slight changes of the interaction potential will destroy the dynamics. Later we will consider examples of stable alignment for this type of models. We first consider the situation where several peaks develop. We start with the simple observation that mass and the first moments on disjoint sets are preserved under Assumption 4.1 also for $k \geq 2$.

Lemma 4.7 *Let $k \geq 2$ be a positive integer and f_0 be an initial distribution satisfying Assumption 4.1. Suppose that $M_w(v)$ and V satisfy Assumption 2.2 and condition* (6). *Assume further that h satisfies Assumption 2.1. Then* $\text{supp } f(\cdot,t) \subset \text{supp } f_0 = \cup_{j=1}^{k} A_j$. *Furthermore*

$$\int_{A_j} f(u,t)du = \int_{A_j} f_0(u)du \quad \textit{and} \quad \int_{A_j} uf(u,t)du = \int_{A_j} uf_0(u)du, \quad \textit{for } j = 1,...,k.$$

Proof. For simplicity and w.l.o.g. we show the result just for $k = 2$. The first part is similar to Lemma 4.3 for the uni-directional case, thus we omit details. We now show conservation of mass on each of the sets.

$$\frac{d}{dt}\int_{A_1} f(u,t)du = -\int_{A_1}\int_{\mathcal{I}} h(w-u)f(w,t)f(u,t)dwdu$$

$$+\int_{A_1}\int_{\mathcal{I}}\int_{\mathcal{I}} h(v-w)\delta_0\left(u - M_w(v)\right) f(w,t)f(v,t)dwdvdu =: I_1(t) + I_2(t).$$

We show that $I_1(t) + I_2(t) = 0$. Note that f is only supported in A_1 and A_2 and

$$M_w(v) = \begin{cases} v + V(w-v) & \text{if } w,v \in A_1 \text{ or if } w,v \in A_2 \\ v & \text{if } v \in A_1, w \in A_2 \text{ or if } w \in A_1, v \in A_2, \end{cases} \tag{14}$$

due to the assumption on the arc-length separating A_1 and A_2. Thus we know that

$$\int_{A_1} \delta_0\left(u - M_w(v)\right) du = 1 \quad \text{if and only if } w,v \in A_1 \text{ or } v \in A_1, w \in A_2.$$

And so $I_2(t)$

$$= \int_{A_1}\int_{A_1} h(v-w)f(w,t)f(v,t)dwdv + \int_{A_1}\int_{A_2} h(v-w)f(w,t)f(v,t)dwdv = -I_1(t).$$

The argument for $\int_{A_2} f(u,t)du$ is the same, and the first part of the proof is completed.

Without loss of generality we assume, that $A_1 = (-3/8, -1/8)$ and $A_2 = (1/8, 3/8)$. Again we consider only $\int_{A_1} uf(u,t)du$. Since f is supported in A_1 and A_2, by arguments of the previous lemma, and due to Assumption 2.1, we get

$$\frac{d}{dt}\int_{A_1} uf(u,t)du = -\int_{A_1}\int_{\mathcal{I}} h(w-u)f(w,t)uf(u,t)dwdu$$

$$+\int_{A_1}\int_{\mathcal{I}}\int_{\mathcal{I}} h(w-v)u\delta_0\left(u - M_w(v)\right)f(w,t)f(v,t)dwdvdu$$

$$= -\int_{A_1}\int_{A_1} h(w-u)f(w,t)uf(u,t)dwdu + \int_{A_1}\int_{A_1} h(w-v)M_w(v)f(w,t)f(v,t)dwdv.$$

$$= \int_{A_1}\int_{A_1} h(w-v)V(w-v)f(w,t)f(v,t)dwdv$$

since $M_w(v) = v + V(w-v)$. The last term equals zero, because V is odd and h is even with respect to 0. This completes the proof. □

Since the first moments are preserved on each of the sets $A_j, j = 1, ..., k$, we define the constants

$$\xi_j = \frac{\int_{A_j} uf(u,t)du}{\int_{A_j} f(u,t)du}, \qquad j = 1, ..., k. \tag{15}$$

Like in the uni-directional case, we show that the second moments are non-increasing.

Theorem 4.8 *Let $k \geq 2$ be a positive integer and f_0 be an initial distribution satisfying Assumption 4.1. Let $\xi_j, j = 1, ..., k$ be defined as in* (15)*. Suppose that $M_w(v)$ and V satisfy Assumption 2.2 and condition* (6)*. Let h satisfy Assumption 2.1. Then*

$$\frac{d}{dt}\int_{A_j} (u-\xi_j)^2 f(u,t)du \leq 0, \qquad j = 1, ..., k. \tag{16}$$

Equality in (16) *only holds for $f(u,t) = \sum_{j=1}^k m_j\delta_{\{u=\xi_j\}}$, where $\int_{A_j} f_0(u)du = m_j$.*

Proof. Again, we only consider the case $\int_{A_1}(u-\xi_1)^2 f(u,t)du =: S(t)$, and show that $S(t)$ is non-increasing. Let $m_1 := \int_{A_1} f_0(u)du$.

$$\frac{d}{dt}S(t) = -\int_{A_1}\int_{\mathcal{I}} h(w-u)f(w,t)(u-\xi_1)^2 f(u,t)dwdu$$

$$+\int_{A_1}(u-\xi_1)^2\int_{\mathcal{I}}\int_{\mathcal{I}} h(w-v)\delta_0\left(u-M_w(v)\right)f(w,t)f(v,t)dwdvdu$$

$$=-\int_{A_1}\int_{A_1} h(w-u)f(w,t)(u-\xi_1)^2 f(u,t)dwdu$$

$$+\int_{A_1}\int_{A_1} h(w-v)\left(M_w(v)-\xi_1\right)^2 f(w,t)f(v,t)dwdv =: I_1(t)+I_2(t).$$

Here we used that $u \in A_1, \operatorname{supp} f \subset \cup_{j=1}^k A_j$ and the assumption on the size of the arc-length between the A_j. As in the uni-directional case, we can compute

$$I_2(t)=\frac{1}{2}\int_{A_1}\int_{A_1} h(w-v)\left((M_w(v)-\xi_1)^2+(M_v(w)-\xi_1)^2\right)f(w,t)f(v,t)dwdv$$

$$=\frac{1}{2}\int_{A_1}\int_{A_1} h(w-v)\left((v+V(w-v)-\xi_1)^2+(w+V(v-w)-\xi_1)^2\right)f(w,t)f(v,t)dwdv$$

$$=\int_{A_1}\int_{A_1} h(w-u)f(w,t)(u-\xi_1)^2 f(u,t)dwdu$$

$$+\int_{A_1}\int_{A_1} h(w-v)V(w-v)\left(V(w-v)-w+v\right)f(w,t)f(v,t)dwdv.$$

Adding up, we obtain

$$\frac{d}{dt}S(t)=\int_{A_1}\int_{A_1} h(w-v)V(w-v)\left(V(w-v)-w+v\right)f(w,t)f(v,t)dwdv.$$

The right hand side is non-positive, and equals zero only if $V(w-v)=0$ for all $v,w \in A_1$.

□

So far we could decouple the system into separate subsystems. The first moment of each of these was locally preserved, due to the fact that each of them had compact support. In the following we consider more general cases.

5 Attractive and Repulsive Optimal Orientation

5.1 The symmetric case

Here we consider the case of symmetric initial distributions and the situation that the main part of the mass is almost concentrated at two opposite positions. The support is supposed to be connected. Details will be given later. We intentionally discuss this simplified situation first, as a particular case of the more relevant non-symmetric case, since some of the basic technical details can be conveyed much easier. We start with conditions for the orientational angle.

Assumption 5.1 *Suppose that V is smooth, satisfies (3), (4) and*

(a) $V(\theta) > 0$ in $(0, 1/4)$ and V is anti-symmetric with respect to $1/4$ in $[0, 1/2]$.

(b) There exists $0 < a < 1$ and $\theta_1 \in (0, 1/4)$ such that $V'(0) = V'(1/2) = a$ and

$$V(\theta) \le a\theta \quad \text{in } [0, \theta_1] \quad , \quad -V(\theta) \le -a\,(\theta - 1/2) \quad \text{in } [1/2 - \theta_1, 1/2]\,.$$

(c) Furthermore, there exists $C_0 > 0$ such that $V'(\theta) < -C_0$ in $[\theta_1, 1/2 - \theta_1]$. □

Assumption 5.1 automatically implies $|V(\theta)| \le a\theta$ and $|V(\theta)| \le a(1/2 - \theta)$ in $[0, 1/2]$. Next we specify the initial distributions.

Assumption 5.2 *Suppose that $f_0 : (-1/2, 1/2] \to \mathbb{R}$ is smooth and nonnegative with $\int_{\mathcal{I}} f_0(u)du = m$. Let f_0 be $1/2$-periodic and symmetric with respect to both 0 and $1/4$, and satisfy*

(a) There exist $0 < \delta < 1/4$ and $\epsilon_1 > 0$ such that

$$\int_{I_\delta(\frac{1}{4})} \left(u - \frac{1}{4}\right)^2 f_0(u)du < \epsilon_1, \qquad \text{where } I_\delta(y) = [y - \delta, y + \delta].$$

(b) There exists $\epsilon_2 > 0$ such that $f_0(u) < \epsilon_2$ for $u \in (1/4 + \delta, 1/2)$.

Now we are ready to state and prove the main result of this section.

Theorem 5.3 *Suppose that Assumptions 5.1 and 5.2 hold. If δ, ϵ_1, and ϵ_2 are sufficiently small, then $\mathcal{S}^{\pm}(t) = \int_{I_\delta(\pm 1/4)} (u \mp 1/4)^2 f(u,t)du$ is non-increasing, and*

$$\int_{I_\delta(\pm 1/4)} (u \mp 1/4)^2 f(u,t)du \le e^{-Ct} \int_{I_\delta(\pm 1/4)} (u \mp 1/4)^2 f_0(u)du.$$

Furthermore, the mass is finally concentrated only at $\pm 1/4$, i.e. for any given $\epsilon > 0$,

$$\lim_{t\to\infty} \int_{I_\epsilon(\pm 1/4)} f(u,t)du = \frac{m}{2}.$$

Proof. We first note that f is symmetric with respect to 0 and to $1/4$, thus f is also $1/2$-periodic for all times t. This can be proved by uniqueness, namely let $\tilde{f}(u) := f(-u)$. Then $\tilde{f}$ is a solution of (8) with the same initial data as f, and therefore, due to the uniqueness of solution, we have $f = \tilde{f}$. Thus f is symmetric with respect to 0. Here we used that V is $1/2$-periodic. Similarly symmetry of f with respect to $\frac{1}{4}$ can be checked. Next we set $\tilde{\theta} := \theta_1 + \delta$, with θ_1, δ such that $\tilde{\theta} < \frac{1}{4} - \delta$ where δ is given in Assumption 5.2 and we define

$$A_1 := I_{\tilde{\theta}}(1/4) = (1/4 - \tilde{\theta}, 1/4 + \tilde{\theta}) \quad , \quad A_2 := \left[0, \frac{1}{4} - \tilde{\theta}\right] \cup \left[\frac{1}{4} + \tilde{\theta}, \frac{1}{2}\right].$$

Since f is 1/2-periodic, it is sufficient to estimate $\mathcal{S}^+(t)$, which we denoted by $\mathcal{S}(t)$ in the following, unless any confusion is to be expected. We analyze

$$\mathcal{S}(t) := \int_{A_1} f(u,t)\left(u-\frac{1}{4}\right)^2 du, \quad \text{and} \quad J(t) := \sup_{u\in A_2} f(u,t).$$

First we compute

$$\frac{d}{dt}\mathcal{S}(t) = -m\mathcal{S}(t) + \int_{\mathcal{I}} dv \int_{\mathcal{I}} dw f(w,t)f(v,t)\left(M_w(v)-\frac{1}{4}\right)^2 \chi_{\hat{A}_1} = -m\mathcal{S}(t) + \mathcal{F}(t),$$

where $\hat{A}_1 = \{(w,v)\in \mathcal{I}\times\mathcal{I} : M_w(v)\in A_1\}$. For convenience, we denote $I_\delta^+ := I_\delta(1/4), I_\delta^- := I_\delta(-1/4)$ and $I_\delta^\pm := I_\delta^+ \cup I_\delta^-$. Consider

$$\begin{aligned}\mathcal{F}(t) &= \int_{I_\delta^\pm} dv \int_{I_\delta^\pm} dw f(w,t)f(v,t)\left(M_w(v)-\frac{1}{4}\right)^2 \chi_{\hat{A}_1}\\
&+ \int_{\mathcal{I}\setminus I_\delta^\pm} dv \int_{\mathcal{I}\setminus I_\delta^\pm} dw f(w,t)f(v,t)\left(M_w(v)-\frac{1}{4}\right)^2 \chi_{\hat{A}_1}\\
&+ \int_{\mathcal{I}\setminus I_\delta^\pm} dv \int_{I_\delta^\pm} dw f(w,t)f(v,t)\left(M_w(v)-\frac{1}{4}\right)^2 \chi_{\hat{A}_1}\\
&+ \int_{I_\delta^\pm} dv \int_{\mathcal{I}\setminus I_\delta^\pm} dw f(w,t)f(v,t)\left(M_w(v)-\frac{1}{4}\right)^2 \chi_{\hat{A}_1}\\
&= \mathcal{F}_1(t)+\mathcal{F}_2(t)+\mathcal{F}_3(t)+\mathcal{F}_4(t).\end{aligned}$$

Estimate for $\mathcal{F}_1(t)$:

Keeping the symmetry of f in mind, the effects of the characteristic function, and finally doing Taylor expansion for $V(0)$, we obtain

$$\begin{aligned}\mathcal{F}_1(t) &= \int_{I_\delta^+} dv \int_{I_\delta^+} dw f(w,t)f(v,t)\left(M_w(v)-\frac{1}{4}\right)^2\\
&+ \int_{I_\delta^+} dv \int_{I_\delta^-} dw f(w,t)f(v,t)\left(M_w(v)-\frac{1}{4}\right)^2\end{aligned}$$

$$\leq \int_{I_\delta^+} dv \int_{I_\delta^+} dw f(w,t) f(v,t) \left((1-a)(v-\frac{1}{4}) + a(w-\frac{1}{4}) + O(v-w)^2 \right)^2$$

$$+ \int_{I_\delta^+} dv \int_{I_\delta^-} dw f(w,t) f(v,t) \left((1-a)(v-\frac{1}{4}) + a(w+\frac{1}{4}) + O(v-w-\frac{1}{2})^2 \right)^2$$

$$\leq m\left((1-a)^2 + a^2\right) \int_{I_\delta^+} \left(v-\frac{1}{4}\right)^2 f(v,t)dv + C\delta \int_{I_\delta^+} \left(v-\frac{1}{4}\right)^2 f(v,t)dv. \qquad (17)$$

Here we have exchanged v and w in some integrations and used that $\int_{I_\delta^+} f(v,t)(v-\frac{1}{4})dv = \int_{I_\delta^-} f(w,t)(w+\frac{1}{4})dw$.

Lower estimate for the mass in I_δ^+: Note that

$$\int_{I_\delta^+} f(u,t)du \geq \frac{m}{2} - J(t) - \frac{1}{\delta^2}\mathcal{S}(t) \quad , \quad \int_{[0,1/2]\setminus I_\delta^+} f(u,t)du \leq J(t) + \frac{1}{\delta^2}\mathcal{S}(t). \qquad (18)$$

Indeed, since $\int_{\mathcal{I}} f(u,t)du = m$, we get

$$2\int_{I_\delta^+} f(u,t)du = \int_{I_\delta^\pm} f(u,t)du = \int_{\mathcal{I}} f(u,t)du - 2\int_{A_2} f(u,t)du - 2\int_{A_1\setminus I_\delta^+} f(u,t)du$$

$$\geq m - 2J(t) - \frac{2}{\delta^2}\int_{A_1\setminus I_\delta^+} f(u,t)\left(u-\frac{1}{4}\right)^2 du \geq m - 2J(t) - \frac{2}{\delta^2}\mathcal{S}(t).$$

Here we used $1 \leq \frac{1}{\delta^2}(u-1/4)^2$ for $u \in A_1 \setminus I_\delta^+$. This completes the first inequality of (18). The second inequality is an immediate consequence of this.

Estimate for $\mathcal{F}_2(t)$: Here we will use (18).

$$\mathcal{F}_2(t) = \int_{\mathcal{I}\setminus I_\delta^\pm} \int_{\mathcal{I}\setminus I_\delta^\pm} f(w,t) f(v,t) \left(M_w(v) - \frac{1}{4}\right)^2 \chi_{\hat{A}_1} dv dw$$

$$\leq C\int_{\mathcal{I}\setminus I_\delta^\pm} f(w,t)dw \int_{\mathcal{I}\setminus I_\delta^\pm} f(v,t)dv \leq C\left(\frac{\mathcal{S}(t)}{\delta^2} + J(t)\right)^2 \leq C(\mathcal{S}^2(t) + J^2(t)). \qquad (19)$$

Estimate for $\mathcal{F}_4(t)$: First we consider $\mathcal{F}_4(t)$, and after this $\mathcal{F}_3(t)$ will be estimated. If $v \in I_\delta^-$, then $M_w(v)$ is not contained in A_1, and we obtain

$$\mathcal{F}_4(t) = \int_{I_\delta^+} dv \int_{\mathcal{I}\setminus I_\delta^\pm} dw f(w,t) f(v,t) \left(M_w(v) - \frac{1}{4} \right)^2 \chi_{\hat{A}_1}$$

$$= \int_{I_\delta^+} dv \left(\int_{A_1\setminus I_\delta^+} dw \cdots + \int_{A_1^r\setminus I_\delta^-} dw \cdots + \int_{A_2} dw \cdots + \int_{A_2^r} dw \cdots \right),$$

where A_1^r and A_2^r are obtained by rotating A_1 and A_2 by the angle π. Using Young's inequality we get for arbitrarily small ε_0

$$\int_{I_\delta^+} dv \int_{A_1\setminus I_\delta^+} dw f(w,t) f(v,t) \left(M_w(v) - \frac{1}{4} \right)^2 \chi_{\hat{A}_1}$$

$$\leq (1 + \frac{1}{4\varepsilon_0}) \int_{I_\delta^+} dv \int_{A_1\setminus I_\delta^+} dw f(w,t) f(v,t) \left(v - \frac{1}{4} \right)^2$$

$$+ (1+\varepsilon_0) \int_{I_\delta^+} dv \int_{A_1\setminus I_\delta^+} dw f(w,t) f(v,t) V^2(w-v),$$

$$\leq C\mathcal{S}^2(t) + (1+\varepsilon_0) \int_{I_\delta^+} dv \int_{A_1\setminus I_\delta^+} dw f(w,t) f(v,t) a^2 (v-w)^2$$

$$\leq C\mathcal{S}^2(t) + a^2(1+\varepsilon_0) \int_{I_\delta^+} dv \int_{A_1\setminus I_\delta^+} dw f(w,t) f(v,t) \left((v-1/4)^2 + (w-1/4)^2 \right)$$

$$\leq C\mathcal{S}^2(t) + \frac{(1+\varepsilon_0)a^2 m}{2} \int_{A_1\setminus I_\delta^+} f(w,t)(w-1/4)^2 dw.$$

Here again we used $1 \leq C_\delta (w - 1/4)^2$, $w \in A_1 \setminus I_\delta^+$, Assumption 5.1, and $\int_{I_\delta^+} f(v,t) dv \leq \frac{m}{2}$. In the fourth inequality, the mixed terms vanished due to the symmetry of f and V. Next, using again symmetry, we obtain

$$\int_{I_\delta^+} dv \int_{A_1^r\setminus I_\delta^-} dw \cdots \leq C\mathcal{S}^2(t) + \frac{(1+\varepsilon_0)a^2 m}{2} \int_{A_1\setminus I_\delta^+} f(w,t)(w-1/4)^2 dw.$$

On the other hand, it is straightforward that

$$\int_{A_2}\int_{I_\delta^+}\cdots\,dvdw+\int_{A_2^r}\int_{I_\delta^+}\cdots\,dvdw\le CJ(t).$$

Summing up, we get

$$\mathcal{F}_4(t)\le CJ(t)+C\mathcal{S}^2(t)+(1+\varepsilon_0)ma^2\int_{A_1\setminus I_\delta^+}f(u,t)(u-1/4)^2du. \tag{20}$$

<u>Estimate for $\mathcal{F}_3(t)$:</u> Let $\bar{A}_2=A_2\cup A_2^r$. We have

$$\mathcal{F}_3(t)=\int_{\mathcal{I}\setminus I_\delta^\pm}dv\int_{I_\delta^\pm}dwf(w,t)f(v,t)\left(M_w(v)-\frac{1}{4}\right)^2\chi_{\hat{A}_1}$$

$$=\int_{A_1\setminus I_\delta^+}dv\int_{I_\delta^\pm}dw\cdots+\int_{\bar{A}_2}dv\int_{I_\delta^\pm}dw\cdots,$$

where we used $\chi_{\hat{A}_1}=0$ for $w\in I_\delta^\pm$ and $v\in A_1^r\setminus I_\delta^-$. The second term on the right hand side is bounded by $CJ(t)$, and therefore, it remains to estimate

$$\int_{A_1\setminus I_\delta^+}dv\int_{I_\delta^+}dw\cdots+\int_{A_1\setminus I_\delta^+}dv\int_{I_\delta^-}dw\cdots.$$

Both terms are equal due to the symmetry of V. Thus it is sufficient to estimate

$$\int_{A_1\setminus I_\delta^+}dv\int_{I_\delta^+}dwf(w,t)f(v,t)\left(M_w(v)-\frac{1}{4}\right)^2$$

$$=\int_{A_1\setminus I_\delta^+}dv\int_{I_\delta^+}dwf(w,t)f(v,t)\left((v-\frac{1}{4})-V\left((v-\frac{1}{4})-(w-\frac{1}{4})\right)\right)^2$$

$$=\int_{A_1\setminus I_\delta^+}dv\int_{I_\delta^+}dwf(w,t)f(v,t)\Big\{\left[(v-\frac{1}{4})-V(v-\frac{1}{4})\right]^2$$

$$-2V'\left(v-\frac{1}{4}\right)\left[\left(v-\frac{1}{4}\right)-V\left(v-\frac{1}{4}\right)\right](w-\frac{1}{4})+O\left((w-\frac{1}{4})^2\right)\Big\}$$

$$\le\frac{m}{2}(1-a)^2\int_{A_1\setminus I_\delta^+}f(v,t)\left(v-\frac{1}{4}\right)^2dv+C\mathcal{S}^2(t),$$

because the second of the three integral terms is zero. Finally

$$\mathcal{F}_3(t) \leq m(1-a)^2 \int_{A_1 \setminus I_\delta^+} f(v,t) \left(v - \frac{1}{4}\right)^2 dv + CJ(t) + C\mathcal{S}^2(t). \tag{21}$$

Summing up (17), (19), (20), and (21), we obtain

$$\frac{d}{dt}\mathcal{S}(t) \leq -m\mathcal{S}(t) + m(1+\varepsilon_0)\left((1-a)^2 + a^2\right)\mathcal{S}(t) + C(J(t) + \mathcal{S}^2(t)). \tag{22}$$

<u>Suppose now $u \in A_2$:</u> (The case for A_2^r is similar by symmetry)
Consider

$$\partial_t f(u,t) = -mf(u,t) + \int_{\mathcal{I}} dv \int_{\mathcal{I}} dw f(w,t) f(v,t) \delta_0(u - M_w(v))$$

$$= -mf(u,t) + \int_{\mathcal{I}} dv \int_{I_\delta^\pm} dw \cdots + \int_{\mathcal{I}} dv \int_{\mathcal{I}\setminus I_\delta^\pm} dw \cdots = -mf(u,t) + I_1(t) + I_2(t).$$

If $w \in I_\delta^\pm$ and $v \in \bar{A}_1 = A_1 \cup A_1^r$, then $M_w(v) \notin A_2$. Therefore

$$I_1(t) = \int_{\mathcal{I}\setminus \bar{A}_1} dv \int_{I_\delta^\pm} dw \cdots = \int_{\bar{A}_2} dv \int_{I_\delta^\pm} dw \cdots,$$

where $\bar{A}_2 = A_2 \cup A_2^r$. For fixed $w \in I_\delta^\pm$, we consider $M_w(v)$ as a function of v in A_2. Due to Assumption 5.1, (c), we can see that $M_w'(v) \geq 1 + C_0$ where $C_0 > 0$, thus, $M_w(v)$ is invertible. Let $M_w(v) = z$, so $v = M_w^{-1}(z)$. Using a change of variables, we get

$$I_1(t) = \int_{I_\delta^\pm} dw \int_{M_w(\bar{A}_2)} dz \frac{f(w) f\left(M_w^{-1}(z)\right)}{M_w'(M_w^{-1}(z))} \leq \frac{m}{1+C_0} J(t). \tag{23}$$

If $v \in I_\delta^+ \cup I_\delta^-$, then $M_w(v) \notin A_2$ and

$$I_2(t) = \int_{\mathcal{I}\setminus I_\delta^\pm} dv \int_{\mathcal{I}\setminus I_\delta^\pm} dw \cdots \leq C\left(\mathcal{S}^2(t) + J^2(t)\right). \tag{24}$$

Here we used similar computations as in (19). Summing up, we obtain

$$\partial_t f(u,t) \leq -mf(u,t) + \frac{m}{1+C_0} J(t) + C\left(\mathcal{S}^2(t) + J^2(t)\right), \qquad u \in A_2.$$

Since the above estimate is uniform for all $u \in A_2$, we have

$$\frac{d}{dt} J(t) \leq -\frac{C_0 m}{1+C_0} J(t) + C\left(\mathcal{S}^2(t) + J^2(t)\right). \tag{25}$$

With ε_0 chosen such that $\gamma = \Big(2(1+\varepsilon_0)a(1-a) - \varepsilon_0\Big)m > 0$, we finally obtain

$$\begin{pmatrix} \mathcal{S}'(t) \\ \\ J'(t) \end{pmatrix} = \begin{pmatrix} -\gamma & 0 \\ 0 & -\dfrac{C_0 m}{1+C_0} \end{pmatrix} \cdot \begin{pmatrix} \mathcal{S}(t) \\ \\ J(t) \end{pmatrix} + \begin{pmatrix} O\left(J(t) + \mathcal{S}^2(t)\right) \\ \\ O\left(\mathcal{S}^2(t) + J^2(t)\right) \end{pmatrix}.$$

Since $\mathcal{S}(0)$ and $J(0)$ are sufficiently small, we obtain that both S and J are exponentially decaying. This can be shown by arguments from ODE-theory. An additional linear order term in the errors, namely J, appears only in the first equation. In the second equation the error term is quadratic, and thus the error of J behaves like $\mathcal{S}^2$. So plugging in the second equation into the first one gives an error term of order S^2, which is much smaller than the linear term. This completes the proof. $\square$

5.2 The non-symmetric case

In this section we consider non-symmetric initial distributions with non-separated support. Most of the initial mass will be concentrated close to opposite positions, say $1/4$ and $-1/4$. We discuss the effect of an attracting and repulsive optimal reorientation. Thus the orientational angle satisfies (7) together with Assumption 5.4.

Assumption 5.4 *Suppose that the smooth, not necessarily symmetric, orientational angle V satisfies* (3)*.*

(a) Let $V \in C^2([0,1])$ and $V'(0) = A, V'(\frac{1}{2}) = B$ for $A, B > 0$.

(b) There exists an $L < \frac{1}{6}$ with $||V||_\infty < L$ such that in $[-\frac{L}{2} + \frac{1}{4}, \frac{L}{2} + \frac{1}{4}]$ we have $V'(\theta) \leq -C_0$ where $C_0 > 0$.

(c) There exist $0 < a < 1$, $0 < b < 1$ and $\theta_1 > -\frac{L}{2} + \frac{1}{4}$, $\theta_2 < \frac{L}{2} + \frac{1}{4}$ such that

$$a\theta \geq V(\theta) \geq b\theta \quad \textit{for} \quad \theta \in [0, \theta_1] \qquad \textit{and}$$

$$a(\theta - 1/2) \leq V(\theta) \leq b(\theta - 1/2) \quad \textit{for} \quad \theta \in [\theta_2, 1/2].$$

These assumptions imply the existence of $\sigma_0 > 0$ such that
(i) $a\theta \geq V(\theta) \geq b\theta$ in $\left[0, \frac{1}{4} - \frac{L}{2} + \sigma_0\right]$, and
$a\left(\theta - \frac{1}{2}\right) \leq V(\theta) \leq b\left(\theta - \frac{1}{2}\right)$ in $\left[\frac{1}{4} + \frac{L}{2} - \sigma_0, \frac{1}{2}\right]$.

(ii) $\|V\|_\infty < L - \sigma_0 < \frac{1}{6}$ in $\left[\frac{1}{4} - \frac{L}{2}, \frac{1}{4} + \frac{L}{2}\right]$

Figure 3 explains our assumption.

Next we specify the initial distribution of the cell bundles.

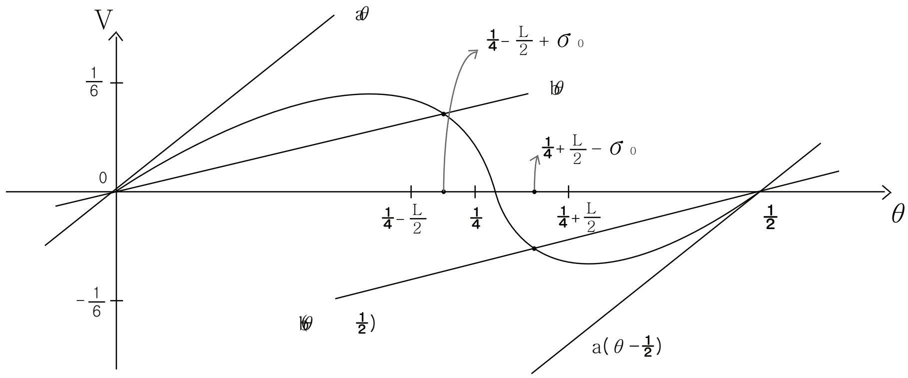

Figure 3: The orientational angle V for the non-symmetric case.

Assumption 5.5 *Let $I_\delta^+ = (1/4-\delta, 1/4+\delta)$ and $I_\delta^- = (-1/4-\delta, -1/4+\delta)$ for fixed $\delta > 0$. We suppose smooth and nonnegative initial conditions $f_0 : (-1/2, 1/2] \rightarrow \mathbb{R}$ with $\int_{\mathcal{I}} f_0(u)du = m$ and*

(a) There exists $\varepsilon_1 > 0$ such that

$$\int_{I_\delta^+} (u-\xi_1(0))^2 f_0(u)du + \int_{I_\delta^-} (u-\xi_2(0))^2 f_0(u)du \leq \varepsilon_1.$$

$$\text{where} \quad \xi_1(0) = \frac{\int_{A_1} u f_0(u)du}{\int_{I_\delta^+} f_0(u)du}, \qquad \xi_2(0) = \frac{\int_{A_1^r} u f_0(u)du}{\int_{I_\delta^-} f_0(u)du}.$$

(b) There exists $\epsilon_2 > 0$ such that $f_0(u) < \epsilon_2$ for $u \in \mathcal{I} \setminus (I_\delta^+ \cup I_\delta^-)$.

(c) There exists $\epsilon_3 > 0$ such that $\xi_1(0) \in I_{\epsilon_3}^+$ and $\xi_2(0) \in I_{\epsilon_3}^-$.

We assume that $\delta < \frac{\sigma_0}{2}$. Define

$$A_1 = \left(\frac{L-\sigma_0}{2}, \frac{1}{2} - \frac{(L-\sigma_0)}{2}\right), \qquad A_2 = \left[-\frac{(L-\sigma_0)}{2}, \frac{L-\sigma_0}{2}\right]$$

and A_1^r , A_2^r are their reflections with respect to the horizontal respectively the vertical axis.

$$A_1^r = \left(-\frac{1}{2} + \frac{L-\sigma_0}{2}, -\frac{(L-\sigma_0)}{2}\right), \qquad A_2^r \ = \left[\frac{1}{2} - \frac{(L-\sigma_0)}{2}, \frac{1}{2}\right] \cup \left[-\frac{1}{2}, -\frac{1}{2} + \frac{L-\sigma_0}{2}\right].$$

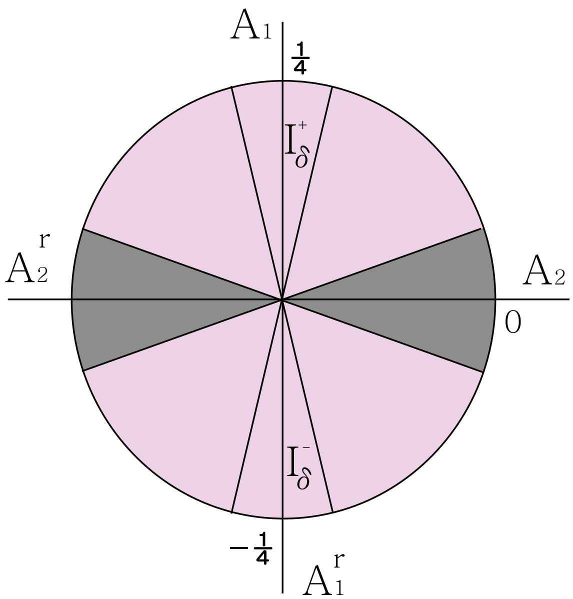

Figure 4: The regions A_1, A_1^r, and A_2, A_2^r, which are separated.

Next we introduce some suitable functionals for our analysis.

$$m_1(t) := \int_{I_\delta^+} f(u,t)du \quad , \quad m_2(t) := \int_{I_\delta^-} f(u,t)du \quad , \quad J(t) := \sup_{A_2 \cup A_2^r} f(u,t) \ ,$$

$$\xi_1(t) := \frac{\int_{A_1} uf(u,t)du}{m_1(t)} \quad , \quad \xi_2(t) := \frac{\int_{A_1^r} uf(u,t)du}{m_2(t)} \ ,$$

$$S_1(t) := \int_{A_1} (u - \xi_1(t))^2 f(u,t)du \quad , \quad S_2(t) := \int_{A_1^r} (u - \xi_2(t))^2 f(u,t)du.$$

The first momentum, denoted by ξ, is preserved over $\mathcal{I}$, i.e.

$$\xi := \int_{\mathcal{I}} uf(u,t)du = \int_{\mathcal{I}} uf(u,0)du. \tag{26}$$

Our main result for the non-symmetric case is

Theorem 5.6 *Let Assumptions 5.4 and 5.5 hold and $m_1(0), m_2(0) > 0$. Then we can define $0 < \epsilon_0 = \epsilon_0(m_1(0), m_2(0), V)$ such that for $\delta, \epsilon_1, \epsilon_2, \epsilon_3 < \epsilon_0$, there exists $\tilde{m}_1, \tilde{m}_2$ with*

$\tilde{m}_1 + \tilde{m}_2 = m$ and $\tilde{\xi}_1, \tilde{\xi}_2$ with $\tilde{\xi}_1 \in I^+_{\frac{\delta}{4}}, \tilde{\xi}_2 \in I^-_{\frac{\delta}{4}}$ such that $m_i(t), \xi_i(t)$ converge to $\tilde{m}_i, \tilde{\xi}_i$, for $t \to \infty$, $i = 1, 2$. Furthermore, $\mathcal{S}_i(t) \le e^{-Ct}\mathcal{S}_i(0)$ and

$$f \to \tilde{m}_1 \delta_{(u-\tilde{\xi}_1)} + \tilde{m}_2 \delta_{(u-\tilde{\xi}_2)} \quad , \quad \tilde{m}_1\tilde{\xi}_1 + \tilde{m}_2\tilde{\xi}_2 = m\xi \quad , \quad \left|\tilde{\xi}_2 - \tilde{\xi}_1\right| = \frac{1}{2} ,$$

where ξ is given as in (26).

Remarks:
Assumption 5.4 ensures the main properties we need for the interactions between particles to prove our theorem.

- The particles in region A_1 cannot jump to region A_1^r or the other way around, since the width of region A_2 is of order L, but the potential V, that measures the size of the jumps, is smaller than that.
- For all interactions between particles v in $I^+_\delta \cup I^-_\delta$ with particles w in $A_1 \cup A_1^r$ the inequalities for the potentials in (c), or the reflected ones, can be applied. Indeed, for these interactions either $v - w \in \left[-\frac{1}{4} + \frac{L}{2} - \sigma_0, \frac{1}{4} - \frac{L}{2} + \sigma_0\right]$ or $v - w \in \left[\frac{1}{4} + \frac{L}{2} - \sigma_0, 1 - \frac{1}{4} - \frac{L}{2} + \sigma_0\right]$.
- The particles in $I^+_\delta \cup I^-_\delta$ cannot reach the region $A_2 \cup A_2^r$. Since $L < \frac{1}{6}$, the width of the region A_1 (or A_1^r) is at least $\frac{1}{3}$, whereas the required jumps have to be at least of order $\frac{1}{6} - \delta > \frac{1}{6} - \sigma_0 > L - \sigma_0 > \|V\|_\infty$, which is not possible.

See figure 4 for illustration.

Proof. (of Theorem 5.6) Several lemmas and propositions have to be proved to get the final result. First we start with

Assumption 5.7 *Suppose first that $\xi_1(t) \in I^+_{\delta/4}$, $\xi_2(t) \in I^-_{\delta/4}$, and $m_1(t), m_2(t) \ge \nu_0 > 0$ for $0 < t \le t^*$.*

Later we will use a continuity argument. First we show a lemma that will be used several times in the following arguments.

Lemma 5.8

$$\int_{\mathcal{I}\setminus(\mathcal{I}^+_\delta \cup \mathcal{I}^-_\delta)} du f(u,t) \le C_\delta(S_1(t) + S_2(t)) + J(t).$$

Proof. As long as Assumption 5.7 is satisfied, we have

$$1 \le C_\delta(u - \xi_1(t))^2 \quad \text{for} \quad \xi_1(t) \in I^+_{\delta/4},\ u \in S_1 \setminus I^+_\delta. \tag{27}$$

$$1 \le C_\delta(u - \xi_2(t))^2 \quad \text{for} \quad \xi_2(t) \in I^-_{\delta/4},\ u \in S_1 \setminus I^-_\delta. \tag{28}$$

Therefore

$$\int_{\mathcal{I}\setminus(\mathcal{I}_\delta^+\cup\mathcal{I}_\delta^-)} du f(u,t) = \int_{A_1\setminus I_\delta^+} du f(u,t) + \int_{A_1^r\setminus I_\delta^-} du f(u,t) + \int_{A_2\cup A_2^r} du f(u,t)$$

$$\leq C_\delta \left[\int_{A_1\setminus I_\delta^+} du (u-\xi_1(t))^2 f(u,t) + \int_{A_1^r\setminus I_\delta^-} du (u-\xi_2(t))^2 f(u,t) \right] + J(t)$$

$$\leq C_\delta(S_1(t)+S_2(t)) + J(t).$$

□

Proposition 5.9

$$\frac{d}{dt}(m_i(t)) = O(J(t)+S_1(t)+S_2(t)) \quad \textit{for} \quad i=1,2 \tag{29}$$

Proof. Let $\chi_1 = \chi_1(v,w) = 1$ on the set $\{(v,w) \in \mathcal{I}\times\mathcal{I} : M_w(v) \in I_\delta^+\}$ and otherwise zero. Then

$$\frac{d}{dt} m_1(t) = -m \int_{I_\delta^+} f(u,t) du + \int_{I_\delta^+} du \int_{\mathcal{I}} dv \int_{\mathcal{I}} dw\, \delta_0(u - M_w(v)) f(w,t) f(v,t)$$

$$= -m m_1(t) + \int_{\mathcal{I}} dv \int_{\mathcal{I}} dw f(w,t) f(v,t) \chi_1$$

$$= -m m_1(t) + \int_{I_\delta^+} dv \int_{\mathcal{I}} dw f(w,t) f(v,t) \chi_1 + \int_{\mathcal{I}\setminus I_\delta^+} dv \int_{\mathcal{I}} dw f(w,t) f(v,t) \chi_1. \tag{30}$$

For the second term on the right hand side we have

$$\int_{I_\delta^+} dv \int_{\mathcal{I}} dw f(w,t) f(v,t) \chi_1 = \int_{I_\delta^+} dv \left(\int_{I_\delta^+} dw \cdots + \int_{I_\delta^-} dw \cdots + \int_{\mathcal{I}\setminus(I_\delta^+\cup I_\delta^-)} dw \cdots \right)$$

$$= m_1(t) m_1(t) + m_2(t) m_1(t) + O(J(t)+S_1(t)+S_2(t)).$$

Here we have used Lemma 5.8. Since $\chi_1 = 0$ if $v \in I_\delta^-$, the third term of the right hand side of (30) can be estimated as

$$\int_{\mathcal{I}\setminus I_\delta^+} dv \int_{\mathcal{I}} dw f(w,t) f(v,t) \chi_1 = \int_{\mathcal{I}} dw \int_{\mathcal{I}\setminus(I_\delta^+\cup I_\delta^-)} dv f(w,t) f(v,t) \chi_1$$

$$\le \int_{\mathcal{I}} f(w,t)dw \int_{\mathcal{I}\setminus(I_\delta^+\cup I_\delta^-)} f(v,t)dv \le mJ(t) + Cm(S_1(t)+S_2(t)).$$

Summing up, we get

$$\frac{d}{dt}m_1(t) = -m_1(t)(m - m_1(t) - m_2(t)) + O(J(t) + S_1(t) + S_2(t)). \tag{31}$$

Similarly one obtains

$$\frac{d}{dt}m_2(t) = -m_2(t)(m - m_1(t) - m_2(t)) + O(J(t) + S_1(t) + S_2(t)). \tag{32}$$

Since

$$m - m_1(t) - m_2(t) \quad = \quad \int_{S_1\setminus I_\delta^+\cup I_\delta^-} f(u,t)du \le C(S_1 + S_2 + J), \tag{33}$$

we immediately obtain (29) by Lemma 5.8. □

Proposition 5.10

$$\frac{d}{dt}\left(S_1(t)+S_2(t)\right) \le -K\left(S_1(t)+S_2(t)\right) + C\left(\xi_1(t) - \xi_2(t) - \frac{1}{2}\right)^2 + C(J + S_1^2 + S_2^2), \tag{34}$$

where $K > 0$ is an absolute constant depending on a, b, δ, and m.

Proof. We first consider $\frac{d}{dt}S_1(t)$. Let χ_1 denote the characteristic function with $\chi_1 = 1$ on $\{(w,v) \in \mathcal{I}\times\mathcal{I} : M_w(v) \in A_1\}$ and otherwise zero. Using $\int_{A_1}(u-\xi_1(t))f(u,t)du = 0$, we have

A. Estimate of $S_1(t)$:

$$\frac{d}{dt}S_1(t) = \int_{A_1}(u-\xi_1(t))^2\,\partial_t f(u,t)du - 2\frac{d\xi_1(t)}{dt}\int_{A_1}(u-\xi_1(t))f(u,t)du$$

$$= \int_{A_1}(u-\xi_1(t))^2\,\partial_t f(u,t)du$$

$$= -m\int_{A_1}(u-\xi_1(t))^2 f(u,t)du + \int_{\mathcal{I}} dv \int_{\mathcal{I}} dw f(w,t)f(v,t)\left(M_w(v)-\xi_1(t)\right)^2\chi_1$$

$$= -mS_1(t) + \int_{\mathcal{I}} dv \int_{\mathcal{I}} dw f(w,t)f(v,t)\left(M_w(v)-\xi_1(t)\right)^2\chi_1$$

$$= -mS_1(t) + \int_{A_1} dv \int_{\mathcal{I}} dw \cdots + \int_{\mathcal{I}\setminus A_1} dv \int_{\mathcal{I}} dw \cdots = -mS_1(t) + I_1(t) + I_2(t).$$

A.1. Estimate of $I_1(t)$:

$$I_1(t) = \int_{A_1} dv \left(\int_{A_1} dw \cdots + \int_{A_1^r} dw \cdots + \int_{A_2 \cup A_2^r} dw \cdots \right) = I_1^a(t) + I_1^b(t) + I_1^c(t).$$

Among these we estimate the first by symmetrization.

A.1.(i) Estimate of $I_1^a(t)$:

$$I_1^a(t) = \frac{1}{2} \int_{A_1} dv \int_{A_1} dw \big(v + V(w-v) - \xi_1(t)\big)^2 f(w) f(v)$$

$$+\frac{1}{2} \int_{A_1} dv \int_{A_1} dw \big(w + V(v-w) - \xi_1(t)\big)^2 f(v) f(w)$$

$$= \frac{1}{2} \int_{A_1} dv \int_{A_1} dw \Big[(v - \xi_1(t))^2 + 2(v - \xi_1(t))V(w-v) + (V(w-v))^2\Big] f(w,t) f(v,t)$$

$$+\frac{1}{2} \int_{A_1} dv \int_{A_1} dw \Big[(w - \xi_1(t))^2 + 2(w - \xi_1(t)V(v-w) + (V(v-w))^2\Big] f(w,t) f(v,t)$$

$$= S_1(t) \int_{A_1} dw f(w,t) - \int_{A_1} dv \int_{A_1} dw V(w-v) \big[(w-v) - V(w-v)\big] f(w,t) f(v,t).$$

The first term can be estimated via

$$\int_{A_1} dw f(w,t) = \int_{I_\delta^+} dw f(w,t) + \int_{A_1 \setminus I_\delta^+} dw f(w,t) = m_1(t) + O(S_1(t)), \tag{35}$$

where we have used Lemma 5.8. Thus

$$I_1^a(t) = m_1(t) S_1(t) + O(S_1^2(t)) - \int_{A_1} dv \int_{A_1} dw Q(w-v) f(w,t) f(v,t),$$

with $Q(\theta) = V(\theta)(\theta - V(\theta))$. We split

$$\int_{A_1} \int_{A_1} dv dw\, Q(w-v) f(w,t) f(v,t) = \int_{I_\delta^+} \int_{I_\delta^+} dv dw\, Q(w-v) f(w,t) f(v,t)$$

$$+\int_{A_1\setminus I_\delta^+}\int_{I_\delta^+}...+\int_{I_\delta^+}\int_{A_1\setminus I_\delta^+}...+\int_{A_1\setminus I_\delta^+}\int_{A_1\setminus I_\delta^+}...=Q_1(t)+Q_2(t)+Q_3(t)+Q_4(t).$$

With (27) used for v and w we obtain

$$|Q_4(t)|\le C(S_1(t))^2. \tag{36}$$

On the other hand, for the domains of integration used for Q_1, Q_2 and Q_3 we have that $|w-v|\le\theta_1$. Therefore, since $Q(\theta)$ is even and V can be estimated in $[0,\theta_1]$ as given in Assumption 5.4, (c), we obtain

$$Q(\theta)\ge b(1-a)\theta^2 \quad \text{for} \quad \theta\in[-\theta_1,\theta_1].$$

Thus

$$\int_{A_1}dv\int_{A_1}dwQ(w-v)f(w,t)f(v,t)\ge b(1-a)\Big[\int_{I_\delta^+}\int_{I_\delta^+}dvdw(w-v)^2f(w,t)f(v,t)$$

$$+\int_{A_1\setminus I_\delta^+}\int_{I_\delta^+}dvdw(w-v)^2f(w,t)f(v,t)+\int_{I_\delta^+}\int_{A_1\setminus I_\delta^+}dvdw(w-v)^2f(w)f(v)\Big]$$

$$+\int_{A_1\setminus I_\delta^+}\int_{A_1\setminus I_\delta^+}dvdw\Big[Q(w-v)-b(1-a)(w-v)^2+b(1-a)(w-v)^2\Big]f(w,t)f(v,t)$$

$$\ge b(1-a)\int_{A_1}\int_{A_1}dvdw(w-v)^2f(w,t)f(v,t)-C_\delta S_1^2(t). \tag{37}$$

For the first term of the right hand side we estimate again by expansion $(w-v)^2=(w-\xi_1(t)-(v-\xi_1(t)))^2$, canceling the cross terms and using (35).

$$b(1-a)\int_{A_1}\int_{A_1}dvdw(w-v)^2f(w,t)f(v,t)=2b(1-a)(m_1(t)+O(S_1(t)))S_1(t).$$

Thus

$$I_1^a(t)\le m_1(t)S_1(t)-2b(1-a)m_1(t)S_1(t)+CS_1^2(t).$$

A.1.(ii) Estimate of $I_1^c(t)$: It is direct that

$$I_1^c(t)\le\int_{A_2\cup A_2^r}f(w,t)dw\int_{A_1}f(v,t)dv\le CJ(t).$$

The estimate for $I_1^b(t)$ is related to an analogous estimate appearing for $\frac{d}{dt}S_2(t)$. We will deal with this later. So first we look at

A.2. Estimate for $I_2(t)$:

$$I_2(t) = \int_{\mathcal{I}\setminus A_1} dv \int_{\mathcal{I}} dw f(w,t) f(v,t) (M_w(v) - \xi_1(t))^2 \chi_1$$

$$= \int_{A_2 \cup A_2^r} dv \int_{\mathcal{I}} dw f(w,t) f(v,t) (M_w(v) - \xi_1(t))^2 \chi_1 \leq CJ(t)$$

due to the definition of the characteristic function and since $||V||_\infty < L$ as given in Assumption 5.4 (b). Combining all estimates we obtain

$$\frac{d}{dt} S_1(t) \leq -mS_1(t) + m_1(t)S_1(t) - 2b(1-a)m_1(t)S_1(t) + I_1^b(t) + CJ(t) + CS_1^2(t).$$

Next we consider

B. Estimate of $S_2(t)$:

$$\frac{d}{dt} S_2(t) = -mS_2(t) + \int_{\mathcal{I}} dw \int_{\mathcal{I}} dv f(w,t) f(v,t) \left(M_v(w) - \xi_2(t)\right)^2 \chi_2,$$

$$= -mS_2(t) + \int_{A_1^r} dv \int_{\mathcal{I}} dw \cdots + \int_{\mathcal{I}\setminus A_1^r} dv \int_{\mathcal{I}} dw \cdots = -mS_2(t) + \tilde{I}_1(t) + \tilde{I}_2(t).$$

Here $\chi_2 = \chi_2(v,w) = 1$ on the set $\{(w,v) \in \mathcal{I} \times \mathcal{I} : M_w(v) \in A_1^r\}$ and otherwise zero. As before we split

$$\tilde{I}_1(t) = \int_{A_1^r} dv \left(\int_{A_1^r} dw \cdots + \int_{A_1} dw \cdots + \int_{A_2 \cup A_2^r} dw \cdots \right) = \tilde{I}_1^a(t) + \tilde{I}_1^b(t) + \tilde{I}_1^c(t).$$

Due to the symmetry of the problem we obtain similarly as before

$$\tilde{I}_1^a(t) \leq m_2(t)S_2(t) - 2b(1-a)m_2(t)S_2(t) + CS_2^2(t),$$

$$\tilde{I}_1^c(t) \leq CJ(t) \quad \text{and} \quad \tilde{I}_2 \leq CJ(t)$$

since $\chi_2 = 0$ if $v \in A_1$. Thus

$$\frac{d}{dt}(S_1(t) + S_2(t)) \leq -m(S_1(t) + S_2(t)) + m_1(t)S_1(t) + m_2(t)S_2(t)$$

$$-2b(1-a)(m_1(t)S_1(t) + m_2(t)S_2(t)) + I_1^b(t) + \tilde{I}_1^b(t) + CJ(t) + C(S_1^2(t) + S_2^2(t)).$$

AB. (iii) Estimate of $I_1^b(t)$ and $\tilde{I}_1^b(t)$:

Exchanging the roles of v and w in the second term in the expressions for $I_1^b(t)$ and $\tilde{I}_1^b(t)$ we obtain

$$I_1^b(t) + \tilde{I}_1^b(t) \leq \int_{A_1} dv \int_{A_1^r} dw f(v,t) f(w,t) \Big[(v + V(w-v) - \xi_1(t))^2 + (w + V(v-w) - \xi_2(t))^2\Big]$$

$$= \int_{A_1} dv \int_{A_1^r} dw f(v,t) f(w,t) \Big[(v - \xi_1(t))^2 + (w - \xi(t))^2$$

$$+2\Big((v-w) - (\xi_1(t) - \xi_2(t))\Big) V(w-v) + 2(V(w-v))^2\Big]$$

$$= m_2(t) S_1(t) + m_1(t) S_2(t) + O(S_1^2(t) + S_2^2(t))$$

$$-2 \int_{A_1} dv \int_{A_1^r} dw f(v,t) f(w,t) \left[(w - v + 1/2) - V(w-v)\right] V(w-v)$$

$$+2(\xi_2(t) - \xi_1(t) + 1/2) \int_{A_1} dv \int_{A_1^r} dw f(v,t) f(w,t) V(w-v)$$

$$= m_2(t) S_1(t) + m_1(t) S_2(t) + O(S_1^2(t) + S_2^2(t)) + K_1(t) + K_2(t).$$

For the first terms we have assumed (35).

Estimate of $K_2(t)$:

The assumptions on the potential V imply $|V(w-v)| = |V(v-w)| \leq C|w - v + 1/2| \leq C(|w - \xi_2(t)| + |v - \xi_1(t)| + |\xi_2(t) - \xi_1(t) + 1/2|)$. With this and the Cauchy-Schwarz inequality we get

$$K_2(t) \leq C\Big(\, |\xi_2(t) - \xi_1(t) + 1/2| + \sqrt{S_1(t)} + \sqrt{S_2(t)}\Big)\, |\xi_2(t) - \xi_1(t) + 1/2|\, .$$

Using Young's inequality we obtain

$$K_2(t) \leq \varepsilon_0 (S_1(t) + S_2(t)) + C(\varepsilon_0) |\xi_2(t) - \xi_1(t) + 1/2|^2.$$

Estimate of $K_1(t)$:

For $K_1(t)$ we first split the domains of integration as usual

$$\int_{A_1} dv \int_{A_1^r} dw f(v,t) f(w,t) \Big[(w - v + 1/2) - V(w-v)\Big] V(w-v)$$

$$= \int_{I_\delta^+} dv \int_{I_\delta^-} dw \cdots + \int_{A_1 \setminus I_\delta^+} dv \int_{I_\delta^-} dw \cdots + \int_{I_\delta^+} dv \int_{A_1^r \setminus I_\delta^-} dw \cdots + \int_{A_1 \setminus I_\delta^+} dv \int_{A_1^r \setminus I_\delta^-} dw \cdots$$

$$= K_2^a(t) + K_2^b(t) + K_2^c(t) + K_2^d(t).$$

We start with the last term

$$|K_2^d(t)| \leq C\left(S_1(t)S_2(t)\right) \leq C\left(S_1^2(t) + S_2^2(t)\right).$$

All other terms are positive because of Assumption 5.4 and the 1-periodicity of V

$$b(\theta + 1/2) \geq V(\theta) \geq a(\theta + 1/2) \quad \text{ in } \theta \in [-3/4 + L/2 - \sigma_0, -1/2],$$

$$b(\theta + 1/2) \leq V(\theta) \leq a(\theta + 1/2) \quad \text{ in } \theta \in [-1/2, -1/4 - L/2 + \sigma_0].$$

Thus $\Big((\theta + 1/2) - V(\theta)\Big)V(\theta) \geq 0$. So

$$I_1^b(t) + \tilde{I}_1^b(t) \leq m_2(t)S_1(t) + m_1(t)S_2(t) + O(S_1^2(t) + S_2^2(t))$$

$$+\varepsilon(S_1(t) + S_2(t)) + C_{\varepsilon_0}|\xi_2(t) - \xi_1(t) + 1/2|^2.$$

Finally, due to several cancelations, we obtain

$$\frac{d}{dt}(S_1(t) + S_2(t)) \leq -2b(1-a)(m_1(t)S_1(t) + m_2(t)S_2(t)) + \varepsilon_0(S_1(t) + S_2(t))$$

$$+C_{\varepsilon_0}|\xi_2(t) - \xi_1(t) + 1/2|^2 + C\left(J(t) + S_1^2(t) + S_2^2(t)\right).$$

Choosing ε_0 sufficiently small, we get the final estimate (34). □

Proposition 5.11

$$\begin{aligned} \frac{d}{dt}\left(\xi_1(t) - \xi_2(t) - 1/2\right) &= -B(m_1(t) + m_2(t))\Big(\xi_1(t) - \xi_2(t) - 1/2\Big) \\ &\quad + O\Big(S_1(t) + S_2(t) + J(t) + (\xi_1(t) - \xi_2(t) - 1/2)^2\Big). \end{aligned} \qquad (38)$$

Proof. We first note that

$$\frac{d}{dt}\xi_1(t) = \frac{\int_{A_1} u\partial_t f(u,t)du}{m_1(t)} - \frac{m_1'(t)\int_{A_1} uf(u,t)du}{m_1^2(t)}.$$

For the first term on the right hand side of this equation we have

$$\frac{1}{m_1(t)}\int_{A_1} u\partial_t f(u,t)du = -m\xi_1(t) + \frac{1}{m_1(t)}\int_{\mathcal{I}} dv \int_{\mathcal{I}} dw M_w(v) f(w,t) f(v,t)\chi_1$$

$$= -m\xi_1(t) + \frac{1}{m_1(t)}\int_{A_1} dv \int_{I_\delta^+\cup I_\delta^-} dw M_w(v) f(v,t) f(w,t)\chi_1$$

$$+\frac{1}{m_1(t)}\int\int_{(\mathcal{I}\times\mathcal{I})\backslash(A_1\times(I_\delta^+\cup I_\delta^-))} dv dw M_w(v) f(v,t) f(w,t)\chi_1$$

$$= -m\xi_1(t) + K_1^a(t) + K_1^b(t).$$

Here, as before, $\chi_1 = 1$ on $\{(w,v)\in\mathcal{I}\times\mathcal{I} : M_w(v)\in A_1\}$ and zero otherwise. Using the fact that $\chi_1 = 0$ if $v\in A_1^r$, we obtain

$$K_1^b(t) = \frac{1}{m_1(t)}\int\int_{((\mathcal{I}\backslash A_1^r)\times\mathcal{I})\backslash(A_1\times(I_\delta^+\cup I_\delta^-))} dv dw M_w(v) f(v,t) f(w,t)\chi_1$$

$$\leq O(J(t) + S_1(t) + S_2(t)).$$

For the inequality we have used (27). Similarly, we get

$$\frac{d}{dt}\xi_2(t) = \frac{\int_{A_1^r} u\partial_t f(u,t)du}{m_2(t)} - \frac{m_2'(t)\int_{A_1^r} uf(u,t)du}{m_2^2(t)}$$

$$= -m\xi_2 + \frac{1}{m_2}\int_{A_1^r} dv \int_{I_\delta^+\cup I_\delta^-} dw M_w(v) f(v,t) f(w,t)\chi_2$$

$$+\frac{1}{m_2(t)}\int\int_{(\mathcal{I}\times\mathcal{I})\backslash(A_1^r\times(I_\delta^+\cup I_\delta^-))} dv dw M_w(v) f(v,t) f(w,t)\chi_2$$

$$= -m\xi_2 + K_2^a(t) + K_2^b(t) - \frac{m_2'(t)}{m_2(t)}\xi_2(t).$$

Thus

$$\frac{d}{dt}(\xi_1(t) - \xi_2(t) - 1/2) = -m(\xi_1(t) - \xi_2(t)) + K_1^a(t) - K_2^a(t) + K_1^b(t) - K_2^b(t)$$

$$-\frac{m_1'(t)}{m_1(t)}\xi_1(t) + \frac{m_2'(t)}{m_2(t)}\xi_2(t).$$

Arguing like for $K_1^b(t)$ we have

$$K_2^b(t) \leq O(J(t) + S_1(t) + S_2(t))$$

and by (31), (32), (33) we obtain

$$\left|\frac{m_1'(t)}{m_1(t)}\xi_1(t) - \frac{m_2'(t)}{m_2(t)}\xi_2(t)\right| \leq O(J(t) + S_1(t) + S_2(t)).$$

Next we estimate

$$K_1^a(t) - K_2^a(t) = \frac{1}{m_1(t)}\int_{A_1} dv \int_{I_\delta^+ \cup I_\delta^-} dw\,(v + V(w-v))\, f(v,t) f(w,t)$$

$$-\frac{1}{m_2(t)}\int_{A_1^r} dv \int_{I_\delta^+ \cup I_\delta^-} dw\,(v + V(w-v))\, f(v,t) f(w,t).$$

Here we dropped χ_1, χ_2, since they are both equal to one in the regions of integration.

$$\begin{aligned} K_1^a(t) - K_2^a(t) &= (m_1(t) + m_2(t))(\xi_1(t) - \xi_2(t)) + \frac{1}{m_1}\int_{A_1} dv \int_{I_\delta^+} dw V(w-v) f(w,t) f(v,t) \\ &+ \frac{1}{m_1}\int_{A_1} dv \int_{I_\delta^-} dw ... - \frac{1}{m_2}\int_{A_1^r} dv \int_{I_\delta^+} dw ... - \frac{1}{m_2}\int_{A_1^r} dv \int_{I_\delta^-} dw .... \end{aligned} \quad (39)$$

The first integral on the right hand side of (39) can be estimated using Assumption 5.4, (a)

$$\frac{1}{m_1}\int_{I_\delta^+} dv \int_{I_\delta^+} dw A(w-v) f(w,t) f(v,t) + \frac{1}{m_1}\int_{A_1 \setminus I_\delta^+} dv \int_{I_\delta^+} dw A(w-v) f(w,t) f(v,t)$$

$$+\frac{1}{m_1}\int_{A_1} dv \int_{I_\delta^+} dw O((v-w))^2 f(w,t) f(v,t) \leq O(S_1(t)).$$

Here we have used that the first term is zero by symmetry. For the second term we estimated $A(w-v) < 1$ and used (27). For the third integral we used $v - w = v - \xi_1(t) + \xi_1(t) - w$ and expanded. For the last integral in (39) the argumentation is the same as the one above. For the second integral in (39) we approximate

$$V(w-v) = B(w - v + \frac{1}{2}) + O\Big((w - v + \frac{1}{2})^2\Big)$$

and for the third integral we use

$$V(w-v) = B(w - v - \frac{1}{2}) + O\Big((w - v - \frac{1}{2})^2\Big).$$

Therefore the sum of these two integrals equals

$$\frac{B}{m_1(t)}\int_{A_1}dv\int_{I_\delta^-}dw\Big(w-v+\frac{1}{2}\Big)f(w,t)f(v,t)-\frac{B}{m_2(t)}\int_{A_1^r}dv\int_{I_\delta^+}dw\Big(w-v-\frac{1}{2}\Big)f(w,t)f(v,t)$$

$$+\int_{A_1}dv\int_{I_\delta^-}dw\;O\Big((w-v+\frac{1}{2})^2\Big)f(w,t)f(v,t)+\int_{A_1^r}dv\int_{I_\delta^+}dw\;O\Big((w-v-\frac{1}{2})^2\Big)f(w,t)f(v,t)$$

$$=B\Big(\frac{m_1(t)m_2(t)\xi_2(t)}{m_1(t)}-\frac{m_1(t)m_2(t)\xi_1(t)}{m_1(t)}+\frac{m_1(t)m_2(t)}{2m_1(t)}\Big)$$

$$-B\Big(\frac{m_1(t)m_2(t)\xi_1(t)}{m_2(t)}-\frac{m_1(t)m_2(t)\xi_2(t)}{m_2(t)}-\frac{m_1(t)m_2(t)}{2m_2(t)}\Big)+O\Big(S_1(t)+S_2(t)+(\xi_2(t)-\xi_1(t)+\frac{1}{2})^2\Big)$$

$$=B(m_1(t)+m_2(t))(\xi_2(t)-\xi_1(t)+\frac{1}{2})+O\Big(S_1(t)+S_2(t)+(\xi_2(t)-\xi_1(t)+\frac{1}{2})^2\Big).$$

Here we have used that $\int_{A_1}G(u,t)du=\int_{I_\delta^+}G(u,t)du+O(S_1(t)+S_2(t))$ for $G(u,t)=uf(u,t)$ and $G(u,t)=f(u,t)$. The errors in the quadratic terms were estimated by expansions like $(w-v+\frac{1}{2})^2=(w-\xi_2(t)-(v-\xi_1(t))+(\xi_2(t)-\xi_1(t)+\frac{1}{2}))^2$ and similar equations. So we finally obtain

$$\frac{d}{dt}\left(\xi_1(t)-\xi_2(t)-\frac{1}{2}\right)=-m(\xi_1(t)-\xi_2(t))+(m_1(t)+m_2(t))(\xi_1(t)-\xi_2(t))$$

$$-B(m_1(t)+m_2(t))\left(\xi_1(t)-\xi_2(t)-\frac{1}{2}\right)+O\Big(S_1(t)+S_2(t)+J(t)+(\xi_1(t)-\xi_2(t)-\frac{1}{2})^2\Big)$$

$$=-B(m_1(t)+m_2(t))\left(\xi_1(t)-\xi_2(t)-\frac{1}{2}\right)+O\Big(S_1(t)+S_2(t)+J(t)+(\xi_1(t)-\xi_2(t)-\frac{1}{2})^2\Big),$$

where we used (33). This completes the proof. □

Proposition 5.12

$$\left|\frac{d}{dt}\left(m_1(t)\xi_1(t)+m_2(t)\xi_2(t)\right)\right|\le C\left(J(t)+S_1(t)+S_2(t)\right). \qquad (40)$$

Proof. Direct calculations show that

$$\frac{d}{dt}(m_1(t)\xi_1(t)+m_2(t)\xi_2(t)) = -m\Big(m_1(t)\xi_1(t)+m_2(t)\xi_2(t)\Big)$$

$$+\int_{\mathcal{I}} dv \int_{\mathcal{I}} dw f(w,t) f(v,t) M_w(v)\chi_1 + \int_{\mathcal{I}} dv \int_{\mathcal{I}} dw f(w,t) f(v,t) M_w(v)\chi_2$$

$$= -m\Big(m_1(t)\xi_1(t)+m_2(t)\xi_2(t)\Big) + I_1(t) + I_2(t).$$

Due to the definition of χ_1 we can split

$$I_1(t) = \int_{I_\delta^+} dv \int_{\mathcal{I}} dw f(w,t) f(v,t) M_w(v)\chi_1 + \int_{\mathcal{I}\setminus(I_\delta^+\cup I_\delta^-)} dv \int_{\mathcal{I}} dw f(w,t) f(v,t) M_w(v)\chi_1$$

$$= \int_{I_\delta^+} dv \left(\int_{I_\delta^+} dw \cdots + \int_{I_\delta^-} dw \cdots + \int_{\mathcal{I}\setminus(I_\delta^+\cup I_\delta^-)} dw \cdots \right) + \int_{\mathcal{I}\setminus(I_\delta^+\cup I_\delta^-)} dv \int_{\mathcal{I}} dw \cdots .$$

The third and fourth term are estimated by $C(J(t)+S_1(t)+S_2(t))$, and therefore,

$$I_1(t) \le \int_{I_\delta^+} dv \int_{I_\delta^+} dw f(w,t) f(v,t) M_w(v) + \int_{I_\delta^+} dv \int_{I_\delta^-} dw f(w,t) f(v,t) M_w(v)$$

$$+O(J(t)+S_1(t)+S_2(t))$$

$$= m_1(t) m_1(t)\xi_1(t) + m_1(t) m_2(t)\xi_1(t) + O(J(t)+S_1(t)+S_2(t))$$

$$+\int_{I_\delta^+} dv \int_{I_\delta^-} dw f(w,t) f(v,t) V(w-v).$$

Due to the symmetry of $V(\theta)$ the respective $\int_{I_\delta^+}\int_{I_\delta^+}$ integration disappears.
For $I_2(t)$ we have by similar arguments

$$I_2(t) \le m_2(t) m_2(t)\xi_2(t) + m_1(t) m_2(t)\xi_2(t) + C(J(t)+S_1(t)+S_2(t))$$

$$+\int_{I_\delta^-} dv \int_{I_\delta^+} dw f(w,t) f(v,t) V(w-v).$$

Adding up, the integral terms cancel and we obtain

$$I_1(t)+I_2(t) \le (m_1(t)+m_2(t))(m_1(t)\xi_1(t)+m_2(t)\xi_2(t)) + O(J(t)+S_1(t)+S_2(t))$$

and thus derive the estimate (40) of our proposition. □

Proposition 5.13

$$\frac{d}{dt}J(t) \leq -\frac{C_0 m}{1+C_0}J(t) + C(S_1^2(t) + S_2^2(t) + J^2(t)). \tag{41}$$

Proof. For $u \in A_2 \cup A_2^r$ we consider

$$\partial_t f(u,t) = -mf(u,t) + \int_{\mathcal{I}} dv \int_{\mathcal{I}} dw f(w,t) f(v,t) G_0(u - M_w(v))$$

$$= -mf(u,t) + \int_{\mathcal{I}} dv \int_{I_\delta^+ \cup I_\delta^-} dw \cdots + \int_{\mathcal{I}} dv \int_{\mathcal{I}\setminus I_\delta^+ \cup I_\delta^-} dw \cdots$$

$$= -mf(u,t) + I_1(t) + I_2(t).$$

If $w \in I_\delta^+ \cup I_\delta^-$ and $v \in \bar{A}_1 = A_1 \cup A_1^r$, then $M_w(v) \notin \bar{A}_2 = A_2 \cup A_2^r$. Therefore

$$I_1(t) = \int_{\mathcal{I}\setminus\bar{A}_1} dv \int_{I_\delta^+ \cup I_\delta^-} dw \cdots = \int_{\bar{A}_2} dv \int_{I_\delta^+ \cup I_\delta^-} dw \cdots .$$

For fixed $w \in I_\delta^+ \cup I_\delta^-$, we consider $M_w(v)$ as a function of v in A_2. Due to Assumption 5.4, (b), we can see that $M_w'(v) \geq 1 + C_0$ where $C_0 > 0$, thus, $M_w(v)$ is invertible. Let $M_w(v) = z$, so $v = M_w^{-1}(z)$. Using a change of variables, we get

$$I_1(t) = \int_{I_\delta^+ \cup I_\delta^-} dw \int_{M_w(A_2)} dz \frac{f(w) f\left(M_w^{-1}(z)\right)}{M_w'(M_w^{-1}(z))} \leq \frac{m}{1+C_0}J(t). \tag{42}$$

If $v \in I_\delta^+ \cup I_\delta^-$, then $M_w(v) \notin A_2$, and since $||V||_\infty < L < \frac{1}{6}$

$$I_2(t) \quad = \quad \int_{\mathcal{I}\setminus I_\delta^+ \cup I_\delta^-} dv \int_{\mathcal{I}\setminus I_\delta^+ \cup I_\delta^-} dw \cdots \leq C(S_1^2(t) + S_2^2(t) + J^2(t)). \tag{43}$$

Here we used similar computations as in (19). Summing (42) and (43), we obtain

$$\partial_t f(u,t) \leq -mf(u,t) + \frac{m}{1+C_0}J(t) + C(S_1^2(t) + S_2^2(t) + J^2(t)), \qquad u \in \bar{A}_2.$$

Since the above estimate is uniform for all $u \in \bar{A}_2$, we have the estimate (41). □

Summary of estimates: So far we have

$$\frac{d}{dt}J(t) \leq -\lambda J(t) + C(S_1^2(t) + S_2^2(t) + J^2(t)),$$

$$\frac{d}{dt}(S_1(t) + S_2(t)) \leq -2\lambda(S_1(t) + S_2(t)) + C\Big((\xi_1(t) - \xi_2(t) - \frac{1}{2})^2 + J(t) + S_1^2(t) + S_2^2(t)\Big),$$

$$\frac{d}{dt}\Big|\xi_1(t) - \xi_2(t) - \frac{1}{2}\Big| \le -4\lambda\Big|\xi_1(t) - \xi_2(t) - \frac{1}{2}\Big| + C\Big(J(t) + S_1(t) + S_2(t) + (\xi_1(t) - \xi_2(t) - \frac{1}{2})^2\Big),$$

$$|\frac{d}{dt}m_1(t)| + |\frac{d}{dt}m_2(t)| \le C(J(t) + S_1(t) + S_2(t)), \tag{44}$$

$$|\frac{d}{dt}(m_1(t)\xi_1(t) + m_2(t)\xi_2(t))| \le C(J(t) + S_1(t) + S_2(t)). \tag{45}$$

And the initial data for our argument are

$$m_1(0) \ge 4\nu_0 \quad , \quad \mu_2(0) \ge 4\nu_0 \quad , \quad J(0) \le \mu_1,$$

$$S_1(0) + S_2(0) \le \mu_2 \quad , \quad |\xi_1(0) - 1/4| \le \frac{\mu_3}{2} \quad , \quad |\xi_2(0) + 1/4| \le \frac{\mu_3}{2}\,.$$

So $\lambda = \lambda(m, \nu_0, V(\cdot))$ is independent of δ, whereas $C = C(m, V(.), \delta, \mu_0)$ depends on δ. Let t^* be maximally chosen such that

$$J(t) \le 4\mu_1 \exp(-\lambda t) \quad , \quad S_1(t) + S_2(t) \le 4\mu_2 \exp(-\lambda t)$$

$$|\xi_1(t) - \xi_2(t) - 1/2| \le 4\mu_3 \exp(-\lambda t)$$

$$\xi_1(t) \in I^+_{\frac{\delta}{4}} \quad , \quad \xi_2(t) \in I^-_{\frac{\delta}{4}} \quad , \quad m_1(t) \ge \nu_0 \quad , \quad m_2(t) \ge \nu_0 \quad \text{for} \quad t \le t^*\,.$$

Local Stability Result
For $\mu_1 = \mu_2^2, \mu_2 = \mu_3^2$ there exists $\mu_0 = \mu_0(\delta, m, \nu_0, V(.))$ such that if $\mu_3 \le \mu_0$ then $t^* = \infty$.

Proof: We integrate the inequalities one after the other. The constant C will change from line to line as usual. Then we have

$$J(t) \le \mu_1 \exp(-\lambda t) + C\int_0^t \exp(-\lambda(t-s))(\mu_1^2 + \mu_2^2)\exp(-2\lambda s)ds$$

$$\le (\mu_1 + C(\mu_1^2 + \mu_2^2))\exp(-\lambda t) \le 2\mu_1 \exp(-\lambda t),$$

$$S_1(t) + S_2(t) \le \mu_2 \exp(-2\lambda t)$$

$$+C\int_0^t \exp(-2\lambda(t-s))\left[\mu_3^2 \exp(-2\lambda s) + \mu_2^2 \exp(-2\lambda s) + \mu_1 \exp(-\lambda s)\right]ds$$

$$\leq (\mu_2 + C(\mu_2^2 + \mu_3^2)) \exp(-2\lambda t) + C\mu_1 \exp(-\lambda t) \leq (\mu_2 + C(\mu_1 + \mu_2^2 + \mu_3^2)) \exp(-\lambda t),$$

$$\left|\xi_1(t) - \xi_2(t) - \frac{1}{2}\right| \leq \mu_3 \exp(-4\lambda t)$$

$$+C \int_0^t \exp(-4\lambda(t-s)) \big[4\mu_2 \exp(-\lambda 2) + \mu_1 \exp(-\lambda s) + \mu_3^2 \exp(-2\lambda s)\big]$$

$$\leq \big(\mu_3 + C(\mu_1 + \mu_2 + \mu_3^2)\big) \exp(-\lambda t).$$

Since $\mu_1 = \mu_3^4$ and $\mu_2 = \mu_3^2$, we obtain for small enough μ_3

$$J(t) \leq 2\mu_3^4 \exp(-\lambda t) < 4\mu_3^4 \exp(-\lambda t),$$

$$S_1(t) + S_2(t) \leq 2\mu_3^2 \exp(-\lambda t) < 4\mu_3^2 \exp(-\lambda t),$$

$$|\xi_1(t) - \xi_2(t) - 1/2| \leq 2\mu_3 \exp(-\lambda t) < 4\mu_3 \exp(-\lambda t). \tag{46}$$

From the previous estimates for the masses (44) we obtain

$$|m_1(t) - m_1(0)| + |m_2(t) - m_2(0)| \leq C\mu_3^2. \tag{47}$$

Due to our assumptions on the initial data this gives $m_1(t) \geq \nu_0$ and $m_2(t) \geq \nu_0$ for sufficiently small μ_3. Integrating (45) we have

$$|m_1(t)\xi_1(t) - m_1(0)\xi_1(0) + m_2(t)\xi_2(t) - m_2(0)\xi_2(0)| \leq C\mu_3.$$

Approximating $m_i(t)\xi_i(t)$ by $m_i(0)\xi_i(t)$ for $i = 1, 2$ in this equation and estimating the resulting error term by using $|\xi_1(0) - 1/4| \leq \mu_3/2$ and $|\xi_2(0) + 1/4| \leq \mu_3/2$ as well as (47), we obtain

$$|m_1(0)(\xi_1(t) - 1/4) + m_2(0)(\xi_2(t) + 1/4)| \leq C\mu_3\ .$$

Combining this inequality with (46), we end up with

$$|\xi_1(t) - 1/4| + |\xi_2(t) + 1/4| \leq C\mu_3\ .$$

For μ_3 sufficiently small, which means choosing $\varepsilon_1, \varepsilon_2, \varepsilon_3$ sufficiently small, we obtain $\xi_1(t) \in I^+_{\frac{\delta}{8}}$ and $\xi_2(t) \in I^-_{\frac{\delta}{8}}$. Therefore we can extend the argument for $t > t^*$, which is our proposition. $\square$

Asymptotic limit of the solution:
After having proved global existence of solutions we can easily derive their long time asymptotics

$$J(t) + S_1(t) + S_2(t) < Ke^{-\lambda t} \text{ as } t \to \infty.$$

Therefore, using $\frac{d}{dt}(m_1(t)\xi_1(t) + m_2(t)\xi_2(t)) = O(J(t) + S_1(t) + S_2(t))$ as well as

$$\left|\frac{d}{dt}m_1(t)\right| + \left|\frac{d}{dt}m_2(t)\right| = O(J(t) + S_1(t) + S_2(t))$$

we obtain:

$$m_1(t) \to m_{1,\infty} \quad , \quad m_2(t) \to m_{2,\infty} \quad , \quad m_1(t)\xi_1(t) + m_2(t)\xi_2(t) \to \ell \text{ as } t \to \infty.$$

The last limit can be combined with

$$\xi_1(t) - \xi_2(t) - \frac{1}{2} \to 0 \text{ as } t \to \infty$$

to obtain

$$\xi_1(t) \to \xi_{1,\infty} \quad , \quad \xi_2(t) \to \xi_{2,\infty} \text{ as } t \to \infty$$

where

$$m_{1,\infty}\xi_{1,\infty} + m_{2,\infty}\xi_{2,\infty} = \ell \quad , \quad \xi_{1,\infty} - \xi_{2,\infty} = \frac{1}{2}.$$

Since the first moment is preserved, compare (26), we have

$$\ell = \int_{\mathcal{I}} u f_{u,0} du\,.$$

Finally, since $S_1(t) \to 0,\ S_2(t) \to 0,\ J(t) \to 0$ as $t \to \infty$ we obtain:

$$f(u,t) \to m_{1,\infty}\delta_{\{u=\xi_{1,\infty}\}} + m_{2,\infty}\delta_{\{u=\xi_{2,\infty}\}} \text{ for } t \to \infty.$$

This finishes the proof of Theorem 5.6.

□

Discussion

In this paper we studied a kinetic model that describes alignment of cells or filaments. This model is a specific case of the integro-differential equations discussed in [4], since we assume deterministic interaction between the cell bundles. We obtained several rigorous results concerning the long time asymptotics of the solutions. If only bundles interact which are close in orientation and these interactions try to align them, then we obtain that the cells or filaments tend to asymptotically align into a finite set of directions. The number of such limiting orientations is larger, the closer the bundles have to be in order to interact with each other.

The technically most involved part of this paper is the analysis of the alignment process in case interactions take place between all orientations. We studied in detail interactions that tend to align bundles of cells or filaments whose orientation is either close or close to nearly opposite directions. An example for this is for instance the behavior of myxobacteria, [2]. We proved rigorously that in this case the bundles become aligned in two exactly opposite directions for large times, thus a quasi-one-dimensional orientation of bundles results.

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