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The Wulff Problem For Diffuse Interface

by

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Abstract. We consider the non local free energy functional defined in [3] and we study its inf over the class of functions with zero average (Wulff problem). We prove that the inf is a minimum achieved on a particular antisymmetric strictly increasing function called the finite volume instanton. The result can be interpreted as an extension of the Wulff theorem to a not sharp interface.

1. Introduction

We will study the "Wulff problem" of minimizing the functional $\mathcal{F}_L(m)$ defined in (2) below, keeping fixed and equal to zero the mean value of m over $[-L, L]$:

$$\inf_{K_L(m)=0} \mathcal{F}_L(m) \qquad (1)$$

where K_L is the averaging operator over $[-L, L]$

$$K_L(m) = \frac{1}{2L} \int_{-L}^{L} m(x)dx,$$

This would be the classical Wulff problem if $\mathcal{F}_L$ were the perimeter functional which describes the free energy when the interface is sharp. Here instead we are interested in the "finite volume corrections" where the interface is diffuse and not sharp. Convergence to the classical Wulff problem in the limit $L \to \infty$ has been proved for the functional (2) by Alberti and Bellettini, [1], [2]. We will prove that for L large enough the inf in (1) is a minimum and that the minimizer is unique. In this way we investigate the fine structure of the interface at finite L's.

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$\mathcal{F}_L$ is given by

$$\mathcal{F}_L(m) = \int_{-L}^{L} \phi_\beta(m)dx + \frac{1}{4}\int_{-L}^{L}\int_{-L}^{L} J^{\text{neum}}(x,x')(m(x)-m(x'))^2 dxdx' \quad (2)$$

with

$$\phi_\beta(m) = \tilde{\phi}_\beta(m) - \min_{|s|\le 1}\tilde{\phi}_\beta(s)$$

$$\tilde{\phi}_\beta(m) = -\frac{m^2}{2} - \frac{1}{\beta}\mathcal{S}(m), \quad \beta > 1$$

$$\mathcal{S}(m) = -\frac{1-m}{2}\ln\frac{1-m}{2} - \frac{1+m}{2}\ln\frac{1+m}{2}$$

$$J^{\text{neum}}(x,y) = J(x,y) + J(x,R_L(y)) + J(x,R_{-L}(y))$$

$R_\xi(y)$ is the reflection of y around ξ. $J(x,y)$ is a smooth, symmetric, translational invariant probability kernel supported in $|x-y| \le 1$. Moreover $J(0,x)$ is a non increasing if restricted to $x \ge 0$.

$\hat{m}_L$ is the finite volume instanton. It is a stationary point for $\mathcal{F}_L$ and it is the unique solution, in a suitable neighbourhood of $\bar{m}$, $\bar{m}$ is defined below in (5) (see also [6]), of the equation

$$m = \tanh\{\beta J^{\text{neum}} * m\}$$

in $[-L, L]$, L large.

Our main result is

Theorem 1. For any L large enough

$$\inf_{K_L(m)=0}\mathcal{F}_L(m) = \mathcal{F}_L(\hat{m}_L) \quad (3)$$

This results can be easily extended to the case of fixed non vanishing mean magnetization. For brevity reasons we do not do that in this paper. Though we work in one dimension, the problem is not trivial because the interaction is non local and the volume is finite.

There is also a relation with tunneling problems. In [4] a penalty functional $I_t(u)$ is defined to evaluate the cost of a trajectory $u(x,t)$ going from a stable phase to another in a ferromagnetic system described by a nonlocal evolution equation, which is the gradient flow associated to $\mathcal{F}_L$. It is proved that the minimal cost is given by $\mathcal{F}_L(\hat{m}_L)$. The proof can be simplified in the part relative to the lower bound by using the result shown in this paper because it is quite easy to prove that $I_T(u) \ge \sup_{0\le t\le T}\mathcal{F}_L(u(\cdot,t))$; but there is a time t_0 when $K_L(u(\cdot,t_0)) = 0$ by continuity, then we get $I_T(u) \ge \mathcal{F}_L(u(\cdot,t_0)) \ge \mathcal{F}_L(\hat{m}_L)$.

Let $\hat{\mathcal{L}}$ be the operator on $L^2([-L, L], d\hat{\nu}_L)$, $\dfrac{d\hat{\nu}_L}{dx} = \hat{p}(x)^{-1}$, $\hat{p}(x) = [\beta(1 - \hat{m}_L^2)]^{-1}$,

$$\hat{\mathcal{L}}\psi = -\psi + \beta(1 - \hat{m}_L^2)J^{\mathrm{neum}} * \psi$$

It is shown in [7] that if L is large enough, $\hat{\mathcal{L}}$ has a positive eigenvalue λ, $c_- e^{-2L} \leq \lambda \leq c_+ e^{-2L}$, $c_\pm$ positive constants, with eigenvector $\hat{e}(x)$, $|x| \leq L$, which is a strictly positive, regular symmetric function. Moreover there is $B > 0$ such that

$$(v, \hat{\mathcal{L}}v)_{\hat{m}_L} \leq -B(v, v)_{\hat{m}_L}, \quad (v, \hat{e})_{\hat{m}_L} = 0$$

$(\cdot, \cdot)_{\hat{m}_L}$ is the scalar product on $L^2([-1, 1], d\hat{\nu}_L)$. We are going to use a slightly different linear operator $\mathcal{L}$ which is linked to $\hat{\mathcal{L}}$ by a simple multiplication

$$\mathcal{L} = [\beta(1 - \hat{m}_L^2)]^{-1}\hat{\mathcal{L}}$$

The spectral properties are preserved and by an abuse of notation we use the same names for the eigenvalues, eigenvectors and spectral constant of $\mathcal{L}$.

2. Sketch Of Proof

In order to show the validity of relation (3) we prove that both the inequalities $\inf\limits_{K_L(m)=0} \mathcal{F}_L(m) \leq \mathcal{F}_L(\hat{m}_L)$, $\inf\limits_{K_L(m)=0} \mathcal{F}_L(m) \geq \mathcal{F}_L(\hat{m}_L)$ are true. The former is obvious because $K_L(\hat{m}_L) = 0$. The proof of the latter is divided in two parts. First we use some results in [3] to show that we can restrict our attention to functions very close in the L^∞ norm to the finite volume instanton. Then we prove that $\hat{m}_L$ is a local minimum for $\mathcal{F}_L$.

Summarizing, the logical steps of the proof are the following

a) $\mathcal{F}_L(\hat{m}_L) \geq \inf\limits_{K_L(m)=0} \mathcal{F}_L(m)$ because $K_L(\hat{m}_L) = 0$

b) $\inf\limits_{K_L(m)=0} \mathcal{F}_L(m) = \inf\limits_{m \in \mathcal{N}_\delta} \mathcal{F}_L(m) \geq \inf\limits_{m \in \mathcal{M}_\epsilon} \mathcal{F}_L(m)$

c) $\inf\limits_{m \in \mathcal{M}_\epsilon} \mathcal{F}_L(m) = \mathcal{F}_L(\hat{m}_L)$

where

$$\mathcal{N}_\delta = \{m \in L^\infty([-L, L]; [-1, 1]) : K_L(m) = 0,$$
$$\mathcal{F}_L(m) \leq \mathcal{F}_L(\hat{m}_L) + \delta, \ ||m'||_\infty \leq \beta||J'||_\infty\}$$

and

$$\mathcal{M}_\epsilon = \{m \in L^\infty([-L, L]; [-1, 1]) : K_L(m) = 0, \ ||m - \hat{m}_L||_\infty < \epsilon\}$$

The first equality in b) is clear for any $\delta > 0$ thanks to a result proved in [3] and stated in (11), which amounts to say that we can consider only regular functions with bounded derivative. The inequality is proved in section (6) where we show that for any $\epsilon > 0$ there is $\delta_\epsilon > 0$ such that for any $\delta < \delta_\epsilon$ and L large enough $\mathcal{N}_\delta \subseteq \mathcal{M}_\epsilon$.

Indeed if $K_L(m) = 0$ and

$$\mathcal{F}_L(m) \leq \mathcal{F}_L(\hat{m}_L) + \delta$$

we prove using contours and some results in [3] that m is close to the restriction to $[-L, L]$ of the translation by ξ of a function $\bar{m}$, called the infinite volume instanton. A lot of properties of $\bar{m}$ are known from literature (see for instance [6]). In particular it is known that for L large, $||\bar{m} - \hat{m}_L||_\infty$ is exponentially small ([5]). We prove that $|\xi|$ is very small, too. Thus we can conclude that m has to be close to $\hat{m}_L$.

Finally the equality in c) is proved in section (3) and (4) for any ϵ less than a suitable $\bar{\epsilon}$. There we use a geometric property of $\hat{e}$, the positive eigenvector of $\mathcal{L}$. In fact, if we move along the direction of $\hat{e}$ starting from $\hat{m}_L$ we decrease the free energy by an amount of order λ. Then we would have a problem if the functions with mean zero were too close to that direction. The minimum angle $\hat{\alpha}$ between $\hat{e}$ and the hyper-plane $\mathcal{C}^\perp = \{\psi \in L^2([-L, L]) : K(\psi) = 0\}$, $\mathcal{C} = \{m \in L^2([-L, L]) : m = c \text{ a.e.}, c \in \mathbb{R}\}$, is given by

$$\sin \hat{\alpha} = \frac{1}{\sqrt{2L}} \int_{-L}^{L} \hat{e}$$

and it is reached when ψ is a multiple of $\hat{e} - K(\hat{e})$, namely the orthogonal projection of $\hat{e}$ on $\mathcal{C}^\perp$. $\hat{\alpha}$ goes to zero when $L \to \infty$ but not as fast as λ and it is sufficient in order to show that $\mathcal{F}_L(\hat{m}_L)$ is a local minimum if we restrict to $\mathcal{C}^\perp$.

3. Local Estimates

Denote by $(\cdot, \cdot)$ the L^2 scalar product and put $m = \hat{m}_L + \psi$, then we have

$$\mathcal{F}_L(m) = \mathcal{F}_L(\hat{m}_L) - \frac{1}{2}(\psi, \mathcal{L}\psi) + R(\psi)$$

because $\hat{m}_L$ is a stationary point for $\mathcal{F}_L$. We recall that the operator $\mathcal{L}$ is given by

$$\mathcal{L}\psi = J^{\text{neum}} * \psi - \frac{1}{\beta(1 - (\hat{m}_L)^2)}\psi$$

and that $\mathcal{L}$ has a positive eigenvalue $\lambda \approx e^{-2L}$, which corresponds to a positive eigenvector $\hat{e}(x)$ whose integral over $[-L, L]$ is different from zero and substantially independent of L. Moreover for any w such that $(\hat{e}, w) = 0$ there is a

spectral gap which in one dimension is independent of L: $(w, \mathcal{L}w) \leq -B(w, w)$. The remainder R does not affect the behaviour of $\mathcal{F}_L$ if the L^∞ norm of ψ can be considered small, as shown in Section 4.

Theorem 2. Let $m \in L^\infty([-L, L]; [-1, 1])$ and $K_L(m) = 0$. Then there is $\bar{\epsilon} > 0$ such that if $||m - \hat{m}_L||_\infty \leq \bar{\epsilon}$ then

$$\mathcal{F}_L(m) \geq \mathcal{F}_L(\hat{m}_L)$$

Proof. For any ψ we can write $\psi = \rho(a\hat{e} + bv)$ where

$$(\hat{e}, \hat{e}) = 1, \; (v, v) = 1, \; (\hat{e}, v) = 0, \; a^2 + b^2 = 1$$

and $\rho \geq 0$, $\rho = ||\psi||_2$. Now we show that there is $c > 0$ such that

$$\sup_{(\psi, \psi)=1} (\psi, \mathcal{L}\psi) < -\frac{c}{L}, \text{ provided } K_L(\psi) = 0$$

The proof relies on the existence of a minimum angle $\hat{\alpha}$ between $\hat{e}$ and the hyper-plane of the functions whose integral is zero. If it was not the case, we could not expect that $\hat{m}_L$ is a point of minimum for the free energy $\mathcal{F}_L$ restricted to functions m such that $K_L(m) = 0$ because moving along the direction of $\hat{e}$ the free energy decreases.

We can compute explicitly $\hat{\alpha}$ because it is simply the angle between $\hat{e}$ and its orthogonal projection on the space of functions with null mean, as proved in appendix. It means that $\hat{\alpha}$ is complementary to the angle between $\hat{e}$ and the orthogonal direction to $\mathcal{C}^\perp$, namely $w := 1/\sqrt{2L}$, because the condition $K_L(m) = 0$ is equivalent to that of orthogonality of m with respect to the constants. Then

$$\sin \hat{\alpha} = \cos(\frac{\pi}{2} - \hat{\alpha}) = (\hat{e}, w) = \frac{1}{\sqrt{2L}} \int_{-L}^{L} \hat{e}$$

We have

$$(a\hat{e} + bv, \mathcal{L}(a\hat{e} + bv)) \leq \lambda a^2 - Bb^2 \tag{4}$$

Call α the angle between $\hat{e}$ and ψ. If $\psi \in \mathcal{C}^\perp$ then $\sin \alpha \geq \frac{1}{\sqrt{2L}} \int_{-L}^{L} \hat{e}$. But $a = (\hat{e}, \psi) = \cos \alpha$. It follows that $b^2 = \sin^2 \alpha$ and $b^2 \geq \frac{1}{2L} \left(\int_{-L}^{L} \hat{e} \right)^2$. Thus

$$\lambda a^2 - Bb^2 \leq \lambda - \frac{B}{2L} \left(\int_{-L}^{L} \hat{e} \right)^2$$

Finally we can write

$$(a\hat{e} + bv, \mathcal{L}(a\hat{e} + bv)) < -\frac{c}{L}$$

for any L large enough. Indeed there is $\delta > 0$ such that $\liminf\limits_{L \to \infty} \int_{-L}^{L} \hat{e} \geq \delta$, thus definitively $\int_{-L}^{L} \hat{e} > \dfrac{\delta}{2}$ and if $\lambda \leq \dfrac{B\delta}{8L}$ we can choose $c = \dfrac{1}{8}B\delta$. For L large, we showed that

$$\sup_{K_L(\psi)=0} (\psi, \mathcal{L}\psi) < -\frac{c}{L}||\psi||_2^2$$

To conclude the proof we need an estimate of the remainder. This is done in the following section.

4. Remainder

The remainder R can be estimated as follows

$$|R(\psi)| \leq \frac{1}{3} \frac{m_\beta + \epsilon_0}{\beta(1 - (m_\beta + \epsilon_0)^2)^2} \int_{-L}^{L} |\psi(x)|^3 dx$$

if we suppose $||\psi||_\infty \leq \epsilon_0$ where $\epsilon_0 = (1 - m_\beta)/2$. Then we have

$$\int |\psi|^3 dx \leq ||\psi||_2^2 ||\psi||_\infty$$

Thus $\mathcal{F}_L(\hat{m}_L)$ is a local minimum if

$$\frac{c}{2L} - \frac{1}{3} \frac{m_\beta + \epsilon_0}{\beta(1 - (m_\beta + \epsilon_0)^2)^2} ||\psi||_\infty > 0$$

In other words for any m such that $K_L(m) = 0$ and

$$||m - \hat{m}_L||_\infty < \min\{\frac{3c\beta(1 - (m_\beta + \epsilon_0)^2)^2}{2(m_\beta + \epsilon)L}, \epsilon_0\}$$

we have

$$\mathcal{F}_L(m) - \mathcal{F}_L(\hat{m}_L) \geq ||m - \hat{m}_L||_2^2 \left(\frac{c}{2L} - \frac{1}{3} \frac{m_\beta + \epsilon_0}{\beta(1 - (m_\beta + \epsilon_0)^2)^2} ||\psi||_\infty \right) \geq 0$$

∎

5. Some Background and Notation

We list some known properties of the finite volume instanton $\hat{m}_L$ and of its counterpart $\bar{m}$ in the case $L = \infty$. We begin with $\bar{m}$.

- $\bar{m}$ is an increasing antisymmetric function which solves

$$\bar{m}(x) = \tanh\{\beta J * \bar{m}(x)\}, \ x \in \mathbb{R} \tag{5}$$

and converges exponentially fast to $\pm m_\beta$ as $x \to \pm\infty$, $\pm m_\beta$ being the constant non vanishing solutions of the mean field equation. In other words there are positive constants a, $\alpha_0 > \alpha$ and c such that

$$|\bar{m}(x) - (m_\beta - ae^{-\alpha x})| \le ce^{-\alpha_0 x}, \ x \ge 0 \tag{6}$$

We introduce also

$$\bar{m}_\xi(x) := \bar{m}(x - \xi)$$

where $\xi \in \mathbb{R}$ is called center of the translated instanton $\bar{m}_\xi$.

- Let $\mathcal{N}$ be the set

$$\mathcal{N} = \left\{ m \in L^\infty(\mathbb{R}; [-1,1]) : \limsup_{x \to -\infty} m(x) < 0, \ \liminf_{x \to \infty} m(x) > 0 \right\}$$

If $m \in \mathcal{N}$, then there is $\xi \in \mathbb{R}$ such that

$$\lim_{t \to \infty} ||S_t(m) - \bar{m}_\xi||_\infty = 0$$

where $S_t(m)$ is the flow solution of the equation

$$u_t = -u + \tanh\{\beta J * u\} \tag{7}$$

with initial datum m.

- The functional $\mathcal{F}$, defined as $\mathcal{F}_L$ but with $L = \infty$ and J^{neum} replaced by J, is decreasing on the solution of (7): $\mathcal{F}(S_t(m)) \le \mathcal{F}(m)$, $t \ge 0$. It follows that $\bar{m}$ is the minimizer of $\mathcal{F}$ in the class $\mathcal{N}$. By the way we notice also that for any ξ $\mathcal{F}(\bar{m}) = \mathcal{F}(\bar{m}_\xi)$.

- ξ is called center of m if

$$(m - \bar{m}_\xi, \bar{m}'_\xi)_\xi = 0$$

where $(\cdot, \cdot)_\xi$ denotes the scalar product in $L^2(\mathbb{R}, d\nu_\xi)$ and

$$\frac{d\nu_\xi}{dx} = p_\xi(x)^{-1}, \ \ p_\xi(x) = \beta[1 - \bar{m}_\xi(x)^2]$$

- Any $m \in \mathcal{N}$ has a center. Moreover there are $c, \delta > 0$ so that if $||m - \bar{m}_{\xi_0}||_\infty < \delta$ then m has a unique center ξ and, defined

$$N_{\xi_0, \xi} = \frac{((m - \bar{m}_{\xi_0}), \bar{m}')_\xi}{(\bar{m}', \bar{m}')_\xi}$$

one has

$$|\xi - (\xi_0 - N_{\xi_0, \xi_0}) \le c||m - \bar{m}_{\xi_0}||_\infty^2 \ \text{ and } \ |N_{\xi_0, \xi_0}| \le c||m - \bar{m}_{\xi_0}||_\infty \tag{8}$$

- Let $\hat{\Omega}_\xi$ be the linear operator on $L^2(\mathbb{R}, d\nu_\xi)$

$$\hat{\Omega}_\xi \psi = -\psi + p_\xi J * \psi$$

$\hat{\Omega}_\xi$ has eigenvalue 0 with eigenvector $\bar{m}'_\xi$ and a strictly positive spectral gap, namely there is $B_{\hat{\Omega}}$ such that

$$(v, \hat{\Omega}_\xi v)_\xi \leq -B_{\hat{\Omega}}(v, v)_\xi, \quad (v, \bar{m}'_\xi)_\xi = 0$$

We can switch to a new operator Ω on $L^2(\mathbb{R}, dx)$ by means of a simple multiplication

$$\Omega = p_\xi^{-1} \hat{\Omega}$$

We note that $(v, \hat{\Omega}_\xi v)_\xi = (v, \Omega v)$ and that

$$\frac{1}{\beta}(v, v) \leq (v, v)_\xi \leq \frac{1}{\beta(1 - m_\beta^2)}(v, v)$$

Now we turn to the finite volume instanton $\hat{m}_L$. It is an antisymmetric function whose absolute value is always less than m_β.

- It is proved in [3] that there are $c' > 0$ and $\omega' > 0$ so that for any L large enough

$$|\mathcal{F}(\bar{m}) - \mathcal{F}_L(\hat{m}_L)| \leq c' e^{-\omega' L} \tag{9}$$

- $\hat{m}_L$ approaches exponentially fast the instanton $\bar{m}$ as L goes to infinity:

$$||\bar{m} - \hat{m}_L|| \leq \epsilon_L$$

and for any γ, $L^\gamma \epsilon_L \to 0$ when $L \to \infty$.

- Both $\bar{m}$ and $\hat{m}_L$ have bounded derivatives as it follows by differentiating the mean field equation:

$$||\bar{m}'||_\infty \leq \beta ||J'||_\infty, \quad ||\hat{m}'_L||_\infty \leq \beta ||J'||_\infty \tag{10}$$

If Λ is a finite union of intervals contained in $[-L, L]$, we write $\Lambda^c = [-L, L] \setminus \Lambda$ for its complement in $[-L, L]$ and m_Λ for the restriction of m to Λ. We define

$$\mathcal{F}_{L;\Lambda}(m_\Lambda) = \int_\Lambda \phi_\beta(m_\Lambda) dx + \frac{1}{4} \int_\Lambda \int_\Lambda J^{\text{neum}}(x, x')(m_\Lambda(x) - m_\Lambda(x'))^2 dx dx'$$

$$\mathcal{F}_{L;\Lambda}(m_\Lambda | m_{\Lambda^c}) = \mathcal{F}_{L;\Lambda}(m_\Lambda) + \frac{1}{2} \int_\Lambda \int_{\Lambda^c} J^{\text{neum}}(x, x')(m_\Lambda(x) - m_{\Lambda^c}(x'))^2 dx dx'$$

Given $l > 0$, we denote by $\mathcal{D}^{(l)}$ the partition of $\mathbb{R}$ into the intervals $[nl, (n+1)l)$, $n \in \mathbb{Z}$, and define

$$m^{(l)}(x) := \fint_{I_x^{(l)}} m(y)dy, \quad \fint_{\Lambda} m(y)dy := \frac{1}{|\Lambda|}\int_{\Lambda} m(y)dy$$

where $I_x^{(l)}$ is the interval in $\mathcal{D}^{(l)}$ which contains the point x. Given an accuracy parameter $\zeta > 0$, we then introduce

$$\eta^{(\zeta,l)}(m;x) = \begin{cases} \pm 1 & \text{if } |m^{(l)} \mp m_\beta| \leq \zeta, \\ 0 & \text{otherwise} \end{cases}$$

Calling l_- and l_+ two values of the parameter l, with l_+ an integer multiple of l_-, we define a phase indicator

$$\Theta^{(\zeta,l_-,l_+)}(m;x) = \begin{cases} \pm 1 & \text{if } \eta^{(\zeta,l_-)}(m;\cdot) = \pm 1 \text{ in } [-L,L] \cap \left(I_{x-l_+}^{(l_+)} \cup I_x^{(l_+)} \cup I_{x+l_+}^{(l_+)} \right), \\ 0 & \text{otherwise} \end{cases}$$

and call contours of m the connected components of the set $\{x : \Theta^{(\zeta,l_-,l_+)}(m;x) = 0\}$. $\Gamma = [x_-, x_+)$ is a plus contour if $\eta^{(\zeta,l_-)}(m;x_\pm) = 1$, a minus contour if $\eta^{(\zeta,l_-)}(m;x_\pm) = -1$, otherwise it is called mixed.

We choose ζ suitably small but fixed and l_- of order ζ^2. L is arbitrarily large and l_+ is of order $L^{1/2}$.

- There is regularizing map $\mathcal{R}$ from $L^\infty([L,-L],[-1,1])$ into itself such that

$$\mathcal{F}_L(m) \geq \mathcal{F}_L(\mathcal{R}(m)), \tag{11}$$

  $[\mathcal{R}(m)]^{(l)} = m^{(l)}$ for l sufficiently small, and

$$\left| \frac{d\mathcal{R}(m)(x)}{dx} \right| \leq \beta \|J'\|_\infty$$

  The map $\mathcal{R}$ allows to work with functions whose derivative is bounded. It is useful in relating different norms as it will be clear later.

- There is $\omega > 0$ so that for any interval $\Lambda = [x', x''] \subset [-L,L]$, union of intervals belonging to $\mathcal{D}^{(l_+)}$, and for any m such that $\Theta^{(\zeta,l_-,l_+)}(m;\cdot) = 1$ on Λ, there is ψ with the following properties. $\mathcal{F}_L(m) \geq \mathcal{F}_L(\psi)$; $\psi = m$ on $[x'+1, x''-1]^c$; $\eta^{(\zeta,l_-)}(\psi;\cdot) = 1$ on Λ;

$$\psi = \tanh\{\beta J^{\text{neum}} * \psi\}, \quad \text{on } [x'+1, x''-1]$$

$$|\psi(x) - m_\beta| \leq c_2 e^{-\omega \operatorname{dist}(x,\Lambda^c)}, \quad x \in [x'+1, x''-1] \tag{12}$$

  An analogous result holds for $\Theta^{(\zeta,l_-,l_+)}(m;\cdot) = -1$ on Λ and $-m_\beta$ replacing m_β and also when $L = \infty$.

6. Global Estimates

Lemma 1. There are constants δ_0 and α_0 such that, if $|m(x) - m_\beta| < \delta_0$ for any $x \in \Omega$, $\Omega \subseteq [-L, L]$, then

$$\mathcal{F}_{L;\Omega}(m_\Omega) \geq \alpha_0 \int_\Omega (m - m_\beta)^2$$

The same result is true replacing everywhere m_β with $-m_\beta$.

Proof. The lemma is a consequence of the structure of ϕ_β which is a double well with quadratic minima.
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Lemma 2. Let $m \in \mathcal{N}$, $||m'||_\infty \leq \beta||J'||_\infty$ and suppose that $\inf_\xi ||m - \bar{m}_\xi||_2^2 \leq L^{-\sigma}$, $\sigma > 0$. Then there is $\delta > 0$ such that

$$\mathcal{F}(m) - \mathcal{F}(\bar{m}) \geq \delta \inf_\xi ||m - \bar{m}_\xi||_2^2$$

Proof. By definition of inf there is ξ_0 such that $||m - \bar{m}_{\xi_0}||_2^2 \leq 2L^{-\sigma}$, it follows that $||m - \bar{m}_{\xi_0}||_\infty \leq 3(\beta||J'||_\infty)^{1/3}L^{-\sigma/3}$. Indeed for any ξ $||m' - \bar{m}'_\xi||_\infty \leq 2\beta||J'||_\infty$ and we use (17) that in our case becomes

$$||f||_\infty \leq \frac{3}{2}||f'||_\infty^{\frac{1}{3}}||f||_2^{\frac{2}{3}}$$

because $\inf |m - \bar{m}_{\xi_0}| = 0$ for $(m - \bar{m}_{\xi_0}) \in L^2(\mathbb{R})$. By (8), there is a unique center ξ and

$$|\xi - \xi_0| \leq |N_{\xi_0, \xi_0}| + c||m - \bar{m}_{\xi_0}||_\infty^2 \leq 6c(\beta||J'||_\infty)^{\frac{1}{3}}L^{-\frac{\sigma}{3}}$$

for L large. Then we can estimate the L^2 distance of m from $\bar{m}_\xi$. Indeed $||m - \bar{m}_\xi||_2 \leq ||m - \bar{m}_{\xi_0}||_2 + ||\bar{m}_{\xi_0} - \bar{m}_\xi||_2$ and

$$||\bar{m}_{\xi_0} - \bar{m}_\xi||_2^2 = ||\bar{m} - \bar{m}_{|\xi - \xi_0|}||_2^2 = \left(\int_{x < -k} + \int_{-k \leq x \leq k} + \int_{x > k} \right) (\bar{m} - \bar{m}_{|\xi - \xi_0|})^2$$

The tails are bounded using (6): $|\bar{m}(x) - \bar{m}_{|\xi - \xi_0|}(x)| \leq a'e^{-\alpha x} + a'e^{-\alpha(x - |\xi - \xi_0|)} \leq 2a'e^{-\alpha(x - |\xi - \xi_0|)}$ for $x > k$ and analogously for $x < -k$. For $|x| \leq k$ we have $|\bar{m} - \bar{m}_{\xi - xi_0}| \leq ||\bar{m}'||_\infty|\xi - \xi_0| \leq \beta||J'||_\infty|\xi - \xi_0|$. Thus

$$||\bar{m}_\xi - \bar{m}_{\xi_0}||_2^2 \leq 2(\beta||J'||_\infty)^2 k|\xi - \xi_0|^2 + \frac{4a'^2}{\alpha}e^{-2\alpha(k - |\xi - \xi_0|)}$$

We can now choose $k = |\xi - \xi_0|^{-1}$, so that, for L large, $||\bar{m}_\xi - \bar{m}_{\xi_0}||_2^2 \leq 24c(\beta||J'||_\infty)^{7/3}L^{-\sigma/3}$, and calling $c_1 = 4\sqrt{6c}(\beta||J'||_\infty)^{7/6}$ we have

$$||m - \bar{m}_\xi||_2 \leq c_1 L^{-\frac{\sigma}{6}}$$

Using again (17) we find that

$$||m - \bar{m}_\xi||_\infty \leq \frac{3}{2}(2\beta||J'||_\infty)^{\frac{1}{3}}c_1^{\frac{2}{3}}L^{-\frac{\sigma}{9}} \tag{13}$$

Now we expand $\mathcal{F}$ around $\bar{m}_\xi$

$$\mathcal{F}(m) - \mathcal{F}(\bar{m}_\xi) = -\frac{1}{2}(\psi, \Omega\psi) + R(\psi), \quad R(\psi) = \frac{1}{3}\int \frac{z}{\beta(1-z^2)^2}\psi^3$$

where $\psi = m - \bar{m}_\xi$ and $z \in [\min\{m(x), \bar{m}_\xi(x)\}, \max\{m(x), \bar{m}_\xi(x)\}]$. The remainder can be estimated as in section (4). On the other hand

$$(\psi, \Omega\psi) = (\psi, \hat{\Omega}_\xi\psi)_\xi \leq -B_{\hat{\Omega}}(\psi, \psi)_\xi \leq -\frac{B_{\hat{\Omega}}}{\beta}(\psi, \psi)$$

because by definition of center $(\psi, \bar{m}_\xi)_\xi = 0$ and we use the spectral gap. Then

$$\mathcal{F}(m) - \mathcal{F}(\bar{m}_\xi) \geq ||\psi||_2^2\left(\frac{B_{\hat{\Omega}}}{2\beta} - \frac{1}{3}\frac{m_\beta + \epsilon_0}{\beta(1-(m_\beta+\epsilon_0)^2)^2}||\psi||_\infty\right)$$

Thanks to (13) it is clear that for any L large enough the round brackets are greater than some $\delta > 0$. So we have

$$\mathcal{F}(m) - \mathcal{F}(\bar{m}_\xi) \geq \delta||\psi||_2^2 \geq \delta\inf_\xi||m - \bar{m}_\xi||_2^2$$

that is the end of the proof because $\mathcal{F}(\bar{m}_\xi) = \mathcal{F}(\bar{m})$.

∎

Corollary 1. If $m \in \mathcal{N}$, $||m'||_\infty \leq \beta||J'||_\infty$ and $\inf_\xi||m - \bar{m}_\xi||_2^2 > L^{-\sigma}$, then

$$\mathcal{F}(m) - \mathcal{F}(\bar{m}) \geq \delta L^{-\sigma}$$

σ and δ the same of lemma (2).

Proof. There is $\hat{\xi}$ such that $\lim_{t\to\infty} S_t(m) = \bar{m}_{\hat{\xi}}$ because $m \in \mathcal{N}$. The L^2 norm is continuous under the flow $S_t(\cdot)$ so that there is t_0 such that $\inf_\xi||S_{t_0}(m) - \bar{m}_\xi||_2^2 = L^{-\sigma}$. But $\mathcal{F}(S_{t_0}(m)) \leq \mathcal{F}(m)$ because $\mathcal{F}$ is a Lyapunov functional for the evolution. It follows that we can use lemma (2) with m replaced by $S_{t_0}(m)$. Then $\mathcal{F}(S_{t_0}(m)) - \mathcal{F}(\bar{m}) \geq \delta\inf_\xi||S_{t_0}(m) - \bar{m}_\xi||_2^2 = \delta L^{-\sigma}$.

∎

Theorem 3. Suppose that m has derivative bounded by $\beta||J'||_\infty$ and it is such that there is only a contour $\Gamma = [x_-, x_+] \subset [-L, L]$, $\eta^{(\zeta, l_-)}(m; x_\pm) = \pm 1$, $x_+ \leq L - 3l_+$, $x_- \geq -L + 3l_+$ and

$$\mathcal{F}(m) < \mathcal{F}(\hat{m}_L) + L^{-100}$$

Then, for any given $r \in (-1, 1)$, there is $\xi \in [x_- + rl_+, x_+ - rl_+]$ such that

$$||m - \bar{m}_\xi||_2^2 < L^{-10}$$

Proof. We divide the proof in two cases.

1) There is $\xi \in [x_- + rl_+, x_+ - rl_+]$ such that

$$\int_{x_--l_+}^{x_++l_+} (m - \bar{m}_\xi)^2 < \frac{L^{-10}}{2}$$

2) For any $\xi \in [x_- + rl_+, x_+ - rl_+]$

$$\int_{x_--l_+}^{x_++l_+} (m - \bar{m}_\xi)^2 \geq \frac{L^{-10}}{2}$$

Case 1. By definition of contour we have $\Theta^{(\zeta,l_-,l_+)}(m;x) = -1$ for $x \in [x_- - l_+, x_-]$ and $\Theta^{(\zeta,l_-,l_+)}(m;x) = 1$ for $x \in [x_+, x_+ + l_+]$. Then we can find a function ψ as in (12) and a slight modification $\tilde{m}$ of it with the following properties. $\tilde{m} = m$ for $x < x_- - l_+ + 1$, $x_- - 1 < x < x_+ + 1$ and $x > x_+ + l_+ - 1$; $\tilde{m} = -m_\beta$ if $x_- - \frac{l_+}{2} - 1 < x < x_- - \frac{l_+}{2} + 1$; $\tilde{m} = m_\beta$ if $x_+ + \frac{l_+}{2} - 1 < x < x_+ + \frac{l_+}{2} + 1$;

$$\mathcal{F}_L(\tilde{m}) \leq \mathcal{F}_L(m) + c_2 e^{-\omega(\frac{l_+}{2}-1)}$$

Let $\Lambda' = [-L, x_- - \frac{l_+}{2}] \cup [x_+ + \frac{l_+}{2}, L]$, then $\mathcal{F}_L(\tilde{m}) = \mathcal{F}_{L;\Lambda'}(\tilde{m}_{\Lambda'}) + \mathcal{F}_{L;\Lambda'^c}(\tilde{m}_{\Lambda'^c}|\tilde{m}_{\Lambda'})$ and

$$\mathcal{F}_{L;\Lambda'^c}(\tilde{m}_{\Lambda'^c}|\tilde{m}_{\Lambda'}) = \mathcal{F}(\hat{m})$$

where $\hat{m} = \tilde{m}1_{x \in \Lambda'^c} + m_\beta(1_{x > x_+ + \frac{l_+}{2}} - 1_{x < x_- - \frac{l_+}{2}})$. Indeed there is not interaction between Λ' and Λ'^c because of the small interval around $x_- - \frac{l_+}{2}$ and $x_+ + \frac{l_+}{2}$ where $\tilde{m}$ is exactly $\pm m_\beta$. We notice that $\hat{m} \in \mathcal{N}$ so that $\mathcal{F}(\hat{m}) \geq \mathcal{F}(\bar{m})$. Moreover, by (9), $\mathcal{F}_L(\hat{m}_L) \leq \mathcal{F}(\bar{m}) + c' e^{-\omega' L}$. Now let $\Lambda = [-L, x_- - l_+] \cup [x_+ + l_+, L]$ then

$$\mathcal{F}_{L;\Lambda'}(\tilde{m}_{\Lambda'}) \geq \mathcal{F}_{L;\Lambda}(\tilde{m}_\Lambda) = \mathcal{F}_{L;\Lambda}(m_\Lambda) \geq \alpha_0 \int_\Lambda (m - \hat{m}_\beta)^2$$

where $\hat{m}_\beta = m_\beta(1_{x > \xi} - 1_{x < \xi})$ and we used for the last inequality lemma (1). Indeed in Λ, $\Theta^{(\zeta,l_-,l_+)}(m;\cdot) = \pm 1$ which means that $|m^{(l_-)} \mp m_\beta| \leq \zeta$. Now, thanks to the assumption on the bound on the derivative of m, we have also that $||m - m^{(l_-)}||_\infty \leq \beta ||J'||_\infty l_-$. Then in order to use lemma (1) it is sufficient to choose ζ and l_- such small that $\zeta + \beta ||J'||_\infty l_- < \delta_0$.

But then

$$\alpha_0 \int_\Lambda (m - \hat{m}_\beta)^2 \leq \mathcal{F}_L(\tilde{m}) - \mathcal{F}(\hat{m}) \leq \mathcal{F}_L(m) + c_2 e^{-\omega(\frac{l_+}{2}-1)} - \mathcal{F}(\bar{m}) \leq$$

$$\leq \mathcal{F}(\hat{m}_L) + L^{-100} + c_2 e^{-\omega(\frac{l_+}{2}-1)} - \mathcal{F}_L(\hat{m}_L) + c' e^{-\omega' L} < \alpha_0 \frac{L^{-10}}{8}$$

for L sufficiently large because l_+ scales as $L^{1/2}$. By (6) there is $k > 0$ such that

$$\int_\Lambda (\bar{m}_\xi - \hat{m}_\beta)^2 \le ke^{-2\alpha(1+r)l_+} < \frac{L^{-10}}{8}$$

Finally we have

$$\left(\int_\Lambda (m - \bar{m}_\xi)^2\right)^{\frac{1}{2}} \le \left(\int_\Lambda (m - \hat{m}_\beta)^2\right)^{\frac{1}{2}} + \left(\int_\Lambda (\hat{m}_\beta - \bar{m}_\xi)^2\right)^{\frac{1}{2}} \le \frac{L^{-5}}{\sqrt{2}}$$

This concludes the proof in case 1).

Case 2. As above we can find a function $\tilde{m}$ such that $\tilde{m} = m$ for $x \in [x_- - l_+ - 1, x_+ + l_+ + 1]$, $\tilde{m} = -m_\beta$ in $[-L, -L+1]$ and $\tilde{m} = m_\beta$ in $[L-1, L]$ and

$$\mathcal{F}_L(m) \ge \mathcal{F}_L(\tilde{m}) - c_2 e^{-\omega(l_+ - 1)} \tag{14}$$

Moreover we introduce $\phi = \tilde{m}1_{x\in[-L,L]} + m_\beta(1_{x>L} - 1_{x<-L})$ and we notice that $\mathcal{F}(\phi) = \mathcal{F}_L(\tilde{m})$. In Λ^c, Λ as in the previous case, we have $\phi = m$, then

$$\inf_{\xi\in[x_-+rl_+,x_+-rl_+]} ||\phi - \bar{m}_\xi||_2^2 \ge \inf_{\xi\in[x_-+rl_+,x_+-rl_+]} \int_{\Lambda^c} (\phi - \bar{m}_\xi)^2 \ge \frac{L^{-10}}{2} \tag{15}$$

where the last inequality follows by the assumption of case 2). Now we want to show that also for ξ outside the interval $[x_-+rl_+, x_+-rl_+]$ the L^2 norm of $\phi - \bar{m}_\xi$ is bounded away from zero. Then let $\xi < x_-+rl_+$ and call $\bar{x} = x_-+rl_+ + \frac{1-r}{2}l_+$. By (6) we have

$$|\bar{m}_\xi(\bar{x}) - m_\beta| \le ke^{-\alpha\left(\frac{1-r}{2}\right)l_+}$$

We note that $\eta^{(\zeta,l_-)}(m; \bar{x}) = -1$ by hypothesis and the choice of $\bar{x}$. It follows that $|m^{(l_-)} + m_\beta| \le \zeta$. Moreover m has bounded derivative so that $||m - m^{(l_-)}||_\infty \le \beta||J'||_\infty l_-$. But $\phi(\bar{x}) = m(\bar{x})$, hence

$$|\phi(\bar{x}) + m_\beta| \le \zeta + \beta||J'||_\infty l_-$$

and

$$|\bar{m}_\xi(\bar{x}) - \phi(\bar{x})| \ge 2m_\beta - \zeta - \beta||J'||_\infty l_- \ge m_\beta$$

for l_+ large and ζ and l_- small. For $\xi > x_+ - rl_+$ one can proceed in the same way. By continuity there is an interval $I_{\bar{x}}$ around $\bar{x}$ where $|\phi - \bar{m}_\xi|$ is strictly positive. The problem is the length of $I_{\bar{x}}$ which could be very small. But it is not the case because the derivatives of ϕ and $\bar{m}_\xi$ are bounded; in particular $||\phi' - \bar{m}'_\xi||_\infty \le 2\beta||J'||_\infty$. Then if $\delta_+ = \inf\{x > \bar{x} : \phi(x) - \bar{m}_\xi(x) = 0\}$ and $\delta_- = \inf\{x < \bar{x} : \phi(x) - \bar{m}_\xi(x) = 0\}$, let $\delta = \min\{\delta_+, \delta_-\}$. Of course δ could be not defined because $\phi - \bar{m}_\xi$ could never vanish in [-L,L]. But this is the

lucky case where no discussion is needed, so we suppose that δ exist. Now by Lagrange $m_\beta/\delta \leq 2\beta||J'||_\infty$, namely

$$\delta > \frac{m_\beta}{2\beta||J'||_\infty}$$

In $[\bar{x}, \bar{x} + \delta]$ the steepest descend with bounded derivative from m_β to zero is provided by the straight line, so we have

$$||\phi - \bar{m}_\xi||_2^2 \geq \int_{I_{\bar{x}}} (\phi - \bar{m}_\xi)^2 \geq 2 \int_0^\delta (m_\beta - \frac{m_\beta}{\delta}x)^2 =$$
$$= \frac{2}{3}m_\beta^2\delta \geq \frac{m_\beta^3}{3\beta||J'||_\infty}$$

This bound together with (15) gives

$$\inf_\xi ||\phi - \bar{m}_\xi||_2^2 \geq \frac{L^{-10}}{2}$$

We can now use lemma (2) and corollary (1) because $\phi \in \mathcal{N}$. We do not know whether $L^{-10}/2 > L^{-\sigma}$ or $\inf_\xi ||\phi - \bar{m}_\xi||_2^2 \leq L^{-\sigma}$. However if $\sigma < 10$ the worst case is the latter because it gives

$$\mathcal{F}(\phi) \geq \mathcal{F}(\bar{m}) + \delta\frac{L^{-10}}{2}$$

The desired contradiction comes from (14) and (9):

$$\mathcal{F}_L(m) \geq \mathcal{F}_L(\hat{m}_L) - c_2 e^{-\omega(l_+-1)} - c' e^{-\omega' L} + \delta\frac{L^{-10}}{2} \geq \mathcal{F}_L(\hat{m}_L) + L^{-100}$$

for L large.

∎

Lemma 3. Let $m \in L^\infty([-L, L]; [-1, 1])$ be such that $K_L(m) = 0$ and $\mathcal{F}_L(m) < \mathcal{F}_L(\hat{m}_L) + L^{-100}$, then there is a unique contour $\Gamma = [x_-, x_+]$ and it is mixed. Moreover

$$-\frac{2\zeta}{m_\beta}L < x_- \leq x_+ < \frac{2\zeta}{m_\beta}L \tag{16}$$

Proof. In [3] it is proved that if $\mathcal{F}_L(m) < \mathcal{F}_L(\hat{m}_L) + L^{-100}$ then

$$|\{\Theta^{(\zeta,l_-,l_+)}(m; \cdot) = 0\}| \leq kl_+, \quad k = \text{smallest integer} \geq \frac{1}{c_1\zeta^2 l_-}(\mathcal{F}_L(\hat{m}_L) + L^{-100})$$

c_1 a constant. We use this estimate to show that there is at least one mixed contour. Let $\mathcal{A}_{\pm 1} = \{\Theta^{(\zeta,l_-,l_+)}(m; \cdot) = \pm 1\}$ and $\mathcal{A}_0 = \{\Theta^{(\zeta,l_-,l_+)}(m; \cdot) = 0\}$. We have

$$0 = \int_{-L}^L m = \int_{-L}^L m^{(l_-)} = \left(\int_{\mathcal{A}_{-1}} + \int_{\mathcal{A}_0} + \int_{\mathcal{A}_1}\right) m^{(l_-)}$$

then, calling $A_{\pm 1} = |\mathcal{A}_{\pm 1}|$ and $A_0 = |\mathcal{A}_0|$,

$$0 \leq (-m_\beta + \zeta)A_{-1} + A_0 + (m_\beta + \zeta)A_1$$
$$0 \geq (-m_\beta - \zeta)A_{-1} - A_0 + (m_\beta - \zeta)A_1$$

But $A_{-1} = 2L - A_1 - A_0$, hence, after some computation and recalling that $A_0 < kl_+$,

$$A_1 \geq L\left(\frac{m_\beta - \zeta}{m_\beta} - \frac{1 + m_\beta - \zeta}{2m_\beta}\frac{kl_+}{L}\right) \geq L\left(1 - \frac{2\zeta}{m_\beta}\right)$$
$$A_1 \leq L\left(\frac{m_\beta + \zeta}{m_\beta} - \frac{1 - m_\beta - \zeta}{2m_\beta}\frac{kl_+}{L}\right) \geq L\left(1 + \frac{2\zeta}{m_\beta}\right)$$

for L large. There is an obvious symmetry between phase 1 and phase -1 so that the same estimates hold for A_{-1}. Thus there is a mixed contour $\Gamma = [x_-, x_+]$. In [3] again is proved that in such a case and if $\mathcal{F}_L(m) < \mathcal{F}_L(\hat{m}_L) + L^{-100}$ the mixed contour is the unique one for L large. It follows that

$$x_+ < L - L\left(1 - \frac{2\zeta}{m_\beta}\right) = \frac{2\zeta}{m_\beta}L$$

and similarly for x_-.

∎

Theorem 4. Suppose that $K_L(m) = 0$, $\mathcal{F}_L(m) < \mathcal{F}_L(\hat{m}_L) + L^{-100}$ and $||m'||_\infty \leq \beta||J'||_\infty$, then there is c such that

$$||m - \hat{m}_L||_\infty \leq cL^{-\frac{10}{3}}$$

Proof. As anticipated in section (2), we estimate $||m - \hat{m}_L||_\infty$ in the following way

$$||m - \hat{m}_L||_\infty \leq ||m - \bar{m}_\xi||_\infty + ||\bar{m}_\xi - \bar{m}||_\infty + ||\bar{m} - \hat{m}_L||_\infty$$

By the assumptions of the theorem we can readily invoke lemma (3) and theorem (3). Then there is $\xi \in [x_- + rl_+, x_+ - rl_+]$ such that $||m - \bar{m}_\xi||_2^2 \leq L^{-10}$. Thus, using (17), we can bound the L^∞ norm of $m - \bar{m}_\xi$ as follows

$$||m - \bar{m}_\xi||_\infty \leq \frac{3}{2}(2\beta||J'||_\infty)^{\frac{1}{3}}L^{-\frac{1}{3}}$$

Moreover

$$||\bar{m} - \bar{m}_\xi||_\infty \leq \beta||J'||_\infty|\xi|$$

and we recall that $||\bar{m} - \hat{m}_L||_\infty \leq \epsilon_L$ where ϵ_L is exponentially small in L. So we need only to estimate $|\xi|$. By (16) we get

$$\xi \leq x_+ + l_+ \leq L\left(\frac{2\zeta}{m_\beta} - \frac{l_+}{L}\right) \leq \frac{L}{2}$$

for L large and ζ small. We proceed in the same way for the lower bound $\xi > x_- - l_+$. Then we can conclude that $|\xi| \leq L/2$. Now we relate $|\xi|$ to the integral of $\bar{m}_\xi$:

$$\int_{-L}^{L} \bar{m}_\xi(x) = \int_{-L-\xi}^{L-\xi} \bar{m}(x) = -\int_{L-\xi}^{L+\xi} \bar{m}(x) = -\text{sign}(\xi) \int_{L-|\xi|}^{L+|\xi|} \bar{m}(x)$$

because $\bar{m}$ is an antisymmetric function. Moreover it is increasing too, so we have

$$\int_{L-|\xi|}^{L+|\xi|} \bar{m}(x) \geq 2|\xi|\bar{m}(L - |\xi|)$$

By (6) and the fact that $|\xi| \leq L/2$, we get $\bar{m}(L - |\xi|) \geq m_\beta - (a + c)e^{-\alpha \frac{L}{2}}$ and, for L large enough, $\left| \int_{-L}^{L} \bar{m}_\xi \right| \geq 2|\xi|(m_\beta - (a + c)e^{-\alpha \frac{L}{2}}) \geq |\xi|m_\beta$. Then

$$|\xi| \leq \frac{1}{m_\beta} \left| \int_{-L}^{L} \bar{m}_\xi - m \right| \leq \frac{(2L)^{1/2}}{m_\beta} ||m - \bar{m}_\xi||_2 \leq \frac{\sqrt{2}}{m_\beta} L^{-\frac{9}{2}}$$

∎

The proof of theorem (1) is a consequence of theorems (2) and (4) because for L sufficiently large $\mathcal{N}_{L^{-100}} \subseteq \mathcal{M}_{\bar{\epsilon}}$.

A. Minimum Angle Between a Vector and a Plane

Let us work in a finite dimensional setting, the extension to infinite dimensions being straightforward. Denote by $\mathcal{V} = V \oplus V^\perp$ a vector space of dimension $n + m$ and with $\{v_1, \ldots, v_n\}$, $\{u_1, \ldots, u_m\}$ two normalized orthogonal basis of V and $V^\perp$ respectively. Call $w = \sum_{i=1}^{n} \alpha_i v_i + \sum_{j=1}^{m} \beta_j u_j$, then we prove that the minimum angle between w and V is given by the angle between w and $\bar{w} = \sum_{i=1}^{n} \alpha_i v_i$, namely the orthogonal projection of w on V. Let $v = \sum_{i=1}^{n} \gamma_i v_i$ be a generic vector of V. We first bound the absolute value of the cosine of the angle between w and v. Let $F(\gamma_1, \ldots, \gamma_n) = \cos v\hat{w}$:

$$|F(\gamma_1, \ldots, \gamma_n)| = \left| \frac{(v, w)}{||v|| \, ||w||} \right| = \left| \frac{\sum_{i=1}^{n} \alpha_i \gamma_i}{\sqrt{\sum_{i=1}^{n} \alpha_i^2 + \sum_{j=1}^{m} \beta_j^2} \sqrt{\sum_{i=1}^{n} \gamma_i^2}} \right| \leq$$

$$\leq \frac{\sqrt{\sum_{i=1}^{n} \alpha_i^2} \sqrt{\sum_{i=1}^{n} \gamma_i^2}}{\sqrt{\sum_{i=1}^{n} \alpha_i^2 + \sum_{j=1}^{m} \beta_j^2} \sqrt{\sum_{i=1}^{n} \gamma_i^2}} = \frac{\sqrt{\sum_{i=1}^{n} \alpha_i^2}}{\sqrt{\sum_{i=1}^{n} \alpha_i^2 + \sum_{j=1}^{m} \beta_j^2}} = F(\alpha_1, \ldots, \alpha_n)$$

Thus
$$F(-\alpha_1, \ldots, -\alpha_n) \leq \cos v\hat{w} \leq F(\alpha_1, \ldots, \alpha_n)$$

because F is antisymmetric. Since $\cos\theta$ is a decreasing function of θ in $[0, \pi]$ we conclude the proof.

B. Bounds on L^∞ in terms of L^2 Norm

Proposition 1. Let $f \in L^2(\Omega; \mathbb{R})$, $\Omega \subseteq \mathbb{R}^n$, f continuous and almost everywhere differentiable with $||\nabla f||_\infty < \infty$. Then there is c_n such that

$$||f||_\infty \leq \inf_{x\in\Omega} |f(x)| + c_n ||\nabla f||_\infty^{\frac{n}{n+2}} ||(|f| - \inf_{x\in\Omega} |f(x)|)||_2^{\frac{2}{n+2}} \qquad (17)$$

$$c_n = \left[\left(\frac{2}{n}\right)^{\frac{n}{n+2}} + \left(\frac{n}{2}\right)^{\frac{2}{n+2}} \right] \left[\left(\frac{1}{\sqrt{\pi}}\right)^{\frac{n}{n+2}} \left(\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\right)^{\frac{1}{n+2}} \right]$$

Proof. The proof is standard and can be found for instance in the book of Fife [8]. ∎

References

[1] Alberti, G. and G. Bellettini: A non-local anisotropic model for phase transitions: asymptotic behavior of rescaled energies. *European J. Appl. Math.* **9** (1998) no. 3 261-284.

[2] Alberti, G. and G. Bellettini: A non local anisotropic model for phase transitions. I. The optimal profile problem. *Math. Ann.* **310** (1998) no. 3 527-560.

[3] Bellettini, G., De Masi, A. and E. Presutti: Energy levels of critical points of a nonlocal functional (2004).

[4] Bellettini, G. De Masi, A. and E. Presutti: Tunneling in nonlocal evolution equations. *J. Nonlinear Math. Phys. (12)* **1** (2005) 1-14.

[5] Buttà, P., De Masi, A. and E. Rosatelli: Slow motion and meta-stability for a non-local evolution equation. *J. Stat. Phys.* **112** (2003) 709-764.

[6] De Masi, A., Orlandi, E., Presutti, E. and L. Triolo: Uniqueness and global stability of the instanton in non local evolution equations. *Rend. Math. Appl. (7)* **14** (1994) 693-723.

[7] De Masi, A., Olivieri, E. and E. Presutti: Spectral properties of integral operators in problems of interface dynamics and meta-stability. *Markov Process. Related Fields* **4** (1998) 27-112.

[8] Fife, P.C.: Mathematical aspects of reacting and diffusing systems. Lectures notes in Biomathematics, 28, *Springer-Verlag, Berlin-New York*, (1979).