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Removal of violations of the Master Ward
Identity in perturbative QFT

by

Ferdinand Brennecke, and Michael Dütsch

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Ferdinand Brennecke

Institut für Quantenelektronik, ETH Zürich

CH-8093 Zürich, Switzerland

`brennecke@phys.ethz.ch`

Michael Dütsch

Institut für Theoretische Physik, Universität Zürich

CH-8057 Zürich, Switzerland

and Max Planck Institute for Mathematics in the Sciences

D-04103 Leipzig, Germany

`duetsch@physik.unizh.ch`

Abstract

We study the appearance of anomalies of the Master Ward Identity, which is a universal renormalization condition in perturbative QFT. The main insight of the present paper is that any violation of the Master Ward Identity can be expressed as a *local* interacting field; this is a version of the well-known Quantum Action Principle of Lowenstein and Lam. Proceeding in a proper field formalism by induction on the order in \hbar , this knowledge about the structure of possible anomalies as well as techniques of algebraic renormalization are used to remove possible anomalies by finite renormalizations. As an example the method is applied to prove the Ward identities of the $O(N)$ scalar field model.

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1 Introduction

In the quantization of a classical field theory symmetries and corresponding conservation laws are in general not maintained: due to the distributional character of quantum fields the arguments valid for classical field theory are not applicable. Therefore, in perturbative quantum field theory (pQFT) symmetries resp. conservation laws play the role of renormalization conditions (the 'Ward identities').

In [1] a universal formulation of Ward identities was studied and termed Master Ward Identity (MWI). This identity – which originally was proposed in [2] – can be derived in the framework of classical field theory simply from the fact that classical fields can be multiplied pointwise. However, in pQFT the MWI serves as a highly non-trivial renormalization condition, which cannot be fulfilled in general due to the well known anomalies appearing in QFT.

In traditional renormalization theory (e.g. BPHZ renormalization or dimensional renormalization) the question whether certain Ward identities can be fulfilled, is usually treated by means of algebraic renormalization¹. This method relies on the Quantum Action Principle (QAP), which was derived by Lowenstein and Lam in the early seventies [3, 4] and proved in several renormalization schemes [5]. The QAP characterizes the possible violations of Ward identities and, hence, allows one to derive algebraic conditions whose solvability guarantees the existence of a renormalization maintaining the Ward identities. These conditions often lead to cohomological problems involving the infinitesimal symmetry operators which appear in the considered Ward identities.

¹There is a huge literature about algebraic renormalization. For shortness we cite only some of the founding articles and a few textbooks and reviews, which should suffice to understand this paper.

The application of this procedure to Yang-Mills theories lead to a detailed study of BRST cohomology [6, 7, 8, 9, 10, 11].

Traditionally, perturbation theory is done in the functional formulation of QFT which starts from the path integral. However, we take the point of view of algebraic pQFT [12, 13, 14, 15, 16], which is based on causal perturbation theory (Bogoliubov [17], Epstein and Glaser [18]) and concentrates on the algebraic structure of interacting fields. Starting with some well defined free QFT one separates UV-problems from IR-problems by considering solely interactions with compact support. Whereas the UV-problem concerns the construction of time ordered resp. retarded products (in terms of which interacting fields are formulated), the IR-problem appears only in the construction of states on the algebra of local observables. The restriction on compactly supported interactions leaves it possible to construct the whole net of local observables [12]. Therefore, this approach seems to be well suited for a rigorous perturbative construction of quantum Yang-Mills theories, for example, where an adiabatic limit seems to be out of reach. However, in the non-Abelian case the construction of the net of local observables is still an open problem within the framework of algebraic pQFT. As it was worked out in [2], the decisive input to reach this goal is the MWI resp. certain cases of it. Motivated by this, the aim of our present work is to transfer techniques from algebraic renormalization theory into the framework of algebraic pQFT in order to gain more insight into the violations of the MWI and to find concrete conditions for the solvability of the MWI in relevant cases.

The paper is organized as follows: Sect. 2 deals with classical field theory for localized interactions. We generalize the treatment given in Sect. 2 of [1] to the off-shell formalism, i.e. the values of the retarded products are off-shell fields. In Sect. 3 we summarize the quantization of perturbative classical field theory worked out in [16] and give some completions. Algebraic renormalization proceeds in terms of the 'vertex functional' (or 'proper function') Γ (which is usually derived in the functional formulation of QFT along a Legendre transformation). Hence, to make accessible techniques of algebraic renormalization, we develop a proper field formalism, which describes the combinatorics of 1-particle-irreducible (1PI) diagrams in a purely algebraic setting (Sect. 4). This is done by reformulating perturbative QFT as a classical field theory with a non-local interaction Γ which is a formal power series in \hbar .

After these preparatory sections we turn in Sect. 5 to the maintenance of the MWI in the process of renormalization. Starting from a derivation of the MWI in the off-shell formalism [16], we prove an identity ('anomalous MWI') which gives a characterization of the possible violations of the MWI. More precisely, we find that the most general violation can be expressed in terms of a local interacting field. Translation of the anomalous MWI into the proper field formalism (introduced in Sect. 4) yields an identity which contains solely the 'quantum part' (loop part) of the original version and which is shown to be formally equivalent to the QAP. Crucial properties of the violating local terms appearing in the anomalous MWI are proved, in particular an upper bound for the mass dimension. Equipped with this new insight, we transfer basic ideas of algebraic renormalization into the setting of causal perturbation theory. Usually, in the latter the maintenance of Ward identities in renormalization is proved by induction on the power of the coupling constant. In contrast, we proceed by induction on the power of \hbar similarly to algebraic renormalization. We find explicit

conditions, whose solvability guarantees the existence of a renormalization prescription satisfying corresponding cases of the MWI. In addition, we prove that the MWI can always be fulfilled to first order in the coupling constant (that is to second order of the corresponding time ordered products). We apply these results to models fulfilling (classically) a localized off-shell version of Noether's Theorem and find simplifications of the mentioned conditions. Finally, as a simple application of the method, we prove the Ward identities of the $O(N)$ scalar field model by using cohomological arguments.

2 Classical field theory for localized interactions

In this section we generalize the formalism developed in Sect. 2 of [1] to off-shell fields. This will provide us with the necessary framework to derive the off-shell version of the MWI in Sect. 5.1. In order to keep the formulas as simple as possible, we study the theory of a real scalar field φ on d dimensional Minkowski space \mathbb{M} , $d > 2$. We interpret φ and partial derivatives thereof as evaluation functionals on the configuration space $\mathcal{C} \equiv \mathcal{C}^\infty(\mathbb{M}, \mathbb{R}) : (\partial^a \varphi)(x)(h) = \partial^a h(x)$, $a \in \mathbb{N}_0^d$. Let $\mathcal{F}(\mathcal{C})$ be the space of all functionals

$$F(\varphi) : \mathcal{C} \longrightarrow \mathbb{C}, \quad F(\varphi)(h) = F(h), \quad (2.1)$$

which are localized polynomials in φ^2 :

$$F(\varphi) = \sum_{n=0}^N \int dx_1 \dots dx_n \varphi(x_1) \dots \varphi(x_n) f_n(x_1, \dots, x_n) =: \sum_n \langle f_n, \varphi^{\otimes n} \rangle, \quad (2.2)$$

where $N < \infty$ and the f_n 's are \mathbb{C} -valued distributions with compact support, which are symmetric under permutations of the arguments and whose wave front sets satisfy the condition³

$$\text{WF}(f_n) \cap (\mathbb{M}^n \times (\overline{V}_+^n \cup \overline{V}_-^n)) = \emptyset \quad (2.3)$$

and $f_0 \in \mathbb{C}$. \mathcal{F} is a commutative algebra with the *classical product* $(F_1 \cdot F_2)(h) := F_1(h) \cdot F_2(h)$. By the support of a functional $F \in \mathcal{F}$ we mean the support of $\frac{\delta F}{\delta \varphi}$.

The space of *local functionals* $\mathcal{F}_{\text{loc}}(\mathcal{C}) \subset \mathcal{F}(\mathcal{C})$ is defined as

$$\mathcal{F}_{\text{loc}}(\mathcal{C}) \stackrel{\text{def}}{=} \left\{ \int dx \sum_{i=1}^N A_i(x) h_i(x) \equiv \sum_{i=1}^N A_i(h_i) \mid A_i \in \mathcal{P}, h_i \in \mathcal{D}(\mathbb{M}) \right\}, \quad (2.4)$$

where \mathcal{P} is the space of all polynomials of the field φ and its partial derivatives. This representation of local functionals as smeared fields can be made unique by introducing the subspace of balanced fields $\mathcal{P}_{\text{bal}} \subset \mathcal{P}$ [16, 20]:

$$\mathcal{P}_{\text{bal}} \stackrel{\text{def}}{=} \left\{ P(\partial_1, \dots, \partial_n) \varphi(x_1) \dots \varphi(x_n) \big|_{x_1=\dots=x_n=x} \mid P(\partial_1, \dots, \partial_n) \in P_n^{\text{rel}}, n \in \mathbb{N}_0 \right\} \quad (2.5)$$

²A generalization to non-polynomial (localized) interactions is possible, see e.g. [19].

³ \overline{V}_\pm^n denotes the closure of the forward and backward light-cones, respectively, and \overline{V}_\pm^n their n -fold direct products.

where P_n^{rel} is the space of all polynomials in the 'relative derivatives' $(\partial_k - \partial_l)$, $1 \leq k < l \leq n$. With that it holds: given $F \in \mathcal{F}_{\text{loc}}$ there exists a *unique* $h \in \mathcal{D}(\mathbb{M}, \mathcal{P}_{\text{bal}})$ with $F = \int dx h(x)$. Proofs are given in Proposition 3.1 of [16] and in Lemma 1 of [20].

Since we are mainly interested in perturbation theory we consider action functionals of the form $S_{\text{tot}} = S_0 + \lambda S$ where $S_0 \stackrel{\text{def}}{=} \int dx \frac{1}{2} (\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2)$ denotes the free action, λ a real parameter and $S \in \mathcal{F}(\mathcal{C})$ is some compactly supported interaction, which may be non-local.⁴ We denote by $\Delta_{S_{\text{tot}}}^{\text{ret}}$ the retarded Green function corresponding to the action S_{tot} , which is defined by

$$\int dy \Delta_{S_{\text{tot}}}^{\text{ret}}(x, y) \frac{\delta^2 S_{\text{tot}}}{\delta \varphi(y) \delta \varphi(z)} = \delta(x - z) = \int dy \frac{\delta^2 S_{\text{tot}}}{\delta \varphi(x) \delta \varphi(y)} \Delta_{S_{\text{tot}}}^{\text{ret}}(y, z) \quad (2.6)$$

and $\Delta_{S_{\text{tot}}}^{\text{ret}}(x, y) = 0$ for x sufficiently early. In the following we will assume that for all actions S_{tot} under consideration the retarded Green function exists and is unique in the sense of formal power series in λ . Analytic expressions for $\Delta_{S_{\text{tot}}}^{\text{ret}}$ are in general unknown. However, perturbatively the retarded solution of (2.6) can be given in terms of the (unique) retarded Green function $\Delta_{S_0}^{\text{ret}}(x, y) = \Delta_m^{\text{ret}}(x - y)$ of the Klein Gordon operator:

Lemma 1. *In the sense of formal power series in λ , the retarded Green function $\Delta_{S_0 + \lambda S}^{\text{ret}}$ is given by the following formula [1]:*

$$\begin{aligned} \Delta_{S_0 + \lambda S}^{\text{ret}}(x, y) &= \Delta_{S_0}^{\text{ret}}(x, y) + \sum_{n=1}^{\infty} (-\lambda)^n \int d(u_1, \dots, u_n) d(v_1, \dots, v_n) \Delta_{S_0}^{\text{ret}}(x, u_1) \\ &\quad \times \frac{\delta^2 S}{\delta \varphi(u_1) \delta \varphi(v_1)} \cdots \Delta_{S_0}^{\text{ret}}(v_{n-1}, u_n) \frac{\delta^2 S}{\delta \varphi(u_n) \delta \varphi(v_n)} \Delta_{S_0}^{\text{ret}}(v_n, y). \end{aligned} \quad (2.7)$$

Its support is contained in the set

$$\{(x, y) | x \in \text{supp}(\frac{\delta S}{\delta \varphi}) + \overline{V}_+ \wedge y \in \text{supp}(\frac{\delta S}{\delta \varphi}) + \overline{V}_-\} \cup \{(x, y) | x \in y + \overline{V}_+\}. \quad (2.8)$$

In case of a local interaction $S \in \mathcal{F}_{\text{loc}}(\mathcal{C})$ the support can be stronger limited:

$$\text{supp}(\frac{\delta \Delta_{S_0 + \lambda S}^{\text{ret}}}{\delta \varphi}) \subset \{(x, y) | x \in y + \overline{V}_+\}. \quad (2.9)$$

Our assumption ensures that the pointwise product of distributions in (2.7) exists.

Proof. We multiply the left equation in (2.6) by $\Delta_{S_0}^{\text{ret}}(z, u)$ and integrate over z afterwards to obtain the relation

$$\Delta_{S_0 + \lambda S}^{\text{ret}}(x, u) = \Delta_{S_0}^{\text{ret}}(x, u) - \lambda \int dy dz \Delta_{S_0 + \lambda S}^{\text{ret}}(x, y) \frac{\delta^2 S}{\delta \varphi(y) \delta \varphi(z)} \Delta_{S_0}^{\text{ret}}(z, u), \quad (2.10)$$

which can be solved by recursion on the powers of λ (2.7). For local interactions $S \in \mathcal{F}_{\text{loc}}(\mathcal{C})$, the support property (2.9) follows immediately from $\text{supp}(\frac{\delta \Delta_{S_0}^{\text{ret}}}{\delta \varphi}) \subset \{(x, y) | x \in y + \overline{V}_+\}$.

⁴Note that the free action is not an element of $\mathcal{F}(\mathcal{C})$. Therefore we interpret S_0 as functional on the subspace of compactly supported functions in configuration space \mathcal{C} . However, this restriction is not necessary (and, hence, will not be done) for the free field equation: $\frac{\delta S_0}{\delta \varphi}$ is an element of \mathcal{P} .

$y + \overline{V}_+\}$ and $\frac{\delta^2 S}{\delta\varphi(u)\delta\varphi(v)} = 0$ for $u \neq v$. In the general case $S \in \mathcal{F}(\mathcal{C})$ we use $\text{supp}(\frac{\delta^2 S}{\delta\varphi^2}) \subset \text{supp}(\frac{\delta S}{\delta\varphi}) \times \text{supp}(\frac{\delta S}{\delta\varphi})$ to conclude that x has to lie in the future of $u_1 \in \text{supp}(\frac{\delta S}{\delta\varphi})$ and y in the past of $v_n \in \text{supp}(\frac{\delta S}{\delta\varphi})$ to get a non vanishing contribution of the integral in (2.7). \square

The space of all smooth solutions of the Euler-Lagrange equation with respect to the action S_{tot} will be denoted by $\mathcal{C}_{S_{\text{tot}}} \subset \mathcal{C}$. Interacting fields F_S , corresponding to some functional $F \in \mathcal{F}(\mathcal{C})$, could be defined by restricting F to the space of solutions \mathcal{C}_{S_0+S} : $F_S \stackrel{\text{def}}{=} F|_{\mathcal{C}_{S_0+S}}$. However, the idea of perturbative algebraic classical field theory is to introduce interacting fields as functionals on the space \mathcal{C}_{S_0} of *free* solutions. This corresponds to the usual interaction picture known from QFT where interacting fields are constructed as operators on the Fock space of the underlying free theory. Therefore one introduces retarded wave operators which map solutions of the free theory to solutions of the interacting theory [1]. As it turns out, it is convenient to construct such a map on the space \mathcal{C} of all field configurations (*off-shell formalism*) and not only on the space of solutions, as it was done in [1]:

Definition 1. A retarded wave operator is a family of maps $(r_{S_0+S, S_0})_{S \in \mathcal{F}(\mathcal{C})}$ from \mathcal{C} into itself with the properties

(i) $r_{S_0+S, S_0}(f)(x) = f(x)$ for x sufficiently early

(ii) $\frac{\delta(S_0+S)}{\delta\varphi} \circ r_{S_0+S, S_0} = \frac{\delta S_0}{\delta\varphi}$.

Lemma 2. The retarded wave operator $(r_{S_0+S, S_0})_{S \in \mathcal{F}(\mathcal{C})}$ exists and is unique and invertible in the sense of formal power series in the interaction S .

Proof. To determine r_{S_0+S, S_0} we multiply S in Def. 1, (ii) by a real parameter λ and differentiate with respect to this parameter, ending up with

$$\int dy \frac{\delta(S_0 + \lambda S)}{\delta\varphi(x)\delta\varphi(y)} \circ r_{S_0+\lambda S, S_0} \cdot \varphi(y) \circ \frac{d}{d\lambda} r_{S_0+\lambda S, S_0} + \frac{\delta S}{\delta\varphi(x)} \circ r_{S_0+\lambda S, S_0} = 0.$$

After multiplication of this equation with $\Delta_{S_0+\lambda S}^{\text{ret}}(z, x)$ and integration over x , we obtain the following differential equation for $r_{S_0+\lambda S, S_0}$

$$\varphi(z) \circ \frac{d}{d\lambda} r_{S_0+\lambda S, S_0} = - \int dx \left(\Delta_{S_0+\lambda S}^{\text{ret}}(z, x) \cdot \frac{\delta S}{\delta\varphi(x)} \right) \circ r_{S_0+\lambda S, S_0}. \quad (2.11)$$

Finally, integration over λ leads to the equation

$$\varphi(z) \circ r_{S_0+S, S_0} = \varphi(z) - \int_0^1 d\lambda \int dx \left(\Delta_{S_0+\lambda S}^{\text{ret}}(z, x) \cdot \frac{\delta S}{\delta\varphi(x)} \right) \circ r_{S_0+\lambda S, S_0}, \quad (2.12)$$

which can be solved iteratively in the sense of formal power series in the interaction S . \square

We define the retarded wave operator $r_{S_0+S_1, S_0+S_2}$ connecting two interacting theories by

$$r_{S_0+S_1, S_0+S_2} := r_{S_0+S_1, S_0} \circ (r_{S_0+S_2, S_0})^{-1}. \quad (2.13)$$

Obviously it fulfills Def. 1(i) and $\frac{\delta(S_0+S_1)}{\delta\varphi} \circ r_{S_0+S_1, S_0+S_2} = \frac{\delta(S_0+S_2)}{\delta\varphi}$ and

$$r_{S_0+S_1, S_0+S_2} \circ r_{S_0+S_2, S_0+S_3} = r_{S_0+S_1, S_0+S_3}. \quad (2.14)$$

Retarded fields (to the interaction S and the free theory S_0 and corresponding to the functional $F \in \mathcal{F}(\mathcal{C})$) are defined by

$$F_{S_0, S}^{\text{ret}} \stackrel{\text{def}}{=} F \circ r_{S_0+S, S_0} : \mathcal{C} \longrightarrow \mathbb{C}. \quad (2.15)$$

A crucial property of classical interacting fields – which does *not* hold anymore for interacting quantum fields – is their factorization with respect to the classical product

$$(F \cdot G)_{S_0, S}^{\text{ret}} = F_{S_0, S}^{\text{ret}} \cdot G_{S_0, S}^{\text{ret}}. \quad (2.16)$$

This is why certain symmetry properties of classical field theory in general cannot be transferred directly into quantum field theory (see Sect. 5.1).

In classical field theory retarded products R_{cl} are defined as coefficients in the expansion (with respect to the interaction) of interacting retarded fields [1]:

$$R_{\text{cl}} : \mathcal{F}(\mathcal{C})^{\otimes n} \otimes \mathcal{F}(\mathcal{C}) \rightarrow \mathcal{F}(\mathcal{C}), \quad R_{\text{cl}}(S^{\otimes n}, F) \stackrel{\text{def}}{=} \left. \frac{d^n}{d\lambda^n} F \circ r_{S_0+\lambda S, S_0} \right|_{\lambda=0}. \quad (2.17)$$

Interacting fields can then be written as

$$F_{S_0, S}^{\text{ret}} \simeq \sum_{n=0}^{\infty} \frac{1}{n!} R_{\text{cl}}(S^{\otimes n}, F) \equiv R_{\text{cl}}(e_S^S, F) \quad (2.18)$$

where the r.h.s. of \simeq is interpreted as a formal power series (we do not care about convergence of the series). In the last expression R_{cl} is viewed as a linear map

$$R_{\text{cl}} : \mathbb{T}\mathcal{F}(\mathcal{C}) \otimes \mathcal{F}(\mathcal{C}) \rightarrow \mathcal{F}(\mathcal{C}), \quad (2.19)$$

where $\mathbb{T}\mathcal{V} \stackrel{\text{def}}{=} \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{V}^{\otimes n}$ denotes the tensor algebra corresponding to some vector space \mathcal{V} .

By introducing the differential operator

$$\mathcal{D}_{S_0, S}(\lambda) \stackrel{\text{def}}{=} - \int dx dy \Delta_{S_0+\lambda S}^{\text{ret}}(x, y) \frac{\delta S}{\delta\varphi(y)} \frac{\delta}{\delta\varphi(x)}, \quad (2.20)$$

we obtain from (2.12) the following explicit expression for the interacting field:

$$F_{S_0, S}^{\text{ret}} \simeq F + \sum_{n=1}^{\infty} \int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \dots \int_0^{\lambda_{n-1}} d\lambda_n \mathcal{D}_{S_0, S}(\lambda_n) \dots \mathcal{D}_{S_0, S}(\lambda_1) F. \quad (2.21)$$

To first order in S this formula reads (see also [1])

$$R_{\text{cl}}(S, F) = - \int dx dy \Delta_{S_0}^{\text{ret}}(x, y) \frac{\delta S}{\delta\varphi(y)} \frac{\delta F}{\delta\varphi(x)}. \quad (2.22)$$

To complete the Poisson structure of classical fields we introduce the (off-shell) Poisson bracket using Peierls definition [21] (see also [22] and [1])

Definition 2. The Poisson bracket associated to the action $S \in \mathcal{F}(\mathcal{C})$ is the map

$$\{\cdot, \cdot\}_S : \mathcal{F}(\mathcal{C}) \otimes \mathcal{F}(\mathcal{C}) \rightarrow \mathcal{F}(\mathcal{C}) \quad (2.23)$$

$$\{F, G\}_S \stackrel{\text{def}}{=} R_S(F, G) - R_S(G, F). \quad (2.24)$$

where

$$R_S(F, G) \stackrel{\text{def}}{=} \left. \frac{d}{d\lambda} \right|_{\lambda=0} G \circ r_{S+\lambda F, S}. \quad (2.25)$$

The properties

- $R_S(F, G) = - \int dx dy \frac{\delta G}{\delta \varphi(x)} \Delta_S^{\text{ret}}(x, y) \frac{\delta F}{\delta \varphi(y)}$ (which is a generalization of (2.22)) and
- $\{\cdot, \cdot\}_S$ is indeed a Poisson bracket, i.e. it satisfies the Leibniz rule and the Jacobi identity,

are proved in [1] for the on-shell restrictions $R_S^{\text{on-shell}}(F, G) = R_S(F, G)|_{\mathcal{C}_S}$ and $\{F, G\}_S^{\text{on-shell}} = \{F, G\}_S|_{\mathcal{C}_S}$. These proofs can easily be generalized to R_S and $\{\cdot, \cdot\}_S$. The Jacobi identity is derived from

$$\{R_S(H, F), G\}_S + \{F, R_S(H, G)\}_S = R_S(H, \{F, G\}_S) + \left. \frac{d}{d\lambda} \right|_{\lambda=0} \{F, G\}_{S+\lambda H}, \quad (2.26)$$

which is the infinitesimal version of the statement that the map $\mathcal{F}(\mathcal{C}) \ni F \mapsto F \circ r_{S_1, S_2} \in \mathcal{F}(\mathcal{C})$ is a canonical transformation:

$$\{F \circ r_{S_1, S_2}, G \circ r_{S_1, S_2}\}_{S_2} = \{F, G\}_{S_1} \circ r_{S_1, S_2}. \quad (2.27)$$

In [1] only the proof of the infinitesimal version (2.26) is given. We are going to show that integration of (2.26), written in a suitable form, yields indeed (2.27).⁵ First note

$$\left. \frac{d}{d\lambda} \right|_{\lambda=0} F \circ r_{S, S+\lambda H} = - \left. \frac{d}{d\lambda} \right|_{\lambda=0} F \circ r_{S+\lambda H, S} = -R_S(H, F). \quad (2.28)$$

Let $H := S_2 - S_1$, $S(\lambda) := S_1 + \lambda H$ and $F_\lambda := F \circ r_{S_1, S(\lambda)}$. With that we obtain

$$\begin{aligned} & \left. \frac{d}{d\lambda} \right|_{\lambda=\lambda_0} \{F \circ r_{S_1, S(\lambda)}, G \circ r_{S_1, S(\lambda)}\}_{S(\lambda)} \circ r_{S(\lambda), S_1} = \\ & \left. \frac{d}{d\lambda} \right|_{\lambda=\lambda_0} \left(\{F_{\lambda_0} \circ r_{S(\lambda_0), S(\lambda)}, G_{\lambda_0} \circ r_{S(\lambda_0), S(\lambda)}\}_{S(\lambda)} \circ r_{S(\lambda), S(\lambda_0)} \right) \circ r_{S(\lambda_0), S_1} = 0 \end{aligned} \quad (2.29)$$

by using (2.25), (2.28) and the infinitesimal version (2.26). Integrating this equation over λ_0 from $\lambda_0 = 0$ to $\lambda_0 = 1$ it results the assertion.

Due to the perturbative expansion around the free theory only the Poisson bracket associated to the free action, $\{\cdot, \cdot\}_{\text{cl}} \equiv \{\cdot, \cdot\}_{S_0}$, will be used in the following sections.

As one can easily check, the retarded products (2.17) have the same properties as the on-shell retarded products in [1] (which are related to (2.17) by $R_{\text{cl}}^{\text{on-shell}}(S^{\otimes n}, F) = R_{\text{cl}}(S^{\otimes n}, F)|_{\mathcal{C}_{S_0}}$). These are the properties which are used to define retarded products in perturbative QFT in an axiomatic way.

⁵This proof is due to Klaus Fredenhagen.

3 Perturbative Quantum Field Theory

We summarize the quantization of perturbative classical fields as it is worked out in [16] on the basis of causal perturbation theory [17], [18] and work of Steinmann [23]. Since the direct quantization of an interacting theory is in general not solved, we quantize the free theory, around which the perturbative expansion is done (see Sect. 2), by using deformation quantization: we replace $\mathcal{F}(\mathcal{C})$ and $\mathcal{F}_{\text{loc}}(\mathcal{C})$ by $\mathcal{F} \stackrel{\text{def}}{=} \mathcal{F}(\mathcal{C})[[\hbar]]$ and $\mathcal{F}_{\text{loc}} \stackrel{\text{def}}{=} \mathcal{F}_{\text{loc}}(\mathcal{C})[[\hbar]]$ resp. (i.e. all functionals are formal power series in \hbar) and deform the classical product into the \star -product, $\star : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$, which is still associative but non-commutative and is defined by

$$(F \star G)(\varphi) := \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \int dx_1 \dots dx_n dy_1 \dots dy_n \frac{\delta^n F}{\delta \varphi(x_1) \dots \delta \varphi(x_n)} \cdot \prod_{i=1}^n H_m(x_i - y_i) \frac{\delta^n G}{\delta \varphi(y_1) \dots \delta \varphi(y_n)} . \quad (3.1)$$

There is a freedom in the choice of the 2-point function $H_m(x)$: it is required to differ from the Wightman 2-point function $\Delta_m^+(x)$ by a smooth and symmetric function of x , to be Lorentz invariant and to satisfy the Klein Gordon equation. The 'vacuum state' is the map (see (2.2))

$$\omega_0 : \mathcal{F} \longrightarrow \mathbb{C}[[\hbar]] , \quad F \mapsto F(0) = f_0 . \quad (3.2)$$

For the interacting quantum field F_G , ($F, G \in \mathcal{F}_{\text{loc}}$) one makes the ansatz of a *formal power series* in the interaction G :⁶

$$F_G = \sum_{n=0}^{\infty} \frac{1}{n!} R_{n,1}(G^{\otimes n}, F) \equiv R(e_{\otimes}^G, F) . \quad (3.3)$$

The 'retarded product' $R_{n,1}$ is a **linear** map, from $\mathcal{F}_{\text{loc}}^{\otimes n} \otimes \mathcal{F}_{\text{loc}}$ into \mathcal{F} which is **symmetric in the first n variables**. The last expression in (3.3) is understood analogously to (2.19). We interpret $R(A_1(x_1), \dots, A_n(x_n))$, $A_1, \dots, A_n \in \mathcal{P}$, as \mathcal{F} -valued distributions on $\mathcal{D}(\mathbb{M}^n)$, which are defined by: $\int dx h(x) R(\dots, A(x), \dots) := R(\dots \otimes A(h) \otimes \dots) \quad \forall h \in \mathcal{D}(\mathbb{M})$.

Interacting fields are defined by the following axioms [16], which are motivated by the principle that we want to maintain as much as possible of the classical structure in the process of quantization:

Basic axioms:

Initial Condition. $R(F) \equiv R_{0,1}(1, F) = F$.

Causality. $F_{G+H} = F_G$ if $\text{supp}(\frac{\delta F}{\delta \varphi}) \cap (\text{supp}(\frac{\delta H}{\delta \varphi}) + \bar{V}_+) = \emptyset$.

⁶With respect to factors of \hbar , our conventions (R) differ from [16] (R^{DF}), namely $R(e_{\otimes}^G, F) = R^{DF}(e_{\otimes}^{G/\hbar}, F)$. However, for the T -products we use the same conventions.

GLZ Relation. In the classical GLZ Relation we replace the Poisson bracket $\{\cdot, \cdot\}_{\text{cl}}$ by

$$\{\cdot, \cdot\} \stackrel{\text{def}}{=} \frac{1}{i\hbar} [\cdot, \cdot]_{\star} = \{\cdot, \cdot\}_{\text{cl}} + \mathcal{O}(\hbar) \quad (3.4)$$

(where $[H, F]_{\star} \equiv H \star F - F \star H$). This gives

$$\{F_G, H_G\} = \left. \frac{d}{d\lambda} \right|_{\lambda=0} (F_{G+\lambda H} - H_{G+\lambda F}) . \quad (3.5)$$

Based on these requirements, the retarded products $R_{n,1}$ can be constructed by induction on n . However, in each inductive step one is free to add an arbitrary *local* functional, which corresponds to the usual renormalization ambiguity. This ambiguity is reduced by imposing the following further axioms:

Renormalization conditions:

Unitarity. $(F_G)^* = F_{G^*}^*$

Poincare Covariance. For $L \in \mathcal{P}_+^{\uparrow}$ we set $\varphi_L(x) := \varphi(L^{-1}x)$ and $h_L(x) := h(L^{-1}x)$, $h \in \mathcal{C}$, and define an automorphism

$$\beta_L : \mathcal{F} \rightarrow \mathcal{F}; \beta_L(\langle f_n, \varphi^{\otimes n} \rangle) = \langle f_n, (\varphi_{L^{-1}})^{\otimes n} \rangle , \quad (3.6)$$

that is $(\beta_L F)(h) = F(h_{L^{-1}})$. \mathcal{P}_+^{\uparrow} -Covariance of the interacting fields means:

$$\beta_L(F_G) = (\beta_L F)_{\beta_L G} , \quad \forall L \in \mathcal{P}_+^{\uparrow}.$$

Field Independence. The interacting field F_G depends on φ only through F and G : $\frac{\delta R}{\delta \varphi(x)} = 0$. This condition is equivalent to the requirement that R fulfills the causal Wick expansion [18], that is

$$\begin{aligned} R_{n-1,1}(A_1(x_1) \otimes \cdots \otimes A_{n-1}(x_{n-1}), A_n(x_n)) &= \sum_{l_1, \dots, l_n} \frac{1}{l_1! \cdots l_n!} \\ \times \omega_0 \left(R_{n-1,1} \left(\cdots \sum_{a_{i1} \dots a_{il_i}} \frac{\partial^{l_i} A_i}{\partial(\partial^{a_{i1}} \varphi) \cdots \partial(\partial^{a_{il_i}} \varphi)}(x_i) \cdots \right) \right) &\prod_{j=1}^{l_i} \partial^{a_{ij}} \varphi(x_i) \end{aligned} \quad (3.7)$$

with multi-indices $a_{ij_i} \in \mathbb{N}_0^d$.

Field Equation.

$$\varphi_G(x) = \varphi(x) - \int \Delta_m^{\text{ret}}(x-y) \left(\frac{\delta G}{\delta \varphi(y)} \right)_G dy , \quad \forall G \in \mathcal{F}_{\text{loc}} . \quad (3.8)$$

Smoothness in m . Through the GLZ condition the interacting fields depend on the 2-point function H_m and with that they depend on the mass m of the free field: $F_G \equiv (F_G)^{H_m}$. We require that the maps

$$0 \leq m \mapsto (F_G)^{H_m} , \quad F, G \in \mathcal{F}_{\text{loc}} , \quad (3.9)$$

are smooth. In even dimensional spacetime this excludes the 2-point function Δ_m^+ due to logarithmic singularities at $m = 0$; more generally, homogeneous scaling of H_m is not compatible with smoothness in $m \geq 0$. As in [16] we work with the 2-point function H_m^μ (and the corresponding Feynman propagator) which is distinguished by almost homogeneous scaling [19]. H_m^μ depends on an additional mass parameter $\mu > 0$ and is explicitly given in Appendix A of [16]. For the corresponding star product, retarded product and interacting fields we write $\star_{m,\mu}$, $R^{(m,\mu)}$ and $(F_G)^{(m,\mu)} \equiv (F_G)^{H_m^\mu}$, respectively.

μ -Covariance. The \star -products $(\star_{m,\mu})_{\mu>0}$ (and the \star -product with respect to Δ_m^+) are *equivalent*, that is there exists an invertible operator $(\mu_2/\mu_1)^\Gamma$ which intertwines these products (see e.g. [14]):

$$F \star_{m,\mu_2} G = (\mu_2/\mu_1)^{\Gamma(m)} \left(\left((\mu_2/\mu_1)^{-\Gamma(m)} F \right) \star_{m,\mu_1} \left((\mu_2/\mu_1)^{-\Gamma(m)} G \right) \right), \quad (3.10)$$

where $r^\Gamma \stackrel{\text{def}}{=} 1 + \sum_{k=1}^{\infty} \frac{1}{k!} (\log(r) \cdot \Gamma)^k$ (for $r > 0$) and

$$\Gamma \equiv \Gamma^{(m)} \stackrel{\text{def}}{=} \int dx dy m^{d-2} f(m^2(x-y)^2) \frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)}. \quad (3.11)$$

(The smooth function f is explicitly given by formula (A.9) in [16].) The axiom μ -Covariance requires that $(\mu_2/\mu_1)^\Gamma$ intertwines also the retarded products:⁷

$$R^{(m,\mu_2)} = (\mu_2/\mu_1)^{\Gamma(m)} \circ R^{(m,\mu_1)} \circ \mathbb{T}(\mu_2/\mu_1)^{-\Gamma(m)}. \quad (3.13)$$

Scaling. The *mass dimension of a monomial* in \mathcal{P} is defined by the conditions

$$\dim(\partial^a \varphi) = \frac{d-2}{2} + |a| \quad \text{and} \quad \dim(A_1 A_2) = \dim(A_1) + \dim(A_2) \quad (3.14)$$

for all monomials $A_1, A_2 \in \mathcal{P}$. The mass dimension of a *polynomial* in \mathcal{P} is the maximum of the mass dimensions of the contributing monomials. We denote by \mathcal{P}_{hom} the set of all field polynomials which are homogeneous in the mass dimension. A scaling transformation σ_ρ is introduced as an automorphism of \mathcal{F} (considered as an algebra with the classical product) by

$$\sigma_\rho(\langle f_n, \varphi^{\otimes n} \rangle) \stackrel{\text{def}}{=} \rho^{\frac{n(2-d)}{2}} \int dx_1 \dots dx_n f_n(x_1, \dots, x_n) \varphi(x_1/\rho) \dots \varphi(x_n/\rho). \quad (3.15)$$

For $A \in \mathcal{P}_{\text{hom}}$ we obtain $\rho^{\dim(A)} \sigma_\rho(A(\rho x)) = A(x)$. Our condition of **almost homogeneous scaling** states that

$$\sigma_\rho \circ R_{n,1}^{(\rho^{-1}m,\mu)} \circ (\sigma_\rho^{-1})^{\otimes(n+1)}, \quad n \in \mathbb{N}_0, \quad m \geq 0, \quad \mu > 0, \quad (3.16)$$

is a polynomial in $(\log \rho)$.

⁷Given a linear map $f : \mathcal{V} \rightarrow \mathcal{V}$ (where \mathcal{V} is a vector space) we define

$$\mathbb{T}f : \mathbb{T}\mathcal{V} \rightarrow \mathbb{T}\mathcal{V}; \quad (\mathbb{T}f)(c \oplus \bigoplus_{j=1}^{\infty} (v_{j1} \otimes \dots \otimes v_{jj})) = c \oplus \bigoplus_{j=1}^{\infty} (f(v_{j1}) \otimes \dots \otimes f(v_{jj})). \quad (3.12)$$

The construction of the retarded products proceeds in terms of the distributions $R(A_1(x_1), \dots; A_n(x_n))$, $A_1, \dots, A_n \in \mathcal{P}$. Since the retarded products depend only on the functionals (and not on how the latter are written as smeared fields (2.4)), they must satisfy the **Action Ward Identity (AWI)** [16]:

$$\partial_\mu^x R_{n-1,1}(\dots A_k(x) \dots) = R_{n-1,1}(\dots, \partial_\mu A_k(x), \dots) . \quad (3.17)$$

The AWI can simply be fulfilled by constructing $R(A_1(x_1), \dots; A_n(x_n))$ first only for balanced fields $A_k \in \mathcal{P}_{\text{bal}} \forall k$, and by using the AWI and linearity for the extension to general fields $A_k \in \mathcal{P}$.

The axioms Smoothness in m , μ -Covariance and Scaling can be replaced by the weaker axiom **Scaling Degree**, which requires that 'renormalization may not make the interacting fields more singular' (in the UV-region). Usually this is formulated in terms of Steinmann's *scaling degree* [24]:

$$\text{sd}(f) \stackrel{\text{def}}{=} \inf\{\delta \in \mathbb{R} \mid \lim_{\rho \downarrow 0} \rho^\delta f(\rho x) = 0\}, \quad f \in \mathcal{D}'(\mathbb{R}^k) \text{ or } f \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\}). \quad (3.18)$$

Namely, one requires

$$\text{sd}\left(\omega_0(R(A_1, \dots; A_n))(x_1 - x_n, \dots)\right) \leq \sum_{j=1}^n \dim(A_j), \quad \forall A_j \in \mathcal{P}_{\text{hom}} . \quad (3.19)$$

In the inductive construction of the sequence $(R_{n,1})_{n \in \mathbb{N}}$ (given in [16]), the problem of renormalization appears as the extension of $\mathbb{C}[[\hbar]]$ -valued distributions from $\mathcal{D}(\mathbb{R}^{dn} \setminus \{0\})$ to $\mathcal{D}(\mathbb{R}^{dn})$. This extension has to be done in the sense of formal power series in \hbar , that is individually in each order in \hbar . With that it holds

$$\lim_{\hbar \rightarrow 0} R = R_{\text{cl}} . \quad (3.20)$$

Namely, the GLZ Relation is the only axiom which depends explicitly on \hbar and in the classical limit it goes over into the classical GLZ Relation, due to (3.4).

The retarded product, having two different kinds of arguments, can be derived from the **time ordered product** (' T -product') $T : \mathbb{T}\mathcal{F}_{\text{loc}} \rightarrow \mathcal{F}$, which is totally symmetric i.e. it has only one kind of arguments. The corresponding relation is Bogoliubov's formula:

$$R(e_\otimes^S, F) = -i \mathbf{S}(S)^{-1} \star \frac{d}{d\tau} \Big|_{\tau=0} \mathbf{S}(S + \tau F), \quad \mathbf{S}(S) \equiv T(e_\otimes^{iS}), \quad (3.21)$$

The axioms for retarded products translate directly into corresponding axioms for T -products, see Appendix E of [16]. There is no axiom corresponding to the GLZ Relation. The latter can be interpreted as 'integrability condition' for the 'vector potential' $R(e_\otimes^S, F)$, that is it ensures the existence of the 'potential' $\mathbf{S}(S)$ fulfilling (3.21); for details see [19]. (A derivation of the GLZ Relation from (3.21) is given in [13].)

In [16] it is shown that there **exist** retarded products which fulfill all axioms. The **non-uniqueness** of solutions is characterized by the 'Main Theorem'; we use the version given in [16]:

Theorem 3. (a) Let R and \hat{R} be retarded products which fulfill the basic axioms and the renormalization conditions Unitarity, \mathcal{P}_+^\dagger -Covariance, Field Independence and Field Equation. Then there exists a unique, symmetric and linear map

$$D : \mathbb{T}\mathcal{F}_{\text{loc}} \longrightarrow \mathcal{F}_{\text{loc}} \quad (3.22)$$

with $D(1) = 0$, $D(F) = F$ ($\forall F \in \mathcal{F}_{\text{loc}}$), such that for all $F, S \in \mathcal{F}_{\text{loc}}$ the following intertwining relation holds (in the sense of formal power series in λ)

$$\hat{R}(e_{\otimes}^{\lambda S}, F) = R(e_{\otimes}^{D(e_{\otimes}^{\lambda S})}, D(e_{\otimes}^{\lambda S} \otimes F)) . \quad (3.23)$$

In addition, D satisfies the conditions:

- (i) $\text{supp}\left(\frac{\delta D(F_1 \otimes \cdots \otimes F_n)}{\delta \varphi}\right) \subset \bigcap_{i \in \underline{n}} \text{supp}\left(\frac{\delta F_i}{\delta \varphi}\right), \quad F_i \in \mathcal{F}_{\text{loc}}$
- (ii) $\frac{\delta D(e_{\otimes}^F)}{\delta \varphi} = D\left(e_{\otimes}^F \otimes \frac{\delta F}{\delta \varphi}\right)$
- (iii) $D(e_{\otimes}^F \otimes \varphi(h)) = \varphi(h), \quad \forall F \in \mathcal{F}_{\text{loc}}$
- (iv) $D(F^{\otimes n})^* = D((F^*)^{\otimes n})$
- (v) $\beta_L \circ D = D \circ \mathbb{T}\beta_L, \quad \forall L \in \mathcal{P}_+^\dagger$
- (vi) (A) If $R^{(m, \mu)}, \hat{R}^{(m, \mu)}$ are smooth in $m \geq 0$ and satisfy the axioms μ -Covariance and Scaling, then the corresponding $D^{(m, \mu)}$ is also smooth in m , invariant under scaling

$$\sigma_\rho \circ D^{(\rho^{-1}m, \mu)} \circ \mathbb{T}\sigma_\rho^{-1} = D^{(m, \mu)} . \quad (3.24)$$

and μ -covariant ⁸

$$D^{(m, \mu_2)} = (\mu_2/\mu_1)^{\Gamma^{(m)}} \circ D^{(m, \mu_1)} \circ \mathbb{T}(\mu_2/\mu_1)^{-\Gamma^{(m)}} . \quad (3.25)$$

(B) Alternatively, if R and \hat{R} satisfy the axiom Scaling degree, then

$$\text{sd}\left(\omega_0(D(A_1, \dots, A_n))(x_1 - x_n, \dots)\right) \leq \sum_{j=1}^n \dim(A_j), \quad \forall A_j \in \mathcal{P}_{\text{hom}} . \quad (3.26)$$

(b) Conversely, given R and D as above, equation (3.23) gives a new retarded product \hat{R} which satisfies the axioms.

Since the classical limit of the axioms has a unique solution (which is R_{cl}), the map D is trivial to lowest order in \hbar , i.e.

$$D(e_{\otimes}^S) = S + \mathcal{O}(\hbar) \quad \text{and} \quad D(e_{\otimes}^S \otimes F) = F + \mathcal{O}(\hbar) . \quad (3.27)$$

The relation (3.23) can equivalently be expressed in terms of time ordered products,

$$\hat{T}(e_{\otimes}^{iS/\hbar}) = T\left(e_{\otimes}^{iD(e_{\otimes}^S)/\hbar}\right), \quad (3.28)$$

where T and \hat{T} are the time ordered products belonging to R and \hat{R} , resp., according to (3.21).

⁸The claim in [16] that $D^{(m, \mu)}$ is independent of μ , is wrong, cf. [19].

4 Proper vertices for T -products

Proper vertices⁹ are an old and standard tool in perturbative QFT. In terms of R -products the basic idea is the following: a perturbative QFT can be rewritten as *classical* field theory with a *non-local interaction* ('proper interaction') which agrees to lowest order in \hbar with the original local interaction. Since R_{cl} is the sum of all (connected) tree diagrams (as explained in Appendix A), this rewriting means that we interpret each diagram as tree diagram with non-local vertices ('proper vertices') given by the 1-particle-irreducible (1PI) subdiagrams. This structural decomposition of Feynman diagrams can just as well be done for T -products and it is this latter form of proper vertices which is well known in the literature. Since T -products are totally symmetric, it is simpler to introduce proper vertices in terms of T -products than in terms of R -products and, hence, we work with the former (for the introduction of proper vertices for R -products see Appendix A).

A main motivation to introduce proper vertices is that the renormalization of an arbitrary diagram reduces to the renormalization of its 1PI-subdiagrams. Indeed, due to the validity of the MWI for tree diagrams (i.e. in classical FT), the MWI can equivalently be formulated in terms of proper vertices (i.e. in terms of 1PI-diagrams), see Sect. 5.3. This 'proper MWI' formally coincides with the usual formulation of Ward identities in the functional approach to QFT (for an overview see e.g. [10]).

4.1 Diagrammatics and definition of the 1-particle-irreducible part $T^{1\text{PI}}$ of the time ordered product

To introduce proper vertices we need the tree part T_{tree} for *non-local entries* and, for later purpose, the 1PI part $T^{1\text{PI}}$ of the time ordered product T . The definition of T_{tree} can obviously be given in terms of Feynman diagrams; but in case of $T^{1\text{PI}}$ we are faced with the problem that for loop diagrams the decomposition of $T(A_1(x_1), \dots)$ into contributions of Feynman diagrams is *non-unique*, due to the local terms coming from renormalization.

To motivate the definition of $T^{1\text{PI}}$ we first study a 'smooth and symmetric \star -product': let $f \in \mathcal{C}^\infty(\mathbb{R}^4, \mathbb{C})$ with $f(x) = f(-x)$, $\forall x$. We define $\star_f : \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F}$ by replacing in the definition (3.1) of the \star -product the 2-point function H_m by f . This product, \star_f , is associative and commutative. By definition \star_f satisfies 'Wick's theorem'. Due to that, $\star_{f,j=1}^n F_j \equiv F_1 \star_f \dots \star_f F_n$ can uniquely be viewed as a sum of diagrams. In spite of the possible non-locality of the F_j 's, we symbolize each F_j by *one* vertex. The contractions are symbolized by inner lines connecting the vertices, which are not orientated due to $f(x) = f(-x)$. (We do not draw any external lines.)

The contribution of the connected diagrams (denoted by $(\star_{f,j=1}^n F_j)^c$) is obtained from $\star_{f,j=1}^n F_j$ by subtraction of the contributions of all disconnected diagrams; this

⁹In the literature proper vertices (or the 'proper interaction') are sometimes called 'effective vertices' (or 'effective interaction' resp.). However, differently to what we are doing here, the notion 'effective field theory' usually means an *approximation* to the perturbation series. For this reason we omit the word 'effective' and use the terminology of Sect. 6-2-2 in [25].

gives the recursion relation (see e.g. [15])

$$(\star_{f,j=1}^n F_j)^c = \star_{f,j=1}^n F_j - \sum_{|P| \geq 2} \prod_{J \in P} (\star_{f,j \in J} F_j)^c, \quad (4.1)$$

where the sum runs over all partitions P of $\{1, \dots, n\}$ in at least two subsets and \prod means the classical product. One easily sees that the linked cluster theorem applies to \star_f :

$$e_{\star_f}^F = \exp_{\bullet}(e_{\star_f}^F), \quad \text{where} \quad e_{\star_f}^F \stackrel{\text{def}}{=} 1 + \sum_{n=1}^{\infty} \frac{F^{\star_f n}}{n!}, \quad e_{\star_f}^F \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{(F^{\star_f n})^c}{n!} \quad (4.2)$$

(with $F^{\star_f n}$ being the n -fold product $F \star_f \dots \star_f F$) and \exp_{\bullet} denotes the exponential function with respect to the classical product.

Analogously to (4.1) the contribution of all 1PI-diagrams to $\star_{f,j=1}^n F_j$ (denoted by $(\star_{f,j=1}^n F_j)^{1\text{PI}}$), is obtained from the connected diagrams $(\star_{f,j=1}^n F_j)^c$ by subtracting the contributions of all connected one-particle-reducible diagrams. To formulate this we need the contribution of all connected tree diagrams to $(\star_{f,j=1}^n F_j)^c$, which we denote by $(\star_{f,j=1}^n F_j)_{\text{tree}}^c$. This diagrammatic definition of $(\star_{f,j=1}^n F_j)_{\text{tree}}^c$ fulfills the following unique and independent characterizations:

- *By recursion:* one easily finds that the connected tree diagrams satisfy the recursion relation

$$\begin{aligned} (\star_{f,j=1}^{n+1} F_j)_{\text{tree}}^c &= \sum_{k=1}^n \int dx_1 \dots dx_k dy_1 \dots dy_k \frac{\delta^k F_{n+1}}{\delta \varphi(x_1) \dots \delta \varphi(x_k)} \\ &\prod_{j=1}^k f(x_j - y_j) \frac{1}{k!} \sum_{I_1 \sqcup \dots \sqcup I_k = \{1, \dots, n\}} \frac{\delta}{\delta \varphi(y_1)} (\star_{f,j \in I_1} F_j)_{\text{tree}}^c \dots \\ &\quad \cdot \frac{\delta}{\delta \varphi(y_k)} (\star_{f,j \in I_k} F_j)_{\text{tree}}^c, \end{aligned} \quad (4.3)$$

where $I_j \neq \emptyset \forall j$, \sqcup means the disjoint union and the product of the $\frac{\delta}{\delta \varphi(y_l)} (\star_{f,j \in I_l} F_j)_{\text{tree}}^c$, $l = 1, \dots, k$, is the classical one. (Note that in the sum over I_1, \dots, I_k the succession of I_1, \dots, I_k is distinguished and, hence, there is a factor of $\frac{1}{k!}$.)

- *By the power in \hbar :* as explained in Sect. 5.2 of [15] it holds

$$(\star_{f,j=1}^n F_j)^c = \mathcal{O}(\hbar^{n-1}) \quad \text{for} \quad F_1, \dots, F_n \sim \hbar^0, \quad (4.4)$$

and the contribution of all tree diagrams is given by the terms of lowest order in \hbar

$$(\star_{f,j=1}^n F_j)_{\text{tree}}^c = \hbar^{n-1} \lim_{\hbar \rightarrow 0} \hbar^{-(n-1)} (\star_{f,j=1}^n F_j)^c. \quad (4.5)$$

The contribution of all tree diagrams to $\star_{f,j=1}^n F_j$, which we denote by $(\star_{f,j=1}^n F_j)_{\text{tree}}$, is related to the connected tree diagrams by the linked cluster theorem: $e_{\star_{f,\text{tree}}}^F = \exp_{\bullet}(e_{\star_{f,\text{tree}}}^F)$. (This follows immediately from (4.2) by selecting all tree diagrams.)

We now give the (above announced) unique recursive characterization of the 1PI-diagrams:

$$(\star_{f,j=1}^n F_j)^{1\text{PI}} = (\star_{f,j=1}^n F_j)^c - \sum_{|P| \geq 2} \left(\star_{f,J \in P} (\star_{f,j \in J} F_j)^{1\text{PI}} \right)_{\text{tree}}^c. \quad (4.6)$$

The formulas (4.1), (4.4) and (4.5) hold also for the usual \star -product (i.e. with H_m instead of f) [15]; but (4.6) needs to be refined, because H_m is not symmetrical and, hence, \star_{H_m} is not commutative (in particular $(F_1 \star_{H_m} \dots \star_{H_m} F_n)_{\text{tree}}^c$ is not symmetrical).

Turning to the *time ordered product* T , we will use (4.1) and (4.6) as motivation for the (recursive) *definition* of the connected part T^c and the 1PI-part $T^{1\text{PI}}$ of T , respectively. So we define [15]

$$T^c(\otimes_{j=1}^n F_j) \stackrel{\text{def}}{=} T(\otimes_{j=1}^n F_j) - \sum_{|P| \geq 2} \prod_{J \in P} T^c(\otimes_{j \in J} F_j). \quad (4.7)$$

It follows that T and T^c are related by the linked cluster theorem (4.2): $T(e_{\otimes}^{iF}) = \exp_{\bullet}(T^c(e_{\otimes}^{iF}))$.

The following definition of the connected tree part T_{tree}^c of T applies also to *non-local entries*.¹⁰

$$T_{\text{tree},n}^c : \mathcal{F}^{\otimes n} \rightarrow \mathcal{F}; \quad T_{\text{tree},n}^c(\otimes_{j=1}^n F_j) = (F_1 \star_{\Delta^F} \dots \star_{\Delta^F} F_n)_{\text{tree}}^c, \quad (4.8)$$

i.e. we replace in the definition of $(\star_f \dots)_{\text{tree}}^c$ the smooth function f by the Feynman propagator

$$\begin{aligned} \Delta^F(z) &\equiv \Delta_m^F(z) = \Theta(z^0) H_m^\mu(z) + \Theta(-z^0) H_m^\mu(-z) \\ &= -i \Delta_m^{\text{ret}}(z) + H_m^\mu(-z) = \Delta^F(-z). \end{aligned} \quad (4.9)$$

For tree diagrams the resulting expressions are well defined, since pointwise products of Feynman propagators do not appear. Obviously T_{tree}^c fulfills the recursion relation (4.3). In case of local entries another unique characterization of T_{tree}^c is possible: doing renormalization individually in each order in \hbar , T^c satisfies (4.4) and $\hbar^{-(n-1)} T_{\text{tree},n}^c$

¹⁰In QCD the interaction $S = \kappa S_1 + \kappa^2 S_2$ is a sum of a term of first order in the coupling constant κ , $S_1 \sim \int g A A \partial A$ ($g \in \mathcal{D}(\mathbb{M}, \mathbb{R})$), and a term of second order in κ , $S_2 \sim \int g^2 A A A A$. One can achieve that the order in κ agrees with the order of the T - (or R -)product [26]. Namely, one starts with $T_1(S_1)$, the term S_2 is generated by a non-trivial renormalization of a certain tree diagram: in $T_2(S_1^{\otimes 2}) \sim \int dx dy g A A(x) g A A(y) \partial \partial \Delta^F(x-y) + \dots$ the propagator $\partial^\mu \partial^\nu \Delta^F(x-y)$ is replaced by $\partial^\mu \partial^\nu \Delta^F(x-y) - 1/2 g^{\mu\nu} \delta(x-y)$. Due to the inductive procedure of causal perturbation theory this additional term propagates to higher orders such that this modified T -product, T^N , yields the same S -matrix: $T^N(e_{\otimes}^{i\kappa S_1}) = T(e_{\otimes}^{i(\kappa S_1 + \kappa^2 S_2)})$ in the sense of formal power series in κ . (The corresponding renormalization map D (3.22) is given in [27].) Our definitions of T_{tree}^c (4.8) and T_{tree} (4.10) do not contain this $1/2 g^{\mu\nu} \delta$ -term, in agreement with the definition of R_{cl} (2.17). Generally, in this paper *all* terms S_n of the interaction $S = \sum_{n \geq 1} \kappa^n S_n$ enter the perturbative construction of the S -matrix (or interacting field) already to first order of the T - (or R -)product.

is the classical limit of $\hbar^{-(n-1)} T_n^c$ similarly to (4.5). (This is the translation of (3.20) into T -products, see Sect. 5.2 of [15].) Analogously one defines the tree part T_{tree} of T by

$$T_{\text{tree},n}(\otimes_{j=1}^n F_j) \stackrel{\text{def}}{=} (F_1 \star_{\Delta^F} \dots \star_{\Delta^F} F_n)_{\text{tree}} , \quad F_j \in \mathcal{F} \ \forall j . \quad (4.10)$$

Obviously the linked cluster theorem for $(\star_{f,\text{tree}}, \star_{f,\text{tree}}^c)$ is valid also for $f = \Delta^F$: $T_{\text{tree}}(e_{\otimes}^{iF}) = \exp_{\bullet}(T_{\text{tree}}^c(e_{\otimes}^{iF}))$.

Since T_{tree}^c is totally symmetric we may define T^{1PI} in analogy to (4.6) by the recursive formula

$$T^{1PI}(F^{\otimes n}) \stackrel{\text{def}}{=} T^c(F^{\otimes n}) - \sum_{k=2}^n \sum_{l_1+\dots+l_k=n, l_j \geq 1 \ \forall j} \frac{n!}{k! l_1! \dots l_k!} \cdot T_{\text{tree},k}^c \left(T^{1PI}(F^{\otimes l_1}) \otimes \dots \otimes T^{1PI}(F^{\otimes l_k}) \right) . \quad (4.11)$$

The renormalization conditions listed in Sect. 3 are satisfied by $T_{\text{tree}}^c \equiv T_{\text{tree}}^{c(m,\mu)}$, $T_{\text{tree}} \equiv T_{\text{tree}}^{(m,\mu)}$ and, provided that T fulfills these conditions, also by $T^{1PI} \equiv T^{1PI(m,\mu)}$ (apart from the Field Equation). This can be verified by using the definitions (4.8), (4.10), (4.11) and corresponding properties of the Feynman propagator. Or, in case of local interactions, these properties can be derived for T_{tree}^c by the classical limit of $\hbar^{-(n-1)} T_n^c$; the linked cluster theorem implies then their validity for T_{tree} .

4.2 Definition and basic properties of the vertex function Γ_T

Note that T and T_{tree} satisfy the relations

$$T_{(\text{tree})}(1) = 1 , \quad T_{(\text{tree})}(F) = F , \quad T_{(\text{tree})n+1}(1 \otimes F_1 \otimes \dots) = T_{(\text{tree})n}(F_1 \otimes \dots) , \quad (4.12)$$

which imply the following conclusions for T_{tree} and T

$$T_{(\text{tree})} \left(e^{\sum_{n=1}^{\infty} F_n \lambda^n} \otimes \sum_{n=0}^{\infty} G_n \lambda^n \right) = 0 \implies G_n = 0 \ \forall n \quad (4.13)$$

$$T_{(\text{tree})} \left(e^{\sum_{n=1}^{\infty} F_n \lambda^n} \right) = T_{(\text{tree})} \left(e^{\sum_{n=1}^{\infty} G_n \lambda^n} \right) \implies F_n = G_n \ \forall n \quad (4.14)$$

(where $F_n, G_n \in \mathcal{F}$ are independent of λ), as one obtains by proceeding by induction on the order in λ .

R - and T -products can be obtained from each other by Bogoliubov's formula (3.21), which we will use in the more explicit form

$$R(e_{\otimes}^S, F) = \bar{T}(e_{\otimes}^{-iS/\hbar}) \star T(e_{\otimes}^{iS/\hbar} \otimes F) , \quad (4.15)$$

where the anti-chronological product \bar{T} is defined by

$$\bar{T}(e_{\otimes}^{-iG}) \equiv T(e_{\otimes}^{iG})^{-1} = \sum_{n=0}^{\infty} \left(1 - T(e_{\otimes}^{iG}) \right)^{\star n} . \quad (4.16)$$

($T(e_{\otimes}^{iG})^{-1}$ is the inverse with respect to the \star -product.) Although R contains solely connected diagrams (see Appendix A), disconnected diagrams of T and \bar{T} contribute to (4.15). Unitarity reads

$$\bar{T}_{(\text{tree})}(e_{\otimes}^{-iG}) = T_{(\text{tree})}(e_{\otimes}^{iG^*})^* ; \quad (4.17)$$

in this form it holds for the tree diagrams separately, where \bar{T}_{tree} is defined by (4.10) with Δ^F replaced by the Anti-Feynman propagator $\Delta^{AF}(x) = \Delta^F(x)^*$. However, note that $T_{\text{tree}}(e_{\otimes}^{iG}) \star T_{\text{tree}}(e_{\otimes}^{iG^*})^*$ is not equal to 1.

We define the 'vertex function' Γ_T implicitly by the following Proposition:

Proposition 4. *There exists a totally symmetric and linear map*

$$\Gamma_T : \mathbb{T}\mathcal{F}_{\text{loc}} \rightarrow \mathcal{F} \quad (4.18)$$

which is uniquely determined by

$$T(e_{\otimes}^{iS/\hbar}) = T_{\text{tree}}\left(e_{\otimes}^{i\Gamma_T(e_{\otimes}^S)/\hbar}\right). \quad (4.19)$$

To zeroth and first order in S we obtain

$$\Gamma_T(1) = 0, \quad \Gamma_T(S) = S. \quad (4.20)$$

The defining relation (4.19) also implies

$$T(e_{\otimes}^{iS/\hbar} \otimes F) = T_{\text{tree}}\left(e_{\otimes}^{i\Gamma_T(e_{\otimes}^S)/\hbar} \otimes \Gamma_T(e_{\otimes}^S \otimes F)\right). \quad (4.21)$$

For $S = 0$ this gives $F = \Gamma_T(1 \otimes F)$. The Proposition remains true if, in (4.19), we replace the time ordered product (T, T_{tree}) by the anti-chronological product $(\bar{T}, \bar{T}_{\text{tree}})$; we denote the corresponding vertex function by $\Gamma_{\bar{T}}$.

Proof. We construct $\Gamma_T(\otimes_{j=1}^n F_j)$ by induction on n , starting with (4.20). Let Γ_T of less than n factors be constructed. Then, (4.19) and the requirements total symmetry and linearity determines $\Gamma_T(\otimes_{j=1}^n F_j)$ uniquely:

$$\Gamma_T(\otimes_{j=1}^n F_j) = (i/\hbar)^{n-1} T(\otimes_{j=1}^n F_j) - \sum_{|P| \geq 2} (i/\hbar)^{|P|-1} T_{\text{tree}}\left(\bigotimes_{J \in P} \Gamma_T(\otimes_{j \in J} F_j)\right), \quad (4.22)$$

where P is a partition of $\{1, \dots, n\}$ in $|P|$ subsets J . Obviously the so constructed Γ_T is totally symmetric and linear. \square

The proof applies also to the connected parts (T^c, T_{tree}^c) . Hence, a totally symmetric and linear map Γ_c can be defined analogously to (4.18)-(4.19), that is by

$$T^c(e_{\otimes}^{iS/\hbar}) = T_{\text{tree}}^c\left(e_{\otimes}^{i\Gamma_c(e_{\otimes}^S)/\hbar}\right). \quad (4.23)$$

The linked cluster theorem for T and T_{tree} and the definitions of Γ_T and Γ_c give

$$T_{\text{tree}}\left(e_{\otimes}^{i\Gamma_c(e_{\otimes}^S)/\hbar}\right) = T_{\text{tree}}\left(e_{\otimes}^{i\Gamma_T(e_{\otimes}^S)/\hbar}\right) \quad (4.24)$$

and with (4.14) we conclude

$$\Gamma_T = \Gamma_c . \quad (4.25)$$

Therefore, on the right-hand side of (4.22) we may replace the time ordered products T and $T_{\text{tree},k}$ by their connected parts:

$$\begin{aligned} \Gamma_T(S^{\otimes n}) &= (i/\hbar)^{n-1} T^c(S^{\otimes n}) - \sum_{k=2}^n \sum_{\substack{l_1 + \dots + l_k = n \\ l_j \geq 1 \forall j}} \frac{(i/\hbar)^{k-1} n!}{k! l_1! \dots l_k!} \\ &\quad \cdot T_{\text{tree},k}^c \left(\Gamma_T(S^{\otimes l_1}) \otimes \dots \otimes \Gamma_T(S^{\otimes l_k}) \right) . \end{aligned} \quad (4.26)$$

Now let $S \sim \hbar^0$ and $F \sim \hbar^0$. From (4.26) and $T^c(S^{\otimes n}) - T_{\text{tree}}^c(S^{\otimes n}) = \mathcal{O}(\hbar^n)$ we inductively conclude

$$\Gamma_T(e_{\otimes}^S) = S + \mathcal{O}(\hbar) , \quad \Gamma_T(e_{\otimes}^S \otimes F) = F + \mathcal{O}(\hbar). \quad (4.27)$$

Motivated by this relation and (4.19) we call $\Gamma_T(e_{\otimes}^S)$ the 'proper interaction'.

By comparing the recursion relation (4.26) for Γ_T with the recursive definition of T^{1PI} (4.11) we conclude:

Corollary 5.

$$\Gamma_T(e_{\otimes}^S) = (\hbar/i) T^{\text{1PI}}(e_{\otimes}^{iS/\hbar}) . \quad (4.28)$$

Analogously to the Main Theorem it holds:

Lemma 6. *The validity of the renormalization conditions for $T(\equiv T^{(m,\mu)})$ implies corresponding properties of $\Gamma_T(\equiv \Gamma_T^{(m,\mu)})$:*

- **\mathcal{P}_+^\dagger -Covariance:** $\beta_L \circ \Gamma_T = \Gamma_T \circ \mathbb{T}\beta_L$ for all $L \in \mathcal{P}_+^\dagger$;
- **Unitarity:** $\Gamma_T(e_{\otimes}^S)^* = \bar{\Gamma}_T(e_{\otimes}^{S^*})$;
- **Field Independence:** $\frac{\delta \Gamma_T(e_{\otimes}^S)}{\delta \varphi} = \Gamma_T\left(\frac{\delta S}{\delta \varphi} \otimes e_{\otimes}^S\right)$;
- **Field Equation:** $\Gamma_T(e_{\otimes}^S \otimes \varphi(h)) = \varphi(h)$;
- **Smoothness in $m \geq 0$:** $\Gamma_T^{(m,\mu)}$ is smooth in $m \geq 0$;
- **μ -Covariance:** $\Gamma_T^{(m,\mu_2)} = (\mu_2/\mu_1)^\Gamma \circ \Gamma_T^{(m,\mu_1)} \circ \mathbb{T}(\mu_2/\mu_1)^{-\Gamma}$;
- **Almost homogeneous Scaling:** In contrast to the map D of the Main Theorem, Γ_T scales only almost homogeneously; $\sigma_\rho \circ \Gamma_T^{(\rho^{-1}m,\mu)} \circ \mathbb{T}\sigma_\rho^{-1} = \Gamma_T^{(m,\mu)} + \mathcal{O}(\log \rho)$ is a polynomial in $\log \rho$.
- If, instead of Smoothness in m , μ -Covariance and Almost homogeneous Scaling, T satisfies the axiom **Scaling Degree**, then

$$\text{sd}\left(\omega_0(\Gamma_T(A_1, \dots, A_n))(x_1 - x_n, \dots)\right) \leq \sum_{j=1}^n \dim(A_j) , \quad \forall A_j \in \mathcal{P}_{\text{hom}} . \quad (4.29)$$

Proof. Each property can be proved for $\Gamma_T(S^{\otimes n})$ (or $\Gamma_T(S^{\otimes n} \otimes \varphi(h))$ resp.) by induction on n : we work with the recursion relation (4.22) and use that T and T_{tree} satisfy the corresponding axiom. In case of the property Scaling Degree we take into account that $\omega_0(T_{\text{tree}}(\dots))$ is a tensor product of distributions t_j and apply $\text{sd}(\otimes_j t_j) = \sum_j \text{sd}(t_j)$.

Only the Field Equation is somewhat more involved. We use that T and T_{tree} fulfil the Field Equation and the Field Independence. This implies

$$T^c(\varphi(h) \otimes F_1 \otimes \dots \otimes F_n) = \hbar \int dx dy h(x) \Delta^F(x-y) \frac{\delta}{\delta \varphi(y)} T^c(F_1 \otimes \dots \otimes F_n) \quad (4.30)$$

and the same equation for T_{tree}^c . In the latter case F_1, \dots, F_n may be non-local. (It is not necessary to work with the connected parts, but this simplifies the formulas.) With the recursion relation (4.26) and the inductive assumption we obtain

$$\begin{aligned} \Gamma_T(\varphi(h) \otimes F_1 \otimes \dots \otimes F_n) &= T^c(\varphi(h) \otimes F_1 \otimes \dots \otimes F_n) - \sum_P T_{\text{tree}}^c \left(\varphi(h) \otimes \bigotimes_{J \in P} \Gamma_T(F_J) \right) \\ &= \hbar \int dx dy h(x) \Delta^F(x-y) \frac{\delta}{\delta \varphi(y)} \left(T^c(F_1 \otimes \dots \otimes F_n) - \sum_P T_{\text{tree}}^c \left(\bigotimes_{J \in P} \Gamma_T(F_J) \right) \right) = 0, \end{aligned} \quad (4.31)$$

where $F_J \equiv \otimes_{j \in J} F_j$ and P runs through all partitions of $\{1, \dots, n\}$. \square

Analogously to the conventions for R - and T -products we sometimes write $\int dx g(x) \Gamma_T(A(x) \otimes F_2 \dots)$ for $\Gamma_T(\int dx g(x) A(x) \otimes F_2 \dots)$. Since Γ_T depends only on the *functionals*, it fulfills the AWI: $\partial_x^\mu \Gamma_T(A(x) \otimes F_2 \dots) = \Gamma_T(\partial^\mu A(x) \otimes F_2 \dots)$.

In the proper vertex formalism a finite renormalization $T \rightarrow \hat{T}$ of the T -products is reflected in a finite renormalization $\Gamma_T \rightarrow \hat{\Gamma}_T$ of the corresponding vertex functions. To derive this we insert $\hat{T}(e_\otimes^{iS/\hbar}) = T_{\text{tree}}(e_\otimes^{i\hat{\Gamma}_T(e_\otimes^S)/\hbar})$ and (4.19) into (3.28) and obtain

$$T_{\text{tree}}(e_\otimes^{i\hat{\Gamma}_T(e_\otimes^S)/\hbar}) = T_{\text{tree}}(e_\otimes^{i\Gamma_T(e_\otimes^{D(e_\otimes^S)})/\hbar}). \quad (4.32)$$

(Note that the tree part T_{tree} is independent of the normalization of the T -product.) By using (4.14) we conclude

$$\hat{\Gamma}_T(e_\otimes^S) = \Gamma_T(e_\otimes^{D(e_\otimes^S)}). \quad (4.33)$$

4.3 Comparison with the literature

Definition of the 'vertex functional' (or 'proper function') Γ in the literature, see e.g. [10]. Usually $\Gamma(h)$, $h \in \mathcal{S}(\mathbb{M}, \mathbb{R})$, is defined as the Legendre transformed $j \rightarrow h$ of the generating functional $Z(j)$ of the connected Green's functions (where j is the 'classical source' of φ). With that $\Gamma(h)$ is the generating functional of the 1PI-diagrams of $T(e_\otimes^{iS})$ (see [28]). We are going to express the latter fact in our formalism. To simplify the notations we study a scalar field φ with free action

$S_0(\varphi) = 1/2 \int dx ((\partial\varphi(x))^2 - m^2 (\varphi(x))^2)$. Green's functions are obtained by the Gell-Mann Low formula [29], which contains the adiabatic limit $g \rightarrow 1$:

$$G(x_1, \dots, x_n) = \lim_{g \rightarrow 1} \frac{\omega_0 \left(T(\varphi(x_1) \dots \varphi(x_n)) e^{iS(g)/\hbar} \right)}{\omega_0 \left(T(e^{iS(g)/\hbar}) \right)}, \quad (4.34)$$

where $S(g) = \sum_{n=1}^{\infty} \kappa^n \int dx (g(x))^n \mathcal{L}_n(x)$ and κ is the coupling constant. All diagrams with vacuum-subdiagrams are divided out. These diagrams are disconnected and, hence, not of interest for our purposes. Namely, to obtain the vertex functional Γ one selects all diagrams of $G(x_1, \dots, x_n)$ which are 1PI after amputation of the external legs. The contribution of these diagrams is given by

$$G^{1\text{PI}}(x_1, \dots, x_n) = \lim_{g \rightarrow 1} \int dy_1 \dots dy_n \Delta_F(x_1 - y_1) \dots \Delta_F(x_n - y_n) \cdot \omega_0 \left(\frac{\delta^n}{\delta\varphi(y_1) \dots \delta\varphi(y_n)} T^{1\text{PI}}(e^{iS(g)/\hbar}) \right) + \delta_{n,2} \Delta^F(x_1 - x_2). \quad (4.35)$$

From this expression $\Gamma(h)$ is obtained by replacing each external leg $\Delta_F(x_l - y_l)$ by the classical field $h(y_l)$. In addition one multiplies with $(-i)/n!$ and sums over $n \geq 1$:¹¹

$$\Gamma(h) = S_0(h) + \frac{\hbar}{i} \lim_{g \rightarrow 1} \sum_{n \geq 1} \frac{1}{n!} \int dy_1 \dots dy_n h(y_1) \dots h(y_n) \cdot \omega_0 \left(\frac{\delta^n}{\delta\varphi(y_1) \dots \delta\varphi(y_n)} T^{1\text{PI}}(e^{iS(g)/\hbar}) \right). \quad (4.36)$$

Note that the term $S_0(h)$ comes from the $\delta_{n,2} \Delta^F$ -term in (4.35).

Relation to our proper interaction $\Gamma_T(e_{\otimes}^{S(g)})$: We compare (4.36) with the Taylor expansion in φ of $\Gamma_T(e_{\otimes}^{S(g)})$:

$$\Gamma_T(e_{\otimes}^{S(g)}) = \sum_{n \geq 0} \frac{1}{n!} \int dy_1 \dots dy_n \varphi(y_1) \dots \varphi(y_n) \omega_0 \left(\frac{\delta^n}{\delta\varphi(y_1) \dots \delta\varphi(y_n)} \Gamma_T(e_{\otimes}^{S(g)}) \right) \quad (4.37)$$

and use Corollary 5. This yields

$$\Gamma(h) = S_0(h) + \lim_{g \rightarrow 1} \left(\Gamma_T(e_{\otimes}^{S(g)})(h) - \omega_0(\Gamma_T(e_{\otimes}^{S(g)})) \right). \quad (4.38)$$

(On the right-hand side the functionals $S_0, \Gamma_T(e_{\otimes}^{S(g)}) \in \mathcal{F}$ are evaluated on the classical field configuration $h \in \mathcal{S}(\mathbb{M}, \mathbb{R})$.)

5 The Master Ward Identity

5.1 The classical MWI in the off-shell formalism

In [1] the MWI for on-shell fields (i.e. the retarded products are restricted to the solutions of the free field equation(s)) was derived in the framework of classical field

¹¹Usually it is assumed that $\lim_{g \rightarrow 1} \omega_0(\varphi_{g\mathcal{L}}(x)) = 0$, which implies $\frac{\delta\Gamma(h)}{\delta h(x)}|_{h=0} = 0$, i.e. the sum runs only over $n \geq 2$.

theory. Since here, we work throughout in a general off-shell formalism [16], we shall derive an off-shell version of the classical MWI. In addition, we give an equivalent formulation of the classical MWI in terms of T_{tree} -products that will be useful for the proper field formulation of the MWI in Sect. 5.3.

The classical off-shell MWI follows from the factorization (2.16) and the definition of the retarded wave operators. Let \mathcal{J} be the ideal generated by the free field equation(s),

$$\mathcal{J} \stackrel{\text{def}}{=} \left\{ \sum_{n=1}^N \int dx_1 \dots dx_n \varphi(x_1) \dots \varphi(x_{n-1}) \frac{\delta S_0}{\delta \varphi(x_n)} f_n(x_1, \dots, x_n) \right\} \subset \mathcal{F},$$

with $N < \infty$ and the f_n 's being defined as in (2.2). Every $A \in \mathcal{J}$ can be written as

$$A \stackrel{\text{def}}{=} \int dx Q(x) \frac{\delta S_0}{\delta \varphi(x)}, \quad (5.1)$$

where Q is of the form

$$Q(x) = \sum_{n=0}^N \int dx_1 \dots dx_n \varphi(x_1) \dots \varphi(x_n) f_{n+1}(x_1, \dots, x_n, x). \quad (5.2)$$

Note that in the present framework of classical field theory Q does not need to be a local functional. Given $A \in \mathcal{J}$ we introduce a corresponding derivation [1]

$$\delta_A \stackrel{\text{def}}{=} \int dx Q(x) \frac{\delta}{\delta \varphi(x)}. \quad (5.3)$$

From the defining property of the retarded wave operators Def. 1 (ii) we obtain

$$\begin{aligned} (A + \delta_A S) \circ r_{S_0+S, S_0} &= \int dx Q(x) \circ r_{S_0+S, S_0} \frac{\delta(S_0 + S)}{\delta \varphi(x)} \circ r_{S_0+S, S_0} \\ &= \int dx Q(x) \circ r_{S_0+S, S_0} \frac{\delta S_0}{\delta \varphi(x)}, \end{aligned} \quad (5.4)$$

which reads perturbatively

$$R_{\text{cl}}(e_{\otimes}^S, A + \delta_A S) = \int dx R_{\text{cl}}(e_{\otimes}^S, Q(x)) \frac{\delta S_0}{\delta \varphi(x)} \in \mathcal{J}. \quad (5.5)$$

This is the MWI written in the general off-shell formalism. Indeed, by restricting (5.5) on solutions of the free field equation, the right-hand side vanishes and we obtain the on-shell version of the MWI, as it was derived in [1]. Note that for the simplest case $Q = 1$ the MWI reduces to the off-shell version of the (interacting) field equation

$$R_{\text{cl}}\left(e_{\otimes}^S, \frac{\delta(S_0 + S)}{\delta \varphi(x)}\right) = \frac{\delta S_0}{\delta \varphi(x)}, \quad (5.6)$$

which is an alternative formulation of the axiom Field Equation in Sect. 3.

The classical field equation (5.6) can be expressed in the time ordered formalism:

$$T_{\text{tree}}\left(e_{\otimes}^{iS} \otimes \frac{\delta(S_0 + S)}{\delta\varphi(x)}\right) = \frac{\delta S_0}{\delta\varphi(x)} . \quad (5.7)$$

This identity holds even for non-local entries and can be obtained easily by using the definition of T_{tree} given in (4.10) and the fact that Δ^F is a Green's function of the Klein Gordon operator. Similarly to R_{cl} , the tree diagrams of the time ordered product factorize, that is

$$T_{\text{tree}}\left(e_{\otimes}^{iS} \otimes F\right) \cdot T_{\text{tree}}\left(e_{\otimes}^{iS} \otimes G\right) = T_{\text{tree}}\left(e_{\otimes}^{iS} \otimes FG\right) . \quad (5.8)$$

We now multiply the field equation for T_{tree} with $T_{\text{tree}}(e_{\otimes}^{iS} \otimes Q(x))$ and integrate afterwards over x . This yields the MWI in the time ordered formalism:

$$T_{\text{tree}}\left(e_{\otimes}^{iS} \otimes (A + \delta_A S)\right) = \int dx T_{\text{tree}}\left(e_{\otimes}^{iS} \otimes Q(x)\right) \cdot \frac{\delta S_0}{\delta\varphi(x)} . \quad (5.9)$$

We point out that the MWI for T_{tree} (5.9) holds also for non-local entries S , $Q(x)$ and A .

5.2 Structure of possible anomalies of the MWI in pQFT

The classical MWI was derived for arbitrary interaction $S \in \mathcal{F}$ and arbitrary $A \in \mathcal{J}$. For *local* functionals $A \in \mathcal{J}_{\text{loc}} \equiv \mathcal{J} \cap \mathcal{F}_{\text{loc}}$ and $S \in \mathcal{F}_{\text{loc}}$ it can be transferred formally into pQFT (by the replacement $R_{\text{cl}} \rightarrow R$), where it serves as an additional, highly non-trivial renormalization condition. It is impossible to fulfill this condition for all $A \in \mathcal{J}_{\text{loc}}$. We aim to find a general expression for the possible violations ('anomalies') of the MWI. Later we will use this result as starting point for a proof of relevant cases of the MWI. This procedure is motivated by algebraic renormalization, where the QAP serves as the crucial input to prove Ward identities (see [10] and references cited therein).

The main insight into the structure of possible anomalies of the MWI is the fact that they can be expressed in terms of a *local* interacting field:

Theorem 7. *Given a retarded product R fulfilling the basic axioms Initial Condition, Causality and GLZ Relation and given a local functional*

$$A = \int dx h(x) Q(x) \frac{\delta S_0}{\delta\varphi(x)} \in \mathcal{J}_{\text{loc}} , \quad h \in \mathcal{D}(\mathbb{M}) , \quad Q \in \mathcal{D}(\mathbb{M}, \mathcal{P}) , \quad (5.10)$$

there exists a unique, linear and symmetric map

$$\begin{aligned} \Delta_A : \mathbb{T}\mathcal{F}_{\text{loc}} &\longrightarrow \mathcal{F}_{\text{loc}} \\ F_1 \otimes \cdots \otimes F_n &\longmapsto \Delta_A(F_1 \otimes \cdots \otimes F_n) \end{aligned} \quad (5.11)$$

which is implicitly defined by the 'anomalous MWI'

$$R(e_{\otimes}^S, A + \delta_A S + \Delta_A(e_{\otimes}^S)) = \int dy h(y) R(e_{\otimes}^S, Q(y)) \frac{\delta S_0}{\delta\varphi(y)} . \quad (5.12)$$

As a consequence of (5.12) the map Δ_A has the following properties:

- (i) Δ_A depends linearly on A ;
- (ii) $\text{supp}\left(\frac{\delta\Delta_A(F_1 \otimes \cdots \otimes F_n)}{\delta\varphi}\right) \subset \bigcap_{i=1}^n \text{supp}\left(\frac{\delta F_i}{\delta\varphi}\right), \quad F_i \in \mathcal{F}_{\text{loc}} \quad (\text{locality})$;
- (iii) $\Delta_A(1) = 0$;
- (iv) $\Delta_A \equiv 0 \Leftrightarrow R(e_{\otimes}^S, A + \delta_A S) = \int dx h(x) R(e_{\otimes}^S, Q(x)) \frac{\delta S_0}{\delta\varphi(x)}, \forall S \in \mathcal{F}_{\text{loc}};$
- (v) $\Delta_A(F_1 \otimes \cdots \otimes F_n) = \mathcal{O}(\hbar) \quad \forall n > 0, F_i \sim \hbar^0$,

(5.13)

and

- (vi) We set $\Delta_A^n \equiv \Delta_A|_{\mathcal{F}_{\text{loc}}^{\otimes n}}$. For $g_j \in \mathcal{D}(\mathbb{M})$, $L_j \in \mathcal{P}$ it holds

$$\begin{aligned} \Delta_A^n(L_1(g_1) \otimes \cdots \otimes L_n(g_n)) &= \int dx_1 \dots dx_n dy g_1(x_1) \dots g_n(x_n) h(y) \\ &\quad \Delta^n(L_1(x_1) \otimes \cdots \otimes L_n(x_n); Q(y)) , \end{aligned} \quad (5.14)$$

where the distributional kernel $\Delta^n(L_1(x_1) \otimes \cdots; Q(y))$ is inductively given by

$$\begin{aligned} \Delta^n(L_1(x_1) \otimes \cdots \otimes L_n(x_n); Q(y)) &= -R\left(\bigotimes_{j=1}^n L_j(x_j); Q(y) \cdot \frac{\delta S_0}{\delta\varphi(y)}\right) \\ &\quad - \sum_{l=1}^n R\left(\bigotimes_{j(\neq l)} L_j(x_j); Q(y) \sum_a (\partial^a \delta)(x_l - y) \frac{\partial L_l}{\partial(\partial^a \varphi)}(x_l)\right) \\ &\quad - \sum_{I \subset \{1, \dots, n\}, I \neq \emptyset} R\left(\bigotimes_{i \in I} L_i(x_i); \Delta^{|I^c|}(\bigotimes_{j \in I^c} L_j(x_j); Q(y))\right) \\ &\quad + R\left(\bigotimes_{j=1}^n L_j(x_j); Q(y)\right) \cdot \frac{\delta S_0}{\delta\varphi(y)} . \end{aligned} \quad (5.15)$$

Note that (5.12) differs from the MWI (5.5) only by the local term $\Delta_A(e_{\otimes}^S)$, which clearly depends on the chosen normalization of the retarded products. Therefore, property (iv) means that the MWI for A is fulfilled if and only if the corresponding map Δ_A vanishes identically.

Proof. To show the existence and uniqueness of Δ_A we construct its components Δ_A^n by induction on n using (5.12). In this inductive procedure we also prove the properties (i), (ii), (iii) and (vi). To lowest order in S the condition (5.12) gives $\Delta_A(1) \stackrel{\text{def}}{=} 0$. Given $n > 0$, we assume the existence and uniqueness of linear and symmetrical maps $\Delta_A^k : \mathcal{F}_{\text{loc}}^{\otimes k} \rightarrow \mathcal{F}_{\text{loc}}$, $0 < k < n$, which depend linearly on A , are local and satisfy (vi), such that (5.12) is fulfilled to all lower orders in S :

$$\begin{aligned} R(S^{\otimes k}, A) + kR(S^{\otimes k-1}, \delta_A S) + \sum_{l=0}^k \binom{k}{l} R(S^{\otimes k-l}, \Delta_A^l(S^{\otimes l})) = \\ \int dx h(x) R(S^{\otimes k}, Q(x)) \frac{\delta S_0}{\delta\varphi(x)} \end{aligned} \quad (5.16)$$

for all $k < n$. We define Δ_A^n in terms of the inductively known Δ_A^k , $k < n$:

$$\begin{aligned} \Delta_A^n(F_1 \otimes \cdots \otimes F_n) &\stackrel{\text{def}}{=} \int dx h(x) R(F_1 \otimes \cdots \otimes F_n, Q(x)) \frac{\delta S_0}{\delta \varphi(x)} \\ &- \left(R(F_1 \otimes \cdots \otimes F_n, A) + \sum_{k=1}^n R(\otimes_{i \in \underline{n} \setminus \{k\}} F_i, \delta_A F_k) \right. \\ &\quad \left. + \sum_{I \sqcup J = \underline{n}, |J| < n} R(\otimes_{i \in I} F_i, \Delta_A^{|J|}(\otimes_{j \in J} F_j)) \right), \end{aligned} \quad (5.17)$$

where we used the notation $\underline{n} \stackrel{\text{def}}{=} \{1, \dots, n\}$ and $I \sqcup J$ for the disjoint union of I and J . For $F_1 = \dots = F_n = S$ the formula (5.17) agrees with (5.12) to order n in S . Obviously, Δ_A^n is linear and symmetrical and it is uniquely determined by these properties and (5.12). We also see that Δ_A^n is linear in Q resp. A .

The main task is to show the locality (ii) of the right-hand side of (5.17) (which also implies $\Delta_A^n(F_1 \otimes \cdots \otimes F_n) \in \mathcal{F}_{\text{loc}}$). To this end we denote by

$$\begin{aligned} M_{n,1}(L_1(x_1), \dots, L_n(x_n); Q(y)) &\stackrel{\text{def}}{=} R\left(\otimes_{i \in \underline{n}} L_i(x_i), Q(y) \frac{\delta S_0}{\delta \varphi(y)}\right) \\ &+ \sum_{k=1}^n R\left(\otimes_{i \in \underline{n} \setminus \{k\}} L_i(x_i), Q(y) \frac{\delta L_k(x_k)}{\delta \varphi(y)}\right) \\ &+ \sum_{I \sqcup J = \underline{n}, |J| < n} R\left(\otimes_{i \in I} L_i(x_i), \Delta^{|J|}(\otimes_{j \in J} L_j(x_j); Q(y))\right) \end{aligned}$$

the distributional kernel of the three terms in brackets in equation (5.17), where $F_i = \int dx g_i(x) L_i(x)$ ($g_i \in \mathcal{D}(\mathbb{M})$, $L_i \in \mathcal{P}$) and we use (5.14) for $\Delta_A^{|J|}$ to lower orders $|J| < n$. We will show that $M_{n,1}(L_1(x_1), \dots; Q(y))$ coincides outside the total diagonal $\mathbb{D}_{n+1} = \{(x_1, \dots, x_{n+1}) \in \mathbb{M}^{n+1}, x_1 = \dots = x_{n+1}\}$ with the distributional kernel $R(L_1(x_1) \otimes \cdots \otimes L_n(x_n), Q(y)) \frac{\delta S_0}{\delta \varphi(y)}$ of the first term in (5.17). This shows the locality of $\Delta_A^n(F_1 \otimes \cdots \otimes F_n)$. Since $M_{n,1}$ has retarded support

$$\text{supp}(\delta M_{n,1}/\delta \varphi) \subset \{(x_1, \dots, x_n, y) \in \mathbb{M}^{n+1}; x_1, \dots, x_n \in y + \bar{V}_-\} \quad (5.18)$$

and is symmetrical under permutation of the first n entries, it is thereby sufficient to consider the particular case where $x_n \in y + \bar{V}_- \setminus \{y\}$. In this case it holds

$$\begin{aligned} M_{n,1}(L_1(x_1), \dots; Q(y)) &= R\left(\otimes_{i \in \underline{n-1}} L_i(x_i) \otimes L_n(x_n), Q(y) \frac{\delta S_0}{\delta \varphi(y)}\right) \\ &+ \sum_{k=1}^{n-1} R\left(\otimes_{i \in \underline{n} \setminus \{k\}} L_i(x_i) \otimes L_n(x_n), Q(y) \frac{\delta L_k(x_k)}{\delta \varphi(y)}\right) \\ &+ \sum_{I \sqcup J = \underline{n-1}} R\left(\otimes_{i \in I} L_i(x_i) \otimes L_n(x_n), \Delta^{|J|}(\otimes_{j \in J} L_j(x_j); Q(y))\right) \end{aligned}$$

where the locality of the distributions $\Delta^{|J|}(\otimes_{j \in J} L_j(x_j); Q(y))$ for $|J| < n$ is used. Now we apply the GLZ Relation to each term and take the support property of the

retarded products into account. This yields

$$\begin{aligned}
M_{n,1}(L(x_1), \dots; Q(y)) &= \sum_{I \sqcup J = \underline{n-1}} \left\{ R(\otimes_{i \in I} L_i(x_i), L_n(x_n)), R(\otimes_{j \in J} L_j(x_j), Q(y) \frac{\delta S_0}{\delta \varphi(y)}) \right\} \\
&\quad + \sum_{k=1}^{n-1} \sum_{I \sqcup J \sqcup \{k\} = \underline{n-1}} \left\{ R(\otimes_{i \in I} L_i(x_i), L_n(x_n)), R(\otimes_{j \in J} L_j(x_j), Q(y) \frac{\delta L_k(x_k)}{\delta \varphi(y)}) \right\} \\
&\quad + \sum_{I \sqcup J = \underline{n-1}} \sum_{K \sqcup L = I} \left\{ R(\otimes_{i \in K} L_i(x_i), L_n(x_n)), R(\otimes_{l \in L} L_l(x_l), \Delta^{|J|}(\otimes_{j \in J} L_j(x_j); Q(y))) \right\}.
\end{aligned} \tag{5.19}$$

To transform the last term we use the induction hypothesis (5.16) written in distributional form,

$$\begin{aligned}
&\sum_{L \sqcup J = H} R(\otimes_{l \in L} L_l(x_l), \Delta^{|J|}(\otimes_{j \in J} L_j(x_j); Q(y))) = R(\otimes_{i \in H} L_i(x_i), Q(y)) \frac{\delta S_0}{\delta \varphi(y)} \\
&- R(\otimes_{i \in H} L_i(x_i), Q(y) \frac{\delta S_0}{\delta \varphi(y)}) - \sum_{j \in H} R(\otimes_{i \in H \setminus \{j\}} L_i(x_i), Q(y) \frac{\delta L_j(x_j)}{\delta \varphi(y)})
\end{aligned} \tag{5.20}$$

for all $H \subset \underline{n-1}$. After rearranging the sums the last term in (5.19) is equal to

$$\begin{aligned}
&- \sum_{I \sqcup J = \underline{n-1}} \left\{ R(\otimes_{i \in I} L_i(x_i), L_n(x_n)), R(\otimes_{j \in J} L_j(x_j), Q(y) \frac{\delta S_0}{\delta \varphi(y)}) \right\} \\
&- \sum_{k=1}^{n-1} \sum_{I \sqcup J \sqcup \{k\} = \underline{n-1}} \left\{ R(\otimes_{i \in I} L_i(x_i), L_n(x_n)), R(\otimes_{j \in J} L_j(x_j), Q(y) \frac{\delta L_k(x_k)}{\delta \varphi(y)}) \right\} \\
&+ \sum_{I \sqcup J = \underline{n-1}} \left\{ R(\otimes_{i \in I} L_i(x_i), L_n(x_n)), R(\otimes_{j \in J} L_j(x_j), Q(y) \frac{\delta S_0}{\delta \varphi(y)}) \right\}.
\end{aligned} \tag{5.21}$$

The last term in (5.21) can be transformed by using the identity

$$\{F, P(x) \frac{\delta S_0}{\delta \varphi(x)}\} = \{F, P(x)\} \frac{\delta S_0}{\delta \varphi(x)}, \quad F, P(x) \in \mathcal{F} \tag{5.22}$$

(which follows from the fact that the 2-point function H_m is a solution of the free field equation) and by applying the GLZ Relation:

$$\begin{aligned}
&\sum_{I \sqcup J = \underline{n-1}} \left\{ R(\otimes_{i \in I} L_i(x_i), L_n(x_n)), R(\otimes_{j \in J} L_j(x_j), Q(y) \frac{\delta S_0}{\delta \varphi(y)}) \right\} \\
&= \sum_{I \sqcup J = \underline{n-1}} \left\{ R(\otimes_{i \in I} L_i(x_i), L_n(x_n)), R(\otimes_{j \in J} L_j(x_j), Q(y)) \right\} \frac{\delta S_0}{\delta \varphi(y)} \\
&= R(L_1(x_1) \otimes \dots \otimes L_n(x_n), Q(y)) \frac{\delta S_0}{\delta \varphi(y)}.
\end{aligned}$$

Summing up the first two terms in (5.19) and the first two terms in (5.21) cancel and we get the desired result

$$M_{n,1}(L_1(x_1), \dots, Q(y)) = R(L_1(x_1) \otimes \dots \otimes L_n(x_n), Q(y)) \frac{\delta S_0}{\delta \varphi(y)} \tag{5.23}$$

$\forall(x_1, \dots, x_n, y) \notin \mathbb{D}_{n+1}$ which proves the locality of Δ_A^n .

The statement (5.14)-(5.15) about the distributional kernel of Δ_A^n is verified as follows: we insert the distributional kernels of the retarded products into the right-hand side of (5.17) and use

$$\delta_A L(g) = \int dx dy g(x) h(y) Q(y) \sum_a (\partial^a \delta)(x-y) \frac{\partial L}{\partial(\partial^a \varphi)}(x) \quad (5.24)$$

and the inductive assumption.

The conclusion ' \Rightarrow ' of property (iv) is obvious from (5.12) and ' \Leftarrow ' follows inductively for $\Delta_A^n(S^{\otimes n})$, because the right-hand side of (5.17) (for $F_1 = \dots = F_n = S$) vanishes in that case if $\Delta_A^k \equiv 0 \quad \forall k < n$. And $\Delta_A(e_\otimes^S) = 0 \quad \forall S$ implies $\Delta_A \equiv 0$ by the polarization identity.

The crucial property that $\Delta_A(F_1 \otimes \dots \otimes F_n) = \mathcal{O}(\hbar)$ for all $n > 0$ and $F_i \sim \hbar^0$, follows immediately from the validity of the classical MWI and $\lim_{\hbar \rightarrow 0} R = R_{\text{cl}}$ by using property (iv). \square

Mostly we will omit the index n of Δ_A^n and its kernel Δ^n .

Up to here we only assumed that the R -product satisfies the basic axioms. If it fulfills renormalization conditions, then corresponding properties of Δ_A are implied.

Lemma 8. (i) *The axioms \mathcal{P}_+^\dagger -Covariance, Unitarity, Field Independence and Field Equation, respectively, imply corresponding properties of Δ_A :*

- **\mathcal{P}_+^\dagger -Covariance** $\beta_L \Delta_A(e_\otimes^S) = \Delta_{\beta_L A}(e_\otimes^{\beta_L S})$, $\forall L \in \mathcal{P}_+^\dagger$;
- **Unitarity** $\Delta_A(e_\otimes^S)^* = \Delta_{A^*}(e_\otimes^{S^*})$;
- **Field Independence**

$$\frac{\delta}{\delta \varphi(x)} \Delta_A(S^{\otimes n}) = \Delta_{A_x}(S^{\otimes n}) + n \Delta_A(S^{\otimes(n-1)} \otimes \frac{\delta S}{\delta \varphi(x)}) , \quad (5.25)$$

where A_x is obtained from A (5.10) by replacing $Q(y)$ by $\frac{\delta Q(y)}{\delta \varphi(x)}$:

$$A_x \stackrel{\text{def}}{=} \int dy h(y) \frac{\delta Q(y)}{\delta \varphi(x)} \frac{\delta S_0}{\delta \varphi(y)} \in \mathcal{I}_{S_0} ; \quad (5.26)$$

- **Field Equation**

$$\Delta_A \equiv 0 \quad \text{if} \quad A = \int dx h(x) \frac{\delta S_0}{\delta \varphi(x)}, \quad \forall h \in \mathcal{D}(\mathbb{M}) ; \quad (5.27)$$

(ii) *Let $g_j \in \mathcal{D}(\mathbb{M})$, $L_j \in \mathcal{P}$. Assuming that the R -products satisfy the axiom Field Independence, there exist linear maps $P_a^n : \mathcal{P}^{\otimes(n+1)} \rightarrow \mathcal{P}$ (where a runs through a finite subset of $(\mathbb{N}_0^d)^n$), which are symmetric in the first n factors, such that Δ_A^n can be written as*

$$\begin{aligned} & \Delta_A^n(L_1(g_1) \otimes \dots \otimes L_n(g_n)) = \\ & \sum_{a \in (\mathbb{N}_0^d)^n} \int dx h(x) \partial^{a_1} g_1(x) \dots \partial^{a_n} g_n(x) P_a^n(L_1 \otimes \dots \otimes L_n; Q)(x) . \end{aligned} \quad (5.28)$$

Proof. (i) We prove all properties for the components Δ_A^n of Δ_A by induction on n . We verify that the right-hand side of (5.17) fulfills the assertion by using the pertinent property of the R -product and the inductive assumption.

In case of the Field Independence we additionally take

$$\frac{\delta A}{\delta \varphi(x)} = A_x + \int dy h(y) Q(y) \frac{\delta^2 S_0}{\delta \varphi(y) \delta \varphi(x)} , \quad (5.29)$$

$$\frac{\delta(\delta_A S)}{\delta \varphi(x)} = \delta_{A_x} S + \delta_A \left(\frac{\delta S}{\delta \varphi(x)} \right) \quad (5.30)$$

into account.

Field Equation: for $A = \int dx h(x) \frac{\delta S_0}{\delta \varphi(x)}$ the MWI reduces to the off-shell field equation in differential form, which is equivalent to the integrated version (3.8). The assertion follows by means of property (iv) of Δ_A in Theorem 7.

(ii) The statement is a simplified version of Proposition 4.3 in [16]. The proof, which is given there in words, is carried out here explicitly, since these formulas will be used below in the proof of part (ii) of Prop. 10.

The Field Independence (5.25) translates into a corresponding relation for the distributional kernel (5.14):

$$\begin{aligned} & \frac{\delta}{\delta \varphi(x)} \Delta(\otimes_{j=1}^n L_j(x_j); Q(y)) = \\ & \sum_{l=1}^n \Delta\left(\otimes_{j(\neq l)} L_j(x_j) \otimes \frac{\delta L_l(x_l)}{\delta \varphi(x)}; Q(y)\right) + \Delta\left(\otimes_{j=1}^n L_j(x_j); \frac{\delta Q(y)}{\delta \varphi(x)}\right) . \end{aligned} \quad (5.31)$$

We insert this identity into the Taylor expansion of $\Delta(\otimes_{j=1}^n L_j(x_j); Q(y))$ with respect to φ . This yields the causal Wick expansion

$$\begin{aligned} \Delta(\otimes_{j=1}^n L_j(x_j); Q(y)) &= \sum_{l_1, \dots, l; a_{ij_i}, a_j} C_{a_{11} \dots a_{1l_1}, \dots, a_1 \dots a_l}^{l_1 \dots l; l} \\ & \cdot \omega_0 \left(\Delta \left(\otimes_{i=1}^n \frac{\partial^{l_i} L_i}{\partial(\partial^{a_{i1}} \varphi) \dots \partial(\partial^{a_{il_i}} \varphi)}(x_i); \frac{\partial^l Q}{\partial(\partial^{a_1} \varphi) \dots \partial(\partial^{a_l} \varphi)}(y) \right) \right) \\ & \cdot \prod_{i=1}^n \left(\prod_{j_i=1}^{l_i} \partial^{a_{ij_i}} \varphi(x_i) \right) \cdot \prod_{j=1}^l \partial^{a_j} \varphi(y) , \end{aligned} \quad (5.32)$$

where $C_{a_{11} \dots a_{1l_1}, \dots, a_1 \dots a_l}^{l_1 \dots l; l}$ is a combinatorial factor. Now we crucially use that $\Delta_A(F_1 \otimes \dots \otimes F_n)$ is a *local* functional. Therefore, the support of $\omega_0 \left(\Delta \left(\frac{\partial^{l_1} L_1}{\partial \dots} (x_1) \otimes \dots; \frac{\partial^l Q}{\partial \dots} (y) \right) \right)$ is located on the diagonal $\{(x_1, \dots, x_n, y) \mid x_j = y \ \forall j\}$. Hence this distribution is of the form

$$\omega_0 \left(\Delta \left(\frac{\partial^{l_1} L_1}{\partial \dots} (x_1) \otimes \dots; \frac{\partial^l Q}{\partial \dots} (y) \right) \right) = \sum_b C_b (\partial^b \delta)(x_1 - y, \dots, x_n - y) , \quad (5.33)$$

with constant numbers C_b which depend on the field polynomials in the argument of Δ . The term with index b gives the following contribution to $\Delta_A(L_1(g_1) \otimes \dots \otimes L_n(g_n))$

(5.14):

$$(-1)^{|b|} \int dx_1 \dots dx_n dy \delta(x_1 - y, \dots, x_n - y) \cdot h(y) \cdot \partial_{x_1 \dots x_n}^b \left(g_1(x_1) \dots g_n(x_n) \cdot \prod_{i=1}^n \left(\prod_{j_i=1}^{l_i} \partial^{a_{ij_i}} \varphi(x_{i_i}) \right) \cdot \prod_{j=1}^l \partial^{a_j} \varphi(y) \right), \quad (5.34)$$

where we have omitted constant factors. By reordering the sums we write $\Delta_A(L_1(g_1) \otimes \dots \otimes L_n(g_n))$ in the form (5.28). Since $\Delta_A(L_1(g_1) \otimes \dots \otimes L_n(g_n))$ is multilinear in the fields L_1, \dots, L_n, Q and symmetric in L_1, \dots, L_n , the maps P_a^n must satisfy corresponding properties. \square

We turn to the scaling behaviour of Δ_A . Assuming that the R -products satisfy the axioms Smoothness in $m \geq 0$, μ -Covariance and almost homogeneous Scaling (3.16), the map $(S^{\otimes n} \otimes A) \mapsto \Delta_A^{(m, \mu)}(S^{\otimes n})$ does not scale almost homogeneously in general. Namely, for non-vanishing mass we are faced with m -dependent inhomogeneous polynomials: $S_0^{(m)} \equiv S_0 \notin \mathcal{P}_{\text{hom}}$, $\frac{\delta S_0^{(m)}}{\delta \varphi} \notin \mathcal{P}_{\text{hom}}$ and, hence, $A^{(m)} \equiv A$ is in general not in \mathcal{P}_{hom} . In typical applications of the MWI (see e.g. the $O(N)$ -model in Sect. (5.4.4) or current conservation in QED) the simplification appears that Q_j is independent of m and the m -dependent terms of $\frac{\delta S_0}{\delta \varphi_j}$ cancel in $A = \int dx \sum_j Q_j(x) \frac{\delta S_0}{\delta \varphi_j(x)}$. But even with that, $\sigma_\rho \Delta_{\sigma_\rho^{-1} A}^{(\rho^{-1} m, \mu)}((\sigma_\rho^{-1} S)^{\otimes n})$ is in general not a polynomial in $(\log \rho)$: proceeding inductively the first term on the right-hand side of (5.17) does not scale almost homogeneously due to the mass term in $\frac{\delta S_0}{\delta \varphi}$.¹²

For this reason, we assume here the axiom **Scaling Degree** (3.19) instead of Smoothness in m , μ -Covariance and almost homogeneous Scaling.

With this weaker assumption we are going to derive a corresponding scaling degree property of Δ_A and an upper bound for the mass dimension of $\Delta_A(S^{\otimes n})$ which does not depend on n if S is a renormalizable interaction. For the latter purpose we have to define the mass dimension of a local *functional*. We use that every $F \in \mathcal{F}_{\text{loc}}$ can uniquely be written in the form

$$F = \sum_i \int dx g_i(x) P_i(x), \quad g_i \in \mathcal{D}(\mathbb{M}), \quad P_i \in \mathcal{P}_{\text{bal}}. \quad (5.35)$$

With that we define

$$\dim F \stackrel{\text{def}}{=} \max_i \dim P_i \quad (5.36)$$

This definition is minimal in the following sense:

¹²In detail: from the derivation of the classical MWI (5.4)-(5.5) it follows that the scaling of the right-hand side of the anomalous MWI (5.12) must be such that the mass m in $R^{(m, \mu)}$ and in $\frac{\delta S_0^{(m)}}{\delta \varphi(y)}$ has the same value that is the scale transformation reads $\int dy h(y) \sigma_\rho \circ R^{(\rho^{-1} m, \mu)}(e_{\otimes}^{\sigma_\rho^{-1} S}, \sigma_\rho^{-1} Q(y)) \frac{\delta S_0^{(\rho^{-1} m)}}{\delta \varphi(y)}$.

Lemma 9. *Let*

$$F = \sum_i \int dx g_i(x) B_i(x) , \quad g_i \in \mathcal{D}(\mathbb{M}) , \quad B_i \in \mathcal{P} . \quad (5.37)$$

Then it holds

$$\dim F \leq \max_i \dim B_i \quad (5.38)$$

Proof. Every B_i can uniquely be written as $B_i = \sum_j p_{ij}(\partial) P_{ij}$, $P_{ij} \in \mathcal{P}_{\text{bal}}$, where $p_{ij}(\partial)$ is a polynomial in the partial derivatives [16]. Inserting this into (5.37) and shifting the derivatives to the test function we obtain the unique representation (5.35) of F . The assertion follows from $\dim P_{ij} \leq \dim B_i \quad \forall i, j$. \square

From $A = \int dy h(y) Q(y) \frac{\delta S_0}{\delta \varphi(y)}$ we conclude

$$\dim(A) \leq \dim(Q) + \dim\left(\frac{\delta S_0}{\delta \varphi}\right) \quad \text{where} \quad \dim\left(\frac{\delta S_0}{\delta \varphi}\right) = \dim((\square + m^2)\varphi) = \frac{d+2}{2} . \quad (5.39)$$

Analogously the relation $\delta_A S = \int dy h(y) Q(y) \frac{\delta S}{\delta \varphi(y)}$ implies

$$\dim(\delta_A S) \leq \dim(Q) + \dim(S) - \dim(\varphi) = \dim(Q) + \dim(S) - \frac{d-2}{2} . \quad (5.40)$$

With these tools we are ready to formulate and prove the following Proposition.

Proposition 10. *(i) **Scaling Degree.** If the R-products fulfill the axiom Scaling Degree (3.19), then the scaling degree of the 'vacuum expectation value' of $\Delta(L_1(x_1) \otimes \dots; Q(y))$ is bounded by*

$$\text{sd } \omega_0\left(\Delta(\otimes_{j=1}^n L_j(x_j); Q(y))\right) \leq \sum_{i=1}^n \dim L_i + \dim Q + \frac{d+2}{2} . \quad (5.41)$$

*(ii) **Mass Dimension.** If the R-products fulfill the axioms Field Independence and Scaling Degree (3.19), then the mass dimension of $\Delta_A(F_1 \otimes \dots \otimes F_n)$ is bounded by*

$$\dim \Delta_A(F_1 \otimes \dots \otimes F_n) \leq \sum_{i=1}^n \dim(F_i) + \dim Q + \frac{d+2}{2} - d n . \quad (5.42)$$

For a renormalizable interaction, that is $\dim(S) \leq d$, this implies

$$\dim \Delta_A(e_{\otimes}^S) \leq \dim Q + \frac{d+2}{2} . \quad (5.43)$$

Note that for a renormalizable interaction the upper bounds on the mass dimension of A (5.39), $\delta_A S$ (5.40) and $\Delta_A(e_{\otimes}^S)$ agree, as one expects because the sum $(A + \delta_A S + \Delta_A(e_{\otimes}^S))$ appears in Theorem 7.

Proof. (i) Proceeding by induction on n we apply ω_0 to (5.15) and estimate the scaling degree of the resulting terms on the right-hand side by using

$$\text{sd } \omega_0 \left(R(\otimes_{j=1}^{n-1} L_j(x_j); L_n(x_n)) \right) \leq \sum_{j=1}^n \dim L_j , \quad (5.44)$$

which follows from (3.19). (In contrast to (3.19) we do not assume $L_j \in \mathcal{P}_{\text{hom}}$.) For example we obtain

$$\begin{aligned} & \text{sd } \omega_0 \left(R(\otimes_{j(\neq l)} L_j(x_j); Q(y) (\partial^a \delta)(x_l - y) \frac{\partial L_l}{\partial(\partial^a \varphi)}(x_l)) \right) \\ & \leq \sum_{j(\neq l)} \dim L_j + \dim Q + \dim \left(\frac{\partial L_l}{\partial(\partial^a \varphi)} \right) + \text{sd}(\partial^a \delta) \\ & = \sum_{j=1}^n \dim L_j + \dim Q + \frac{d+2}{2} . \end{aligned} \quad (5.45)$$

The same bound results for all other terms. (The vacuum expectation value of the last term vanishes.)

(iv) We may assume $F_j = L_j(g_j)$ with L_j a balanced field, $L_j \in \mathcal{P}_{\text{bal}}$. With that it holds $\dim(L_j) = \dim(F_j)$. We use the formulas derived in the proof of part (ii) of Lemma 8, in particular the causal Wick expansion of $\Delta(\otimes_j L_j(x_j); Q(y))$ (5.32). (In the formulas given below the indices have precisely the same meaning as in that proof.) Looking at (5.33), we get an upper bound for the scaling degree of the left-hand side by means of (5.41). This implies that the range of b on the right-hand side of (5.33) is restricted by

$$\begin{aligned} |b| + d n & \leq \sum_{i=1}^n \left(\dim B_i - \sum_{j_i=1}^{l_i} \left(|a_{ij_i}| + \frac{d-2}{2} \right) \right) \\ & + \dim Q - \sum_{j=1}^l \left(|a_j| + \frac{d-2}{2} \right) + \frac{d+2}{2} . \end{aligned} \quad (5.46)$$

All terms of $\Delta_A(L_1(g_1) \otimes \dots \otimes L_n(g_n))$ are of the form (5.34) where $|b|$ is bounded by (5.46). Due to Lemma 9 the mass dimension of the functional (5.34) is bounded by

$$\begin{aligned} & \sum_{i=1}^n \sum_{j_i=1}^{l_i} \left(|a_{ij_i}| + \frac{d-2}{2} \right) + \sum_{j=1}^l \left(|a_j| + \frac{d-2}{2} \right) + |b| \\ & \leq \sum_{i=1}^n \dim(B_i) + \dim Q + \frac{d+2}{2} - d n . \end{aligned} \quad (5.47)$$

This implies the assertion (5.42). \square

5.3 The MWI in the proper field formalism, and the Quantum Action Principle

In the literature the Quantum Action Principle (QAP) is formulated in terms of time ordered products. Hence, to be able to compare our results with the QAP, we first rewrite Theorem 7 using T -products:

Lemma 11. *Using the assumptions and notations of Theorem 7 the anomalous MWI (5.12) can equivalently be written in terms of the time-ordered product:*

$$T(e_{\otimes}^{iS/\hbar} \otimes (A + \delta_A S + \Delta_A(e_{\otimes}^S))) = \int dy h(y) T(e_{\otimes}^{iS/\hbar} \otimes Q(y)) \frac{\delta S_0}{\delta \varphi(y)}. \quad (5.48)$$

Proof. Using the formula of Bogoliubov (4.15) and the identity

$$(F \star G) \cdot \frac{\delta S_0}{\delta \varphi} = F \star (G \cdot \frac{\delta S_0}{\delta \varphi}) \quad \forall F, G \in \mathcal{F} \quad (5.49)$$

(cf. (5.22)), where \cdot denotes the classical product, we obtain

$$\begin{aligned} \bar{T}(e_{\otimes}^{-iS/\hbar}) \star T(e_{\otimes}^{iS/\hbar} \otimes (A + \delta_A S + \Delta_A(e_{\otimes}^S))) &= \\ \bar{T}(e_{\otimes}^{-iS/\hbar}) \star \left(\int dy h(y) T(e_{\otimes}^{iS/\hbar} \otimes Q(y)) \cdot \frac{\delta S_0}{\delta \varphi(y)} \right). \end{aligned} \quad (5.50)$$

Multiplication by $T(e_{\otimes}^{iS/\hbar})$ from the left yields (5.48). \square

As shown in (5.9) the MWI is satisfied by T_{tree} , i.e. it can be violated only by the contribution of the loop diagrams. To concentrate the study of the solvability of the MWI to that terms for which the MWI is a non-trivial condition, we reformulate the anomalous MWI (5.12) within the proper field formalism introduced in Sect. 4.

Corollary 12. *The anomalous MWI in Theorem 7 can equivalently be expressed in terms of the vertex function Γ_T (defined in Prop. 4):*

$$\int dx h(x) \Gamma_T(e_{\otimes}^S \otimes Q(x)) \frac{\delta(S_0 + \Gamma_T(e_{\otimes}^S))}{\delta \varphi(x)} = \Gamma_T(e_{\otimes}^S \otimes (A + \delta_A S + \Delta_A(e_{\otimes}^S))). \quad (5.51)$$

Proof. Applying (4.19) on both sides of (5.48) we obtain

$$\begin{aligned} T_{\text{tree}} \left(e_{\otimes}^{i\Gamma_T(e_{\otimes}^S)/\hbar} \otimes \Gamma_T(e_{\otimes}^S \otimes (A + \delta_A S + \Delta_A(e_{\otimes}^S))) \right) \\ = \int dy h(y) T_{\text{tree}} \left(e_{\otimes}^{i\Gamma_T(e_{\otimes}^S)/\hbar} \otimes \Gamma_T(e_{\otimes}^S \otimes Q(y)) \right) \frac{\delta S_0}{\delta \varphi(y)}. \end{aligned}$$

On the right-hand side we use the classical MWI in terms of T -products (5.9) and obtain

$$\dots = T_{\text{tree}} \left(e_{\otimes}^{i\Gamma_T(e_{\otimes}^S)/\hbar} \otimes \int dy g(y) \Gamma_T(e_{\otimes}^S \otimes Q(y)) \frac{\delta(S_0 + \Gamma_T(e_{\otimes}^S))}{\delta \varphi(y)} \right),$$

which leads, by means of (4.13), to the assertion (5.51). \square

To compare this result with the literature, we introduce $\Gamma(S_0, S) \stackrel{\text{def}}{=} S_0 + \Gamma_T(e_\otimes^S)$ and, motivated by (4.27) and (4.38), interpret $\Gamma(S_0, S)$ as the *proper total action* associated with the total classical action $S_{\text{tot}} = S_0 + S$. In addition, we introduce 'insertions' (see [10]) in our algebraic formalism: let $P \in \mathcal{P}$ be some field polynomial and $\rho \in \mathcal{D}(\mathbb{M})$ some testfunction (in the literature often called 'external field'). We define the 'insertion of $P(x)$ ', usually denoted¹³ by $P(x) \cdot \Gamma(S_0, S)$, as follows:¹⁴

$$P(x) \cdot \Gamma(S_0, S) \stackrel{\text{def}}{=} \frac{\delta}{\delta \rho(x)} \Big|_{\rho=0} \Gamma\left(S_0, S + \int dx \rho(x) P(x)\right) = \Gamma_T(e_\otimes^S \otimes P(x)). \quad (5.52)$$

We denote by $S' \stackrel{\text{def}}{=} S + \int dx \rho(x) Q(x)$, $\rho \in \mathcal{D}(\mathbb{M})$ the modified classical interaction containing a coupling to the external field ρ , and write $\Delta_A(e_\otimes^S) = \int dx h(x) \tilde{\Delta}(e_\otimes^S; Q(x))$ with $\tilde{\Delta}(e_\otimes^S; Q(x)) \in \mathcal{D}(\mathbb{M}, \mathcal{P})$ (Lemma 8(ii)). Using this notation and the fact that the anomalous MWI (5.51) is valid for all $h \in \mathcal{D}(\mathbb{M})$, we rewrite (5.51) in terms of the proper total action:

$$\frac{\delta \Gamma(S_0, S')}{\delta \rho(x)} \frac{\delta \Gamma(S_0, S')}{\delta \varphi(x)} \Big|_{\rho=0} = \Delta(x) \cdot \Gamma(S_0, S), \quad (5.53)$$

where the local field $\Delta(x)$ is given by $\Delta(x) \stackrel{\text{def}}{=} Q(x) \frac{\delta(S_0 + S)}{\delta \varphi(x)} + \tilde{\Delta}(e_\otimes^S; Q(x))$. This formulation of the anomalous MWI is formally equivalent to the general formulation of the QAP in [10]. Note that due to property (v) in Theorem 7 and (4.27), the \hbar -expansion of the right-hand side of (5.53) starts with

$$\Delta(x) \cdot \Gamma(S_0, S) = Q(x) \frac{\delta(S_0 + S)}{\delta \varphi(x)} + \mathcal{O}(\hbar) \equiv \frac{\delta(S_0 + S')}{\delta \rho(x)} \frac{\delta(S_0 + S')}{\delta \varphi(x)} \Big|_{\rho=0} + \mathcal{O}(\hbar), \quad (5.54)$$

which is completely analogous to the expansion of the right-hand side of the QAP in [10]. Moreover, for a renormalizable interaction S , the mass dimension of the local insertion $\Delta(x)$ is bounded by $\dim Q + \frac{d+2}{2} = \dim Q - \dim \varphi + d$ (Prop. 10), in agreement with [10].

5.4 Removal of violations of the MWI

5.4.1 Fulfillment of the MWI to first order in S

Before we are going to investigate the question whether the MWI can be fulfilled in a concrete model (i.e. for a given $A \in \mathcal{J}_{\text{loc}}$ and a given interaction $S \in \mathcal{F}_{\text{loc}}$), we will show that the MWI can always be fulfilled to first order in the interaction S .¹⁵ Apart from the interest in its own, this result will be needed in our proof of the Ward identities of the scalar $O(N)$ -model (Sect. 5.4.4).

¹³The dot does not mean the classical product here!

¹⁴Note, that in the setting of causal perturbation theory the introduction of external fields in order to express 'insertions' or nonlinear symmetry transformations is – in contrast to conventional perturbation theory – not necessary.

¹⁵E.g. the axial anomaly of QED is of second order in the interaction.

Proposition 13. *Given $A = \int dy h(y) Q(y) \frac{\delta S_0}{\delta \varphi(y)}$ (with $Q \in \mathcal{D}(\mathbb{M}, \mathcal{P})$) and $S \in \mathcal{F}_{\text{loc}}$ there exists a retarded product $R_{1,1}$ which fulfills all renormalization conditions (specified in Sect. 3) and*

$$R_{1,1}(S, A) + \delta_A S = \int dy h(y) R_{1,1}(S, Q(y)) \frac{\delta S_0}{\delta \varphi(y)} . \quad (5.55)$$

Proof. We start with T -products resp. retarded products satisfying all renormalization conditions. For simplicity we work in the proper field formalism. From the anomalous MWI (Cor. 12) we obtain to first order in S

$$\Delta_A(S) = - \int dy h(y) \left(\Gamma_T(S \otimes Q(y)) \frac{\delta S_0}{\delta \varphi(y)} - \Gamma_T(S \otimes Q(y)) \frac{\delta S_0}{\delta \varphi(y)} \right) . \quad (5.56)$$

By inserting the Wick expansion (3.7) of Γ_T (which holds since Γ_T fulfills Field Independence), we find that the terms containing no contraction of $\frac{\delta S_0}{\delta \varphi}$ cancel. It remains:

$$\Delta_A(S) = - \int dx dy g(x) h(y) \sum_{\overline{\mathcal{L}} \subset \mathcal{L}, \overline{Q} \subset Q} \omega_0 \left(\Gamma_T(\overline{\mathcal{L}}(x) \otimes \overline{Q}(y)) \frac{\delta S_0}{\delta \varphi(y)} \right) \underline{\mathcal{L}}(x) \underline{Q}(y) \quad (5.57)$$

where we write $S = \int dx g(x) \mathcal{L}(x)$, $g \in \mathcal{D}(\mathbb{M})$ and sum over all subpolynomials $\overline{P} = \frac{\partial^l P}{\partial(\partial^{a_1} \varphi) \dots \partial(\partial^{a_l} \varphi)}$ of $P \in \mathcal{P}$ (for $P = \mathcal{L}, Q$), denoting by $\underline{P} = \frac{1}{l!} \prod_{j=1}^l \partial^{a_j} \varphi$ the corresponding ‘counterpart’ (see (3.7)). Due to locality of $\Delta_A(S)$ the distributions $\omega_0 \left(\Gamma_T(\overline{\mathcal{L}}(x) \otimes \overline{Q}(y)) \frac{\delta S_0}{\delta \varphi(y)} \right)$ are supported on the diagonal $\{(x, y) | x = y\}$, cancellations of non-local terms on the right-hand side of (5.57) are impossible. We are searching a finite renormalization $\Gamma_T \rightarrow \hat{\Gamma}_T$ which removes the violating term $\Delta_A(S)$. Due to (4.33) a renormalization of $\Gamma_T(S \otimes F)$ must be of the form

$$\hat{\Gamma}_T(S \otimes F) = \Gamma_T(S \otimes F) + D(S \otimes F), \quad \forall F \in \mathcal{F}_{\text{loc}}, \quad (5.58)$$

with D satisfying the properties listed in the Main Theorem. To obtain a field independent D we construct it from its vacuum expectation values by the causal Wick expansion (3.7). Obviously, setting

$$\omega_0 \left(D(\overline{\mathcal{L}}(x) \otimes \overline{Q}(y)) \frac{\delta S_0}{\delta \varphi(y)} \right) \stackrel{\text{def}}{=} -\omega_0 \left(\Gamma_T(\overline{\mathcal{L}}(x) \otimes \overline{Q}(y)) \frac{\delta S_0}{\delta \varphi(y)} \right), \quad (5.59)$$

the corresponding $\hat{\Gamma}_T$ satisfies the MWI to first order in S . Here we essentially use that the distribution on the right-hand side has local support. One verifies easily that D (and hence the resulting $\hat{\Gamma}_T$) satisfies all renormalization conditions. \square

5.4.2 Removal of possible anomalies by induction on the order in \hbar

The formal equivalence of the anomalous MWI (written in the form of Cor. 12) and the QAP makes it possible to apply basic techniques of algebraic renormalization within the framework of causal perturbation theory. The main idea underlying algebraic renormalization is to start with an arbitrary renormalization prescription; with that

the considered Ward identities are in general broken. Proceeding by induction on the power of \hbar , the question whether one can find a finite renormalization which removes the possible anomaly (i.e. the local terms violating the Ward identity) then amounts – by means of the QAP – to purely algebraic (often cohomological) problems.

We will apply this strategy to the MWI in the proper field formalism (which is given by (5.51) with $\Delta_A(e_\otimes^S) = 0$): given $A \in \mathcal{F}_{\text{loc}}$ and $S \in \mathcal{F}_{\text{loc}}$, we are going to investigate whether possible anomalies of the corresponding MWI can be removed by finite renormalizations of Γ_T (4.33). Proceeding by induction on the power of \hbar , we assume that for a given map $\Gamma_T^{(k)}$ (which satisfies the renormalization conditions Unitarity, \mathcal{P}_+^\dagger -Covariance, Field Independence, Field Equation and Scaling Degree, see Lemma 6) the MWI is violated only by terms of order \hbar^k ; that is in the anomalous MWI (Cor. 12) for $\Gamma_T^{(k)}$,

$$\begin{aligned} & \Gamma_T^{(k)}(e_\otimes^S \otimes (A + \delta_A S)) + \Gamma_T^{(k)}(e_\otimes^S \otimes \Delta_A^{(k)}(e_\otimes^S)) = \\ & \int dy h(y) \Gamma_T^{(k)}(e_\otimes^S \otimes Q(y)) \frac{\delta(S_0 + \Gamma_T^{(k)}(e_\otimes^S))}{\delta\varphi(y)}, \end{aligned} \quad (5.60)$$

the violating term can be written as

$$\Gamma_T^{(k)}(e_\otimes^S \otimes \Delta_A^{(k)}(e_\otimes^S)) = \Delta_A^{(k)}(e_\otimes^S) + \mathcal{O}(\hbar^{k+1}), \quad \Delta_A^{(k)}(e_\otimes^S) = \mathcal{O}(\hbar^k), \quad (5.61)$$

where (4.27) is used. Note that, due to Theorem 7 (v) and (4.27), any vertex function Γ_T fulfills this assumption (5.60)-(5.61) for $k = 1$ and, hence, can be used as $\Gamma_T^{(1)}$.

To fulfill the MWI to k^{th} order in \hbar (and to maintain the other renormalization conditions), we have to find a renormalization map $D^{(k)}$ (with the properties (i)-(v) and (vi)(B) of the Main Theorem) such that

$$\begin{aligned} & \Gamma_T^{(k+1)}(e_\otimes^S \otimes (A + \delta_A S)) + \mathcal{O}(\hbar^{k+1}) = \\ & \int dy h(y) \Gamma_T^{(k+1)}(e_\otimes^S \otimes Q(y)) \frac{\delta(S_0 + \Gamma_T^{(k+1)}(e_\otimes^S))}{\delta\varphi(y)}, \end{aligned} \quad (5.62)$$

with $\Gamma_T^{(k+1)}$ given by

$$\Gamma_T^{(k+1)}(e_\otimes^F) = \Gamma_T^{(k)}(e_\otimes^{D^{(k)}(e_\otimes^F)}) \quad \forall F \in \mathcal{F}_{\text{loc}}. \quad (5.63)$$

To maintain the validity of the MWI to lower orders in \hbar we choose $D^{(k)}$ to be of the form

$$D^{(k)}(e_\otimes^F) = F + D_{>1}^{(k)}(e_\otimes^F), \quad D_{>1}^{(k)}(e_\otimes^F) = \mathcal{O}(\hbar^k), \quad \forall F \in \mathcal{F}_{\text{loc}}, \quad F \sim \hbar^0. \quad (5.64)$$

This implies

$$e_\otimes^{D^{(k)}(e_\otimes^F)} = e_\otimes^F + e_\otimes^F \otimes_{\text{sym}} D_{>1}^{(k)}(e_\otimes^F) + \mathcal{O}(\hbar^{k+1}) \quad (5.65)$$

(where \otimes_{sym} denotes the symmetrized tensor product) and

$$\Gamma_T^{(k+1)}(e_\otimes^S) = \Gamma_T^{(k)}(e_\otimes^S) + D_{>1}^{(k)}(e_\otimes^S) + \mathcal{O}(\hbar^{k+1}), \quad (5.66)$$

$$\Gamma_T^{(k+1)}(e_\otimes^S \otimes F) = \Gamma_T^{(k)}(e_\otimes^S \otimes F) + D_{>1}^{(k)}(e_\otimes^S \otimes F) + \mathcal{O}(\hbar^{k+1}). \quad (5.67)$$

We insert the latter two equations into our requirement (5.62) and use the inductive assumption (5.60)-(5.61). It results

$$\begin{aligned} \Delta_A^{(k)}(e_\otimes^S) &= D_{>1}^{(k)}(e_\otimes^S \otimes (A + \delta_A S)) - \delta_A D_{>1}^{(k)}(e_\otimes^S) \\ &\quad - \int dy h(y) D_{>1}^{(k)}(e_\otimes^S \otimes Q(y)) \frac{\delta(S_0 + S)}{\delta\varphi(y)} + \mathcal{O}(\hbar^{k+1}). \end{aligned} \quad (5.68)$$

The violating term $\Delta_A^{(k)}(e_\otimes^S)$ is inductively given by $\Gamma_T^{(k)}$. If we succeed to find a corresponding map $D_{>1}^{(k)}$ fulfilling (5.68) and the properties of a renormalization map, then the pertinent finite renormalization removes the 'anomaly' $\Delta_A^{(k)}(e_\otimes^S)$. However, since $D_{>1}^{(k)}$ appears in (5.68) several times with different arguments it seems almost impossible to discuss the existence of solutions in general.

5.4.3 Assumption: localized off-shell version of Noether's Theorem

In various important applications of the MWI the search for solutions $D_{>1}^{(k)}$ of (5.68) is simplified due to the validity of the following assumption. In the given model the total action¹⁶

$$S_0 + S(g) \quad (\text{with } S(g) = \sum_{n \geq 1} \kappa^n S_n(g), \quad S_n(g) = \int dx (g(x))^n \mathcal{L}_n(x), \quad g \in \mathcal{D}(\mathbb{M}), \quad \mathcal{L}_n \in \mathcal{P})$$

is invariant with respect to the symmetry transformation

$$\delta_A = \int dy h(y) Q(y) \frac{\delta}{\delta\varphi(y)} = \sum_{n \geq 0} \kappa^n \delta_{A_n} \quad (\text{corresponding to } A = \sum_{n \geq 0} \kappa^n A_n, \quad A_n \in \mathcal{J}_{\text{loc}})$$

in the following way: there exist

- a current $j^\mu(g) = \sum_{n \geq 0} \kappa^n j_n^\mu(g)$, $j_n^\mu(g)(x) = (g(x))^n j_n^\mu(x)$ (with $j_n^\mu \in \mathcal{P}$) and
- a 'Q-vertex'¹⁷ $\mathcal{L}^{(1)\mu}(g) = \sum_{n \geq 1} \kappa^n \mathcal{L}_n^{(1)\mu}(g)$, $\mathcal{L}_n^{(1)\mu}(g)(x) = (g(x))^{n-1} \mathcal{L}_n^{(1)\mu}(x)$ (with $\mathcal{L}_n^{(1)\mu} \in \mathcal{P}$)

such that

$$\delta_A(S_0 + S(g)) \equiv A + \delta_A S(g) = \int dx h(x) \left(\partial_\mu j^\mu(g)(x) - \mathcal{L}^{(1)\mu}(g)(x) \partial_\mu g(x) \right) \quad (5.69)$$

¹⁶ κ denotes the coupling constant.

¹⁷The name 'Q-vertex' is due to 'perturbative gauge invariance' [30], which is related to BRST-symmetry.

in the sense of formal power series in κ . To zeroth order in κ the assumption (5.69) reads

$$\delta_{A_0} S_0 \equiv A_0 = \int dx h(x) \partial_\mu j_0^\mu(x) , \quad (5.70)$$

where j_0 is the symmetry current of the underlying free theory. This simplifying assumption can be interpreted as the validity of an off-shell version of Noether's Theorem for the case that the interaction and the symmetry transformation are localized. It is satisfied e.g. for the scalar $O(N)$ -model treated in Sect. 5.4.4 and for the BRST-symmetry of (massless) Yang-Mills theories, massive spin-1 fields and massless spin-2 fields (gravity). For the interacting scalar $O(N)$ -model (see Sect. 5.4.4) the simplifications $A = A_0$, $\delta_A S = 0$, $\mathcal{L}^{(1)} = 0$ and $j = j_0$ appear. However, for the BRST-symmetry δ_A and j are generically non-trivial deformations of δ_{A_0} and j_0 , and $\delta_A S$ and $\mathcal{L}^{(1)}$ do not vanish.

Example: BRST-symmetry. We are going to verify that the above mentioned models satisfy the assumption (5.69). Our argumentation is based on conservation of the classical BRST-current. For constant couplings (i.e. $g(x) = 1 \ \forall x$) there is a conserved Noether current $j = \sum_{n \geq 0} \kappa^n j_n$, due to the BRST-invariance of the total action. We use these j_n 's to construct the BRST-current of the corresponding model with localized coupling $\kappa g(x)$ ($g \in \mathcal{D}(\mathbb{M})$): we set $j(g)(x) := \sum_{n \geq 0} (\kappa g(x))^n j_n(x)$. The violation of the conservation of $j(g)$ is expressed in terms of the Q -vertex [13, 2, 1]: in Sects. 3 and 4 of [27] it is shown that for the considered models there exists a Q -vertex $\mathcal{L}^{(1)}(g)$ such that

$$R_{\text{cl}}\left(e_\otimes^{S(g)}, \int dx h(x) (\partial j(g)(x) - \mathcal{L}^{(1)\mu}(g)(x) \partial_\mu g(x))\right) \in \mathcal{J} . \quad (5.71)$$

Proceeding by induction on the order in κ and using the MWI (as it is worked out in formulas (190)-(191) and (152)-(157) of [1]) one finds that (5.71) is equivalent to the sequence of relations

$$A_0 := \int dx h(x) \partial j_0(x) \in \mathcal{J}_{\text{loc}} \quad (5.72)$$

and

$$A_n := - \sum_{l=0}^{n-1} \delta_{A_l} S_{n-l}(g) + \int dx h(x) \left(\partial j_n(g)(x) - \mathcal{L}_n^{(1)}(g)(x) \partial g(x) \right) \in \mathcal{J}_{\text{loc}} \quad (5.73)$$

for $n \geq 1$. This yields our assumption: with (5.72) the condition (5.70) holds true and (5.73) implies $A_n = \delta_{A_n} S_0$ and with that (5.73) gives (5.69) to n th order in κ . $\delta_A = \sum_{n \geq 0} \kappa^n \delta_{A_n}$ is a localized version of the usual BRST-transformation.

We point out that the nilpotency of the BRST-transformation is nowhere used. Every model fulfilling the local current conservation (5.71) (with $S(g)$, $j(g)$ and $\mathcal{L}^{(1)}(g)$ of the above form) satisfies our assumption (5.69) if the symmetry transformation δ_A is defined by (5.72)-(5.73).

Assuming now the validity of (5.69), we are going to derive a simplified version of (5.68). For shortness and coincidence with the notations of the preceding Sects. we

write S , j and $\mathcal{L}^{(1)}$ for $S(g)$, $j(g)$ and $\mathcal{L}^{(1)}(g)$, respectively. For the time being, we additionally assume that the testfunction h satisfies $h(x) = 1$ for all $x \in \text{supp}(\delta S/\delta\varphi)$. With that $\Delta_A^{(k)}(e_\otimes^S)$ is independent of the choice of h within this class and h can be replaced by the number 1 (see Lemma 8(ii)). h does not appear also on the right-hand side of (5.68): namely, due to our assumption and the locality of the map $D_{>1}^{(k)}$ (see Theorem 3,(i)) we obtain

$$D_{>1}^{(k)}(e_\otimes^S \otimes (A + \delta_A S)) = \int dy \left(\underbrace{\partial_\mu^y D_{>1}^{(k)}(e_\otimes^S \otimes j^\mu(y))}_{=0 \text{ for } y \notin \text{supp}(\delta S/\delta\varphi)} - D_{>1}^{(k)}(e_\otimes^S \otimes \mathcal{L}^{(1)\mu}(y)) \partial_\mu g(y) \right) \quad (5.74)$$

and, using again the locality of $D_{>1}^{(k)}$, (5.68) simplifies to the condition

$$\begin{aligned} \Delta^{(k)}(e_\otimes^S) = & - \int dy \left(D_{>1}^{(k)}(e_\otimes^S \otimes Q(y)) \frac{\delta(S_0+S)}{\delta\varphi(y)} + D_{>1}^{(k)}(e_\otimes^S \otimes \mathcal{L}^{(1)}(y)) \partial g(y) \right) \\ & - \delta D_{>1}^{(k)}(e_\otimes^S) + \mathcal{O}(\hbar^{k+1}), \end{aligned} \quad (5.75)$$

where $\Delta^{(k)}(e_\otimes^S) \stackrel{\text{def}}{=} \Delta_A^{(k)}(e_\otimes^S)|_{h \equiv 1}$ and δ is the non-localized version of δ_A :

$$\delta \stackrel{\text{def}}{=} \int dx Q(x) \frac{\delta}{\delta\varphi(x)}. \quad (5.76)$$

(5.75) is a simpler condition than (5.68) for the following reasons: Since in (5.75) the localization of the derivation δ_A is removed (i.e. h is replaced by 1), techniques of algebraic renormalization can be used to find a $D_{>1}^{(k)}$ which solves (5.75) (see Sect. 5.4.4). In addition the $D_{>1}^{(k)}(e_\otimes^S \otimes \partial j)$ -term vanishes.

However, we want to solve the MWI for general $h \in \mathcal{D}(\mathbb{M})$.¹⁸ To investigate whether (5.75) is also sufficient for the more involved condition (5.68) for arbitrary h , let a solution $D_{>1}^{(k)}$ of (5.75) be given. We point out that the map $D_{>1}^{(k)}$ is not completely determined by that, only the combination of $D_{>1}^{(k)}(e_\otimes^S)$, $D_{>1}^{(k)}(e_\otimes^S \otimes Q)$ and $D_{>1}^{(k)}(e_\otimes^S \otimes \mathcal{L}^{(1)})$ appearing on the right-hand side of (5.75) is fixed. We claim that, given such a $D_{>1}^{(k)}$, there exists a linear map $K^\mu : \mathbb{T}\mathcal{F}_{\text{loc}} \rightarrow \mathcal{D}(\mathbb{M}, \mathcal{P})$ with the property

$$\begin{aligned} \Delta_A^{(k)}(e_\otimes^S) = & \int dy h(y) \left(\partial_\mu K^\mu(e_\otimes^S)(y) - D_{>1}^{(k)}(e_\otimes^S \otimes Q(y)) \frac{\delta(S_0+S)}{\delta\varphi(y)} \right. \\ & \left. - D_{>1}^{(k)}(e_\otimes^S \otimes \mathcal{L}^{(1)}(y)) \partial g(y) \right) - \delta_A D_{>1}^{(k)}(e_\otimes^S) + \mathcal{O}(\hbar^{k+1}) \end{aligned} \quad (5.77)$$

for all $h \in \mathcal{D}(\mathbb{M})$ and with $K^\mu(1) = 0$. The latter condition is compatible with (5.77), because to zeroth order in κ the condition (5.77) reduces to $\int h \partial K(1) = \mathcal{O}(\hbar^{k+1})$, due to $D_{>1}^{(k)}(1 \otimes F) = D_{>1}^{(k)}(F) = 0$ ($\forall F$, see (5.64)).

To show the existence of K^μ , we first prove the following Lemma, which describes the difference between δ_A and δ with respect to their action on local functionals:

¹⁸For example this is used in the derivation of our version of the QAP (5.53) from the anomalous MWI.

Lemma 14. *Let be given $F \in \mathcal{F}_{\text{loc}}$, $l \in \mathcal{D}(\mathbb{M}, \mathcal{P})$ and a localized derivation $\delta_h = \int dx h(x) Q(x) \frac{\delta}{\delta \varphi(x)}$, $h \in \mathcal{D}(\mathbb{M})$, $Q \in \mathcal{P}$, such that the corresponding non-localized derivation $\delta = \int dx Q(x) \frac{\delta}{\delta \varphi(x)}$ satisfies*

$$\delta F = \int dx l(x). \quad (5.78)$$

Then there exists a $k^\mu \in \mathcal{D}(\mathbb{M}, \mathcal{P})$ such that the following localized version of (5.78) holds true:

$$\delta_h F = \int dx h(x) (l(x) + \partial_\mu k^\mu(x)). \quad (5.79)$$

Proof. Let $f \in \mathcal{D}(\mathbb{M}, \mathcal{P})$ with $F = \int dy f(y)$. Carrying out the functional derivative in (5.78) we conclude

$$l(x) = \sum_{a \in \mathbb{N}_0^d} \partial^a Q(x) \frac{\partial f}{\partial(\partial^a \varphi)}(x) + \partial_\mu k_1^\mu(x) \quad (5.80)$$

for some $k_1^\mu \in \mathcal{D}(\mathbb{M}, \mathcal{P})$. On the other hand we obtain

$$\begin{aligned} \delta_h F &= \int dx \sum_{a \in \mathbb{N}_0^d} \partial^a (h(x) Q(x)) \frac{\partial f}{\partial(\partial^a \varphi)}(x) \\ &= \int dx h(x) \sum_{a \in \mathbb{N}_0^d} \partial^a Q(x) \frac{\partial f}{\partial(\partial^a \varphi)}(x) + \int dx h(x) \partial_\mu k_2^\mu(x) \end{aligned} \quad (5.81)$$

for some other $k_2^\mu \in \mathcal{D}(\mathbb{M}, \mathcal{P})$. Hence, setting $k^\mu = -k_1^\mu + k_2^\mu$ we obtain the assertion (5.79). \square

To prove (5.77) we use that $\Delta_A^{(k)}(e_\otimes^S)$ can be written as

$$\Delta_A^{(k)}(e_\otimes^S) = \int dx h(x) \tilde{\Delta}^{(k)}(e_\otimes^S; Q(x)) \quad \text{with} \quad \tilde{\Delta}^{(k)}(e_\otimes^S; Q(x)) \in \mathcal{D}(\mathbb{M}, \mathcal{P}), \quad (5.82)$$

due to Lemma 8(ii). Hence, (5.75) can be written in the form $\delta F = \int l + \mathcal{O}(\hbar^{k+1})$ (5.78) with $F = D_{>1}^{(k)}(e_\otimes^S)$ and

$$-l(x) = D_{>1}^{(k)}(e_\otimes^S \otimes Q(x)) \frac{\delta(S_0 + S)}{\delta \varphi(x)} + D_{>1}^{(k)}(e_\otimes^S \otimes \mathcal{L}^{(1)}(x)) \partial g(x) + \tilde{\Delta}^{(k)}(e_\otimes^S; Q(x)) \in \mathcal{D}(\mathbb{M}, \mathcal{P}). \quad (5.83)$$

With that Lemma 14 yields our assertion (5.77).

We conclude that a solution of (5.68) can be obtained from a solution of (5.75) by setting

$$D_{>1}^{(k)}(e_\otimes^S \otimes j^\mu(y)) \stackrel{\text{def}}{=} K^\mu(e_\otimes^S)(y), \quad (5.84)$$

provided this does not lead to any contradictions with the partial fixing of D in terms of $D_{>1}^{(k)}(e_\otimes^S)$, $D_{>1}^{(k)}(e_\otimes^S \otimes Q)$ and $D_{>1}^{(k)}(e_\otimes^S \otimes \mathcal{L}^{(1)})$. Due to the causal Wick expansion (3.7) and the Field equation $D_{>1}^{(k)}(e_\otimes^S \otimes \partial^a \varphi) = 0$ (see Theorem 3,(iii)), this is the

case whenever the intersection of the subpolynomials of j with the subpolynomials of Q , $\mathcal{L}^{(1)}$ or \mathcal{L}_n ($\forall n \in \mathbb{N}$) contains only numbers and terms which are linear in the field φ itself or partial derivatives thereof.¹⁹ Note that (5.84) satisfies the condition $D_{>1}^{(k)}(j) = 0$.

Remark: If Q is linear in φ , the maintenance of the Field Equation requires $D_{>1}^{(k)}(e_{\otimes}^S \otimes Q) = 0$ (Theorem 3). With that the right-handed side of (5.75) vanishes to first order in κ up to terms of order \hbar^{k+1} . That is the condition (5.75) can only be satisfied if $\Delta^{(k)}(S_1) = \mathcal{O}(\hbar^{k+1})$. Hence, following the proof of Proposition 13, we first perform a finite renormalization which maintains the considered renormalization conditions and removes the term $\sim \hbar^k$ of $\Delta_A^{(k)}(S_1)$. This can be done such that $\Delta_A^{(k)}(e_{\otimes}^S) = \mathcal{O}(\hbar^k)$ is preserved. Namely, since $\Delta_A^{(k)}(S_1) = \mathcal{O}(\hbar^k)$ the vacuum expectation values on the right-hand side of (5.57) are of order \hbar^k and, hence, the pertinent renormalization map D (5.59) can be chosen of the form (5.64).

5.4.4 Proof of the Ward identities in the $O(N)$ scalar field model

As an application of the developed formalism we will prove that an off-shell generalization of the Ward identities expressing current conservation in a scalar $O(N)$ -model can be fulfilled to all orders of perturbation theory. Our strategy is based partially on techniques of algebraic renormalization described in detail in [10].

We consider a multiplet of N scalar fields $\varphi_i(x)$, $i = 1, \dots, N$, transforming under the defining representation of $O(N)$ – the group of orthogonal $N \times N$ matrices –²⁰

$$\varphi_i \longrightarrow A_{ij} \varphi_j \quad , \quad A \in O(N).^{21} \quad (5.85)$$

Let $\{X^a \mid a = 1, \dots, \frac{1}{2}N(N-1)\}$ be a basis of the Lie algebra $\mathfrak{o}(N)$ of $O(N)$, and f^{abc} the corresponding structure constants,

$$[X^a, X^b] = f^{abc} X^c . \quad (5.86)$$

The dynamics of our model is given by the free action

$$S_0 = \frac{1}{2} \int dx (\partial^\mu \varphi_i(x) \partial_\mu \varphi_i(x) - m^2 \varphi_i(x) \varphi_i(x)) \quad (5.87)$$

and the localized, $O(N)$ -invariant interaction

$$S = \int dx g(x) (\varphi_i(x) \varphi_i(x))^2 \quad \text{with} \quad g \in \mathcal{D}(\mathbb{M}) . \quad (5.88)$$

Since the free action S_0 is invariant under the transformation (5.85), there exist $\frac{1}{2}N(N-1)$ conserved Noether currents $j_\mu^a = X_{ij}^a \varphi_j \partial_\mu \varphi_i$, i.e. the local functionals

¹⁹In QED this condition is not satisfied; one has to discuss the individual cases to see that the definition (5.84) does not lead to contradictions (see e.g. [13]).

²⁰By setting $\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$ and $\bar{\varphi} = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2)$ the $O(2)$ model can be seen to be equivalent to the well known $U(1)$ model of a complex scalar field φ .

²¹repeated indices are summed over;

$A^a \stackrel{\text{def}}{=} \int dx h(x) \partial^\mu j_\mu^a(x)$ (with arbitrary $h \in \mathcal{D}(\mathbb{M})$) are elements of the ideal \mathcal{J} generated by the free field equations:

$$A^a = \delta_{A^a} S_0 \in \mathcal{J}, \quad \text{with} \quad \delta_{A^a} = \int dx h(x) X_{ij}^a \varphi_j(x) \frac{\delta}{\delta \varphi_i(x)}. \quad (5.89)$$

Essential simplifications of this model are the validity of (5.69) in simplified form and additionally that Q (5.10) is linear in φ .

The conservation of the interacting currents $(j_\mu^a)_S = R(e_\otimes^S, j_\mu^a)$ follows from the MWI for the given A^a and interaction S :

$$\int dx h(x) \partial^\mu R(e_\otimes^S, j_\mu^a(x)) \equiv R(e_\otimes^S, A^a) = \int dx h(x) R(e_\otimes^S, X_{ij}^a \varphi_j(x)) \frac{\delta S_0}{\delta \varphi_i(x)}. \quad (5.90)$$

Regarding the question whether (5.90) can be fulfilled to all orders, we start with an R -product satisfying the renormalization conditions Unitarity, \mathcal{P}_+^\dagger -Covariance, Field Independence, Field Equation and Scaling Degree. Then, the Ward identities (5.90) may be violated; however, Theorem 7 guarantees the existence of local maps $\Delta_{A^a} : \mathbb{T}\mathcal{F}_{\text{loc}} \rightarrow \mathcal{F}_{\text{loc}}$ such that

$$R(e_\otimes^S, A^a + \Delta_{A^a}(e_\otimes^S)) = \int dx h(x) R(e_\otimes^S, X_{ij}^a \varphi_j(x)) \frac{\delta S_0}{\delta \varphi_i(x)}. \quad (5.91)$$

To find a finite renormalization of the R -product which removes $R(e_\otimes^S, \Delta_{A^a}(e_\otimes^S))$, we follow the technique described in the preceding sections: we assume the existence of a $\Gamma_T^{(k)}$ such that (5.90) is fulfilled up to terms of order \hbar^k , i.e.

$$\Gamma_T^{(k)}(e_\otimes^S \otimes A^a) + \Delta_{A^a}^{(k)}(e_\otimes^S) + \mathcal{O}(\hbar^{k+1}) = \int dy h(y) X_{ij}^a \varphi_j(y) \frac{\delta(S_0 + \Gamma_T^{(k)}(e_\otimes^S))}{\delta \varphi_i(y)} \quad (5.92)$$

where $\Delta_{A^a}^{(k)}(e_\otimes^S) = \mathcal{O}(\hbar^k)$. (The Field Equation for $\Gamma_T^{(k)}$ (Lemma 6) is taken into account.) First we perform a finite renormalization which maintains $\Delta_{A^a}^{(k)}(e_\otimes^S) = \mathcal{O}(\hbar^k)$ and the mentioned renormalization conditions and which removes the terms $\sim \hbar^k$ of $\Delta_{A^a}^{(k)}(S)$. Due to the requirement $D_{>1}^{(k)}(e_\otimes^S \otimes \varphi) = 0$ (Theorem 3, (iii)), the condition (5.75) simplifies to

$$\Delta_a^{(k)}(e_\otimes^S) = -\delta_a D_{>1}^{(k)}(e_\otimes^S) + \mathcal{O}(\hbar^{k+1}), \quad (5.93)$$

where we set $\Delta_a^{(k)}(e_\otimes^S) \stackrel{\text{def}}{=} \Delta_{A^a}^{(k)}(e_\otimes^S)|_{h \equiv 1}$ and $\delta_a \stackrel{\text{def}}{=} \int dx X_{ij}^a \varphi_j(x) \frac{\delta}{\delta \varphi_i(x)}$. To fulfill (5.90) up to terms of order \hbar^{k+1} , we have to solve (5.93) and to extend the definition of this $D_{>1}^{(k)}$ in such a way that condition (5.68) (for general h) holds true. The latter can be done by means of (5.84), because the intersection of the non-trivial subpolynomials of j_μ^a with the subpolynomials of Q or $(\varphi_i \varphi_i)^2$ is a subset of $\mathbb{C} \varphi$.

It remains to show the solvability of (5.93). For this purpose we temporarily restrict the functionals (2.1) in (5.92) to the space $\mathcal{D}(\mathbb{M})$ of compactly supported test functions on Minkowski space. This permits us to perform the limit²² $h \rightarrow 1$ in (5.92), ending

²²This limit is done as follows: let $h \in \mathcal{D}(\mathbb{M})$ such that there is a neighbourhood U of $0 \in \mathbb{M}$ with $h|_U = 1$. Then we replace $h(x)$ by $h_\epsilon(x) \equiv h(\epsilon x)$ ($\epsilon > 0$) and perform the limit $\epsilon \rightarrow 0$.

up with the equation

$$\delta_a(S_0 + \Gamma_T^{(k)}(e_\otimes^S)) = \Delta_a^{(k)}(e_\otimes^S) + \mathcal{O}(\hbar^{k+1}). \quad (5.94)$$

Furthermore, using (5.86) we obtain the identity

$$[\delta_a, \delta_b] = f_{abc} \delta_c, \quad (5.95)$$

which we insert into $[\delta_a, \delta_b](S_0 + \Gamma_T^{(k)}(e_\otimes^S))$. This yields the *consistency condition*

$$\delta_a \Delta_b^{(k)}(e_\otimes^S) - \delta_b \Delta_a^{(k)}(e_\otimes^S) = f_{abc} \Delta_c^{(k)}(e_\otimes^S) + \mathcal{O}(\hbar^{k+1}). \quad (5.96)$$

Due to the compact support of S , and the locality of $\Delta_a^{(k)}$, each term in (5.96) has compact support as well. Therefore, this equation holds true on the entire configuration space $\mathcal{C}(\mathbb{M}, \mathbb{R})$, i.e. the restriction of the functionals to $\mathcal{D}(\mathbb{M})$ can be omitted.

The consistency condition (5.96) is the cocycle condition in the Lie algebra cohomology corresponding to the Lie algebra generated by the derivations $\{\delta_a\}$ acting on \mathcal{F}_{loc} . Trivial solutions are the coboundaries,

$$\Delta_a^{(k)}(e_\otimes^S) = -\delta_a \hat{\Delta}^{(k)}(e_\otimes^S) + \mathcal{O}(\hbar^{k+1}) \quad (5.97)$$

for some linear, symmetric and local map $\hat{\Delta}^{(k)} : \mathbb{T}\mathcal{F}_{\text{loc}} \rightarrow \mathcal{F}_{\text{loc}}$. If $\Delta_a^{(k)}(e_\otimes^S)$ is a coboundary, the condition (5.93) can be solved by setting

$$D_{>1}^{(k)}(1) \stackrel{\text{def}}{=} 0, \quad D_{>1}^{(k)}(S) \stackrel{\text{def}}{=} 0 \quad \text{and} \quad D_{>1}^{(k)}(S^{\otimes j}) \stackrel{\text{def}}{=} \hat{\Delta}^{(k)}(S^{\otimes j}) \quad \forall j \geq 2, \quad (5.98)$$

due to $\Delta_a^{(k)}(S) = \mathcal{O}(\hbar^{k+1})$.

Hence, we only have to show that the present cohomology is trivial. In the literature (see e.g. [10] and references therein) it is shown that every Lie algebra cohomology corresponding to some semi-simple Lie group and some finite representation is trivial. This result applies to our problem. Namely, $O(N)$ is semi-simple and, since the mass dimension of $\Delta_a^{(k)}(e_\otimes^S)$ is bounded (5.43), the anomaly terms indeed span a finite representation of $\mathfrak{o}(N)$. It does not matter that, in contrast to the literature, our functionals are *local*. Note that $D_{>1}^{(k)}(e_\otimes^S)$ is not uniquely defined by this procedure.

6 Conclusions and Outlook

In algebraic renormalization the QAP is used to remove possible anomalies of Ward identities by induction on \hbar . We have worked out an analogous procedure for the MWI in the different framework of causal perturbation theory. The main difference is that we work solely with compactly supported interactions S and localized symmetry transformations δ_A . Our main result gives a crucial insight into the structure of possible anomalies of the MWI, in particular with respect to the deformation parameter \hbar , and allows the transfer of techniques from algebraic renormalization into causal perturbation theory.

This yields a general method to fulfill the MWI for a given model. A first non-trivial application is worked out (Sect. 5.4.4). The developed method seems to be applicable to many models (as suggested by [2], [11] and [10]). Together with the powerful tool of BRST cohomology it should make possible a proof of that cases of the MWI which are needed for the construction of the net of local observables of Yang-Mills type QFTs. (This would complete the construction given in [2].) A main advantage of this approach to quantum Yang-Mills theories is that there seems to be no serious obstacle for the generalization to curved spacetimes where the techniques developed for scalar fields in [12] and [31] can be used.

A Proper vertices for R -products

A.1 Definition and basic properties

Before we introduce proper vertices in terms of R -products, we shortly consider the diagrammatics of retarded products $R(A_1(x_1), \dots; A_n(x_n))$.

For the **unrenormalized** expressions (i.e. for $x_i \neq x_j$, $\forall i \neq j$) the diagrammatic interpretation is unique. One can show that there are two kinds of inner lines which symbolize Δ^{ret} and H_m^μ , respectively, and are orientated. For tree diagrams only Δ^{ret} appears and all inner lines are pointing to the distinguished vertex $A_n(x_n)$. Solely connected diagrams contribute to R ; and R_{cl} is precisely the contribution of all tree diagrams. Both statements follow from the inductive construction of the $(R_{n,1})_{n \in \mathbb{N}}$ [16]. The decisive step is the GLZ Relation: $\{F_G, H_G\} = \dots$. In the quantum case there is at least one contraction between F_G and H_G , and in classical FT there is precisely one contraction in $\{F_G, H_G\}_{\text{cl}}$.

For the **renormalized** retarded product we use these results as definition of the connected and tree part: $R^c \equiv R$ and $R_{\text{tree}} \equiv R_{\text{cl}}$.

Analogously to (4.12)-(4.14) the property $R_{(\text{cl})}(1, F) = F$ implies the following conclusions

$$R_{(\text{cl})}\left(e^{\sum_{n=1}^{\infty} F_n \lambda^n}, \sum_{n=0}^{\infty} G_n \lambda^n\right) = 0 \implies G_n = 0 \quad \forall n, \quad (\text{A.1})$$

$$\begin{aligned} R_{(\text{cl})}\left(e^{\sum_{n=1}^{\infty} F_n \lambda^n}, \sum_{n=0}^{\infty} \frac{\delta F_n}{\delta \varphi} \lambda^n\right) &= R_{(\text{cl})}\left(e^{\sum_{n=1}^{\infty} G_n \lambda^n}, \sum_{n=0}^{\infty} \frac{\delta G_n}{\delta \varphi} \lambda^n\right) \\ \wedge \quad \omega_0(F_n) &= \omega_0(G_n) \quad \forall n \implies F_n = G_n \quad \forall n, \end{aligned} \quad (\text{A.2})$$

which hold for the R -products of classical FT (R_{cl}) and of QFT (R). In classical FT the statement (A.1) can be proved also non-perturbatively: $0 = R_{\text{cl}}(e_{\otimes}^{F(\lambda)}, G(\lambda)) = G(\lambda) \circ r^{S_0 + F(\lambda), S_0}$ implies $G(\lambda) = G(\lambda) \circ r^{S_0 + F(\lambda), S_0} \circ r^{S_0, S_0 + F(\lambda)} = 0$.

The concept of proper vertices has a clear physical interpretation when applied to R -products, since $R_{\text{tree}} = R_{\text{cl}}$. As explained in Sect. 2 the entries of R_{cl} may be *non-local*. We want to rewrite an interacting QFT-field $R(e_{\otimes}^S, F)$ as a classical field $R_{\text{cl}}(e_{\otimes}^{\Gamma_R(e_{\otimes}^S)}, \Gamma_{\text{ret}}(e_{\otimes}^S, F))$ where the 'proper interaction' $\Gamma_R(e_{\otimes}^S)$ and the 'proper

retarded field' $\Gamma_{\text{ret}}(e_{\otimes}^S, F)$ are non-local and agree to lowest order in \hbar with the original local functionals S and F respectively. This is indeed possible:

Proposition 15. (a) *There exist*

– *a totally symmetric and linear map*

$$\Gamma_R : \mathbb{T}\mathcal{F}_{\text{loc}} \longrightarrow \mathcal{F} \quad (\text{A.3})$$

– *and a linear map*

$$\Gamma_{\text{ret}} : \mathbb{T}\mathcal{F}_{\text{loc}} \otimes \mathcal{F}_{\text{loc}} \longrightarrow \mathcal{F} \quad (\text{A.4})$$

which is totally symmetric in the former entries (i.e. $\Gamma_{\text{ret}}(\otimes_{j=1}^n F_{\pi j}, F) = \Gamma_{\text{ret}}(\otimes_{j=1}^n F_j, F)$),

which are uniquely determined by the conditions

$$R(e_{\otimes}^S, F) = R_{\text{cl}}\left(e_{\otimes}^{\Gamma_R(e_{\otimes}^S)}, \Gamma_{\text{ret}}(e_{\otimes}^S, F)\right) , \quad (\text{A.5})$$

$$\Gamma_{\text{ret}}(e_{\otimes}^S, \varphi(h)) = \varphi(h) , \quad h \in \mathcal{D}(\mathbb{M}) , \quad (\text{A.6})$$

$$\Gamma_R(\mathbf{1}) = 0 , \quad \omega_0(\Gamma_R(e_{\otimes}^S)) = 0 . \quad (\text{A.7})$$

(b) Γ_R and Γ_{ret} are related by

$$\frac{\delta \Gamma_R(e_{\otimes}^S)}{\delta \varphi(x)} = \Gamma_{\text{ret}}\left(e_{\otimes}^S, \frac{\delta S}{\delta \varphi(x)}\right) , \quad (\text{A.8})$$

that is, with (A.7), Γ_R is uniquely determined by Γ_{ret} .

Compared with the defining condition (4.19) for Γ_T , there is more flexibility in (A.5) since it contains two kinds of 'vertex functions', Γ_R and Γ_{ret} . To define the latter *uniquely*, we additionally require (A.6) and (A.7).

Proof. (b) First we show that the defining conditions for Γ_R , Γ_{ret} given in part (a) imply the statement in part (b). The off-shell field equation

$$R_{(\text{cl})}\left(e_{\otimes}^S, \frac{\delta S_0}{\delta \varphi(x)}\right) = \frac{\delta S_0}{\delta \varphi(x)} - R_{(\text{cl})}\left(e_{\otimes}^S, \frac{\delta S}{\delta \varphi(x)}\right) , \quad (\text{A.9})$$

holds for R (QFT) and R_{cl} (classical FT), in the latter case even for non-local entries. With that and using the conditions (A.5) and (A.6) and finally (A.1) we obtain the assertion (A.8).

(a) By expanding (A.5) in powers of S and using (A.8) and (A.7) we find an inductive construction of Γ_R and Γ_{ret} in terms of R and R_{cl} :

$$\Gamma_{\text{ret}}(\mathbf{1}, F) = R(F) \equiv F , \quad (\text{A.10a})$$

$$\frac{\delta \Gamma_R(S)}{\delta \varphi(x)} = \Gamma_{\text{ret}}\left(\mathbf{1}, \frac{\delta S}{\delta \varphi(x)}\right) = \frac{\delta S}{\delta \varphi(x)} \implies \Gamma_R(S) = S , \quad (\text{A.10b})$$

$$\Gamma_{\text{ret}}(S, F) = R(S, F) - R_{\text{cl}}(S, F) \quad (\text{A.10c})$$

$$\frac{\delta \Gamma_R(S^{\otimes 2})}{\delta \varphi(x)} = 2\Gamma_{\text{ret}}\left(S, \frac{\delta S}{\delta \varphi(x)}\right) = 2R\left(S, \frac{\delta S}{\delta \varphi(x)}\right) - 2R_{\text{cl}}\left(S, \frac{\delta S}{\delta \varphi(x)}\right), \quad (\text{A.10d})$$

$$\Gamma_{\text{ret}}(S^{\otimes 2}, F) = R(S^{\otimes 2}, F) - R_{\text{cl}}(S^{\otimes 2}, F) - R_{\text{cl}}(\Gamma_R(S^{\otimes 2}), F) - 2R_{\text{cl}}(S, \Gamma_{\text{ret}}(S, F)). \quad (\text{A.10e})$$

We explicitly see that Γ_{ret} is not totally symmetric, it is retarded with respect to the last entry.

Now let $\Gamma_R(e_{\otimes}^S)$ and $\Gamma_{\text{ret}}(e_{\otimes}^S, F)$ be constructed up to order n and $(n-1)$ respectively in S . Then, the condition (A.5) determines $\Gamma_{\text{ret}}(S^{\otimes n}, F)$ uniquely:

$$\begin{aligned} \Gamma_{\text{ret}}(S^{\otimes n}, F) &= R(S^{\otimes n}, F) - \sum_{k=1}^n \binom{n}{k} R_{\text{cl}}\left(\frac{d^k}{d\lambda^k}\Big|_{\lambda=0} e_{\otimes}^{\Gamma_R(e_{\otimes}^{\lambda S})}, \Gamma_{\text{ret}}(S^{\otimes n-k}, F)\right) \\ &= R(S^{\otimes n}, F) - \sum_{k=1}^n \binom{n}{k} \sum_{j=1}^k \sum_{\substack{l_1, \dots, l_j=1 \\ l_1 + \dots + l_j = k}} \frac{1}{j! l_1! \dots l_j!} R_{\text{cl}}\left(\bigotimes_{i=1}^j \Gamma_R(S^{\otimes l_i}), \Gamma_{\text{ret}}(S^{\otimes n-k}, F)\right). \end{aligned} \quad (\text{A.11})$$

From that and with (A.8) and (A.7) we uniquely get $\Gamma_R(S^{\otimes n+1})$. \square

Remark: The roles of the conditions (A.6) and (A.8) can be exchanged. In the list (A.5)-(A.7) of defining conditions, (A.6) can be replaced by (A.8). Then, (A.6) can be derived from (A.5), (A.7) and (A.8) analogously to (4.30) and (4.31): proceeding inductively we use (A.11), the integrated field equation for R and R_{cl} and (A.8).

Following the construction (A.10)-(A.11) we inductively prove the following properties of $\Gamma_R, \Gamma_{\text{ret}}$:

- **\hbar -dependence:**

$$\left. \begin{aligned} \Gamma_{\text{ret}}(e_{\otimes}^S, F) &= F + \mathcal{O}(\hbar) \\ \Gamma_R(e_{\otimes}^S) &= S + \mathcal{O}(\hbar) \end{aligned} \right\} \quad \text{if } F, S \sim \hbar^0. \quad (\text{A.12})$$

- **\mathcal{P}_+^\dagger -Covariance, Unitarity** ($\Gamma_R(e_{\otimes}^S)^* = \Gamma_R(e_{\otimes}^{S^*})$ and similarly for Γ_{ret}), **Field Independence, Smoothness in $m \geq 0$, μ -Covariance and Almost homogeneous Scaling of $\Gamma_R \equiv \Gamma_R^{(m, \mu)}$ and $\Gamma_{\text{ret}} \equiv \Gamma_{\text{ret}}^{(m, \mu)}$ (or, alternatively, Scaling Degree).**

In the proof of (A.12) we use $R = R_{\text{cl}} + \mathcal{O}(\hbar)$, $R_{\text{cl}} \sim \hbar^0$. The other properties rely on the validity of the corresponding axioms for R and R_{cl} , analogously to Lemma 6.

We point out that $\Gamma_R(e_{\otimes}^S \otimes \varphi(h))$ differs in general from $\varphi(h)$, in contrast to the Field equation for Γ_T and (A.6). Namely, inserting (A.5)-(A.6) into the GLZ-relation for $[R(e_{\otimes}^S, \varphi(h)), R(e_{\otimes}^S, \varphi(g))]$ we obtain

$$\begin{aligned} &\frac{i}{\hbar} \left[R_{\text{cl}}\left(e_{\otimes}^{\Gamma_R(e_{\otimes}^S)}, \varphi(h)\right), R_{\text{cl}}\left(e_{\otimes}^{\Gamma_R(e_{\otimes}^S)}, \varphi(g)\right) \right] = \\ &R_{\text{cl}}\left(e_{\otimes}^{\Gamma_R(e_{\otimes}^S)} \otimes \Gamma_R(e_{\otimes}^S \otimes \varphi(g)), \varphi(h)\right) - R_{\text{cl}}\left(e_{\otimes}^{\Gamma_R(e_{\otimes}^S)} \otimes \Gamma_R(e_{\otimes}^S \otimes \varphi(h)), \varphi(g)\right). \end{aligned} \quad (\text{A.13})$$

Due to the GLZ Relation for R_{cl} , the left-hand side is equal to

$$R_{\text{cl}}\left(e_{\otimes}^{\Gamma_R(e_{\otimes}^S)} \otimes \varphi(g), \varphi(h)\right) - R_{\text{cl}}\left(e_{\otimes}^{\Gamma_R(e_{\otimes}^S)} \otimes \varphi(h), \varphi(g)\right) \\ + \frac{i}{\hbar} \left[R_{\text{cl}}\left(e_{\otimes}^{\Gamma_R(e_{\otimes}^S)}, \varphi(h)\right), R_{\text{cl}}\left(e_{\otimes}^{\Gamma_R(e_{\otimes}^S)}, \varphi(g)\right) \right]_{\star(\geq 2)}, \quad (\text{A.14})$$

where $\frac{i}{\hbar}[\cdot, \cdot]_{\star(\geq 2)} \equiv \frac{i}{\hbar}[\cdot, \cdot]_{\star} - \{\cdot, \cdot\}_{\text{cl}}$. The assertion follows from the non-vanishing of the $[\cdot, \cdot]_{\star(\geq 2)}$ -term.

Finally we study a finite renormalization $R \rightarrow \hat{R}$. To express the corresponding renormalizations of Γ_R and Γ_{ret} in terms of the corresponding map D of the Main Theorem we insert the defining relation (A.5) into both sides of (3.23). In the resulting equation

$$R_{\text{cl}}\left(e_{\otimes}^{\hat{\Gamma}_R(e_{\otimes}^S)}, \hat{\Gamma}_{\text{ret}}(e_{\otimes}^S, F)\right) = R_{\text{cl}}\left(e_{\otimes}^{\Gamma_R(e_{\otimes}^{D(e_{\otimes}^S)})}, \Gamma_{\text{ret}}\left(e_{\otimes}^{D(e_{\otimes}^S)}, D(e_{\otimes}^S \otimes F)\right)\right) \quad (\text{A.15})$$

we choose $F = \frac{\delta S}{\delta \varphi}$. With that we may replace $\hat{\Gamma}_{\text{ret}}(e_{\otimes}^S, F)$ by $\frac{\delta \hat{\Gamma}_R(e_{\otimes}^S)}{\delta \varphi}$ and $\Gamma_{\text{ret}}\left(e_{\otimes}^{D(e_{\otimes}^S)}, D(e_{\otimes}^S \otimes F)\right)$ by $\frac{\delta \Gamma_R(e_{\otimes}^{D(e_{\otimes}^S)})}{\delta \varphi}$. By means of (A.2) we conclude

$$\hat{\Gamma}_R(e_{\otimes}^S) = \Gamma_R(e_{\otimes}^{D(e_{\otimes}^S)}) . \quad (\text{A.16})$$

We insert this into (A.15) and apply (A.1). This yields

$$\hat{\Gamma}_{\text{ret}}(e_{\otimes}^S, F) = \Gamma_{\text{ret}}\left(e_{\otimes}^{D(e_{\otimes}^S)}, D(e_{\otimes}^S \otimes F)\right) . \quad (\text{A.17})$$

A.2 Comparison of the vertex functions in terms of T - and R -products

The vertex functions Γ_T and Γ_R defined in terms of T - and R -products, respectively, are both totally symmetric, nevertheless they do not agree. This follows from the different forms of the unitarity property or, alternatively, from the non-validity of $\Gamma_T(e_{\otimes}^S \otimes \varphi(h)) = \varphi(h)$ for Γ_R . We are going to compare Γ_T with Γ_R to lowest orders in S .

By using the definitions of $\Gamma_R, \Gamma_{\text{ret}}$ (A.5) and Γ_T (4.19), as well as (4.15)-(4.16) we obtain

$$R_{\text{cl}}\left(e_{\otimes}^{\Gamma_R(e_{\otimes}^S)}, \Gamma_{\text{ret}}(e_{\otimes}^S, F)\right) = \sum_{n=0}^{\infty} \left(1 - T_{\text{tree}}\left(e_{\otimes}^{i\Gamma_T(e_{\otimes}^S)/\hbar}\right)\right)^{\star n} \star T_{\text{tree}}\left(e_{\otimes}^{i\Gamma_T(e_{\otimes}^S)/\hbar} \otimes \Gamma_T(e_{\otimes}^S \otimes F)\right) . \quad (\text{A.18})$$

If we interpret $\Gamma_R, \Gamma_{\text{ret}}$ and Γ_T as one vertex, then the l.h.s. contains solely tree diagrams, but on the r.h.s. there appear also loop diagrams! This indicates that the

relation of Γ_R to Γ_T is rather involved. To zeroth and first order in S we obtain

$$\Gamma_{\text{ret}}(1, F) = \Gamma_T(1 \otimes F) = F, \quad (\text{A.19})$$

$$\Gamma_{\text{ret}}(S, F) + R_{\text{cl}}(S, F) = i/\hbar T_{\text{tree}}(S \otimes F) + \Gamma_T(S \otimes F) - i/\hbar S \star F \quad (\text{A.20})$$

The terms $\sim \hbar^{-1}$ and $\sim \hbar^0$ of (4.15) read

$$R_{\text{cl}}(S, F) = i/\hbar T_{\text{tree}}(S \otimes F) - i/\hbar S \star^{(\leq 1)} F, \quad (\text{A.21})$$

where

$$\begin{aligned} S \star^{(\leq 1)} F &= \sum_{n \leq 1} \frac{\hbar^n}{n!} \int dx_1 \dots dx_n dy_1 \dots dy_n \frac{\delta^n S}{\delta \varphi(x_1) \dots \delta \varphi(x_n)} \\ &\quad \cdot \prod_{i=1}^n H_m^\mu(x_i - y_i) \frac{\delta^n F}{\delta \varphi(y_1) \dots \delta \varphi(y_n)}. \end{aligned} \quad (\text{A.22})$$

In the same way we define $S \star^{(\geq 2)} F$ (i.e. $S \star F = S \star^{(\leq 1)} F + S \star^{(\geq 2)} F$) and e.g. $S \star^{(2)} F$. With that (A.20) reads

$$\begin{aligned} \Gamma_T(S \otimes F) &= \Gamma_{\text{ret}}(S, F) + i/\hbar S \star^{(\geq 2)} F \\ &= \frac{1}{2} \left(\Gamma_{\text{ret}}(S, F) + \Gamma_{\text{ret}}(F, S) + i/\hbar (S \star^{(\geq 2)} F + F \star^{(\geq 2)} S) \right). \end{aligned} \quad (\text{A.23})$$

Using additionally the Field Independence of Γ_T and (A.10d) we find

$$\frac{\delta}{\delta \varphi} \left(\Gamma_T(S \otimes F) - \Gamma_R(S \otimes F) \right) = \frac{i}{\hbar} \left(S \star^{(\geq 2)} \frac{\delta F}{\delta \varphi} + F \star^{(\geq 2)} \frac{\delta S}{\delta \varphi} \right). \quad (\text{A.24})$$

Selecting the terms of second order in S from (A.18) we find

$$\begin{aligned} 1/2 \Gamma_{\text{ret}}(S^{\otimes 2}, F) + R_{\text{cl}}(S, \Gamma_{\text{ret}}(S, F)) + 1/2 R_{\text{cl}}(S^{\otimes 2}, F) + 1/2 R_{\text{cl}}(\Gamma_R(S^{\otimes 2}), F) = \\ 1/2 \Gamma_T(S^{\otimes 2} \otimes F) + i/\hbar T_{\text{tree}}(S \otimes \Gamma_T(S \otimes F)) + i/2\hbar T_{\text{tree}}(\Gamma_T(S^{\otimes 2}) \otimes F) \\ - 1/2\hbar^2 T_{\text{tree}}(S^{\otimes 2} \otimes F) - i/\hbar S \star \Gamma_T(S \otimes F) + 1/\hbar^2 S \star T_{\text{tree}}(S \otimes F) \\ - i/2\hbar \Gamma_T(S^{\otimes 2}) \star F + 1/2\hbar^2 T_{\text{tree}}(S^{\otimes 2}) \star F - 1/\hbar^2 S \star S \star F. \end{aligned} \quad (\text{A.25})$$

To simplify this formula and to eliminate all vertex functions Γ_{ret} , Γ_R , Γ_T with two arguments, we use (A.23), (A.10d), the Field Independence, as well as

$$R_{\text{cl}}(G, H) = \int dx dy \frac{\delta H}{\delta \varphi(x)} \Delta^{\text{ret}}(x - y) \frac{\delta G}{\delta \varphi(y)} \quad (\text{A.26})$$

and the corresponding expression for $T_{\text{tree}}^c(G, H)$, in addition (A.21) and the corresponding identity

$$\begin{aligned} 1/2 R_{\text{cl}}(S^{\otimes 2}, F) &= \frac{1}{\hbar^2} \left(-1/2 T_{\text{tree}}(S^{\otimes 2} \otimes F) \right. \\ &\quad \left. + (S \star T(S \otimes F))_{\text{tree}} + 1/2 (T(S^{\otimes 2}) \star F)_{\text{tree}} + (S \star S \star F)_{\text{tree}} \right), \end{aligned} \quad (\text{A.27})$$

and also

$$T_{\text{tree}}(S \otimes F) - i\hbar \Gamma_T(S \otimes F) = T(S \otimes F) \quad (\text{A.28})$$

(which follows from Corollary 5 and (4.11)). It results

$$\begin{aligned} & 1/2 \left(\Gamma_{\text{ret}}(S^{\otimes 2}, F) - \Gamma_T(S^{\otimes 2} \otimes F) \right) = \\ & \frac{i}{\hbar} \int dx dy \frac{\delta F}{\delta \varphi(x)} \Delta^{\text{ret}}(x-y) \left(S \star^{(\geq 2)} \frac{\delta S}{\delta \varphi(y)} \right) + \frac{i}{\hbar} R_{\text{cl}}(S, S \star^{(\geq 2)} F) \\ & - \frac{1}{\hbar^2} \sum_{a+b=3} (S \star^{(\geq a)} S \star^{(\geq b)} F)^c + \frac{1}{\hbar^2} \left((S \star^{(\geq 2)} T(S \otimes F))^c - (S \star^{(2)} (S \cdot F))_{\text{tree}} \right) \\ & + \frac{1}{2\hbar^2} \left((T(S^{\otimes 2}) \star^{(\geq 2)} F)^c - ((S \cdot S) \star^{(2)} F)_{\text{tree}} \right). \end{aligned} \quad (\text{A.29})$$

In comparison with (A.25) a main simplification is that on the right-hand side solely connected diagrams contribute and the cancellation of all tree diagrams is obvious (i.e. the right-hand side is manifestly of order \hbar).

We have not succeeded to generalize the results (A.23), (A.24) and (A.29) to a general formula relating Γ_T to Γ_{ret} or Γ_R .

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References

- [1] Dütsch, M. and Fredenhagen, K. “The Master Ward identity and generalized Schwinger-Dyson equation in classical field theory”. *Commun. Math. Phys.* **243** (2003) 275–314.
- [2] Dütsch, M. and Boas, F. M. “The Master Ward identity”. *Rev. Math. Phys.* **14** (2002) 977.
- [3] Lowenstein, J. H. “Differential vertex operations in Lagrangian field theory”. *Commun. Math. Phys.* **24** (1971) 1–21.
- [4] Lam, Y.-M. P. “Perturbation Lagrangian theory for scalar fields: Ward- Takahasi identity and current algebra”. *Phys. Rev.* **D6** (1972) 2145–2161.
- [5] Breitenlohner, P. and Maison, D. “Dimensional renormalization and the action principle”. *Commun. Math. Phys.* **52** (1977) 11–38.
- [6] Becchi, C., Rouet, A. and Stora, R. “Renormalization of the abelian Higgs-Kibble Model”. *Commun. Math. Phys.* **42** (1975) 127–162.
- [7] Becchi, C., Rouet, A. and Stora, R. “Renormalization of gauge theories”. *Annals Phys.* **98** (1976) 287–321.

- [8] Zinn-Justin, J. “Renormalization of gauge theories and master equation”. *Mod. Phys. Lett. A* **14** (1999) 1227–1236.
- [9] Henneaux, M. and Teitelboim, C. “Quantization of gauge systems” (1992) 1–520. Princeton University Press, Princeton, New Jersey.
- [10] Piguet, O. and Sorella, S. P. *Algebraic renormalization: Perturbative renormalization, symmetries and anomalies*, vol. M28 of *Lect. Notes Phys.*. 1995.
- [11] Barnich, G., Brandt, F. and Henneaux, M. “Local BRST cohomology in gauge theories”. *Phys. Rept.* **338** (2000) 439–569.
- [12] Brunetti, R. and Fredenhagen, K. “Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds”. *Commun. Math. Phys.* **208** (2000) 623–661.
- [13] Dütsch, M. and Fredenhagen, K. “A local (perturbative) construction of observables in gauge theories: The example of QED”. *Commun. Math. Phys.* **203** (1999) 71–105.
- [14] Dütsch, M. and Fredenhagen, K. “Perturbative algebraic field theory, and deformation quantization”. *Fields Institute Communications* **30** (2001) 151–160.
- [15] Dütsch, M. and Fredenhagen, K. “Algebraic quantum field theory, perturbation theory, and the loop expansion”. *Commun. Math. Phys.* **219** (2001) 5–30.
- [16] Dütsch, M. and Fredenhagen, K. “Causal perturbation theory in terms of retarded products, and a proof of the Action Ward identity”. *Rev. Math. Phys.* **16** (2004) 1291–1348.
- [17] Bogoliubov, N. and Shirkov, D. *Introduction to the Theory of Quantized Fields*. 1959. New York.
- [18] Epstein, H. and Glaser, V. “The role of locality in perturbation theory”. *Annales Poincaré Phys. Theor.* **A19** (1973) 211.
- [19] Brunetti, R., Dütsch, M. and Fredenhagen, K. “Perturbative Algebraic Quantum Field Theory and the Renormalization Groups” work in progress.
- [20] Dütsch, M. and Fredenhagen, K. “Action Ward identity and the Stückelberg-Petermann renormalization group”. In Boutet de Monvel, A., Iagolnitzer, D. and Moschella, U. (Eds.) “Rigorous Quantum Field Theory”, pp. 113–123. Birkhäuser Verlag, 2006. hep-th/0501228.
- [21] Peierls, R. “The commutation laws of relativistic field theory”. *Proc. Roy. Soc.(London) A* **214** (1952) 143.
- [22] Marolf, D. M. “The Generalized Peierls bracket”. *Ann. Phys.* **236** (1994) 392–412.
- [23] Steinmann, O. *Perturbative Expansion in axiomatic field theory*, vol. 11 of *Lect. Notes Phys.*. 1971.

- [24] Steinmann, O. *Perturbative quantum electrodynamics and axiomatic field theory*. Berlin, Germany: Springer (2000) 355 p.
- [25] Itzykson, C. and Zuber, J.-B. *Quantum Field Theory*. New York: Mc Graw-Hill (1980).
- [26] Dütsch, M., Hurth, T., Krahe, F. and Scharf, G. “Causal Construction of Yang-Mills theories. I.” *N. Cimento A* **106** (1993) 1029.
- [27] Dütsch, M. “Proof of perturbative gauge invariance for tree diagrams to all orders”. *Ann. Phys. (Leipzig)* **14** (2005) 438–461.
- [28] Jona-Lasinio, G. “Relativistic Field Theories with Symmetry-Breaking Solutions”. *Nuovo Cimento* **34** (1964) 1790–1795.
- [29] Gell-Mann, M. and Low, F. “Bound States in Quantum Field Theory”. *Physical Review* **84** (1951) 350–354.
- [30] Dütsch, M., Hurth, T., Krahe, F. and Scharf, G. “Causal Construction of Yang-Mills theories. II.” *N. Cimento A* **107** (1994) 375.
- [31] Hollands, S. and Wald, R. M. “Conservation of the stress tensor in interacting quantum field theory in curved spacetimes”. *Rev. Math. Phys.* **17** (2005) 227–312.
- [32] Brennecke, F. “Investigations to the anomaly problem of the Master Ward Identity”. *diploma thesis (in german)* (2005). <http://www.desy.de/unith/lqp/psfiles/dipl-brennecke.ps.gz>.