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by

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A new approach to interior regularity of elliptic systems with quadratic growth in dimension two. *

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Abstract

We provide a new proof of a regularity theorem for systems of nonlinear elliptic equations with the quadratic nonlinearity in dimension two.

1 Introduction

Rivière [17] proved the following remarkable result.

Theorem 1 *Let $D \subset \mathbb{R}^2$ be an open set. If $\Omega^i_j \in L^2(D, \mathbb{R}^2)$, $\Omega^i_j = -\Omega^j_i$, $i, j = 1, 2, \ldots, m$ and $u = (u^1, u^2, \ldots, u^m) \in W^{1,2}(D, \mathbb{R}^m)$ solves the system of equations*

$$-\Delta u^i = \sum_{j=1}^{m} \Omega^i_j \cdot \nabla u^j, \quad i = 1, 2, \ldots, m, \tag{1}$$

then u is continuous.

This result solves a conjecture of Heinz about regularity of solutions to the prescribed bounded mean curvature equation and a conjecture of Hildebrandt about regularity of all critical points of continuously differentiable elliptic conformally invariant Lagrangians in dimension two. In particular it provides a new proof of Hélein's theorem [11, 13] about regularity of two dimensional harmonic mappings into arbitrary compact manifolds.

An important example is provided by the equation of prescribed mean curvature

$$\Delta u = 2H(u) u_{x_1} \wedge u_{x_2}, \tag{2}$$

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where $u \in W^{1,2}(D,\mathbb{R}^3)$, $D \subset \mathbb{R}^2$ and $H \in L^\infty(\mathbb{R}^3)$. Heinz conjectured that under these assumptions u is continuous. Let $\nabla^\perp = (-\partial_y, \partial_x)$. It is easy to see that (2) can be rewritten in the form

$$-\Delta u^i = \sum_{j=1}^{3} \Omega^i_j \cdot \nabla u^j, \quad i = 1, 2, 3,$$

where

$$\Omega = (\Omega^i_j)_{i,j=1,2,3} = H(u) \begin{bmatrix} 0 & \nabla^\perp u^3 & -\nabla^\perp u^2 \\ -\nabla^\perp u^3 & 0 & \nabla^\perp u^1 \\ \nabla^\perp u^2 & -\nabla^\perp u^1 & 0 \end{bmatrix}$$

and therefore the Heinz conjecture directly follows from Rivière's theorem.

The antisymmetry condition $\Omega^j_i = -\Omega^i_j$ is crucial in Theorem 1 because a well known example of Frehse [8] (cf. [17]) shows that without this condition solutions to the system (1) may be discontinuous.

Our aim is to prove the following result.

Theorem 2 *Let $D \subset \mathbb{R}^2$ be an open set. Let $H_{jk} = (H^i_{jk})_{i=1\ldots,m} : \mathbb{R}^m \to \mathbb{R}^m$, $1 \le j < k \le m$, be a family of bounded Lipschitz mappings. If $u \in W^{1,2}(D,\mathbb{R}^m)$ is a solution to the system*

$$-\Delta u = \sum_{1\le j<k\le m} H_{jk}(u)\, du^j \wedge du^k \tag{3}$$

i.e.

$$-\Delta u^i = \sum_{1\le j<k\le m} H^i_{jk}(u)\, du^j \wedge du^k \quad i = 1, 2, \ldots, m \tag{4}$$

then $u \in C^{2,\alpha}_{\mathrm{loc}}$ for all $0 < \alpha < 1$.

Here $du^j \wedge du^k = u^j_{x_1} u^k_{x_2} - u^k_{x_1} u^j_{x_2}$. It is well known that to prove Theorem 2, it is enough to prove continuity of u. Once it is known that u is continuous, one proves first higher integrability of $|\nabla u|$, using Gehring's lemma. A routine bootstrap argument gives then the claim of Theorem 2.

Theorem 2 cannot be deduced from that of Rivière because the system of equations does not possess antisymmetric structure. On the other hand the Lipschitz continuity of functions H^i_{jk} is a very strong condition. This is a price we have to pay for the lack of the antisymmetry; in the case of the H-surface equation (2) Theorem 2 gives the following result which is, however, weaker than that of Rivière.

Corollary 3 (Bethuel [2]) *Let $H : \mathbb{R}^3 \to \mathbb{R}$ be a bounded Lipschitz function. Assume that $u \in W^{1,2}(D,\mathbb{R}^3)$ is a weak solution of the H-surface equation (2). Then, $u \in C^{2,\alpha}_{\mathrm{loc}}(D)$ for every $\alpha < 1$.*

Two different proofs of Bethuel's theorem presented in [2], [18] can easily be generalized to cover Theorem 2, so the result is not really new, but what is new is the proof. The common feature of all proofs is a heavy use of delicate analytic tools: the duality of Hardy space and BMO (inspired by Coifman et al. [5]), L^p estimates for Hodge decomposition and its variants, interpolation in Lorentz spaces etc. Our proof is more elementary. It still employs the the duality of Hardy space and BMO, but even that can be replaced by an elementary argument (we will comment on it later on).

All known proofs seem to be purely 2-dimensional (including Rivière's result). That is, they all break down when one tries to adapt them to the case of higher-dimensional H-systems,

$$-\mathrm{div}\,(|\nabla u|^{n-2}\nabla u) = H(u)u_{x_1} \wedge u_{x_2} \wedge \ldots \wedge u_{x_n}\,, \qquad (5)$$

where $u \in W^{1,n}(\Omega, \mathbb{R}^{n+1})$ for some domain $\Omega \subset \mathbb{R}^n$, or to the system of n-harmonic maps into compact manifolds,

$$-\mathrm{div}\,(|\nabla u|^{n-2}\nabla u) \perp T_{u(x)}N \quad \text{a.e.}, \qquad u(x) \in N \text{ a.e.}$$

Our motivation was to give one more argument, fairly general, and to see whether it can be generalized to obtain full regularity of $W^{1,n}$ weak solutions of H-system (5) for $n > 2$.

The main difficulty in proving regularity of the solutions to the system (3) stems from the fact that the right hand side of (3) is only in L^1 and we cannot use u as a test function. Instead, we follow an idea of Lewis [14] (cf. [6], [7], [15], [16], [19]) and we built a ct-Lipschitz test function which coincides with u on the set where the maximal function of the gradient is less than or equal to t, see [1]. This method is combined here with the proof given in [18].

The notation is mostly standard. The integral average over a ball will be denoted by

$$u_B = \fint_B u\,dx = \frac{1}{|B|}\int_B u\,dx$$

and C will denote a general constant that can change its value in a single string of estimates. The symbol B will be used to denote a ball.

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2 Proof of Theorem 2

Some of the steps of the proof are similar to analogous steps in [18] and they will be sketched only.

Lemma 4 *Assume that $u \in W^{1,2}(D, \mathbb{R}^m)$ is a weak solution of the system (3). There exist numbers $r_0 > 0$, $\varepsilon \in (0,1)$ and $\lambda \in (0,1)$ such that for all $a \in D$ and all radii $r < \min(r_0, \frac{1}{4}\mathrm{dist}\,(a, \partial D))$ the following decay inequality holds:*

$$M_{2-\varepsilon}(a,r) \le \lambda M_{2-\varepsilon}(a,4r), \tag{6}$$

where

$$M_{2-\varepsilon}(a,r) \colon = \sup \frac{1}{\varrho^{\varepsilon}} \int_{B(z,\varrho)} |\nabla u|^{2-\varepsilon}\, dx\,,$$

the supremum being taken over all z, ϱ such that $B(z,\varrho) \subset B(a,r)$.

Once this lemma is proved, iterations of inequality (6) lead to

$$\int_{B(a,r)} |\nabla u|^{2-\varepsilon}\, dx \le C \left(\frac{r}{R}\right)^{\varepsilon+\gamma} \int_{B(a,R)} |\nabla u|^{2-\varepsilon}\, dx, \qquad \gamma > 0,$$

where γ is some positive constant depending only on λ. Thus, by Dirichlet Growth Theorem, u is locally Hölder continuous. Therefore it remains to prove the lemma.

The proof of Lemma 4 has two separate stages. First, we test system (3) with functions that are good Lipschitz approximations of u (i.e., they agree with u on the set where the maximal function of the gradient of u is not too large). This yields an estimate for the integral of $|\nabla u|^2$ on, roughly speaking, sets of the form $\{x \colon M|\nabla u|(x) \le t\}$.

The second stage is to average this estimate w.r.t. t, with weight equal to $t^{-1-\varepsilon}$, and to obtain an *averaged Caccioppoli inequality*. Then, we show that any function u satisfying this averaged Caccioppoli inequality must also satisfy (6). In this last step, is not at all important that u solves (2).

3 Proof of Lemma 4

Fix a and $r > 0$ such that $B_r \equiv B(a,r) \subset B_{4r} = B(a,4r) \Subset D$. The choice of r_0, ε and λ shall be specified later on.

It suffices to prove that

$$\frac{1}{r^{\varepsilon}} \int_{B(a,r)} |\nabla u|^{2-\varepsilon} \le \lambda M_{2-\varepsilon}(a,4r). \tag{7}$$

Indeed, for $B(z,\varrho) \subset B(a,r)$, (7) gives

$$\frac{1}{\varrho^\varepsilon} \int_{B(z,\varrho)} |\nabla u|^{2-\varepsilon} \le \lambda M_{2-\varepsilon}(z, 4\varrho) \le \lambda M_{2-\varepsilon}(a, 4r)$$

and hence (6) follows after taking supremum over all $B(z,\varrho) \subset B(a,r)$. If

$$\int_{B_{2r}} |\nabla u|^{2-\varepsilon} > 8 \int_{B_r} |\nabla u|^{2-\varepsilon} \,,$$

then

$$\frac{1}{r^\varepsilon} \int_{B_r} |\nabla u|^{2-\varepsilon} < \frac{2^\varepsilon}{8} \frac{1}{(2r)^\varepsilon} \int_{B_{2r}} |\nabla u|^{2-\varepsilon} \le \frac{1}{4} M_{2-\varepsilon}(a, 2r)$$

and hence (7) follows with $\lambda = 1/4$. Therefore we can assume that

$$\int_{B_{2r}} |\nabla u|^{2-\varepsilon} \le 8 \int_{B_r} |\nabla u|^{2-\varepsilon}. \tag{8}$$

We will frequently use the following well known lemma.

Lemma 5 *If $u \in W^{1,p}_{\rm loc}(\mathbb{R}^n)$, then*

$$|u(x) - u(y)| \le C|x-y|(M|\nabla u|(x) + M|\nabla u|(y)) \ \textit{a.e.}$$

and

$$|u(x) - u_B| \le CrM|\nabla u|(x) \quad \textit{for a.e. } x \in B,$$

where r is the radius of the ball B and $M|\nabla u|$ is the Hardy-Littlewood maximal function of $|\nabla u|$.

For the proof see for example [7], [9], [14], [15].

Step 1. Choice of test functions. Fix $t > 0$ and a cutoff function $\varphi \in C_0^\infty(B_{2r})$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on B_r and $|\nabla\varphi| \le C/r$.

Set

$$g(x) = |\varphi(x)|\,|\nabla u(x)| + |u(x) - u_{B_{2r}}|\,|\nabla\varphi(x)|\,.$$

We define $g \equiv 0$ in $\mathbb{R}^2 \setminus B_{2r}$. Let

$$F_t \colon = \{x \in B_{2r} \colon Mg(x) \le t\}$$

and $\tilde u(x) = \varphi(x)\big(u(x) - u_{B_{2r}}\big)$. We claim that $\tilde u$ is Lipschitz continuous with constant ct on $(\mathbb{R}^2 \setminus B_{2r}) \cup F_t$.

Case 1. Let $x, y \in F_t$. Then, since $|\nabla \tilde u| \le g$, we have

$$|\tilde u(x) - \tilde u(y)| \le C|x-y|\big(M|\nabla\tilde u|(x) + M|\nabla \tilde u|(y)\big) \le Ct|x-y|$$

by definition of F_t.

Case 2. Assume that $x \in F_t$, $y \in \mathbb{R}^2 \setminus B_{2r}$. Let $\varrho\colon = 2\mathrm{dist}\,(x, \partial B_{2r})$. Since $\tilde u$ equals zero on a large part of the ball $B(x,\varrho)$, Poincaré inequality yields

$$|\tilde u_{B(x,\varrho)}| \le C\varrho \fint_{B(x,\varrho)} |\nabla \tilde u| \le C\varrho M g(x) \le C|x-y|t\,.$$

Therefore

$$\begin{aligned} |\tilde u(x) - \tilde u(y)| = |\tilde u(x)| &\le |\tilde u(x) - \tilde u_{B(x,\varrho)}| + |\tilde u_{B(x,\varrho)}| \\ &\le C\varrho M|\nabla \tilde u|(x) + Ct|x-y| \\ &\le Ct|x-y|\,. \end{aligned}$$

This proves the claim. We now extend $\tilde u\colon F_t \cup (\mathbb{R}^2 \setminus B_{2r}) \to \mathbb{R}^m$ to a Lipschitz continuous function $u_t\colon \mathbb{R}^2 \to \mathbb{R}^m$ such that $u_t \in \mathrm{Lip}(Ct)$, $|\nabla u_t| \le Ct$, $u_t \equiv \tilde u$ in $F_t \cup (\mathbb{R}^2 \setminus B_{2r})$ — so that, in particular, $u_t \equiv 0$ off B_{2r}.

Step 2. We use u_t as a testing function for system (3). This gives

$$\begin{aligned} \int_{F_t} \nabla u \cdot \nabla u_t \, dx &\le Ct \int_{B_{2r}\setminus F_t} |\nabla u|\, dx \\ &\quad + \left| \sum_{1\le j<k\le m} \int_{\mathbb{R}^2} H_{jk}(u)\cdot u_t \, du^j \wedge du^k \right| . \end{aligned}$$

and next

$$\begin{aligned} \int_{F_t} |\nabla u|^2 \varphi \, dx &\le \int_{F_t} |\nabla u|\, |\nabla\varphi|\, |u - u_{B_{2r}}|\, dx \\ &\quad + Ct \int_{B_{2r}\setminus F_t} |\nabla u|\, dx + |I_t|, \end{aligned} \tag{9}$$

where

$$I_t\colon = \sum_{1\le j<k\le m} \int_{\mathbb{R}^2} H_{jk}(u)\cdot u_t \, du^j \wedge du^k\,. \tag{10}$$

Inequality (9) holds for all $t > 0$. To obtain estimates for (3) involving local norms of $|\nabla u|$ in Morrey spaces, we multiply (9) by $t^{-1-\varepsilon}$ and integrate with respect to $t \in (t_0, \infty)$, for an appropriately chosen number t_0. Before doing that, however, we record a crucial estimate for I_t.

Step 3. Estimating the critical nonlinearity. We claim that

$$|I_t| \le C \cdot K_\varepsilon \cdot \left(\int_{B_{2r}} |\nabla u_t|^{2+\varepsilon}\, dx \right)^{1/(2+\varepsilon)}, \tag{11}$$

where

$$K_\varepsilon : \; = M_{2-\varepsilon}(a, 4r)^{1/(2-\varepsilon)} r^{\varepsilon/(2+\varepsilon)} \|\nabla u\|_{L^2(B_{2r})}. \tag{12}$$

This estimate follows from the duality of Hardy space $\mathcal{H}^1$ and the space BMO of functions of bounded mean oscillation. Here are some details. I_t is the sum of expressions I^i_{jk}, where

$$I^i_{jk} = \int_{\mathbb{R}^2} H^i_{jk}(u)\, u^i_t \, du^j \wedge du^k \,.$$

We estimate each such integral, integrating by parts. Let $\zeta_1 \equiv 1$ on B_{2r}, $|\nabla \zeta_1| \le 2/r$, $\zeta_1 \equiv 0$ off B_{3r}. We have

$$\begin{aligned} |I^i_{jk}| &\le \left| \int_{\mathbb{R}^2} \zeta_1 (u^j - u^j_{B_{2r}}) d[H^i_{jk}(u) u^i_t] \wedge du^k \right| \\ &\le C \|\zeta_1 (u^j - u^j_{B_{2r}})\|_{\mathrm{BMO}} \, \|\nabla [H^i_{jk}(u) u^i_t]\|_{L^2(B_{2r})} \, \|\nabla u^k\|_{L^2(B_{2r})} \\ &\le C M_{2-\varepsilon}(a, 4r)^{1/(2-\varepsilon)} \, \|\nabla [H^i_{jk}(u) u^i_t]\|_{L^2(B_{2r})} \, \|\nabla u^k\|_{L^2(B_{2r})}. \end{aligned}$$

The first inequality follows from Fefferman's duality theorem and the result of Coifman Lions Meyer and Semmes [5]. The second inequality is an elementary estimate of the local BMO norm of u; see [18] for details.

Since we estimate the BMO norm in terms of the Morrey norm of the gradient, the above inequality can be proved in an elementary way bypassing Fefferman's theorem, see [3], [4], [10].

Further,

$$\begin{aligned} \|\nabla [H^i_{jk}(u) u^i_t]\|_{L^2} &\le \|H\|_\infty \|\nabla u_t\|_{L^2(B_{2r})} + \|\nabla H\|_\infty \|\nabla u\|_{L^2(B_{2r})} \|u_t\|_{L^\infty(B_{2r})} \\ &\le C r^{\varepsilon/(2+\varepsilon)} (1 + \|\nabla u\|_{L^2(B_{2r})}) \left(\int_{B_{2r}} |\nabla u_t|^{2+\varepsilon} \, dx \right)^{1/(2+\varepsilon)} . \end{aligned}$$

(To obtain the last line, we apply Hölder inequality to deal with $\|\nabla u_t\|_{L^2}$, and Sobolev imbedding theorem to deal with $\|u_t\|_{L^\infty}$. The point here is that u_t is a priori more regular than u is.)

One can fix $r_0 > 0$ such that $\|\nabla u\|_{L^2(B_{2r})} \le 1$ for all $r < r_0$; claim (11) follows.

Step 4. Averaging. We now rewrite (9) as

$$\begin{aligned} \int_{F_t} |\nabla u|^2 \varphi \, dx &\le \int_{F_t} |\nabla u| \, |\nabla \varphi| \, |u - u_{B(a,2r)}| \, dx \\ &\quad + Ct \int_{B_{2r} \setminus F_t} |\nabla u| \, dx \\ &\quad + C K_\varepsilon \cdot \left(\int_{B_{2r}} |\nabla u_t|^{2+\varepsilon} \, dx \right)^{1/(2+\varepsilon)} , \end{aligned} \tag{13}$$

multiply both sides of (13) by $t^{-1-\varepsilon}$ and integrate w.r.t. $t \in (t_0, \infty)$, setting

$$t_0 \colon = \delta\Big(\fint_{B_r} |\nabla u|^{2-\varepsilon}\, dx\Big)^{1/(2-\varepsilon)} . \tag{14}$$

(Here, δ is a small constant independent of ε.) We obtain an *averaged Caccioppoli inequality* of the form

$$J_1 \le C_1(J_2 + J_3 + J_4), \tag{15}$$

where

$$\begin{aligned} J_1 &= \int_{t_0}^{\infty} t^{-1-\varepsilon} \int_{F_t} |\nabla u|^2 \varphi\, dx\, dt\,, &(16)\\ J_2 &= \int_{t_0}^{\infty} t^{-1-\varepsilon} \int_{F_t} |\nabla u|\, |\nabla\varphi|\, |u - u_{B_{2r}}|\, dx\, dt\,, &(17)\\ J_3 &= \int_{t_0}^{\infty} t^{-\varepsilon} \int_{B_{2r}\setminus F_t} |\nabla u|\, dx\, dt\,, &(18)\\ J_4 &= K_\varepsilon \int_{t_0}^{\infty} t^{-1-\varepsilon} \Big(\int_{B_{2r}} |\nabla u_t|^{2+\varepsilon}\, dx\Big)^{1/(2+\varepsilon)} dt\,. &(19) \end{aligned}$$

Step 5. Estimates of J_1–J_4. Tedious but elementary estimates of J_1–J_4 (involving only the Fubini theorem, Hölder, Young and Poincaré inequalities, and the Hardy–Littlewood maximal theorem) yield the following inequalities:

$$\begin{aligned} J_1 &\ge \frac{C_2}{\varepsilon} \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx\,, &(20)\\ J_2 &\le \frac{C_3}{\varepsilon} \Big(\int_{B_{2r}\setminus B_r} |\nabla u|^{2-\varepsilon}\, dx\Big)^{\frac{1}{2-\varepsilon}} \Big(\int_{B_r} |\nabla u|^{2-\varepsilon}\, dx\Big)^{\frac{1-\varepsilon}{2-\varepsilon}} \\ &\le \frac{C_4}{\varepsilon} \int_{B_{2r}\setminus B_r} |\nabla u|^{2-\varepsilon}\, dx + \frac{C_2}{4C_1\varepsilon} \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx\,, &(21)\\ J_3 &\le C_5 \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx\,, &(22)\\ J_4 &\le C_6(\varepsilon) r^{\varepsilon} \|Du\|_{L^2(B_{2r})} M_{2-\varepsilon}(a, 4r)\,, &(23) \end{aligned}$$

where the constants C_1, C_2, C_3, C_4, C_5 *do not* depend on ε, whereas $C_6 = C_6(\varepsilon)$ does.

The details of these estimates are given in the next Section. Here we just show how to conclude the proof of Lemma 4, assuming these estimates.

Step 6. Conclusion. Inserting the above estimates into (15), we obtain

$$\begin{aligned} \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx \;\le\; & C_7 \int_{B_{2r}\setminus B_r} |\nabla u|^{2-\varepsilon}\, dx + \frac{1}{4} \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx \\ & + C_8 \varepsilon \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx + C_9(\varepsilon) r^{\varepsilon} \|Du\|_{L^2(B_{2r})} M_{2-\varepsilon}(a, 4r)\,. \end{aligned} \tag{24}$$

Now we add $C_7 \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx$ to both sides to "fill the hole" on the right hand side and after elementary calculations we arrive at

$$\begin{aligned}
\frac{1}{r^\varepsilon} \int_{B_r} |\nabla u|^{2-\varepsilon} &\le \frac{C_7 2^\varepsilon}{C_7+1} \frac{1}{(2r)^\varepsilon} \int_{B_{2r}} |\nabla u|^{2-\varepsilon} + \frac{\frac{1}{4} + C_8\varepsilon}{C_7+1} \frac{1}{r^\varepsilon} \int_{B_r} |\nabla u|^{2-\varepsilon} \\
&\quad + \frac{C_9(\varepsilon) \|Du\|_{L^2(B_{2r})}}{C_7+1} M_{2-\varepsilon}(a, 4r) \\
&\le \frac{C_7 2^\varepsilon + \frac{1}{4} + C_8\varepsilon + C_9(\varepsilon)\|Du\|_{L^2(B_{2r})}}{C_7+1} M_{2-\varepsilon}(a, 4r).
\end{aligned}$$

We now fix ε so small that

$$C_7 2^\varepsilon + \frac{1}{4} + C_8\varepsilon < C_7 + \frac{1}{2}$$

and then $r_0 = r_0(\varepsilon)$ so small that

$$C_9(\varepsilon)\|Du\|_{L^2(B_{2r})} < \frac{1}{4}$$

for all points $a \in D$ and all radii $r < r_0(\varepsilon)$. Now (7) follows with $\lambda = (C_7{+}3/4)/(C_7{+}1)$. This completes the proof of the lemma. □

4 Averaged Caccioppoli inequality: proofs of (20)–(23)

In this Section we provide details of Step 5 of the proof from the previous Section. Numerous estimates are based on the inequalities

$$\int_{B_{2r}} (Mg)^{2-\varepsilon}\, dx \le C \int_{B_{2r}} g^{2-\varepsilon}\, dx \le C \int_{B_{2r}} |\nabla u|^{2-\varepsilon}\, dx \le C \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx\,. \qquad (25)$$

All constant can be chosen independently of ε. The first estimate follows from Hardy–Littlewood maximal theorem, the second one — from Poincaré inequality. The last one is just the assumption (8).

Estimate of J_1. Recall that $F_t = \{x \in B(a,2r) \colon Mg(x) \le t\}$. Since $\varphi \equiv 1$ on B_r, Fubini's theorem yields

$$\begin{aligned}
J_1 &= \int_{t_0}^\infty t^{-1-\varepsilon} \int_{F_t} |\nabla u|^2 \varphi\, dx\, dt \ge \int_{B_r \cap \{Mg > t_0\}} |\nabla u|^2 \varphi \int_{Mg(x)}^\infty t^{-1-\varepsilon}\, dt\, dx \\
&= \frac{1}{\varepsilon} \int_{B_r} |\nabla u|^2 (Mg)^{-\varepsilon}\, dx - \frac{1}{\varepsilon} \int_{B_r \cap \{Mg \le t_0\}} |\nabla u|^2 (Mg)^{-\varepsilon}\, dx \\
&=: J_{11} - J_{12}\,.
\end{aligned}$$

We apply Hölder inequality and (25) to estimate J_{11}. We have

$$\begin{aligned}\int_{B_r} |\nabla u|^{2-\varepsilon}\,dx &\overset{(H)}{\leq} \left(\int_{B_r} |\nabla u|^2 (Mg)^{-\varepsilon}\,dx\right)^{\frac{2-\varepsilon}{2}} \left(\int_{B_r} (Mg)^{2-\varepsilon}\,dx\right)^{\frac{\varepsilon}{2}} \\ &\overset{(25)}{\leq} C\left(\int_{B_r} |\nabla u|^2 (Mg)^{-\varepsilon}\,dx\right)^{\frac{2-\varepsilon}{2}} \left(\int_{B_r} |\nabla u|^{2-\varepsilon}\,dx\right)^{\frac{\varepsilon}{2}}.\end{aligned}$$

(with some constant C that is independent from ε.) Thus,

$$J_{11} \geq \frac{C_0}{\varepsilon} \int_{B_r} |\nabla u|^{2-\varepsilon}\,dx\,.$$

To estimate J_{12} we note that $|\nabla u| \leq g \leq Mg$ in B_r. Hence,

$$|J_{12}| \leq \frac{1}{\varepsilon} t_0^{2-\varepsilon} |B_r| = \frac{1}{\varepsilon} \delta^{2-\varepsilon} \int_{B_r} |\nabla u|^{2-\varepsilon}\,dx\,.$$

Choosing $\delta < \min(\frac{1}{2}, C_0/2)$, we obtain $\delta^{2-\varepsilon} < \delta < C_0/2$. Combining the estimates of J_{11} and J_{12}, we finish the proof of (20).

Estimate of J_2. Using Fubini's theorem, we have

$$\begin{aligned}J_2 &\leq \int_0^\infty t^{-1-\varepsilon} \int_{F_t} |\nabla u|\,|\nabla\varphi|\,|u - u_{B_{2r}}|\,dx\,dt \\ &= \int_{B_{2r}} |\nabla u|\,|\nabla\varphi|\;|u - u_{B_{2r}}| \int_{Mg(x)}^\infty t^{-1-\varepsilon}\,dt\,dx \\ &= \frac{1}{\varepsilon} \int_{B_{2r}} |\nabla u|\,|\nabla\varphi|\;|u - u_{B_{2r}}|\,(Mg)^{-\varepsilon}\,dx \\ &\leq \frac{1}{\varepsilon} \int_{B_{2r}} |\nabla u|\,|\nabla\varphi|^{1-\varepsilon}\;|u - u_{B_{2r}}|^{1-\varepsilon}\,dx \\ &\leq \frac{C}{\varepsilon} \left(\int_{B_{2r}\setminus B_r} |\nabla u|^{2-\varepsilon}\,dx\right)^{\frac{1}{2-\varepsilon}} \left(\int_{B_r} |\nabla u|^{2-\varepsilon}\,dx\right)^{\frac{1-\varepsilon}{2-\varepsilon}}.\end{aligned}$$

(Note that $|\nabla\varphi|\,|u - u_{B_{2r}}| \leq g \leq Mg$. In the last line, we apply Hölder and Poincaré inequalities combined with assumption (8).) By a standard application of Young's inequality, (21) follows.

Estimate of J_3. Since $t < Mg(x)$ in the complement of F_t, we obtain

$$\begin{aligned} J_3 &\leq \int_0^\infty t^{-\varepsilon} \int_{B_{2r}\setminus F_t} |\nabla u|\, dx\, dt \\ &= \frac{1}{1-\varepsilon} \int_{B_{2r}} |\nabla u|\, (Mg)^{1-\varepsilon}\, dx \\ &\overset{(H)}{\leq} \frac{1}{1-\varepsilon} \Big(\int_{B_{2r}} |\nabla u|^{2-\varepsilon}\, dx \Big)^{\frac{1}{2-\varepsilon}} \Big(\int_{B_{2r}} (Mg)^{2-\varepsilon}\, dx \Big)^{\frac{1-\varepsilon}{2-\varepsilon}} \\ &\leq C \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx\, . \end{aligned}$$

To obtain the last line, one applies inequalities (25).

Estimate of J_4. This is the heart of the matter. We split

$$\begin{aligned} J_4 &= K_\varepsilon \int_{t_0}^\infty t^{-1-\varepsilon} \Big(\int_{B_{2r}} |\nabla u_t|^{2+\varepsilon}\, dx \Big)^{1/(2+\varepsilon)} dt \\ &\leq K_\varepsilon \int_{t_0}^\infty t^{-1-\varepsilon} \Big(\int_{F_t} |\nabla \tilde u|^{2+\varepsilon}\, dx \Big)^{1/(2+\varepsilon)} dt \\ &\quad + K_\varepsilon \int_{t_0}^\infty t^{-1-\varepsilon} \cdot Ct \cdot |B_{2r} \setminus F_t|^{1/(2+\varepsilon)}\, dt \\ &=: K_\varepsilon (J_{41} + J_{42}). \end{aligned}$$

We used the fact that $\nabla u_t = \nabla \tilde u$ in F_t and $|\nabla u_t| \leq Ct$ everywhere, in particular in $B_{2r} \setminus F_t$. To estimate J_{41} observe that $|\nabla \tilde u| \leq g \leq Mg \leq t$ in F_t and hence $|\nabla \tilde u|^{2+\varepsilon} \leq t^{2\varepsilon} |\nabla \tilde u|^{2-\varepsilon}$. Moreover the Poincaré inequality gives

$$\Big(\int_{F_t} |\nabla \tilde u|^{2-\varepsilon} \Big)^{\frac{1}{2+\varepsilon}} \leq C \Big(\int_{B_{2r}} |\nabla u|^{2-\varepsilon} \Big)^{\frac{1}{2+\varepsilon}}\, .$$

Hence

$$\begin{aligned} J_{41} &\leq \int_{t_0}^\infty t^{-1-\varepsilon} t^{\frac{2\varepsilon}{2+\varepsilon}} \Big(\int_{F_t} |\nabla \tilde u|^{2-\varepsilon}\, dx \Big)^{\frac{1}{2+\varepsilon}} dt \\ &\leq C(\varepsilon) t_0^{-\varepsilon + \frac{2\varepsilon}{2+\varepsilon}} \Big(\int_{B_{2r}} |\nabla u|^{2-\varepsilon} \Big)^{\frac{1}{2+\varepsilon}} \\ &\leq C(\varepsilon) r^{\frac{2\varepsilon^2}{4-\varepsilon^2}} \Big(\int_{B_r} |\nabla u|^{2-\varepsilon} \Big)^{\frac{1-\varepsilon}{2-\varepsilon}}\, . \end{aligned}$$

The last constant depends also on δ, but δ depends on general constants only, so there is no need to write dependence on δ explicitly.

To estimate J_{42} first observe that Cavalieri's principle and (25) give

$$(2-\varepsilon)\int_0^\infty t^{1-\varepsilon}|B_{2r}\setminus F_t|\,dt = \int_{B_{2r}}(Mg)^{2-\varepsilon}\,dx \le C\int_{B_r}|\nabla u|^{2-\varepsilon}\,dx.$$

Hence

$$\begin{aligned}
J_{42} &\le C\int_{t_0}^\infty t^{-\varepsilon}|B_{2r}\setminus F_t|^{\frac{1}{2+\varepsilon}}\,dt\\
&\le C\left(\int_{t_0}^\infty \left(t^{-\varepsilon-\frac{1-\varepsilon}{2+\varepsilon}}\right)^{\frac{2+\varepsilon}{1+\varepsilon}}\right)^{\frac{1+\varepsilon}{2+\varepsilon}}\left(\int_{t_0}^\infty t^{1-\varepsilon}|B_{2r}\setminus F_t|\,dt\right)^{\frac{1}{2+\varepsilon}}\\
&\le C(\varepsilon)\left(t_0^{1+(-\varepsilon-\frac{1-\varepsilon}{2+\varepsilon})\frac{2+\varepsilon}{1+\varepsilon}}\right)^{\frac{1+\varepsilon}{2+\varepsilon}}\left(\int_{B_r}|\nabla u|^{2-\varepsilon}\,dx\right)^{\frac{1}{2+\varepsilon}}\\
&= C(\varepsilon)r^{\frac{2\varepsilon^2}{4-\varepsilon^2}}\left(\int_{B_r}|\nabla u|^{2-\varepsilon}\right)^{\frac{1-\varepsilon}{2-\varepsilon}}.
\end{aligned}$$

Now (23) follows from the definition of K_ε. This completes the whole proof. □

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