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by

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Abstract

We consider a model of a binary mixture of compressible, viscous, and macroscopically immiscible fluids based on the diffuse interface approximation, where the difference in concentrations of the two fluids plays the role of the order parameter. The resulting system consists of the compressible Navier-Stokes equations governing the motion of the mixture coupled with the Cahn-Hilliard equation for the order parameter. We prove existence of global-in-time weak (distributional) solutions of the problem without any restriction on the size of initial data.

Key words: Two-phase flow, free boundary value problems, diffuse interface model, mixtures of viscous fluids, Cahn-Hilliard equation, Navier-Stokes equation
AMS-Classification: 35Q30, 35Q35, 76N10, 76T99

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1 Introduction and Main Result

We consider the flow of two macroscopically immiscible, viscous compressible Newtonian fluids filling a bounded domain $\Omega \subset \mathbb{R}^3$. Classical models assume that both fluids are separated by a surface $\Gamma(t)$ and capillary forces are modeled by the so-called Young-Laplace law

$$[n \cdot \mathbb{T}] = \sigma H \mathbf{n} \qquad \text{on } \Gamma(t)$$

which relates the jump of the norm-component of the stress tensor $n \cdot \mathbb{T}$ to the mean-curvature vector $H\mathbf{n}$ of the interface $\Gamma(t)$. This is a well-accepted law for situations sufficiently close to equilibrium. On the other hand, when the interface $\Gamma(t)$ develops singularities during the flow e.g. due to droplet formation or coalescence, this classical model breaks down. In order to describe a general two-phase flow with droplet formation and coalescence of several droplet, diffuse interface models were developed, which take a (partial) mixing of the two macroscopically immiscible fluids and a small mesoscopic length-scale into account. We refer to Anderson and McFadden [3] for a review on that topic.

In the present contribution we study a variant of a model by Lowengrub and Truskinovsky [20], which can also be found in [3, Pages 151-152]. This model consists of a system of equations

$$\varrho\partial_t \mathbf{u} + \varrho\mathbf{u} \cdot \nabla\mathbf{u} - \operatorname{div} \mathbb{S} + \nabla p = -\operatorname{div}\left(\nabla c \otimes \nabla c - \frac{|\nabla c|^2}{2}\mathbb{I}\right) \tag{1.1}$$

$$\partial_t \varrho + \operatorname{div}(\varrho\mathbf{u}) = 0 \tag{1.2}$$

$$\varrho\partial_t c + \varrho\mathbf{u} \cdot \nabla c = \Delta\mu \tag{1.3}$$

$$\varrho\mu = \varrho\frac{\partial f}{\partial c} - \Delta c \tag{1.4}$$

where $p = \varrho^2 \frac{\partial f}{\partial \varrho}(\varrho, c)$ and

$$\begin{aligned} \mathbb{S} &= 2\nu(c)\mathbb{D}(\mathbf{u}) + \eta(c) \operatorname{div} \mathbf{u}\, \mathbb{I}, \\ \mathbb{D}(\mathbf{u}) &= \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) - \frac{1}{3}\operatorname{div} \mathbf{u}\mathbb{I} \end{aligned} \tag{1.5}$$

for some suitable functions $\lambda(c) > 0, \eta(c) \geq 0$ and the free energy density $f(\varrho, c)$ to be specified later. Here $\mathbf{u}$ is the mean velocity of the fluid mixture, p is the pressure, c is the (mass) concentration difference of the two components and μ is the chemical potential. The first equation (1.1) describes the conservation of linear momentum. In comparison with the compressible Navier-Stokes equation for a single fluid, there is an extra stress contribution in the stress tensor $\nabla c \otimes \nabla c - \frac{|\nabla c|^2}{2}\mathbb{I}$, which describes capillary effect related to the free energy

$$E_{\text{free}}(\varrho, c) = \int_\Omega \left(\varrho f(\varrho, c) + \frac{1}{2}|\nabla c|^2 \right) \, dx, \tag{1.6}$$

representing here the surface energy penalizing mixing of the fluids as well as large variations of the concentration difference c. The second equation (1.2) is the usual conservation of mass. Moreover, (1.3)-(1.4) is a diffusion-convection equation for the concentration difference of Cahn-Hilliard type. The model is derived and explained in more detail in Section 2.2 below.

The system is closed by the initial and boundary conditions

$$\begin{aligned} \mathbf{u}|_{\partial\Omega} &= \nabla c \cdot \mathbf{n}|_{\partial\Omega} = \nabla \mu \cdot \mathbf{n}|_{\partial\Omega} = 0, && (1.7) \\ (\mathbf{u}, c)|_{t=0} &= (\mathbf{u}_0, c_0) && (1.8) \end{aligned}$$

Keeping in mind that system (1.1) - (1.5) was proposed as an alternative to the classical model when the latter fails, meaning when singularities occur, we focus on the large data giving rise to solutions defined on an arbitrarily long time interval. Accordingly, a suitable framework is provided by the class of weak (distributional) solutions introduced by Leray [17] in the context of a single incompressible fluid and generalized by P.-L.Lions [18] to the compressible case. As is well-known, a peculiar and rather unpleasant feature of this approach is a (hypothetical) appearance of the vacuum zones, that means, the density ϱ may vanish on a set of positive measure due to the lack of sufficiently strong *a priori* estimates. This fact represents the main technical stumbling block that prevents us from considering the model proposed by Lowengrub and Truskinovsky in [20, Section 3, Equation (3.34)], where the total free energy is given by

$$E_{\text{free}}(\varrho, c) = \int_\Omega \left(\varrho f(\varrho, c) + \frac{\varrho}{2}|\nabla c|^2 \right) \, dx. \tag{1.9}$$

Indeed in this case the energy estimates do not provide any bound on ∇c in the vacuum zone, which seems to be an unsurmountable problem for all available techniques based on compactness arguments in the spirit of Rellich-Kondrashov theorem.

A similar model for incompressible fluids was studied by Boyer [6], Liu and Shen [19], Starovoitov [22], and the first author [1]. Finally, we let us remark that a model for non-isothermal fluids undergoing a change of phase was derived and studied by Blesgen [4]. The model leads to a compressible (non-barotropic) Navier-Stokes system coupled with a modified Allen-Cahn equation.

Before we introduce our main result, let us summarize the principal hypotheses: We suppose that $\Omega \subset \mathbb{R}^3$ is a bounded domain with C^2-boundary. The viscosity coefficients ν, η are assumed to be continuously differentiable functions of c satisfying

$$0 < \underline{\nu} \leq \nu(c) \leq \overline{\nu}, \; 0 \leq \eta(c) \leq \overline{\eta} \text{ for all } c. \tag{1.10}$$

The specific (homogeneous) free energy f takes the form

$$f(\varrho, c) = f_{\mathrm{e}}(\varrho) + f_{\mathrm{mix}}(\varrho, c), \; f_{\mathrm{mix}}(\varrho, c) = H(c)\log(\varrho) + G(c) \tag{1.11}$$

and is interrelated to the pressure through the equation of state

$$p(\varrho, c) = \varrho^2 \frac{\partial f(\varrho, c)}{\partial \varrho} = p_{\mathrm{e}}(\varrho) + \varrho H(c), \; f_{\mathrm{e}}(\varrho) = \int_1^{\varrho} \frac{p_{\mathrm{e}}(z)}{z^2} \, \mathrm{d}z \tag{1.12}$$

where $p_{\mathrm{e}} \in C([0,\infty) \cap C^1(0,\infty)$. In what follows, we shall assume that

$$p_{\mathrm{e}}(0) = 0, \; \underline{p}_1 \varrho^{\gamma-1} - \underline{p}_2 \leq p_{\mathrm{e}}'(\varrho) \leq \overline{p}(1 + \varrho^{\gamma-1}) \tag{1.13}$$

for a certain $\gamma > \frac{3}{2}$ and

$$-\underline{H} \leq H'(c), H(c) \leq \overline{H}, \; \underline{G}_1 c - \underline{G}_2 \leq G'(c) \leq \overline{G}(1 + c) \tag{1.14}$$

for all $c \in \mathbb{R}$.

Remark 1.1 *Let us remark that the assumptions on the free energy are motivated by the free energy density of the form*

$$\begin{aligned} \varrho f(\varrho, c) &= \alpha_1 \varrho \frac{1-c}{2} \ln\left(\varrho \frac{1-c}{2}\right) + \alpha_2 \varrho \frac{1+c}{2} \ln\left(\varrho \frac{1+c}{2}\right) - \beta c^2 \\ &= \varrho \left(\alpha_1 \frac{1-c}{2} \ln \frac{1-c}{2} + \alpha_2 \frac{1+c}{2} \ln \frac{1+c}{2}\right) \\ &\quad + \varrho \ln \varrho \left(\alpha_1 \frac{1-c}{2} + \alpha_2 \frac{1+c}{2}\right) - \beta c^2, \end{aligned}$$

where $\alpha_1, \alpha_2, \beta > 0$ and the logarithmic terms are related to the entropy of the system. Typically, in the case of the Cahn-Hilliard equation, $\alpha_1 \frac{1-c}{2} \ln \frac{1-c}{2} + \alpha_2 \frac{1+c}{2} \ln \frac{1+c}{2}$ is approximated by a suitable smooth bi-stable function. Finally, adding a cold pressure *of the form $p_e(\varrho) = \theta^{\frac{5}{2}} P\left(\frac{\varrho}{\theta^{\frac{3}{2}}}\right) \approx \varrho^{\frac{5}{3}}$, we end up with a free energy of the form above.*

In what follows we denote by

$$E(t) = \int_\Omega \varrho(t)|\mathbf{u}(t)|^2\,dx + E_{\mathrm{free}}(\varrho(t),c(t)), \tag{1.15}$$

$$E_0 = \int_\Omega \varrho_0^{-1}|\mathbf{m}_0|^2\,dx + E_{\mathrm{free}}(\varrho_0,c_0) \tag{1.16}$$

the total energy of the system at time $t\in(0,T)$, $t=0$, respectively. In addition we set $Q_{(s,t)}=\Omega\times(s,t)$ and $Q_T=Q_{(0,T)}$.

Our main result reads as follows:

THEOREM 1.2 *Let $0<T<\infty$, let $\gamma>\frac{3}{2}$, and let the above assumptions be satisfied.*

Then for every non-negative $\varrho_0\in L^\gamma(\Omega)$, measurable $\mathbf{m}_0\colon\Omega\to\mathbb{R}^3$ with $\varrho_0^{-1}|\mathbf{m}_0|^2\in L^1(\Omega)$, and $c_0\in H^1(\Omega)$ there is a weak solution $\varrho\in L^\infty(0,T;L^\gamma(\Omega))$, $\varrho\geq 0$, $\mathbf{u}\in L^2(0,T;H^1(\Omega;\mathbb{R}^3))$, $c\in L^\infty(0,T;H^1(\Omega))$ in the following sense:

1. *For every $\boldsymbol{\varphi}\in\mathcal{D}(\Omega\times(0,T);\mathbb{R}^3)$*

$$\begin{aligned}&-\int_{Q_T}\Big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi}+(\varrho\mathbf{u}\otimes\mathbf{u}+p\,\mathbb{I}-\mathbb{S}):\nabla\boldsymbol{\varphi}\Big)\,d(x,t)\\ &\quad=\int_{Q_T}\Big((\nabla c\otimes\nabla c):\nabla\boldsymbol{\varphi}-\frac{|\nabla c|^2}{2}\operatorname{div}\boldsymbol{\varphi}\Big)\,d(x,t)\end{aligned} \tag{1.17}$$

2. *ϱ is a renormalized solution of (1.2) in the sense of DiPerna and Lions [8], i.e.,*

$$\int_{Q_T}\Big(\varrho B(\varrho)\partial_t\varphi+\varrho B(\varrho)\mathbf{u}\cdot\nabla\varphi-b(\varrho)\operatorname{div}\mathbf{u}\;\varphi\Big)\,d(x,t)=0 \tag{1.18}$$

   for any test function $\varphi\in\mathcal{D}(\overline{\Omega}\times(0,T))$, and any

$$B(\varrho)=B(1)+\int_1^\varrho\frac{b(z)}{z^2}\;dz, \tag{1.19}$$

   where $b\in C^0([0,\infty))$ is a bounded function.

3. *For every $\varphi\in\mathcal{D}(\Omega\times(0,T))$*

$$\int_{Q_T}\left(\varrho c\,\partial_t\varphi+\varrho c\mathbf{u}\cdot\nabla v\right)\,d(x,t)=\int_{Q_T}\nabla\mu\cdot\nabla\varphi\,d(x,t) \tag{1.20}$$

   and

$$\int_{Q_T}\varrho\mu\varphi\,d(x,t)=\int_{Q_T}\left(\varrho\frac{\partial f(\varrho,c)}{\partial c}\varphi+\nabla c\cdot\nabla\varphi\right)\,d(x,t). \tag{1.21}$$

4. *The* **energy inequality**

$$E(t) + \int_{Q_{(s,t)}} \left(\mathbb{S} : \nabla \mathbf{u} + |\nabla \mu|^2 \right) \, d(x,\tau) \leq E(s) \tag{1.22}$$

holds for almost every $0 \leq s \leq t \leq T$ *including* $s = 0$, *where* $E(t), E(0) = E_0$ *are determined through (1.15)-(1.16).*

5. $\varrho, \varrho \mathbf{u}, c$ *are weakly continuous with respect to* $t \in [0, T]$ *with values in* $L^1(\Omega)$ *and* $\varrho|_{t=0} = \varrho_0$, $\varrho \mathbf{u}|_{t=0} = \mathbf{m}_0$, $c|_{t=0} = c_0$.

Remark 1.3 *Note that the class of test functions in (1.18) already includes (implicitly) the satisfaction of the impermeability boundary condition* $\varrho \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$.

Let us comment on the main novelties of this study: To begin with, it is quite natural to assume that viscosity depends effectively on the order parameter c as the transport coefficients may be rather different for each component. On the other hand, this fact modifies considerably the relation satisfied by the effective viscous flux that must be handled in the spirit of [11].

Another intrinsic feature of the problem is the dependence of the pressure $p = p(\varrho, c)$ on the order parameter c, where, in addition, p need not be a monotone function of ϱ for all range of c. This difficulty is overcome by means of the technique introduced in Chapter 6 in [10].

As already pointed out above, the main obstacle in the theory of weak solutions to systems describing compressible fluids is the appearance of vacuum zones. In particular, the *extensive* quantities like ϱc, that means, those whose total amount is proportional to the distribution of mass, may exhibit large time oscillations on the vacuum as there is no control on their time derivative. The main new ingredient of the present theory is therefore *compactness* of c or even ∇c over the whole space-time cylinder regardless the (hypothetical) presence of the areas of zero density. This rather surprising property follows from a simple observation that, in accordance with (1.4), Δc is small as soon as ϱ is small, in particular, Δc_n approaches zero whenever $\varrho_n \to 0$ for any sequence of approximate solutions. On the other hand, as c_n can be shown to converge pointwise out of the vacuum, we conclude

$$\int_0^T \int_\Omega |\nabla c_n|^2 \, dx = - \int_0^T \int_\Omega c_n \Delta c_n \, dx \to \int_0^T \int_\Omega |\nabla c|^2 \, dx,$$

which is equivalent to strong convergence of ∇c_n in $L^2(Q_T)$.

The outline of the article is as follows: In Section 2 we derive the model leading to our system (1.1)-(1.4) on the basis of a local dissipation inequality, which plays the role of the second law of thermodynamics. Moreover, we discuss some preliminary consequences of the a priori estimates obtained from the local dissipation inequality. In order to construct the weak solution, we use a two-level approximation scheme. More precisely, in Section 3, we construct solutions to an approximate system to (1.1)-(1.4), where an extra term is added to the free energy in order to get a better integrability of the density. This is done by using an implicit time discretization of the approximate system. Finally, in Section 4, we consider the limit of the approximate system to show our main result.

2 Modeling, A Priori Estimates and Preliminary Results

2.1 Notation

If $\mathbb{A}, \mathbb{B} \in \mathbb{R}^{n\times n}$ are two matrices, then $\mathbb{A} : \mathbb{B} = \sum_{i,j=1}^{n} \mathbb{A}_{ij}\mathbb{B}_{ij}$ denotes their scalar product. If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then $\mathbf{a}\otimes\mathbf{b} \in \mathbb{R}^{n\times n}$ is defined by $(a\otimes b)_{ij} = \mathbf{a}_i\mathbf{b}_j$. The characteristic function of a set A is denoted by χ_A. If $\Omega \subseteq \mathbb{R}^n$ is a domain, then $C_0^\infty(\Omega;\mathbb{R}^N)$ is the set of all smooth and compactly supported functions $f\colon \Omega \to \mathbb{R}^N$ and $C_0^\infty(\Omega) = C_0^\infty(\Omega;\mathbb{R})$. Moreover, for a general set $A \subseteq \mathbb{R}^n$ we denote $C_{(0)}^\infty(A;\mathbb{R}^N) = \{f \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^N)) : \operatorname{supp} f \subseteq A\}$ and $C_{(0)}^\infty(A;\mathbb{R}) = C_{(0)}^\infty(A)$. For short we also write $\mathcal{D}(A;\mathbb{R}^N) = C_{(0)}^\infty(A;\mathbb{R}^N)$ and $\mathcal{D}(A) = C_{(0)}^\infty(A)$. The usual Lebesgue spaces are denoted by $L^q(\Omega), 1 \le q \le \infty$, $\|\cdot\|_q$, denotes its norm, and $L^q(\Omega;X)$ denotes the corresponding space of q-integrable X-valued functions. The $L^2(\Omega)$-scalar product is denoted by $(.,.)_\Omega$. Furthermore, $W^{s,q}(\Omega;\mathbb{R}^N), W^{s,q}(\Omega)$, $s \ge 0$, are the Sobolev-Slobodetskii spaces, cf. e.g. [2]. As usual $W_0^{m,q}(\Omega)$, $m \in \mathbb{N}_0$, is the closure of $C_0^\infty(\Omega)$ in $W^{m,q}(\Omega)$, $W^{-m,q}(\Omega) = (W_0^{m,q'}(\Omega))'$, $1 = \frac{1}{q} + \frac{1}{q'}$, $H^m(\Omega) = W^{m,2}(\Omega)$ and $H_0^m(\Omega) = W_0^{m,2}(\Omega)$. Finally, $C_{weak}([0,T];X)$ is the space of all weakly continuous $f\colon [0,T] \to X$ and $f_n \to f$ in $C_{weak}([0,T];X)$ if and only if $\langle f_n(t), x'\rangle_{X,X'} \to_{n\to\infty} \langle f(t), x'\rangle$ uniformly in $t \in [0,T]$ for all $x' \in X'$. Here $\langle\cdot,\cdot\rangle_{X,X'}$ denotes the duality product of X and X'.

2.2 Deduction of the Model

In the following part we sketch a brief deduction of the model leading to system (1.1)-(1.4), which is a variant of the model discussed in [20, Section 3]. Our arguments are based on a local dissipation inequality, which plays the role of the second law of thermodynamics in the present context. It is similar to some parts of the discussion in Gurtin et. al. [16], where a diffuse interface model for two incompressible, viscous fluids of the same density is obtained.

We consider two fluids filling a domain $\Omega \subseteq \mathbb{R}^3$. The mass concentration of the fluid $j = 1, 2$ is denoted by $c_j = \frac{M_j}{M}$. Moreover, $\varrho_j = \frac{M_j}{V}$ denotes the apparent mass density of the fluid j and $\varrho = \varrho_1 + \varrho_2$ the total density. Moreover, $\mathbf{u}_j$ denotes the velocity of the fluid $j = 1, 2$ and $\mathbf{u}$ is defined as the *average velocity* given by $\varrho\mathbf{u} = \varrho_1\mathbf{u}_1 + \varrho_2\mathbf{u}_2$, cf. [20]. In addition to the principle of mass conservation, we assume conservation of linear and angular momentum with respect to the mean velocity, i.e., we suppose that

$$\varrho\dot{\mathbf{u}} \equiv \varrho\partial_t\mathbf{u} + \varrho\mathbf{u}\cdot\nabla\mathbf{u} = \operatorname{div}\mathbb{T} \tag{2.1}$$

$$\varrho_t + \operatorname{div}(\varrho\mathbf{u}) = 0 \tag{2.2}$$

for a symmetric stress tensor $\mathbb{T} = \mathbb{T}(\varrho, c, \nabla c, \mathbb{D}(\mathbf{u}))$, where $\mathbb{D}(u) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$, describing the material behavior of the mixture to be specified through a set of constitutive assumptions below. Exterior forces are for simplicity chosen to be zero. Here and always in what follows, $\dot{f} = \partial_t f + \mathbf{u}\nabla f$ denotes the material time derivative of a quantity f.

Furthermore, we denote by $\mathbf{J}_j$ the mass flux of the fluid j relative to the mean velocity $\mathbf{u}$, i.e.,

$$\partial_t\varrho_j + \operatorname{div}(\varrho_j\mathbf{u}) = \operatorname{div}\mathbf{J}_j.$$

In order to obtain conservation of mass (2.2), we assume that $\mathbf{J}_1 + \mathbf{J}_2 = 0$. Let $c = c_1 - c_2 = 2c_1 - 1$ be the concentration difference. Then

$$\varrho\partial_t c + \varrho\mathbf{u}\cdot\nabla c = \operatorname{div}\mathbf{J} \tag{2.3}$$

where $\mathbf{J} = 2\mathbf{J}_1$ since $\varrho_j = \varrho c_j$.

Diffusion, free energy: Now the relative motion of the fluids is assumed to be driven by diffusion. To this end we introduce the Helmholtz free energy of a given volume V in the form

$$\int_V F(\varrho, c(x), \nabla c(x))\,dx. \tag{2.4}$$

Then the chemical potential is defined as

$$\mu = \varrho^{-1}\frac{\delta F}{\delta c} = \varrho^{-1}\left(\frac{\partial F}{\partial c} - \operatorname{div}\frac{\partial F}{\partial\nabla c}\right). \tag{2.5}$$

In the following we will use F in the form $F(\varrho, c, \nabla c) = \varrho f(\varrho, c, \nabla c)$ that coincides with that of [20].

As in [20], we assume that the mass flux $\mathbf{J}$ is given by the generalized Fick's law

$$\mathbf{J} = m\nabla c,$$

where the mobility $m > 0$ is assumed to be constant. Hence we end up with a Cahn-Hilliard type diffusion equation for c:

$$\varrho\dot{c} = m\Delta\mu \tag{2.6}$$

$$\varrho\mu = \frac{\partial F}{\partial c} - \operatorname{div}\frac{\partial F}{\partial \nabla c}. \tag{2.7}$$

Second law of thermodynamics/local dissipation inequality:
Let $V(t)$ be an arbitrary volume that is transported with the flow. Then the total energy in $V(t)$ is given by

$$E(t) = \int_{V(t)} \varrho\frac{|\mathbf{u}|^2}{2}\,dx + \int_{V(t)} F(\varrho, c, \nabla c)\,dx = \int_{V(t)} e(\mathbf{u}, c, \nabla c)\,dx,$$

where $e(\mathbf{u}, c, \nabla c) = \varrho(c)\frac{|\mathbf{u}|^2}{2} + F(\varrho, c, \nabla c)$. Similarly to [16], we assume the dissipation inequality

$$\frac{d}{dt}\int_{V(t)} e(\varrho, \mathbf{u}, c, \nabla c)\,dx \leq \int_{\partial V(t)} \mathbb{T}n\cdot\mathbf{u}\,d\sigma + \int_{\partial V(t)} \dot{c}\,\mathbf{t}\cdot n\,d\sigma + \int_{\partial V(t)} \mu\,\mathbf{J}\cdot n\,d\sigma \tag{2.8}$$

for every control volume $V(t)$ transported with flow, where σ denotes the two-dimensional surface measure. Here the energy carried into $V(t)$ due to the working of the (macroscopic) stresses is given by $\int_{\partial V(t)} \mathbb{T}n\cdot\mathbf{u}\,d\sigma$. The energy carried into $V(t)$ due to diffusion is $\int_{\partial V(t)} \mu\,\mathbf{J}\cdot n\,d\sigma$. Finally,

$$\int_{\partial V(t)} \dot{c}\,\mathbf{t}\cdot n\,d\sigma$$

represents a generalized surface force to be specified later. We note that in [16] $\mathbf{t} = \boldsymbol{\xi}$ is called *microscopic stress* and is related to forces on a microscopic length scale (*micro-force*) π by the micro-force balance

$$\int_V \pi\,dx + \int_{\partial V} \boldsymbol{\xi}\cdot n\,d\sigma = 0$$

for each control volume V. In [9] the quantity $\dot{c}\,\mathbf{t}$ is called *interstitial work flux*,

The equivalent local form of (2.8) is

$$\partial_t e + \operatorname{div}(\mathbf{u}e) - \operatorname{div}(\mathbb{T}\cdot\mathbf{u}) - \operatorname{div}(\dot{c}\mathbf{t}) - \operatorname{div}(\mu\mathbf{J}) =: D \leq 0. \tag{2.9}$$

Using that

$$\partial_t\left(\varrho\frac{|\mathbf{u}|^2}{2}\right) + \operatorname{div}\left(\mathbf{u}\varrho\frac{|\mathbf{u}|^2}{2}\right) = \operatorname{div}(\mathbb{T}\cdot\mathbf{u}) - \mathbb{T}:\nabla\mathbf{u}$$

due to (2.1)-(2.2),

$$(\varrho f)_t + \operatorname{div}(\varrho\mathbf{u}f) = \varrho f_t + \varrho\mathbf{u}\cdot\nabla f = \varrho\dot{f},$$

and

$$\begin{aligned}\operatorname{div}(\dot{c}\mathbf{t}) + \operatorname{div}(\mu\mathbf{J}) &= \dot{c}\,\operatorname{div}\mathbf{t} + \nabla(\dot{c})\cdot\mathbf{t} + \mu\varrho\dot{c} + m|\nabla\mu|^2\\ &= (\varrho\mu + \operatorname{div}\mathbf{t})\dot{c} + (\nabla c)^{\cdot}\cdot\mathbf{t} - \nabla\mathbf{u}:(\mathbf{t}\otimes\nabla c) + m|\nabla\mu|^2;\end{aligned}$$

whence we conclude

$$\begin{aligned}D = {}& \varrho\frac{\partial f}{\partial\varrho}\dot{\varrho} + \left(\frac{\partial F}{\partial c} - \operatorname{div}\mathbf{t} - \varrho\mu\right)\dot{c}\\ &+ \left(\frac{\partial F}{\partial\nabla c} - \mathbf{t}\right)(\nabla c)^{\cdot} - (\mathbb{T} - \mathbf{t}\otimes\nabla c):\nabla\mathbf{u} - m|\nabla\mu|^2,\end{aligned}$$

where f is defined by $F(\varrho, c, \nabla c) = \varrho f(\varrho, c, \nabla c)$. Finally, using $\dot{\varrho} = -\varrho\operatorname{div}\mathbf{u}$ and the definition of μ, we obtain

$$\begin{aligned}D = {}& \left(\operatorname{div}\frac{\partial F}{\partial\nabla c} - \operatorname{div}\mathbf{t}\right)\dot{c} + \left(\frac{\partial F}{\partial\nabla c} - \mathbf{t}\right)(\nabla c)^{\cdot}\\ &- \left(\mathbb{T} + \varrho^2\frac{\partial(\varrho^{-1}F)}{\partial\varrho}\mathbb{I} - \mathbf{t}\otimes\nabla c\right):\nabla\mathbf{u} - m|\nabla\mu|^2.\end{aligned}$$

Hence making the constitutive assumptions

$$\begin{aligned}\mathbf{t} &= \frac{\partial F}{\partial\nabla c},\\ \mathbb{S} &:= \mathbb{T} + P(\varrho, c, \nabla c)\mathbb{I} - \mathbf{t}\otimes\nabla c = 2\nu(c)\mathbb{D}(\mathbf{u}) + \eta(c)\operatorname{div}\mathbf{u}\,\mathbb{I}\end{aligned} \tag{2.10}$$

with $P(\varrho, c, \nabla c) = \varrho^2\frac{\partial(\varrho^{-1}F)}{\partial\varrho}(\varrho, c, \nabla c)$ and $\nu(c), \eta(c) \geq 0$, we have

$$D = -2\nu(c)|\mathbb{D}(\mathbf{u})|^2 - \eta(c)|\operatorname{div}\mathbf{u}|^2 - m|\nabla\mu|^2 \leq 0 \tag{2.11}$$

and the local dissipation inequality (2.8), (2.9), respectively, is satisfied. Hence the stress tensor $\mathbb{T}$ differs from the stress tensor for a single compressible Newtonian fluid by the extra stress $\frac{\partial F}{\partial \nabla c} \otimes \nabla c$, which is often called Ericksen's stress. Here $\mathbb{S}$ is the viscous stress corresponding to inner forces leading to an irreversible loss of energy through dissipation. The part $p(\varrho, c, \nabla c)\mathbb{I} - \mathbf{t} \otimes \nabla c$ of the stress tensor corresponds to inner forces due to (reversible) changes of the energy $E_{\text{free}}(\varrho, c)$. Hence (2.10) reflects Newton's rheological law.

Finally, if we specify F to be of the form

$$F(\varrho, c, \nabla c) = \varrho f(\varrho, c) + \frac{1}{2}|\nabla c|^2,$$

we have

$$P(\varrho) = \varrho^2 \frac{\partial f}{\partial \varrho}(\varrho, c) - \frac{|\nabla c|^2}{2}. \tag{2.12}$$

Hence we obtain (1.1)-(1.4) because of (2.1),(2.2), (2.6), (2.7), and (2.10), where we have put for simplicity $m = 1$. On the point of conclusion, note that the latter equation and (2.7) are consistent with the Gibb's equation

$$DF + PD\Big(\frac{1}{\varrho}\Big) = \mu Dc. \tag{2.13}$$

2.3 Total Mass Conservation

By integration of (1.2) over Ω or choosing $\varphi = \psi(t)$ with $\psi \in \mathcal{D}(0, T)$, we get

$$\int_\Omega \varrho(t)\, dx = \int_\Omega \varrho_0\, dx \equiv M_0 \quad \text{for almost all } t \in (0, T). \tag{2.14}$$

In the same way one gets from (1.3)

$$\int_\Omega \varrho(t)c(t)\, dx = \int_\Omega \varrho_0 c_0\, dx \qquad \text{for almost all } t \in (0, T). \tag{2.15}$$

2.4 Total Energy Balance

We note that integrating (2.9) with respect to Ω, using (1.7) and (2.11) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \int_\Omega \Big(2\mu(c)|\mathbb{D}(\mathbf{u})|^2 + \nu(c)|\operatorname{div}\mathbf{u}|^2 + |\nabla\mu|^2\Big)(t)\, dx = 0$$

for sufficiently smooth solutions, where $E(t)$ is as in (1.15). As usual, this equality will turn into an inequality for the weak solutions constructed, which is nothing other than (1.22).

The total energy inequality (1.22) together with (1.10) give rise to the uniform estimates

$$\begin{aligned} \text{ess sup}_{t\in(0,T)}\|\varrho(t)\|_{L^\gamma(\Omega)} &\leq C(M_0,E_0), &(2.16)\\ \text{ess sup}_{t\in(0,T)}\|\nabla c(t)\|_{L^2\left(\Omega;\mathbb{R}^3\right)} &\leq C(M_0,E_0), &(2.17)\\ \text{ess sup}_{t\in(0,T)}\|\sqrt{\varrho}\mathbf{u}(t)\|_{L^2(\Omega;\mathbb{R}^3)} &\leq C(M_0,E_0), &(2.18)\\ \int_0^T \|\nabla\mu\|^2_{L^2\left(\Omega;\mathbb{R}^3\right)}\ dt &\leq C(M_0,E_0), &(2.19) \end{aligned}$$

where E_0 denotes the initial energy defined in (1.16) and M_0 is the total mass as in (2.14).

Moreover, by means of Korn's inequality and hypothesis (1.10),

$$\int_0^T \|\nabla\mathbf{u}\|^2_{L^2\left(\Omega;\mathbb{R}^{3\times 3}\right)}\ dt \leq C(M_0,E_0), \tag{2.20}$$

2.5 Cahn-Hilliard Type Equation

A weak formulation of (1.3)-(1.4), taking the boundary conditions for (c,μ) in (1.7) into account, reads

$$\int_{Q_T} (\varrho c\,\partial_t\varphi + \varrho c\mathbf{u}\cdot\nabla v)\ d(x,t) = \int_{Q_T} \nabla\mu\cdot\nabla\varphi\, d(x,t), \tag{2.21}$$

$$\int_{Q_T} \varrho\mu\varphi\, d(x,t) = \int_{Q_T}\left(\varrho\frac{\partial f(\varrho,c)}{\partial c}\varphi + \nabla c\cdot\nabla\varphi\right) d(x,t) \tag{2.22}$$

for any test function $\varphi\in\mathcal{D}((0,T)\times\overline{\Omega})$.

In order to estimate $\|c(t)\|_{L^2}$ and $\|\mathbf{u}(t)\|_{L^2}$, we use the following simple variant of Poincare's inequality (cf. Lemma 3.1 in [12]):

Lemma 2.1 *Let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain. Assume that ϱ is a non-negative function such that*

$$0 < M = \int_\Omega \varrho\, dx,\ \int_\Omega \varrho^\gamma\, dx \leq K,\ \text{with } \gamma > \frac{6}{5}.$$

Then there exists a constant $C = C(\gamma, M, K)$ such that

$$\left\| w - \frac{1}{|\Omega|}\int_\Omega \varrho w\, dx\right\|_{L^1(\Omega)} \leq C(\gamma,M,K)\|\nabla w\|_{L^2(\Omega;\mathbb{R}^3)}$$

for any $w\in W^{1,2}(\Omega)$.

As a straightforward consequence of Lemma 2.1, together with estimates (2.16), (2.18), (2.20), we obtain

$$\int_0^T \|\mathbf{u}\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)} \ dt \leq C(M_0, E_0). \tag{2.23}$$

Similarly, by virtue of (2.15), (2.17),

$$\mathrm{ess} \sup_{t \in (0,T)} \|c(t)\|_{W^{1,2}(\Omega)} \leq C(c_0, E_0, M_0). \tag{2.24}$$

Finally, choosing $\varphi = \varphi(t)$ independent of x in (2.22), we get

$$\int_\Omega \varrho(t)\mu(t)\, dx = \int_\Omega \varrho(t) \frac{\partial f(\varrho(t), c(t))}{\partial c}\, dx \qquad \text{for a.a. } t \in (0,T),$$

where the integral on the right-hand side is essentially bounded in $t \in (0,T)$ because of

$$\left| \frac{\partial f(\varrho, c)}{\partial c} \right| \leq C(1 + |c|) \text{ for all } c.$$

due to (1.11), (1.14), and (2.24). Consequently, in accordance with (2.19), we conclude that

$$\int_0^T \|\mu\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)} \ dt \leq C(c_0, M_0, E_0). \tag{2.25}$$

2.6 Strong Compactness of the Concentration Gradients

This is one of the main ingredients of the proof. Assume that $\varrho_n \geq 0$,

$$\varrho_n \to \varrho \text{ in } C_{\mathrm{weak}}([0,T]; L^\gamma(\Omega)), \tag{2.26}$$

$$c_n \to c \text{ weakly in } L^\infty(0,T; W^{1,2}(\Omega)), \tag{2.27}$$

$$\partial_t(\varrho_n c_n) \text{ is bounded in } L^q(0,T; W^{-1,q}(\Omega)) \text{ for a certain } q > 1, \tag{2.28}$$

and, in addition,

$$\int_0^T \int_\Omega \nabla c_n \cdot \nabla \varphi \, dx \ dt = \int_0^T \int_\Omega \sqrt{\varrho_n} f_n \varphi \, dx \ dt + \int_0^T \int_\Omega g_n \varphi \, dx \ dt, \tag{2.29}$$

for any $\varphi \in \mathcal{D}((0,T) \times \overline{\Omega})$, where

$$\left\{ \begin{array}{c} f_n \to f \text{ weakly in } L^2((0,T) \times \Omega), \\ \\ g_n \to g \text{ (strongly) in } L^1(0,T; L^{\frac{6}{5}}(\Omega)). \end{array} \right\} \tag{2.30}$$

Our aim is to show that

$$\int_0^T \int_\Omega |\nabla c_n|^2 \, dx \ dt \to \int_0^T \int_\Omega |\nabla c|^2 \, dx \ dt \tag{2.31}$$

yielding, together with (2.27),

$$c_n \to c \text{ in } L^2(0, T; W^{1,2}(\Omega)).$$

To this end, we observe first that $\varrho \geq 0$, and

$$c_n \to c \text{ a.a. on the set } \{\varrho > 0\} \tag{2.32}$$

passing to a suitable subsequence as the case may be. Indeed it follows from (2.26), (2.27) that

$$\varrho_n c_n \to \varrho c \text{ weakly-(*) in } L^\infty(0, T; L^q(\Omega) \text{ for a certain } q > \frac{6}{5},$$

which, together with (2.28), gives rise to

$$\varrho_n c_n^2 \to \varrho c^2 \text{ weakly-(*) in } L^\infty(0, T; L^r(\Omega)) \text{ for a certain } r > 1.$$

Since, by the same token,

$$(\varrho_n - \varrho) c_n^2 \to 0 \text{ weakly-(*) in } L^\infty(0, T; L^r(\Omega)) \text{ for a certain } r > 1,$$

we get

$$\int_0^T \int_\Omega \varrho c_n^2 \, dx \ dt \to \int_0^T \int_\Omega \varrho c^2 \, dx \ dt,$$

in particular, (2.32) follows.

On the other hand, letting $n \to \infty$ in (2.29) yields

$$\int_0^T \int_\Omega \nabla c \cdot \nabla \varphi \, dx \ dt = \int_0^T \int_\Omega \overline{\sqrt{\varrho} f} \varphi \, dx \ dt + \int_0^T \int_\Omega g \varphi \, dx \ dt,$$

for any $\varphi \in \mathcal{D}((0, T) \times \overline{\Omega})$, where

$$\sqrt{\varrho_n} f_n \to \overline{\sqrt{\varrho} f} \text{ weakly in } L^2(0, T; L^q(\Omega)) \text{ for a certain } q > \frac{6}{5}.$$

In particular, by means of a standard density argument,

$$\int_0^T \int_\Omega |\nabla c|^2 \, dx \ dt = \int_0^T \int_\Omega \overline{\sqrt{\varrho} f} c \, dx \ dt + \int_0^T \int_\Omega g c \, dx \ dt. \tag{2.33}$$

Finally, taking $\varphi = c_n$ and letting $n \to \infty$ in (2.29), we obtain

$$\lim_{n\to\infty} \int_\Omega |\nabla c_n|^2 \, dx \ \ dt = \int_0^T \int_\Omega \overline{\sqrt{\varrho} f c} \, dx \ \ dt + \int_0^T \int_\Omega gc \, dx \ \ dt,$$

which, combined with (2.33), gives rise to the desired conclusion (2.31) as soon as we observe that

$$\overline{\sqrt{\varrho} f c} = \overline{\sqrt{\varrho} f} c, \tag{2.34}$$

where, according to the standard notation convention adopted in this paper, the bar stands for a weak limit in L^1.

In accordance with (2.32), relation (2.34) is satisfied on the set $\{\varrho > 0\}$ where $c_n \to c$ strongly in $L^1(\Omega)$.

On the other hand, since ϱ_n are non-negative,

$$\varrho_n \to 0 \text{ (strongly) in } L^q(\{\varrho = 0\}) \text{ for any } 1 \le q < \gamma;$$

whence (2.34) holds on the set $\{\varrho = 0\}$ as well. The proof of (2.31) is now complete.

3 Existence for the System with Artificial Pressure

In order to construct weak solutions, we use a two-level approximation. At the first approximation level we add an artificial pressure term that ensures better integrability of ϱ. This technique is well-known and can be found e.g. in [11, 18, 21]. More precisely, we start with the approximate system

$$\begin{aligned} \int_{Q_T} & \left(\varrho_\delta \mathbf{u}_\delta \cdot \partial_t \boldsymbol{\varphi} + \varrho_\delta (\mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \nabla \boldsymbol{\varphi} + \left(p(\varrho_\delta, c_\delta) + \delta \varrho_\delta^\Gamma \right) \operatorname{div} \boldsymbol{\varphi} \right) \, d(x,t) \\ = & \int_0^T \int_\Omega \Big(\mathbb{S}_\delta - \mathbb{P}_\delta \Big) : \nabla \boldsymbol{\varphi} \, dx \ \ dt - \int_\Omega \varrho_{0,\delta} \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0) \, dx \end{aligned} \tag{3.1}$$

for any $\boldsymbol{\varphi} \in \mathcal{D}([0,T) \times \Omega; \mathbb{R}^3)$,

$$\int_{Q_T} \Big(\varrho_\delta \partial_t \varphi + \varrho_\delta \mathbf{u}_\delta \cdot \nabla \varphi \Big) \, d(x,t) = - \int_\Omega \varrho_{0,\delta} \varphi|_{t=0} \, dx \tag{3.2}$$

$$\int_{Q_T} \left(\varrho_\delta c_\delta \partial_t \varphi + (\varrho_\delta c_\delta \mathbf{u}_\delta - \nabla \mu) \cdot \nabla \varphi \right) \, d(x,t) = - \int_\Omega \varrho_{0,\delta} c_0 \varphi|_{t=0} \, dx \tag{3.3}$$

$$\int_{Q_T} \varrho\mu\varphi \, d(x,t) = \int_{Q_T} \left(\varrho \frac{\partial f(\varrho, c)}{\partial c} \varphi + \nabla c \cdot \nabla \varphi \right) d(x,t). \tag{3.4}$$

for any $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega})$, where

$$\mathbb{S}_\delta = 2\nu(c_\delta)\mathbb{D}(\mathbf{u}_\delta) + \eta(c_\delta) \operatorname{div} \mathbf{u}_\delta \ \mathbb{I}, \tag{3.5}$$

$$\mathbb{P}_\delta = \nabla c_\delta \otimes \nabla c_\delta - \frac{|\nabla c_\delta|^2}{2} \mathbb{I}. \tag{3.6}$$

Here $\varrho_{0,\delta} \in L^\Gamma(\Omega)$ such that $\varrho_{0,\delta} \geq 0$, and $\varrho_{0,\delta} \to_{\delta\to 0} \varrho_0$ in $L^\gamma(\Omega)$, $\varrho_{0,\delta}|\mathbf{u}_0|^2 \to_{\delta\to 0} \varrho_0|\mathbf{u}_0|^2$ in $L^1(\Omega)$, $\mathbf{u}_0$ satisfies $\varrho_0\mathbf{u}_0 = \mathbf{m}_0$, and $(\mathbf{m}_0, \varrho_0, c_0)$ are as in Theorem 1.2. In addition, the approximate solutions $(\varrho_\delta, \mathbf{u}_\delta, c_\delta)_{\delta>0}$ will satisfy the energy inequality

$$E_\delta(t) + \int_{Q_{(s,t)}} \left(\mathbb{S}_\delta : \nabla \mathbf{u}_\delta + |\nabla \mu_\delta|^2 \right) d(x,\tau) \leq E_\delta(s) \tag{3.7}$$

for almost all $0 \leq s \leq t \leq T$ including $s = 0$, where

$$E_\delta(t) = \int_\Omega \left(\frac{1}{2}\varrho_\delta|\mathbf{u}_\delta|^2 + \varrho_\delta f(\varrho_\delta, c_\delta) + \frac{\delta}{\Gamma - 1}\varrho_\delta^\Gamma + \frac{|\nabla c_\delta|^2}{2} \right)(t) \, dx. \tag{3.8}$$

The main purpose of this section is to prove:

THEOREM 3.1 *Let $\Gamma > 3$, $\delta > 0$, and let $0 < T < \infty$. Then for every non-negative $\varrho_{0,\delta} \in L^\Gamma(\Omega)$, measurable $\mathbf{u}_0 \colon \Omega \to \mathbb{R}^3$ with $\varrho_{0,\delta}|\mathbf{u}_0|^2 \in L^1(\Omega)$, and $c_0 \in H^1(\Omega)$ there are some $\varrho_\delta \in L^\infty(0,T; L^\Gamma(\Omega)) \cap L^{\Gamma+1}(Q_T)$, $\varrho_\delta \geq 0$, $\mathbf{u}_\delta \in L^2(0,T; H^1(\Omega;\mathbb{R}^3))$, $c_\delta \in L^\infty(0,T; H^1(\Omega))$ solving (3.1)-(3.6) and satisfying (3.7).*

In order to prove this theorem, we approximate (3.1)-(3.4) via a suitable time discretization. For simplicity we will drop the subscript δ in most quantities for the rest of this section.

3.1 Implicit Time Discretization

Let $h > 0$. Given $(\mathbf{u}_k, \varrho_k, c_k) \in L^2(\Omega)^3 \times L^\Gamma(\Omega) \times H^1(\Omega)$ we determine $(\mathbf{u}_{k+1}, \varrho_{k+1}, c_{k+1}, \mu_{k+1})$ as a solution of the system

$$\begin{aligned} &\frac{\varrho\mathbf{u} - \varrho_k\mathbf{u}_k}{h} + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbb{S} \\ &\qquad + \varepsilon(\nabla\varrho\cdot\nabla)\mathbf{u} + \nabla p_\delta = \varrho\mu\nabla c - \varrho\frac{\partial f}{\partial c}(\varrho, c)\nabla c \end{aligned} \tag{3.9}$$

$$\frac{\varrho - \varrho_k}{h} + \operatorname{div}(\varrho\mathbf{u}) = \varepsilon\Delta\varrho \tag{3.10}$$

$$\varrho_k\frac{c - c_k}{h} + \varrho\mathbf{u}\cdot\nabla c = \Delta\mu \tag{3.11}$$

$$\varrho_k\mu = \varrho_k\frac{f(\varrho_k, c) - f(\varrho_k, c_k)}{c - c_k} - \Delta c \tag{3.12}$$

where $p_\delta = p(\varrho, c) + \delta\varrho^\Gamma$, $p(\varrho, c) = \varrho^2\frac{\partial f_e}{\partial\varrho} + \varrho H(c)$, and

$$\mathbb{S} \;=\; S(c_k, \nabla\mathbf{u}) \equiv 2\nu(c_k)\mathbb{D}(\mathbf{u}) + \eta(c_k)\operatorname{div} u\,\mathbb{I} \tag{3.13}$$

together with the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = \partial_n\varrho|_{\partial\Omega} = \partial_n c|_{\partial\Omega} = \partial_n\mu|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega. \tag{3.14}$$

Here (3.9)-(3.12) will be understood in the sense of distributions and (3.14) in the sense of traces of Sobolev functions. Note that (3.10)-(3.11) implies that

$$\int_\Omega \varrho\,dx = \int_\Omega \varrho_k\,dx, \qquad \int_\Omega \varrho c\,dx = \int_\Omega \varrho_k c_k\,dx,$$

which is the time discrete version of (2.14)-(2.15).

Remark 3.2 *The right-hand side of (3.9) is motivated by the identity*

$$-\operatorname{div}\left(\nabla c\otimes\nabla c - \frac{|\nabla c|^2}{2}\mathbb{I}\right) = \varrho\mu\nabla c - \varrho\frac{\partial f}{\partial c}(\varrho, c)\nabla c$$

provided that (1.4) holds. For solutions of the discrete system above the latter identity does not hold; but the form of (3.9)-(3.12) ensures that a similar energy estimate holds.

Because of the assumption on $p_e(\varrho) = \varrho^2\frac{\partial f_e}{\partial\varrho}$, we can decompose $p_e(\varrho) = \tilde{p}_m(\varrho) + p_b(\varrho)$, where $p_b \in C^2([0,\infty))$, $p_b \leq 0$, has compact support, $\tilde{p}_m(0) = 0$ and

$$\underline{\tilde{p}_m}(1 + \varrho^{\Gamma-1}) \leq \tilde{p}_m'(\varrho) \leq \overline{\tilde{p}_m}(1 + \varrho^{\Gamma-1}) \tag{3.15}$$

for some constants $\underline{p_m}, \overline{p_m} > 0$. Moreover, we assume without loss of generality that $H(c) \geq 0$. This can always be achieved by adding $\underline{H}\varrho$ to $f_e(\varrho, c)$, where $\underline{H} = \inf_{c\in\mathbb{R}} H(c)$, and replacing $H(c)$ by $H(c) - \underline{H}$.

As a consequence of these assumptions we have that

$$p_\delta(\varrho, c) = p_m(\varrho, c) + p_b(\varrho), \tag{3.16}$$

where $p_m(\varrho, c) = \tilde{p}_m(\varrho) + \varrho H(c) \geq 0$ is again monotone with respect to ϱ. This decomposition of p induces a decomposition

$$f(\varrho, c) = f_m(\varrho, c) + f_b(\varrho),$$

where $f_m(\varrho, c) = \int_0^\varrho \frac{p_m(s,c)}{s^2}\, ds + G(c)$ and $\varrho \mapsto \varrho f_m(\varrho, c)$ is convex and monotone. Moreover, we define

$$E_m(\varrho, \mathbf{u}, c) = \int_\Omega \left(\frac{\varrho |\mathbf{u}|^2}{2} + \varrho f_m(\varrho, c) + \frac{|\nabla c|^2}{2} \right) dx.$$

Lemma 3.3 *Let $(\mathbf{u}_k, \varrho_k, c_k) \in L^2(\Omega; \mathbb{R}^3) \times L^\Gamma(\Omega) \times H^1(\Omega)$, $\varrho \geq 0$, and let $0 < \varepsilon \leq 1$. Then every $(\mathbf{u}, \varrho, c, \mu) \in H^1(\Omega; \mathbb{R}^3) \times H^2(\Omega)^3$, $\varrho \geq 0$, solving (3.9)-(3.14) satisfies the discrete energy estimate*

$$\begin{aligned} E_m(\varrho, \mathbf{u}, c) + \varepsilon h \int_\Omega \frac{\partial_\varrho p_m}{\varrho} |\nabla \varrho|^2 \, dx + \int_\Omega \frac{\varrho_k |\mathbf{u} - \mathbf{u}_k|^2}{2} \, dx + \frac{\|\nabla (c - c_k)\|_2^2}{2} \\ + \alpha \|\varrho - \varrho_k\|_2^2 + h \int_\Omega \mathbb{S} : \nabla \mathbf{u} \, dx + h \|\nabla \mu\|_2^2 \leq E_m(\varrho_k, \mathbf{u}_k, c_k) + R_k \end{aligned} \tag{3.17}$$

for some $\alpha > 0$ depending only on f_m. Here $p_m = p_m(\varrho, c)$ and

$$R_k = h \int_\Omega p_b(\varrho) \operatorname{div} \mathbf{u} \, dx - \varepsilon h \int_\Omega \nabla \varrho \cdot \nabla c \frac{\partial^2 (\varrho f_m(\varrho, c))}{\partial \varrho \partial c} \, dx.$$

Moreover, there is some $h_0 > 0$ independent of $(\mathbf{u}_k, \varrho_k, c_k)$ and $\varepsilon > 0$ such that any solution $(\mathbf{u}, \varrho, c, \mu) \in H^1(\Omega; \mathbb{R}^3) \times H^2(\Omega)^3$ with $\varrho \geq 0$ satisfies

$$\begin{aligned} &\|(\varrho^{\frac{1}{2}} \mathbf{u}, c, \nabla c)\|_2^2 + \|\varrho\|_\Gamma^\Gamma \\ &\quad + h \|(\mathbf{u}, \nabla \mathbf{u}, \mu, \nabla \mu, \varepsilon^{\frac{1}{2}} \nabla \varrho)\|_2^2 \leq C \left(E_m(\varrho_k, \mathbf{u}_k, c_k) + 1 \right) \end{aligned} \tag{3.18}$$

where C is independent of h with $0 < h \leq h_0$, $0 < \varepsilon \leq 1$ and $\varrho_k, \mathbf{u}_k, c_k$, but depends on $\int_\Omega \varrho_k \, dx$ and $\int_\Omega \varrho_k c_k \, dx$. Finally, for all $0 < h \leq h_0$ there is some $(\mathbf{u}, \varrho, c, \mu) \in H^1(\Omega; \mathbb{R}^3) \times H^2(\Omega)^3$ with $\varrho \geq 0$ solving (3.9)-(3.14).

Proof: We first show the energy estimate (3.17). First of all, because of (3.10) multiplied on $\frac{1}{2}|\mathbf{u}|^2$ and integrated by parts, we have

$$\begin{aligned}
&\int_\Omega \frac{\varrho|\mathbf{u}|^2 - \varrho_k \mathbf{u}_k \cdot \mathbf{u}}{h}\, dx + \int_\Omega \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})\cdot\mathbf{u}\, dx + \varepsilon\int_\Omega (\nabla\varrho\cdot\nabla)\mathbf{u}\cdot\mathbf{u}\, dx \\
&= \int_\Omega \varrho\frac{|\mathbf{u}|^2}{2h}\, dx - \int_\Omega \varrho_k \frac{|\mathbf{u}_k|^2}{2h}\, dx + \int_\Omega \varrho_k\frac{|\mathbf{u}-\mathbf{u}_k|^2}{2h}\, dx.
\end{aligned}$$

Hence, testing (3.9) with $\mathbf{u}$ we obtain

$$\begin{aligned}
&\int_\Omega \varrho\frac{|\mathbf{u}|^2}{2h}\, dx - \int_\Omega \varrho_k\frac{|\mathbf{u}_k|^2}{2h}\, dx + \int_\Omega \varrho_k\frac{|\mathbf{u}-\mathbf{u}_k|^2}{2h}\, dx + \int_\Omega \mathbb{S}:\nabla\mathbf{u}\, dx \\
&= \int_\Omega p_\delta \operatorname{div}\mathbf{u}\, dx + \int_\Omega \varrho\mu\nabla c\cdot\mathbf{u}\, dx - \int_\Omega \varrho\frac{\partial f}{\partial c}(\varrho,c)\nabla c\cdot\mathbf{u}\, dx. \quad (3.19)
\end{aligned}$$

Moreover, multiplying (3.10) with $\partial_\varrho F(\varrho)$, where $F(\varrho,c) = \varrho f_m(\varrho,c)$, we obtain

$$\begin{aligned}
&\frac{\varrho-\varrho_k}{h}\partial_\varrho F(\varrho,c) + \operatorname{div}(F(\varrho,c)\mathbf{u}) \\
&\quad + p_m(\varrho)\operatorname{div}\mathbf{u} = \varepsilon\Delta\varrho\partial_\varrho F(\varrho,c) + \varrho\partial_c f(\varrho,c)\nabla c\cdot\mathbf{u}
\end{aligned}$$

since $\varrho\partial_\varrho F(\varrho,c) - F(\varrho,c) = \varrho^2\partial_\varrho f_m(\varrho,c) = p_m(\varrho,c)$. Furthermore, since $\frac{\partial^2 F}{\partial\varrho^2}(\varrho,c) = \varrho^{-1}\partial_\varrho p_m(\varrho,c) \geq \frac{\alpha}{2} > 0$ for some $\alpha > 0$ due to (3.15), we have

$$\frac{\partial F}{\partial c}(\varrho,c)(\varrho-\varrho_k) \;\geq\; \varrho f_m(\varrho,c) - \varrho_k f_m(\varrho_k,c) + \alpha(\varrho-\varrho_k)^2.$$

Therefore

$$\begin{aligned}
&\frac{1}{h}\int_\Omega (\varrho f_m - \varrho_k f_m(\varrho_k,c))\, dx + \alpha\|\varrho-\varrho_k\|_2^2 \leq -\int_\Omega p_m \operatorname{div}\mathbf{u}\, dx \quad (3.20)\\
&\quad -\varepsilon\int_\Omega \frac{\partial_\varrho p_m}{\varrho}|\nabla\varrho|^2\, dx - \varepsilon\int_\Omega \nabla\varrho\cdot\nabla c\frac{\partial^2(\varrho f)}{\partial\varrho\partial c}\, dx + \int_\Omega \varrho\frac{\partial f}{\partial c}\nabla c\cdot\mathbf{u}\, dx,
\end{aligned}$$

where f_m, p_m and their derivatives depend on ϱ, c if not stated differently.

Moreover, multiplying (3.11) with μ and (3.12) with $\frac{c-c_k}{h}$, we obtain

$$\begin{aligned}
&\frac{1}{h}\int_\Omega \varrho_k(f_m(\varrho_k,c) - f_m(\varrho_k,c_k))\, dx + \frac{\|\nabla c\|_2^2}{2h} \\
&\quad + \frac{\|\nabla(c-c_k)\|_2^2}{2h} + \int_\Omega |\nabla\mu|^2\, dx \leq \frac{\|\nabla c_k\|_2^2}{2h} - \int_\Omega \varrho\mu\nabla c\cdot\mathbf{u}\, dx \quad (3.21)
\end{aligned}$$

since $(a-b)\cdot a = \frac{|a|^2}{2} + \frac{|a-b|^2}{2} - \frac{|b|^2}{2}$ for all $a,b\in\mathbb{R}^3$ and $f(\varrho_k,c_k) - f(\varrho_k,c) = f_m(\varrho_k,c_k) - f_m(\varrho_k,c)$. Combining (3.19)-(3.21) we obtain (3.17).

In order to estimate $R_k(\varrho, \mathbf{u}, c)$ we note that

$$\left|\varepsilon \int_\Omega \nabla\varrho \cdot \nabla c \frac{\partial^2(\varrho f_m)}{\partial\varrho\partial c}(\varrho, c)\, dx\right| \le C\varepsilon \|(\log\varrho + 1)\nabla\varrho\|_2 \|\nabla c\|_2$$

since $\frac{\partial^2(\varrho f_m)}{\partial\varrho\partial c}(\varrho, c) = (\log\varrho + 1)H'(c)$. Hence

$$|R_k(\varrho, \mathbf{u}, c)| \le C\left(h\|\operatorname{div}\mathbf{u}\|_2 + \varepsilon h\|(1+\log\varrho)\nabla\varrho\|_2\|\nabla c\|_2\right)$$

Moreover, (3.17) and $\varrho^{-1}\partial_\varrho p_m(\varrho) \ge \underline{\tilde{p}_m}(\varrho^{-1} + \varrho^{\Gamma-2})$ due to (3.15) imply that

$$\begin{aligned} &\|(\varrho^{\frac12}\mathbf{u}, \nabla c)\|_2^2 + \|\varrho\|_\Gamma^\Gamma + h\|(\mathbf{u}, \nabla\mu, \varepsilon^{\frac12}(1+\log\varrho)\nabla\varrho)\|_2^2 \\ &\le\ C\left(E_m(\varrho_k, \mathbf{u}_k, c_k) + |R_k(\varrho, \mathbf{u}, c)|\right) \end{aligned}$$

Therefore, combining the last two estimates and using Young's inequality, we conclude

$$\begin{aligned} &\|(\varrho^{\frac12}\mathbf{u}, \nabla c)\|_2^2 + \|\varrho\|_\Gamma^\Gamma + h\|(\mathbf{u}, \nabla\mu, \varepsilon^{\frac12}(1+\log\varrho)\nabla\varrho)\|_2^2 \\ &\le\ C\left(E_m(\varrho_k, \mathbf{u}_k, c_k) + 1 + \varepsilon^{\frac32}h\|(1+\log\varrho)\nabla\varrho\|_2^2 + h^{\frac12}\|\nabla c\|_2^2\right) \end{aligned}$$

where C is independent of $\varrho, \mathbf{u}, c, \varrho_k, \mathbf{u}_k, \varepsilon, h$. Therefore there is some $h_0 > 0$ such that

$$\|(\varrho^{\frac12}\mathbf{u}, \nabla c)\|_2^2 + \|\varrho\|_\Gamma^\Gamma + h\|(\mathbf{u}, \nabla\mu, \varepsilon^{\frac12}(1+\log\varrho)\nabla\varrho)\|_2^2 \le C\left(E_m(\varrho_k, \mathbf{u}_k, c_k) + 1\right)$$

for all $0 < h \le h_0$. Finally, by the same estimates as in Section 2.5, Lemma 2.1 and (3.11) imply

$$\begin{aligned} \|c\|_2^2 + h\|\mu\|_2^2 &\le C\left(\|\nabla c\|_2^2 + h\|\nabla\mu\|_2^2 + \left|\int_\Omega \varrho c\, dx\right|^2 + h\left|\int_\Omega \varrho_k\mu\, dx\right|^2\right) \\ &\le C'(E_m(\varrho_k, \mathbf{u}_k, c_k) + 1) \end{aligned}$$

where C, C' depend on $\int_\Omega \varrho\, dx = \int_\Omega \varrho_k\, dx$ and $\int_\Omega \varrho c\, dx = \int_\Omega \varrho_k c_k$. This completes the proof of the uniform estimate (3.18).

Next we prove existence of solutions (for a fixed $0 < h \le h_0$) with the aid of a homotopy argument and the Leray-Schauder degree. To this end we introduce operators $\mathcal{L}_k, \mathcal{F}_k \colon X \to Y$ with

$$\begin{aligned} &X = H_0^1(\Omega; \mathbb{R}^3) \times H_N^2(\Omega)^3, \quad H_N^2(\Omega) = \{u \in H^2(\Omega) : \partial_n u|_{\partial\Omega} = 0\}, \\ &Y = H^{-1}(\Omega; \mathbb{R}^3) \times L^2(\Omega)^3, \end{aligned}$$

and

$$\mathcal{L}_k(\mathbf{u},\varrho,c,\mu) = \begin{pmatrix} \operatorname{div} S(c_k,\nabla\mathbf{u}) \\ \lambda\varrho + \operatorname{div}(\varrho\mathbf{u}) - \varepsilon\Delta\varrho \\ \Delta\mu + \int_\Omega \mu\,dx \\ \Delta c + \int_\Omega c\,dx \end{pmatrix},$$

$$\mathcal{F}_k(\mathbf{u},\varrho,c,\mu) = $$
$$\begin{pmatrix} \frac{\varrho\mathbf{u}-\varrho_k\mathbf{u}_k}{h} + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \varepsilon(\nabla\varrho\cdot\nabla)\mathbf{u} + \nabla p_\delta - \varrho\mu\nabla c + \varrho\frac{\partial f}{\partial c}(\varrho,c)\nabla c \\ (\lambda - \frac{1}{h})[\varrho]_+ + \frac{1}{h}\varrho_k \\ \varrho_k\frac{c-c_k}{h} + \varrho\mathbf{u}\cdot\nabla c + \int_\Omega \mu\,dx \\ \varrho_k\frac{f(\varrho_k,c)-f(\varrho_k,c_k)}{c-c_k} - \varrho_k\mu + \int_\Omega c\,dx \end{pmatrix}.$$

Here $\lambda \geq \max(\lambda_0, \frac{1}{h})$, where $\lambda_0 = \lambda_0(\varepsilon, K)$ is the constant in the statement of Lemma 3.4 below with K so large that $\|v\|_6 \leq K$ for any solution of (3.9)-(3.14). Then by Lemma 3.4 below and standard results on elliptic partial differential equations $\mathcal{L}_k \colon X \to Y$ is invertible. Moreover, if $\mathcal{L}_k(\mathbf{u},\varrho,c,\mu) = \mathcal{F}_k(\mathbf{u},\varrho,c,\mu)$ for some $(\mathbf{u},\varrho,c,\mu) \in X$, then $\varrho \geq 0$ by Lemma 3.4 and therefore $[\varrho]_+ = \varrho$. Hence $(\mathbf{u},\varrho,c,\mu) \in X$ with $\varrho \geq 0$ is a solution of (3.9)-(3.14) if and only if

$$\mathcal{L}_k(\mathbf{u},\varrho,c,\mu) = \mathcal{F}_k(\mathbf{u},\varrho,c,\mu) \quad\Leftrightarrow\quad (\mathbf{u},\varrho,c,\mu) = \mathcal{L}_k^{-1}(\mathcal{F}_k(\mathbf{u},\varrho,c,\mu)),$$

i.e., $w = (\mathbf{u},\varrho,c,\mu)$ solves $w - \mathcal{L}_k^{-1}(\mathcal{F}_k(w)) = 0$. . Moreover, the operator norms of $\mathcal{L}_k$ and $\mathcal{L}_k^{-1}$ can be bounded by a constant independent of c_k. Furthermore, it is easy to check that $\mathcal{L}_k^{-1}\mathcal{F}_k \colon X \to X$ is a continuous and compact mapping. Hence the Leray-Schauder degree of $I - \mathcal{L}_k^{-1}\mathcal{F}_k$ is well-defined, cf. e.g. [7]. In order to show that $\deg(I - \mathcal{L}_k^{-1}\mathcal{F}_k, B_R(0), 0) = 1$ for sufficiently large $R > 0$, let $\mathcal{F}_k^\tau(\mathbf{u},\varrho,c,\mu)$, $\tau \in [0,1]$, be the operator obtained by replacing for $\mathbf{u}_k, \varrho_k, c_k, f$ in the definition of $\mathcal{F}_k(\mathbf{u},\varrho,c,\mu)$ by $\mathbf{u}_k^\tau = (1-\tau)\mathbf{u}_k$, $\varrho_k^\tau = (1-\tau)\varrho_k + \tau$, $c_k^\tau = (1-\tau)c_k$ and

$$f^\tau(\varrho,c) = \tau(\varrho^{\Gamma-1} + \log\varrho) + (1-\tau)f(\varrho,c).$$

Then $w = (\mathbf{u},\varrho,c,\mu) \in X$ solves $w - \mathcal{L}_k^{-1}(\mathcal{F}_k^\tau(w)) = 0$ if and only if $(\mathbf{u},\varrho,c,\mu)$ solve (3.9)-(3.14) with $\mathbf{u}_k, \varrho_k, c_k, f$ replaced by $\mathbf{u}_k^\tau, \varrho_k^\tau, c_k^\tau, f^\tau$. Moreover, it is not difficult to check that for each fixed $\varepsilon > 0$, $0 < h \leq h_0$ $\|\mathcal{F}_k^\tau(\mathbf{u},\varrho,c,\mu)\|_Y$ can be estimated by the terms on the left-hand side of (3.18). Hence, if $w = (\mathbf{u},\varrho,c,\mu) \in X$ solves $w - \mathcal{L}_k^{-1}(\mathcal{F}_k^\tau(w)) = 0$, then

$$\|(\mathbf{u},\varrho,c,\mu)\|_X \leq C\|\mathcal{F}_k^\tau(\mathbf{u},\varrho,c,\mu)\|_Y \leq M(E_m(\varrho_k,\mathbf{u}_k,c_k),\varepsilon,h)$$

for some continuous function M independent of $\tau \in [0,1]$. Hence there is some $R > 1$ such that any solution of $w - \mathcal{L}_k^{-1}\mathcal{F}_k^\tau(w) = 0$ with $0 < h \leq h_0$,

$\tau \in [0,1]$ satisfies $\|w\|_X \leq R-1$. Moreover, if $\tau = 1$, (3.17) and the strict convexity of $\varrho f^1(\varrho, c) = \varrho^\Gamma + \varrho \log \varrho$ imply that $w = (0,1,0,0)$ is the unique solution of $w - \mathcal{L}_k^{-1}\mathcal{F}_k^1(w) = 0$. Thus

$$\deg(I - \mathcal{L}_k^{-1}\mathcal{F}_k, B_R(0), 0) = \deg(I - \mathcal{L}_k^{-1}\mathcal{F}_k^1, B_R(0), 0) = 1,$$

which proves the lemma. ■

Lemma 3.4 *Let $K, \varepsilon > 0$, and let $\mathbf{v} \in H^1(\Omega;\mathbb{R}^3)$ with $\|v\|_6 \leq K$. Then there is some $\lambda_0 = \lambda_0(\varepsilon, K) > 0$ such that for any $\lambda \geq \lambda_0$ and any $f \in L^2(\Omega)$ there is a unique $\varrho \in H^1(\Omega)$ solving*

$$\lambda(\varrho, \varphi)_\Omega - (\mathbf{v}\varrho, \nabla\varphi)_\Omega + \varepsilon(\nabla\varrho, \nabla\varphi)_\Omega = (f, \varphi)_\Omega \tag{3.22}$$

for all $\varphi \in H^1(\Omega)$. Moreover, if $f \geq 0$, then $\varrho \geq 0$.

Proof: We can assume without loss of generality that $\varepsilon = 1$. We first show an a priori estimate, which implies uniqueness. To this end let $\varrho \in H^1(\Omega)$ be a solution of (3.22). Then choosing $\varphi = \varrho$ in (3.22) gives

$$\begin{aligned} \lambda\|\varrho\|_2^2 + \|\nabla\varrho\|_2^2 &\leq \|\mathbf{v}\|_6\|\varrho\|_3\|\nabla\varrho\|_2 + \|f\|_2\|\varrho\|_2 \\ &\leq C\|\mathbf{v}\|_6\|\varrho\|_2^{\frac{1}{2}}\|\varrho\|_{H^1}^{\frac{3}{2}} + \|f\|_2\|\varrho\|_2. \end{aligned}$$

Thus, if $\lambda \geq 1$,

$$\lambda\|\varrho\|_2^2 + \|\nabla\varrho\|_2^2 \leq CK^4\|\varrho\|_2^2 + C\|f\|_2^2$$

by Young's inequality. Choosing $\lambda_0 = \max\{1, 2CK^4\}$ yields

$$\lambda\|\varrho\|_2^2 + \|\nabla\varrho\|_2^2 \leq C\|f\|_2^2 \tag{3.23}$$

for all $\lambda \geq \lambda_0$ and some $C > 0$ independent of λ. Hence the solution is unique. In order to prove existence of a solution with $\varrho \geq 0$ if $f \geq 0$ one approximates $\mathbf{v}, f$ by smooth $\mathbf{v}_k, f_k$ such that $\mathbf{v}_k \to_{k\to\infty} \mathbf{v}$ in $L^6(\Omega)$ and $f_k \to_{k\to\infty} f \in L^2(\Omega)$ and $f_k \geq 0$ if $f \geq 0$. Then e.g. by [21, Proposition 4.29] there is some $\varrho_k \in W_p^2(\Omega)$ for any $1 < p < \infty$ which solves (3.22) with $\mathbf{v}, f$ replaced by $\mathbf{v}_k, f_k$ and which is non-negative if f_k are non-negative. Because of (3.23) $(\varrho_k)_{k\in\mathbb{N}}$ are bounded in $H^1(\Omega)$. Hence there is a weakly convergent subsequence that converges to a solution $\varrho \in H^1(\Omega)$ of (3.22) and that is non-negative if $f \geq 0$. ■

Now let $N \in \mathbb{N}$ be given and set $h = \frac{T}{N}$ and $\varepsilon = h$. If h_0 is the constant appearing in Lemma 3.3, then there is some N_0 such that $N \geq$

N_0 implies $h \leq h_0$. Hence, if $N \geq N_0$, we can define $(\mathbf{u}_k, \varrho_k, c_k, \mu_k)$, $k = 1, \ldots, N$, successively as solution of (3.9)-(3.14) with $(\mathbf{u}_0, \varrho_0, c_0)$ as fixed initial values. Moreover, define $g^N(t)\colon (-h, \infty)$ by $g^N(t) = g_k$ for $t \in ((k-1)h, kh]$, $k = 1, \ldots, N$, where $g \in \{\mathbf{u}, \varrho, c, \mu\}$ (setting $\mu_0 = 0$) as well as $p_\delta^N = p(\varrho^N, c^N) + \delta(\varrho^N)^\Gamma$. In what follows we denote

$$
\begin{aligned}
(\Delta_h^+ f)(t) &= f(t+h) - f(t), \quad & (\Delta_h^- f)(t) &= f(t) - f(t-h), \\
(\tau_h g)(t) &= g(t-h), & \partial_{t,h}^{\pm} f &= \frac{1}{h}\Delta_h^{\pm} f.
\end{aligned}
$$

Multiplication of (3.9) by $\varrho_k^{-1} \int_{kh}^{k(h+1)} \varphi(x,t)\,dt$, integration in space, and summation over all $k \in \mathbb{N}_0$ gives

$$
\begin{aligned}
(\partial_{t,h}^-(\varrho^N \mathbf{u}^N), \boldsymbol{\varphi})_{Q_T} &- \left(\varrho^N \mathbf{u}^N \otimes \mathbf{u}^N - \mathbb{S}^N + p_\delta^N \mathbb{I}, \nabla \boldsymbol{\varphi}\right)_{Q_T} \\
&+ h(\nabla \varrho^N \cdot \nabla \mathbf{u}^N, \boldsymbol{\varphi})_{Q_T} = \left(\varrho^N \left(\mu^N - \partial_c f^N\right) \nabla c^N, \boldsymbol{\varphi}\right)_{Q_T} \qquad (3.24)
\end{aligned}
$$

where $\boldsymbol{\varphi} \in C_{(0)}^\infty(\Omega \times [0,T); \mathbb{R}^3)$ is arbitrary, $\partial_c f^N = \frac{\partial f}{\partial c}(\varrho^N, c^N)$, and

$$
\mathbb{S}^N = 2\nu(\tau_h c^N)\mathbb{D}(\mathbf{u}^N) + \eta(\tau_h c^N) \operatorname{div} \mathbf{u}^N \, \mathbb{I} \qquad (3.25)
$$

Moreover, using summation by parts, i.e.,

$$
(\partial_{t,h}^-(\varrho^N \mathbf{u}^N), \varphi)_{Q_T} = -(\varrho^N \mathbf{u}^N, \partial_{t,h}^+ \varphi)_{Q_T} + (\mathbf{u}_0, \varphi(0))_\Omega,
$$

we conclude

$$
\begin{aligned}
-(\varrho^N \mathbf{u}^N, \partial_{t,h}^+ \boldsymbol{\varphi})_{Q_T} - (\varrho_0 \mathbf{u}_0, \boldsymbol{\varphi}|_{t=0})_\Omega &- \left(\varrho^N \mathbf{u}^N \otimes \mathbf{u}^N - \mathbb{S}^N + p_\delta^N \mathbb{I}, \nabla \boldsymbol{\varphi}\right)_{Q_T} \\
+ h(\nabla \varrho^N \cdot \nabla \mathbf{u}^N, \boldsymbol{\varphi})_{Q_T} &= \left(\varrho^N \left(\mu^N - \partial_c f^N\right) \nabla c^N, \boldsymbol{\varphi}\right)_{Q_T} \qquad (3.26)
\end{aligned}
$$

for all $\boldsymbol{\varphi} \in C_{(0)}^\infty(\Omega \times [0,T); \mathbb{R}^3)$. In the same way, one obtains

$$
(\varrho^N, \partial_{t,h}^+ \psi)_{Q_T} + (\varrho_0, \psi|_{t=0})_\Omega + (\varrho^N \mathbf{u}^N, \nabla \psi)_{Q_T} = h(\nabla \varrho, \nabla \psi)_{Q_T}, \qquad (3.27)
$$

$$
\begin{aligned}
(\varrho^N c^N, \partial_{t,h}^+ \psi)_{Q_T} &+ (\varrho_0 c_0, \psi|_{t=0})_\Omega \\
&+ (\varrho^N c^N \mathbf{u}^N, \nabla \psi)_{Q_T} = (\nabla \mu^N, \nabla \psi)_{Q_T} \qquad (3.28)
\end{aligned}
$$

for all $\psi \in C_{(0)}^\infty(\overline{\Omega} \times [0, \infty))$, where we have used that (3.10)-(3.11) implies

$$
\frac{\varrho_{k+1} c_{k+1} - \varrho_k c_k}{h} + \operatorname{div}(\varrho_k \mathbf{u}_{k+1} c_k) = \Delta \mu_{k+1}.
$$

Moreover,

$$
(\tau_h \varrho^N)\mu^N = \tau_h \varrho^N \left(\frac{f(\tau_h \varrho, c^N) - f(\tau_h \varrho, \tau_h c^N)}{\Delta_h^- c^N}\right) - \Delta c^N. \qquad (3.29)
$$

Finally, summation of (3.17) with respect to $k \in \mathbb{N}$ yields

$$\begin{aligned} & E_m(\varrho^N(t), \mathbf{u}^N(t), c^N(t)) + h \int_{Q_{(s,t)}} \frac{\partial_\varrho p_m(\varrho^N, c^N)}{\varrho^N} |\nabla \varrho^N|^2 \, d(x,\tau) \\ + & \int_{Q_{(s,t)}} \tau_h \varrho^N \frac{|\Delta_h^- \mathbf{u}^N|^2}{2h} \, d(x,\tau) + \frac{1}{2h} \|\nabla \Delta_h^- c^N\|^2_{L^2(Q_{(s,t)})} \\ + & \int_{Q_{(s,t)}} \mathbb{S}^N : \nabla \mathbf{u}^N \, d(x,\tau) + \|\nabla \mu^N\|^2_{L^2(Q_{(s,t)})} \\ \leq & \ E_m(\varrho^N(s), \mathbf{u}^N(s), c^N(s)) + R_{t,s}(\varrho^N, \mathbf{u}^N, c^N) \end{aligned} \tag{3.30}$$

for all $0 \leq s \leq t \leq T$ with $s, t \in h\mathbb{N}_0$, where

$$\begin{aligned} & R_{t,s}(\varrho^N, \mathbf{u}^N, c^N) = \\ & \int_{Q_{(t,s)}} \left(p_b(\varrho^N) \operatorname{div} \mathbf{u}^N - h \nabla \varrho^N \cdot \nabla c^N \frac{\partial^2(\varrho f_m)}{\partial \varrho \partial c}(\varrho^N, c^N) \right) dx. \end{aligned}$$

Since $E_m(\varrho^N(t), \mathbf{u}^N(t), c^N(t)) = E_m(\varrho^N(t_k), \mathbf{u}^N(t_k), c^N(t_k))$ for all $t \in (t_k - h, t_k]$ if $t_k \in h\mathbb{N}_0 \cap (0,T)$, we conclude that (3.30) holds for all $0 \leq s \leq t \leq T$ with

$$\begin{aligned} & R_{t,s}(\varrho^N, \mathbf{u}^N, c^N) = \\ & \int_{Q_{(t_k,s_k)}} \left(p_b(\varrho^N) \operatorname{div} \mathbf{u}^N - h \nabla \varrho^N \cdot \nabla c^N \frac{\partial^2(\varrho f_m)}{\partial \varrho \partial c}(\varrho^N, c^N) \right) dx, \end{aligned} \tag{3.31}$$

where $t_k, s_k \in h\mathbb{N}_0 \cap [0,T)$ are determined by the condition $t \in (t_k - h, t_k]$ and $s \in (s_k - h, s_k]$.

Lemma 3.5 *There is some $h_1 > 0$ independent of $\varrho^N, \mathbf{u}^N, c^N$ and a constant $C(\varrho_0, u_0, c_0)$ depending only Ω, d, $\varrho_0, \mathbf{u}_0, c_0$ such that*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left(\|\varrho^N(t)\|_\Gamma + \int_\Omega \varrho^N(t) |u^N(t)|^2 \, dx + \|c^N(t)\|_{H^1} \right) \\ + & \ h^{-\frac{1}{2}} \|(\nabla \Delta_h^- c^N, \Delta_h^- \varrho^N)\|_{L^2(Q_T)} + \|(\mathbf{u}^N, \mu^N, h^{\frac{1}{2}} \varrho^N \log \varrho^N)\|_{L^2(0,T;H^1)} \\ \leq & \ C(\varrho_0, \mathbf{u}_0, c_0) \end{aligned}$$

provided that $h = \frac{T}{N} \leq h_1$.

Proof: First of all, since $\frac{\partial^2(\varrho f_m)}{\partial \varrho \partial c}(\varrho, c) = (\varrho \log \varrho)' H'(c)$ and since $p_b(\varrho)$ is uniformly bounded, we have

$$\begin{aligned} & \left| R_{0,T}(\varrho^N, \mathbf{u}^N, c^N) \right| \\ \leq & \ C \left(\|\operatorname{div} \mathbf{u}^N\|_{L^2(Q_T)} + h^{\frac{3}{2}} \|\nabla(\varrho \log \varrho)\|^2_{L^2(Q_T)} + h^{\frac{1}{2}} \|\nabla c\|^2_{L^2(Q_T)} \right). \end{aligned}$$

On the other hand (3.30) and $\varrho^{-1}\partial_\varrho p_m(\varrho) \geq C|1+\log\varrho|^2$ due to (3.15) imply

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left(\|\varrho^N(t)\|_\Gamma^\Gamma + \int_\Omega \varrho^N(t)|\mathbf{u}^N(t)|^2\,dx + \|\nabla c^N(t)\|_2^2\right)\\
&+\ h^{-1}\|(\nabla\Delta_h^- c^N, \Delta_h^-\varrho^N)\|_{L^2(Q_T)}^2 + \|\nabla\mathbf{u}^N, \nabla\mu^N, h^{\frac12}\nabla(\varrho^N\log\varrho^N)\|_{L^2(Q_T)}^2\\
&\leq\ C\left(E_m(\varrho_0,\mathbf{u}_0,c_0) + R_{0,T}(\varrho^N,\mathbf{u}^N,c^N)\right).
\end{aligned}$$

Combining this with the previous estimate, choosing $0 < h \leq h_1$ sufficiently small, and using Young's inequality yields

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left(\|\varrho^N(t)\|_\Gamma^\Gamma + \int_\Omega \varrho^N(t)|\mathbf{u}^N(t)|^2\,dx + \|\nabla c^N(t)\|_2^2\right)\\
&+\ h^{-\frac12}\|(\nabla\Delta_h^- c^N, \Delta_h^-\varrho^N)\|_{L^2(Q_T)}^2 + \|\nabla\mathbf{u}^N, \nabla\mu^N, h^{\frac12}\nabla(\varrho^N\log\varrho^N)\|_{L^2(Q_T)}^2\\
&\leq\ C\left(E_m(\varrho_0,\mathbf{u}_0,c_0) + 1\right).
\end{aligned}$$

The remaining estimates of $\|c^N\|_{L^\infty(0,T;L^2)}$ and $\|\mu^N\|_{L^2(Q_T)}$ are done in the same way as in the proof of Lemma 3.3. ■

3.2 Improved Density Estimate

In order to show that ϱ^N, $N \geq N_0$, is uniformly bounded in $L^{\Gamma+1}(Q_T)$ we choose

$$\varphi = \psi(t)B\left[P_0\varrho^N\right], \quad \text{where } P_0\varrho^N = \varrho^N - \frac{1}{|\Omega|}\int_\Omega \varrho^N\,dx$$

and $\psi \in C_0^\infty(0,T)$, in (3.26). Here B is the well-known Bogovskii operator, cf. Bogovskii [5] or Galdi [15, Chapter III.3]. In particular, $B\colon L^p_{(0)}(\Omega) \to W_0^{1,p}(\Omega;\mathbb{R}^3)$ is a bounded operator for all $1 < p < \infty$, where $L^p_{(0)}(\Omega) = P_0L^p(\Omega)$, provided that Ω is a Lipschitz domain. Moreover, if $g \in L^p$, $g = \operatorname{div}\mathbf{v}$, $\mathbf{v} \in L^q(\Omega;\mathbb{R}^3)$ such that $\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0$, then

$$\|B[g]\|_{L^q(\Omega;\mathbb{R}^3)} \leq C(p,q)\|\mathbf{v}\|_{L^q(\Omega;\mathbb{R}^3)} \quad \text{for } 1 < p,q < \infty. \tag{3.32}$$

Then a direct computation yields

$$\begin{aligned}
&\int_{Q_T} \psi(t) p_m(\varrho^N, c^N)\varrho^N \, d(x,t) - \int_0^T \psi(t) \int_\Omega p_m(\varrho^N)\, dx \, \frac{1}{|\Omega|} \int_\Omega \varrho^N \, dx\, dt \\
&= \int_{Q_T} \psi(t) \left(\mathbb{S}^N - \varrho^N \mathbf{u}^N \otimes \mathbf{u}^N\right) : \nabla B\left[P_0 \varrho^N\right] \, d(x,t) \\
&\quad + \int_{Q_T} \psi(t) \left(\varrho^N \left(\partial_c f^N - \mu^N\right) \nabla c^N + h \nabla \mathbf{u}^N \cdot \nabla \varrho^N\right) \cdot B\left[P_0\varrho^N\right] \, d(x,t) \\
&\quad + \int_{Q_T} \psi(t) \varrho^N \mathbf{u}^N \tau_{-h} B\left[\operatorname{div}(\varrho^N \mathbf{u}^N - h\nabla \varrho^N)\right] \, d(x,t) \\
&\quad - \int_{Q_T} \varrho^N \mathbf{u}^N (\partial_{t,h}^+ \psi) B[P_0 \varrho^N] \, d(x,t) \equiv \sum_{j=1}^4 I_j,
\end{aligned}$$

where

$$\begin{aligned}
|I_1| &\leq C(T, \varrho_0, c_0, \mathbf{u}_0) \|\nabla B[P_0\varrho^N]\|_{L^2(Q_T)} \|\psi\|_\infty \leq C'(T, \varrho_0, c_0, \mathbf{u}_0)\|\psi\|_\infty, \\
|I_2| &\leq C(T, \varrho_0, c_0, \mathbf{u}_0) \|\varrho^N\|_{L^\infty(0,T;L^\Gamma)} \|\psi\|_\infty \leq C'(T, \varrho_0, c_0, \mathbf{u}_0)\|\psi\|_\infty, \\
|I_3| &\leq \|\varrho^N \mathbf{u}^N\|_{L^2(Q_T)} \|B\left[\operatorname{div}(\varrho^N \mathbf{u}^N - h\nabla\varrho^N)\right]\|_{L^2(Q_T)} \|\psi\|_\infty \\
&\leq C'(T, \varrho_0, c_0, \mathbf{u}_0)\|\psi\|_\infty, \\
|I_4| &\leq C(T, \varrho_0, c_0, \mathbf{u}_0) \|\partial_t \psi\|_{L^1(0,T)}
\end{aligned}$$

since $\Gamma > 3$. Letting ψ to approach 1 we conclude that

$$\delta \int_{Q_T} (\varrho^N)^{\Gamma+1} \, d(x,t) \leq C(T, \varrho_0, c_0, \mathbf{u}_0) \tag{3.33}$$

uniformly in $N \geq N_0$.

3.3 Passing to the Limit

Using the a priori bounds given by Lemma 3.5 and by (3.33), we can pass to a subsequence again denoted by $(\varrho_N, \mathbf{u}^N, c^N, \mu^N)$ such that

$$\begin{aligned}
(\varrho^N, c^N) &\rightharpoonup^*_{N\to\infty} (\varrho, c) && \text{in } L^\infty(0,T; L^\Gamma(\Omega) \times H^1(\Omega)), \\
\varrho^N &\rightharpoonup_{N\to\infty} \varrho && \text{in } L^{\Gamma+1}(Q_T), \\
p_\delta(\varrho^N, c^N) &\rightharpoonup_{N\to\infty} \overline{p_\delta(\varrho, c)} && \text{in } L^{(\Gamma+1)/\Gamma}(Q_T), \\
(\varrho^N \mathbf{u}^N, \varrho^N c^N, \mathbb{S}^N) &\rightharpoonup_{N\to\infty} (\overline{\varrho \mathbf{u}}, \overline{\varrho c}, \overline{\mathbb{S}}) && \text{in } L^2(Q_T; \mathbb{R}^4 \times \mathbb{R}^{3\times 3}), \\
(\mathbf{u}^N, \mu^N) &\rightharpoonup_{N\to\infty} (\mathbf{u}, \mu) && \text{in } L^2(0,T; H^1(\Omega; \mathbb{R}^4)),
\end{aligned}$$

as well as

$$(h\nabla\varrho^N, h\nabla(\varrho^N\log\varrho^N), \Delta_h^- c^N) \to_{N\to\infty} 0 \quad \text{in } L^2(Q_T;\mathbb{R}^7).$$

Next we define $\tilde{\varrho}^N$ and $\widetilde{\varrho c}^N$ as a piecewise linear interpolation of $\varrho^N(t_k)$, $\varrho^N(t_k)c^N(t_k)$, resp., where $t_k = kh$, $k = 0,\ldots,N$. More precisely, $\tilde{\varrho}^N = \frac{1}{h}\chi_{[0,h]} *_t \varrho^N$ and $\widetilde{\varrho c}^N = \frac{1}{h}\chi_{[0,h]} *_t (\varrho^N c^N)$, where the convolution is only taken with respect to the time variable t. Then

$$\partial_t\tilde{\varrho}^N = \partial_{t,h}^-\varrho^N \quad \text{and} \quad \partial_t\widetilde{\varrho c}^N = \partial_{t,h}^-(\varrho^N c^N) \quad \text{almost everywhere.}$$

Thus (3.27) yields that $\partial_t\tilde{\varrho}^N$ is bounded in $L^2(0,T;H^{-1}(\Omega))$, which implies that

$$\tilde{\varrho}^N \to_{N\to\infty} \varrho \qquad \text{in } L^r(0,T;H^{-\varepsilon}(\Omega)) \text{ for all } 1\leq r<\infty, \varepsilon>0$$

by the Aubin-Lions Lemma. In particular, this implies

$$\tilde{\varrho}^N \to \varrho \text{ in } C_{\text{weak}}([0,T];L^\Gamma(\Omega)),$$

since $\|\tilde{\varrho}^N\|_{L^\infty(0,T;L^\Gamma(\Omega))}$ is uniformly bounded in $N\geq N_0$. Moreover,

$$\|\varrho^N - \tilde{\varrho}^N\|_{L^2(Q_T)} \leq \|\Delta_h^-\varrho^N\|_{L^2(Q_T)} \to_{N\to\infty} 0.$$

Hence $w-\lim_{N\to\infty}\tilde{\varrho}^N = w-\lim_{N\to\infty}\varrho^N = \varrho$ and

$$\begin{aligned}
\overline{\varrho u} &= \lim_{N\to\infty}\varrho^N\mathbf{u}^N = \lim_{N\to\infty}\tilde{\varrho}^N\mathbf{u}^N = \varrho\mathbf{u} &&\text{weakly in } L^2(Q_T),\\
\overline{\varrho c} &= \lim_{N\to\infty}\varrho^N c^N = \lim_{N\to\infty}\tilde{\varrho}^N c^N = \varrho c &&\text{weakly in } L^2(Q_T),\\
&\lim_{N\to\infty}\varrho^N\mu^N = \lim_{N\to\infty}\tilde{\varrho}^N\mu^N = \varrho\mu &&\text{weakly in } L^2(Q_T)
\end{aligned}$$

since $\mathbf{u}^N$ and c^N converge weakly in $L^2(0,T;H^1(\Omega))$.

Moreover, we denote

$$\sqrt{\tau_h\varrho^N}\frac{f(\tau_h\varrho^N,c^N) - f(\tau_h\varrho^N,\tau_h c^N)}{c^N - \tau_h c^N} = F(\tau_h\varrho_N, c^N, \tau_h c^N),$$

using the convention $F(\varrho,c,c) = \frac{\partial f}{\partial c}(\varrho,c)$. Then $F(\varrho,c_1,c_2)$ is a continuous function with respect to $(\varrho,c_1,c_2)\in[0,\infty)\times\mathbb{R}^2$ satisfying

$$|F(\varrho,c_1,c_2)| \leq C(1+\varrho^{\frac{1}{2}}|\log\varrho|)(1+|c_1|+|c_2|).$$

Hence

$$\sqrt{\tau_h\varrho^N}\frac{f(\tau_h\varrho^N,c^N) - f(\tau_h\varrho^N,\tau_h c^N)}{c^N - \tau_h c^N} \qquad \text{is bounded in } L^2(Q_T)$$

and we can apply the result of Section 2.6 to $(\tilde{\varrho}^N, c^N)$ using (3.29) together with the fact that $\tau_h \varrho^N - \tilde{\varrho}^N$ converges strongly to zero in $L^\beta(Q_T)$ for all $1 \le \beta < \Gamma + 1$. In this way one concludes that

$$c^N \to_{N\to\infty} c \qquad \text{in } L^2(0,T;H^1(\Omega)).$$

In particular, $c^N \to_{N\to\infty} c$ almost everywhere in Q_T and therefore

$$\begin{aligned} \overline{\mathbb{S}} &= \lim_{N\to\infty} \left(2\nu(\tau_h c^N)\mathbb{D}(\mathbf{u}^N) + \eta(\tau_h c^N)\operatorname{div} \mathbf{u}^N \mathbb{I}\right) \\ &= 2\nu(c)\mathbb{D}(\mathbf{u}) + \eta(c)\operatorname{div} \mathbf{u}\,\mathbb{I} = \mathbb{S}. \end{aligned}$$

Furthermore, because of the growth estimate of F, we conclude that

$$\varrho^N \frac{f(\tau_h \varrho^N, c^N) - f(\tau_h \varrho^N, \tau_h c^N)}{c^N - \tau_h c^N} \rightharpoonup^*_{N\to\infty} \overline{\varrho \frac{\partial f}{\partial c}} \qquad \text{in } L^\infty(0,T;L^{\frac{6}{5}}(\Omega))$$

for a suitable subsequence.

Having all necessary results at hand, we see that $(\mathbf{u}, \varrho, c, \mu)$ solve

$$\begin{aligned} -(\varrho\mathbf{u}, \partial_t \boldsymbol{\varphi})_{Q_T} + (\varrho_0 \mathbf{u}_0, \boldsymbol{\varphi}|_{t=0})_\Omega - (\varrho\mathbf{u}\otimes\mathbf{u} - \mathbb{S}, \nabla\boldsymbol{\varphi})_{Q_T} \\ = (\overline{p_\delta}, \operatorname{div}\boldsymbol{\varphi})_{Q_T} + \left(\varrho\mu\nabla c - \overline{\varrho\frac{\partial f}{\partial c}}\nabla c, \boldsymbol{\varphi}\right)_{Q_T} \end{aligned} \tag{3.34}$$

for all $\boldsymbol{\varphi} \in C^\infty_{(0)}([0,T)\times\Omega);\mathbb{R}^3)$, as well as

$$(\varrho, \partial_t \psi)_{Q_T} + (\varrho_0, \psi|_{t=0})_\Omega + (\varrho\mathbf{u}, \nabla\psi)_{Q_T} = 0 \tag{3.35}$$

$$(\varrho c, \partial_t \psi)_{Q_T} + (\varrho_0 c_0, \psi|_{t=0})_\Omega + (\varrho c\mathbf{u}, \nabla\psi)_{Q_T} = (\nabla\mu, \nabla\psi)_{Q_T} \tag{3.36}$$

$$\left(\varrho\mu - \overline{\varrho\frac{\partial f}{\partial c}}, \psi\right)_{Q_T} = (\nabla c, \nabla\psi)_{Q_T} \tag{3.37}$$

for all $\psi \in C^\infty_{(0)}([0,T)\times\overline{\Omega}))$.

Moreover, since $\varrho \in L^2(Q_T), \mathbf{u} \in L^2(0,T;H^1(\Omega))$, we can use the regularizing procedure of DiPerna and Lions [8] or [21, Lemma 6.9], to conclude that ϱ is a renormalized solution of the transport equation (1.2) as in (1.18) for all $B(\varrho)$ such that $\tilde{b}(\varrho) = \varrho B(\varrho) \in C^0([0,\infty)) \cap C^1(0,\infty)$ and

$$|\tilde{b}'(\varrho)| \le \begin{cases} Ct^{-\lambda_0} & \text{if } t \in (0,1], \\ Ct^{\lambda_1} & \text{if } t > 1 \end{cases}$$

for some $\lambda_0 < 1$ and $\lambda_1 \le 1$. In particular, we can choose $B(\varrho) = \log\varrho$, which implies that

$$\partial_t(\varrho\log\varrho) + \operatorname{div}(\varrho\log(\varrho)\mathbf{u}) + \varrho\operatorname{div}\mathbf{u} = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^3\times(0,T)). \tag{3.38}$$

Furthermore, choosing $\psi = \Psi'(\varrho^N)\chi_{[0,t]}$ in (3.27), where $\Psi\colon \mathbb{R} \to \mathbb{R}_+$ is a smooth and convex function, we obtain

$$
\begin{aligned}
&\int_\Omega \Psi(\tilde\varrho^N(t))\,dx - \int_\Omega \Psi(\varrho_0)\,dx \\
&\quad\le\ \frac{1}{h}\int_{t-h}^t \int_\Omega \Psi(\varrho^N(\tau))\,dx\,d\tau - \int_\Omega \Psi(\varrho_0)\,dx \\
&\quad=\ \int_{Q_t} \partial^-_{\tau,h}\Psi(\varrho^N(\tau))\,d(x,\tau) \le \int_{Q_t} \Psi'(\varrho^N)\partial^-_{\tau,h}\varrho^N(\tau)\,d(x,\tau) \\
&\quad=\ -\int_{Q_t} \Psi'(\varrho^N)\operatorname{div}(\varrho^N\mathbf{u}^N)\,d(x,\tau) + h\int_{Q_t} \Delta\varrho^N\Psi'(\varrho^N)\,d(x,\tau) \\
&\quad=\ -\int_{Q_t} \left((\Psi'(\varrho^N)\varrho^N - \Psi(\varrho^N))\operatorname{div}\mathbf{u}^N + h\Psi''(\varrho^N)|\nabla\varrho^N|^2\right)\,d(x,\tau) \\
&\quad\le\ -\int_{Q_t} (\Psi'(\varrho^N)\varrho^N - \Psi(\varrho^N))\operatorname{div}\mathbf{u}^N\,d(x,\tau)
\end{aligned}
$$

because of Jensen's inequality and $\tilde\varrho^N = \frac{1}{h}\chi_{[0,h]} *_t \varrho^N$. After a simple approximation we can replace $\Psi(s)$ by $s\log s$. Hence, passing to the limit $N \to \infty$ and using (3.38), we obtain

$$
\int_\Omega (\overline{\varrho\log\varrho} - \varrho\log\varrho)(t)\,dx \le \int_{Q_t} \left(\varrho\operatorname{div}\mathbf{u} - \overline{\varrho\operatorname{div}\mathbf{u}}\right)\,d(x,\tau) \tag{3.39}
$$

for almost all $t \in (0,T)$.

In what follows, the symbol $\Delta^{-1}f = K * f$ denotes the convolution of f with the fundamental solution of the Laplacean on $\mathbb{R}^3$, where functions defined on Ω are extended by zero to functions on $\mathbb{R}^3$. We choose $\varphi = \psi\nabla\Delta^{-1}[\varrho^N]$, $\psi \in C_0^\infty(Q_T)$ in (3.26) and obtain

$$
\begin{aligned}
&\int_{Q_T} \psi\left(p_\delta^N I - \mathbb{S}^N\right)\mathcal{R}[\varrho^N]\,d(x,t) \\
&\quad=\ \int_{Q_T} \psi\mathbf{u}^N\left(\varrho^N\nabla\operatorname{div}\Delta^{-1}(\varrho^N\mathbf{u}^N) - \varrho^N\mathbf{u}^N\cdot\mathcal{R}[\varrho^N]\right)\,d(x,t) \\
&\qquad+ \int_{Q_T} \psi\varrho^N\mathbf{u}^N\tau_{-h}\nabla\operatorname{div}\Delta^{-1}(\varrho^N\mathbf{u}^N - h\nabla\varrho^N)\,d(x,t) \\
&\qquad+ \int_{Q_T} \left(-\varrho^N\mathbf{u}^N(\partial^+_{t,h}\psi)\tau_{-h}(\psi\nabla\Delta^{-1}\varrho^N) + g^N\cdot\nabla\Delta^{-1}\varrho^N\right)\,d(x,t)
\end{aligned}
$$

where $\mathcal{R} = \nabla^2\Delta^{-1}$ and

$$
\begin{aligned}
g^N \;=\; & -\left(p_\delta^N I - \mathbb{S}^N + \varrho^N\mathbf{u}^N\otimes\mathbf{u}^N\right)\cdot\nabla\psi \\
& + h\psi\nabla\varrho^N\cdot\nabla\mathbf{u}^N + \varrho^N\left(\partial_c f^N - \mu^N\right)\nabla c^N\psi.
\end{aligned}
$$

Using [10, Corollary 6.1], we conclude

$$\begin{aligned}
&\lim_{N\to\infty}\int_{Q_T}\psi\mathbf{u}^N\left(\varrho^N\Delta^{-1}\nabla\operatorname{div}(\varrho^N\mathbf{u}^N)-\varrho^N\mathbf{u}^N\cdot\mathcal{R}[\varrho^N]\right)\,d(x,t)\\
&=\int_{Q_T}\psi u\left(\varrho\Delta^{-1}\nabla\operatorname{div}(\varrho\mathbf{u})-\varrho u\cdot\mathcal{R}[\varrho]\right)\,d(x,t).
\end{aligned}$$

Moreover, using the previous results on strong and weak convergence, it is easy to pass to the limit in all remaining terms to conclude that

$$\begin{aligned}
&\lim_{N\to\infty}\int_{Q_T}\psi\left(p_\delta^N I-\mathbb{S}^N\right)\mathcal{R}[\varrho^N]\,d(x,t)\\
&=\int_{Q_T}\psi\mathbf{u}\left(\varrho\nabla\operatorname{div}\Delta^{-1}(\varrho\mathbf{u})-\varrho u\cdot\mathcal{R}[\varrho]\right)\,d(x,t)\\
&\quad-\int_{Q_T}\psi\varrho\mathbf{u}\nabla\operatorname{div}\Delta^{-1}(\varrho\mathbf{u})\,d(x,t)\\
&\quad+\int_{Q_T}\left(-\varrho\mathbf{u}(\partial_t\psi)\tau_h(\psi\nabla\Delta^{-1}\varrho)+g\cdot\nabla\Delta^{-1}\varrho\right)\,d(x,t)
\end{aligned}$$

where

$$g=-\left(p_\delta I-\mathbb{S}+\varrho\mathbf{u}\otimes\mathbf{u}\right)\cdot\nabla\psi+h\psi\nabla\varrho\cdot\nabla\mathbf{u}+\overline{\varrho\frac{\partial f}{\partial c}}\nabla c-\varrho\mu\nabla c.$$

On the other hand, choosing $\boldsymbol{\varphi}=\psi\nabla\Delta^{-1}\varrho$ in (3.34) and comparing it with the latter identity, we obtain

$$\begin{aligned}
&\int_{Q_T}\psi\left(\overline{p_\delta}I-\mathbb{S}\right)\mathcal{R}[\varrho]\,d(x,t)\\
&=\lim_{N\to\infty}\int_{Q_T}\psi\left(p_\delta^N I-\mathbb{S}^N\right)\mathcal{R}[\varrho^N]\,d(x,t)
\end{aligned}\tag{3.40}$$

for all $\psi\in C_0^\infty(Q_T)$. Next we show that

$$\begin{aligned}
&\lim_{N\to\infty}\left(\mathcal{R}:[\psi2\nu(c^N)(\nabla\mathbf{u}^N+(\nabla\mathbf{u}^N)^T)]-\psi2\nu(c^N)\operatorname{div}\mathbf{u}^N\right)\\
&=\mathcal{R}:[\psi2\nu(c)(\nabla\mathbf{u}+\nabla\mathbf{u}^T)]-\psi2\nu(c)\operatorname{div}\mathbf{u}
\end{aligned}\tag{3.41}$$

weakly in $L^2(0,T;W^{\omega,q}(\Omega))$ for some $\omega>0, q>1$.

In order to see (3.41), we adapt the technique of [11]. In particular, we report the following result [11, Lemma 4.2].

Lemma 3.1 *Let $w \in W^{1,r}(\mathbb{R}^d)$ and $\mathbf{V} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ be given, where $r > \frac{2d}{d+2}$. Then there exists $\omega = \omega(r) > 0$ and $q = q(r) > 1$ such that*

$$\left\| \mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}] \right\|_{W^{\omega,q}(\mathbb{R}^d;\mathbb{R}^d)} \leq C(r) \|w\|_{W^{1,r}(\mathbb{R}^d)} \|\mathbf{V}\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}.$$

Extending c^N, $\partial_{x_j}\mathbf{u}^N$ to be zero outside Ω we intend to apply Lemma 3.1 to

$$w = \nu(c_\delta),\ \mathbf{V} = [V_1, V_2, V_3],\ V_i = \partial_{x_i} u_{\delta,j} + \partial_{x_j} u_{\delta,i},\ i = 1,2,3,$$

where $j = 1,2,3$ is fixed. Indeed as the shear viscosity coefficient ν is (globally) Lipschitz in c, the uniform estimate stated in Lemma 3.5 allows us to apply Lemma 3.1, with $r = 2$.

Consequently, in accordance with (4.6), (4.7), we get (3.41). Combining this with (3.40), we obtain the essential relation

$$\int_{Q_t} \psi \left(\overline{p_\delta \varrho} - \overline{p_\delta} \varrho \right) d(x,\tau) = \int_{Q_t} \psi \left(\frac{4}{3}\nu(c) + \eta(c) \right) \left(\overline{\varrho \operatorname{div} \mathbf{u}} - \varrho \operatorname{div} \mathbf{u} \right) d(x,\tau).$$

Choosing $\psi = \left(\frac{4}{3}\nu(c) + \eta(c) \right)^{-1}$ above and using (3.39), we obtain

$$\int_\Omega \left(\overline{\varrho \log \varrho} - \varrho \log \varrho \right) dx \leq \int_{Q_t} \left(\frac{4}{3}\nu(c) + \eta(c) \right)^{-1} \left(\overline{p_\delta \varrho} - \overline{p_\delta} \varrho \right) d(x,\tau)$$

for some $\Lambda > 0$, where, because of the decomposition (3.16),

$$\int_{Q_t} \left(\frac{4}{3}\nu(c) + \eta(c) \right)^{-1} \left(\overline{p_\delta \varrho} - \overline{p_\delta} \varrho \right) d(x,\tau) \leq \Lambda \int_{Q_t} \left(\overline{\varrho \log \varrho} - \varrho \log \varrho \right)$$

by the same arguments as in [11, Section 6.6.3]. Hence

$$\int_\Omega \left(\overline{\varrho \log \varrho} - \varrho \log \varrho \right)(t)\, dx \leq \Lambda \int_{Q_t} \left(\overline{\varrho \log \varrho} - \varrho \log \varrho \right) d(x,t),$$

which implies

$$\int_\Omega \left(\overline{\varrho \log \varrho} - \varrho \log \varrho \right)(t)\, dx \equiv 0$$

for all $t \in (0,T)$ because of Gronwall's lemma. Thus ϱ^N converges almost everywhere to ϱ and

$$\overline{p_\delta} = p_\delta(\varrho), \quad \overline{\varrho \frac{\partial f}{\partial c}} = \varrho \frac{\partial f}{\partial c}(\varrho, c).$$

Finally, passing to the limit in (3.30) and (3.31), we obtain that

$$
\begin{aligned}
&E_m(\varrho(t), \mathbf{u}(t), c(t)) + \int_{Q_{(s,t)}} \left(\mathbb{S} : \nabla \mathbf{u} + |\nabla \mu|^2\right) \, d(x,\tau) \\
&\leq \;\; E_m(\varrho(s), \mathbf{u}(s), c(s)) + \int_{Q_{s,t}} p_b(\varrho) \operatorname{div} \mathbf{u} \, d(x,\tau).
\end{aligned}
$$

Now, using the renormalized transport equation (1.18) for $b(\varrho) = p_b(\varrho)$ and $\varphi = \chi_{[s,t]}$ (after a simple approximation), we conclude that

$$
\int_{Q_{(s,t)}} p_b(\varrho) \operatorname{div} \mathbf{u} \, d(x,\tau) = - \int_{\Omega} \varrho(\tau) f_b(\varrho(\tau)) \, dx \Big|_{\tau=s}^{t} .
$$

Summing up, we have proved (3.7), which completes the proof.

4 Vanishing Artificial Pressure Limit

4.1 Uniform Bounds

By virtue of the coercivity of the functions f_{e}, H postulated in (1.13), (1.14), the specific free energy E_δ is bounded from below, and, by the same arguments as in Sections 2.3-2.5, the energy inequality (3.7) gives rise to the estimates (2.16)-(2.25) with $(\mathbf{u}, \varrho, c, \mu)$ replaced by $(\mathbf{u}_\delta, \varrho_\delta, c_\delta, \mu_\delta)$ uniformly in $\delta > 0$. Moreover, (3.7) and (3.8) imply that

$$
\delta \operatorname*{ess\,sup}_{t \in (0,T)} \|\varrho_\delta\|^{\Gamma}_{L^\Gamma(\Omega)} \leq C, \tag{4.1}
$$

4.2 Refined Pressure Estimates

Following [14] we derive a uniform bound on the pressure in the *reflexive* Lebesgue space $L^p((0,T) \times \Omega))$, $p > 1$, in particular, we show that the pressure family $\{p(\varrho_\delta, c_\delta)\}_{\delta > 0}$ is equi-integrable.

Let B be the Bogovskii operator as introduced in Section 3.2. Pursuing the main idea of [14] we use quantities

$$
\boldsymbol{\varphi}(t,x) = \psi(t) B \left[\varrho_\delta^\alpha - \frac{1}{|\Omega|} \int_\Omega \varrho_\delta^\alpha \, dx \right], \;\; \psi \in \mathcal{D}(0,T)
$$

as test functions in the momentum balance (3.1). This procedure, described in detail in [14] and rather similar to the arguments in Section 3.2, gives rise to a uniform estimate

$$
\int_0^T \int_\Omega \left(p(\varrho_\delta, c_\delta) \varrho_\delta^\alpha + \delta \varrho_\delta^{\Gamma + \alpha} \right) dx \;\; dt \leq C \tag{4.2}
$$

provided (i) $\alpha > 0$ is small enough, and (ii) there is a uniform bound

$$\|\mathbb{P}_\delta\|_{L^p(0,T;L^p(\Omega;\mathbb{R}^{3\times 3}))} \leq C \text{ for a certain } p > 1. \tag{4.3}$$

In order to see (4.3), the constitutive relation (3.4), hypothesis (1.14), and estimates (3.25), (2.16), (2.24), (2.25) imply

$$\|\Delta c_\delta\|_{L^\infty(0,T;L^q(\Omega))} \leq C \text{ for a certain } q > \frac{6}{5}; \tag{4.4}$$

whence, by virtue of the standard elliptic estimates,

$$\|\nabla c_\delta\|_{L^\infty(0,T;L^r(\Omega))} \leq C \text{ for a certain } r > 2, \tag{4.5}$$

in particular (4.3) follows.

4.3 Strong Compactness of the Concentration Gradients

Following step by step the arguments of Section 2.6 we obtain that

$$c_\delta \to c \text{ in } L^2(0,T;W^{1,2}(\Omega)). \tag{4.6}$$

4.4 Asymptotic Limit for $\delta \to 0$

To begin with, in accordance with (2.20), we can assume that

$$\mathbf{u}_\delta \to \mathbf{u} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)). \tag{4.7}$$

Moreover, as in Section 3.3, we verify

$$\varrho_\delta \to \varrho \text{ in } C_{\text{weak}}([0,T];L^\gamma(\Omega))$$

for a suitable subsequence of $\delta \to 0$ using (2.16) and strong convergence in $L^r(0,T;H^{-\varepsilon}(\Omega))$, $r < \infty$. This fact together with the momentum equation (3.1) imply

$$\varrho_\delta \mathbf{u}_\delta \to \varrho \mathbf{u} \text{ in } C_{\text{weak}}([0,T];L^q(\Omega;\mathbb{R}^3)),\ q = \frac{2\gamma}{1+\gamma}; \tag{4.8}$$

whence, finally,

$$\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \to \varrho \mathbf{u} \otimes \mathbf{u} \text{ weakly in } L^2(0,T;L^q(\Omega;\mathbb{R}^{3\times 3})) \tag{4.9}$$

with $q = \frac{6\gamma}{3+4\gamma}$.

Similarly, by virtue of (2.24), (4.6),

$$\varrho_\delta c_\delta \to \varrho c \quad \text{in } C_{\rm weak}(0,T;L^q(\Omega;\mathbb{R}^3)) \text{ for } q=\frac{6\gamma}{6+\gamma}, \tag{4.10}$$

$$\varrho_\delta c_\delta \mathbf{u}_\delta \rightharpoonup^* \varrho c\mathbf{u} \text{ in } L^\infty(0,T;L^q(\Omega;\mathbb{R}^3)) \quad \text{for } q=\frac{3\gamma}{3+\gamma}, \tag{4.11}$$

and, in view of (2.25),

$$\mu_\delta \to \mu \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)). \tag{4.12}$$

Finally, it follows from the refined pressure estimates established in (4.2) that

$$\left.\begin{array}{r} p(\varrho_\delta, c_\delta) \rightharpoonup \overline{p(\varrho,c)}, \\ \\ \delta\varrho_\delta^\Gamma \rightharpoonup 0 \end{array}\right\} \text{ in } L^q((0,T)\times\Omega) \text{ for a certain } q>1. \tag{4.13}$$

At this stage, it is easy to let $\delta\to 0$ in (3.1) - (3.6) in order to obtain

$$\int_0^T\int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla\varphi\Big)\,dx\ dt + \int_\Omega \varrho_0\varphi|_{t=0}\,dx = 0 \tag{4.14}$$

for any test function $\varphi\in\mathcal{D}([0,T)\times\overline{\Omega})$,

$$\int_0^T\int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u}:\nabla\boldsymbol{\varphi} + \overline{p(\varrho,c)}\,{\rm div}\,\boldsymbol{\varphi}\Big)\,dx\ dt \tag{4.15}$$

$$= \int_0^T\int_\Omega \Big(\mathbb{S}-\mathbb{P}\Big):\nabla\boldsymbol{\varphi}\,dx\ dt + \int_\Omega \mathbf{m}_0\cdot\boldsymbol{\varphi}|_{t=0}\,dx$$

for any $\boldsymbol{\varphi}\in\mathcal{D}([0,T)\times\Omega;\mathbb{R}^3)$,

$$\int_0^T\int_\Omega \Big(\varrho c\partial_t\varphi + \varrho c\mathbf{u}\cdot\nabla\varphi - \nabla\mu\cdot\nabla\varphi\Big)\,dx\ dt - \int_\Omega \varrho_0 c_0\varphi|_{t=0}\,dx = 0 \tag{4.16}$$

for any $\varphi\in\mathcal{D}([0,T)\times\overline{\Omega})$, where $\mathbb{S}$ satisfies (1.5),

$$\mathbb{P} \;=\; \Big(\nabla c\otimes\nabla c - \frac{|\nabla c|^2}{2}\mathbb{I}\Big), \text{ and} \tag{4.17}$$

$$\varrho\mu \;=\; \overline{\varrho\frac{\partial f(\varrho,c)}{\partial c}} - \Delta c \tag{4.18}$$

provided the family of initial data $\{\varrho_{0,\delta}, (\varrho\mathbf{u})_{0,\delta}, (\varrho c)_{0,\delta}\}_{\delta>0}$ converges at least weakly in L^1.

Thus our ultimate goal consists in removing the bar indicating the weak limits of composed functions in (4.15), (4.18), or, equivalently, showing

$$\varrho_\delta \to \varrho \text{ (strongly) in } L^1((0,T)\times\Omega).$$

This will be done in the last section.

4.5 Strong Pointwise Convergence of the Approximate Densities

In order to show strong convergence of the sequence $\{\varrho_\delta\}_{\delta>0}$ we evoke the method based on certain fine properties of the so-called effective viscous flux established by P.-L.Lions [18], further developed in [11] for the case of non-constant viscosity coefficients.

To this end, observe first that the functions ϱ_δ, $\mathbf{u}_\delta$ satisfy (3.2) in the sense of renormalized solutions introduced by DiPerna and P.-L.Lions [8], cf. (1.18); specifically, the integral identity

$$\int_0^T \int_\Omega \Big[b(\varrho_\delta)\partial_t \varphi + b(\varrho_\delta)\mathbf{u}_\delta \cdot \nabla\varphi + \Big(b(\varrho_\delta) - b'(\varrho_\delta)\varrho_\delta \Big) \operatorname{div} \mathbf{u}_\delta \varphi \Big] \, dx \, dt$$
$$= \; - \int_\Omega b(\varrho_{0,\delta})\varphi(0)\, dx \tag{4.19}$$

holds for any test function $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega})$, and any $b \in C^1[0,\infty)$ such that $b'(\varrho) \equiv 0$ for $\varrho \geq M_b$ large enough.

As in Section 3.3, relation (4.19) can be deduced from (3.2) by means of the regularization technique developed by DiPerna and P.-L.Lions [8] or [21, Lemma 6.9]. Note that this step requires $\varrho_\delta \in L^2((0,T)\times\Omega)$, $\mathbf{u}_\delta \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$, where the former condition holds as a consequence of the artificial pressure term $(\delta/\Gamma - 1)\varrho_\delta^\Gamma$ in (3.8).

The next step is to take the quantity

$$\boldsymbol{\varphi}(t,x) = \psi(t)\omega(x)\nabla\Delta^{-1}[\chi_\Omega b(\varrho_\delta)], \;\; \psi \in \mathcal{D}(0,T), \;\; \omega \in \mathcal{D}(\Omega)$$

as a test function in (3.1), and

$$\boldsymbol{\varphi}(t,x) = \psi(t)\omega(x)\nabla\Delta^{-1}[\chi_\Omega \overline{b(\varrho)}], \;\; \psi \in \mathcal{D}(0,T), \;\; \omega \in \mathcal{D}(\Omega)$$

in (4.15). Here, as always, the symbol $\overline{b(\varrho)}$ stands for a weak limit of $\{b(\varrho_\delta)\}_{\delta>0}$, while $\Delta^{-1}f$ denotes the convolution of f with the fundamental solution of the Laplacian as before. Letting $\delta \to 0$ we deduce a remarkable identity

$$\overline{\Big(p(\varrho,c) - \mathcal{R} : \mathbb{S}\Big) b(\varrho)} = \overline{\Big(p(\varrho,c) - \mathcal{R} : \mathbb{S}\Big)} \; \overline{b(\varrho)}, \tag{4.20}$$

where $\mathcal{R} = (\partial_{x_i}\partial_{x_j}\Delta^{-1})_{i,j}$.

Relation (4.20) is the heart of the existence theory for the barotropic Navier-Stokes system developed by P.-L.Lions [18]. The proof is quite involved but nowadays well-understood and will be omitted. An alternative proof based on Div-Curl lemma is given in [10, Chapter 6, Proposition 6.1]. The proof is similar to the corresponding arguments in Section 3.3. Let us remark that the extra "pressure" term $\mathbb{P}_\delta$ satisfies

$$\mathbb{P}_\delta : \nabla\Delta^{-1}\nabla[b(\varrho_\delta)] \to \mathbb{P} : \nabla\Delta^{-1}\nabla[\overline{b(\varrho)}] \text{ weakly in } L^1((0,T)\times\Omega)$$

as a direct consequence of (4.6).

Our next goal is to deduce from (4.20) a relation

$$\overline{p(\varrho,c)b(\varrho)} - \overline{p(\varrho,c)}\ \overline{b(\varrho)} = (\frac{4}{3}\nu(c)+\eta(c))\Big(\overline{\operatorname{div} b(\varrho)\mathbf{u}} - \overline{b(\varrho)}\ \operatorname{div}\mathbf{u}\Big), \quad (4.21)$$

where the quantity $p-(\frac{4}{3}\nu+\eta)\operatorname{div}\mathbf{u}$ is usually termed the effective viscous flux.

In order to get (4.21), we apply Lemma 3.2 as in Section 3.3 and obtain

$$\begin{gathered} \mathcal{R} : [\nu(c_\delta)(\nabla\mathbf{u}_\delta + \nabla\mathbf{u}_\delta^T)] - 2\nu(c_\delta)\operatorname{div}\mathbf{u}_\delta \\ \to \\ \mathcal{R} : [\nu(c)(\nabla\mathbf{u} + \nabla\mathbf{u}^T)] - 2\nu(c)\operatorname{div}\mathbf{u} \end{gathered} \quad (4.22)$$

weakly in $L^2(0,T;W^{\omega,q}(\Omega))$ for a certain $\omega > 0$. On the other hand, as ϱ_δ satisfies the renormalized equation (4.19),

$$b(\varrho_\delta) \to \overline{b(\varrho)} \text{ in } C_{\text{weak}}([0,T];L^q(\Omega)) \text{ for any finite } q \geq 1 \quad (4.23)$$

as soon as b is uniformly bounded. Combining relation (4.20) with (4.22), (4.23) we obtain (4.21) (see [11] for details).

In order to show strong convergence of the density, we use the renormalized equation (4.19) for $b(\varrho) = \varrho L_k(\varrho)$, where

$$\begin{gathered} L_k(\varrho) = \textstyle\int_1^\varrho \frac{T_k(z)}{z^2}\ dz, \\ T_k(\varrho) = \min\{\varrho, k\},\ \ \varrho \geq 0. \end{gathered} \quad (4.24)$$

Accordingly, we obtain

$$\int_\Omega \varrho_\delta L_k(\varrho_\delta)(\tau)\,dx + \int_{Q_T} T_k(\varrho_\delta)\operatorname{div}\mathbf{u}_\delta\,d(x,t) = \int_\Omega \varrho_{0,\delta}L_k(\varrho_{0,\delta})\,dx. \quad (4.25)$$

At this stage, we have to show that the limit quantities ϱ, $\mathbf{u}$ represent a renormalized solution of (4.14). Following the approach of [13] we introduce the concept of *oscillations defect measure* associated to the family $\{\varrho_\delta\}_{\delta>0}$:

$$\mathbf{osc}_p[\varrho_\delta \to \varrho](O) = \sup_{k\geq 1}\Big(\limsup_{\delta\to 0}\int_O |T_k(\varrho_\delta) - T_k(\varrho)|^p\ dx\ dt\Big).$$

We report the following result [10, Chapter 6, Proposition 6.3].

Lemma 4.1 *Let*

$$\mathbf{osc}_p[\varrho_\delta \to \varrho]((0,T)\times\Omega) < \infty \textit{ for a certain } p > 2. \tag{4.26}$$

Then ϱ, $\mathbf{u}$ *represent a renormalized solution of (4.14).*

In order to show (4.26), we make use of relation (4.21) for $b = T_k$. To begin with, as the pressure p is given through the constitutive relation (1.12) and $\{c_\delta\}_\delta$ converges strongly, we observe that

$$\overline{p(\varrho,c)T_k(\varrho)} = \overline{p(\varrho,\cdot)T_k(\varrho)},\ \overline{p(\varrho,c)} = \overline{p(\varrho,\cdot)},$$

where

$$\overline{p(\varrho,\cdot)T_k(\varrho)} = \mathrm{weak}_{L^1} \lim_{\delta\to 0} p(\varrho_\delta,c)T_k(\varrho_\delta),$$

and, similarly,

$$\overline{p(\varrho,\cdot)} = \mathrm{weak}_{L^1} \lim_{\delta\to 0} p(\varrho_\delta,c).$$

On the other hand, in accordance with hypotheses (1.13), (1.14), the pressure can be written in the form

$$p(\varrho,c) = a\varrho^\gamma + p_m(\varrho,c) + p_b(\varrho),\ a > 0, \tag{4.27}$$

where p_m is non-decreasing in ϱ and $p_b \in C^2[0,\infty)$ has compact support in $[0,\infty)$.

As p_m is non-decreasing in ϱ and $0 \le \overline{T_k(\varrho)} \le k$, it is easy to check that

$$\Big(p_m(\varrho_n,c) - p_m(\overline{T_k(\varrho)},c)\Big)\Big(T_k(\varrho_n) - \overline{T_k(\varrho)}\Big) \ge 0;$$

whence letting $n \to \infty$ we get

$$\overline{p_m(\varrho,\cdot)T_k(\varrho)} - \overline{p_m(\varrho,\cdot)}\ \overline{T_k(\varrho)} \ge 0, \tag{4.28}$$

while, exactly as in [10, Proposition 6.2], we can show that

$$\begin{aligned}&\int_0^T\int_\Omega \Big(\overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\ \overline{T_k(\varrho)}\Big)\,dx\ dt\\ &\quad\ge \limsup_{\delta\to 0}\int_0^T\int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1}\,dx\ dt.\end{aligned} \tag{4.29}$$

Combining (4.28), (4.29), together with (4.21) and Young's inequality, we conclude

$$\mathbf{osc}_{\gamma+1}[\varrho_\delta \to \varrho]((0,T)\times\Omega) < \infty. \tag{4.30}$$

In particular, in view of Lemma 4.1, the limit functions ϱ, $\mathbf{u}$ represent a renormalized solution of (3.2).

Thus we get

$$\int_\Omega \varrho_\delta L_k(\varrho)(\tau)\,dx + \int_0^\tau \int_\Omega T_k(\varrho)\operatorname{div}\mathbf{u}\,dx\ \ dt = \int_\Omega \varrho_0 L_k(\varrho_0)\,dx, \quad (4.31)$$

which, together with (4.25), gives rise to

$$\int_\Omega \Big(\overline{L_k(\varrho)} - L_k(\varrho)\Big)(\tau)\,dx + \int_0^\tau \int_\Omega \Big(\overline{T_k(\varrho)\operatorname{div}\mathbf{u}} - \overline{T_k(\varrho)}\ \operatorname{div}\mathbf{u}\Big)\,dx\ dt$$
$$= \int_0^\tau \int_\Omega \Big(T_k(\varrho) - \overline{T_k(\varrho)}\Big)\operatorname{div}\mathbf{u}\,dx\ dt \quad (4.32)$$

for any $\tau \in [0,T]$ since $\varrho_{0,\delta} \to \varrho_0$ in $L^1(\Omega)$.

Finally, as a consequence of (4.30),

$$\int_0^\tau \int_\Omega \Big(T_k(\varrho) - \overline{T_k(\varrho)}\Big)\operatorname{div}\mathbf{u}\,dx\ \ dt \to 0 \text{ as } k \to \infty;$$

whence, by virtue of (4.21), (4.27 - 4.29), we can let $k \to \infty$ in (4.32) in order to obtain

$$\int_\Omega \Big(\overline{\varrho\log(\varrho)} - \varrho\log(\varrho)\Big)(\tau)\,dx \le \Lambda \int_0^\tau \int_\Omega \Big(\overline{\varrho\log(\varrho)} - \varrho\log(\varrho)\Big)(\tau)\,dx\ dt$$

for a certain $\Lambda > 0$ (see Section 6.6 in Chapter 6 in [10] for details).

Thus, by means of Gronwall's lemma,

$$\overline{\varrho\log(\varrho)} = \varrho\log(\varrho) \text{ a.a. in } (0,T)\times\Omega,$$

in particular

$$\varrho_\delta \to \varrho \text{ in } L^1((0,T)\times\Omega).$$

The proof is now complete.

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