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by

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Representation Class and Geometrical Invariants of Quantum States under Local Unitary Transformations

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Abstract

We investigate the equivalence of bipartite quantum mixed states under local unitary transformations by introducing representation classes from a geometrical approach. It is shown that two bipartite mixed states are equivalent under local unitary transformations if and only if they have the same representation class. Detailed examples are given on calculating representation classes.

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As a key physical resource in quantum information processing such as quantum cryptography, quantum teleportation and quantum computation [1], quantum entanglement has been recently extensively investigated. Due the fact that the properties of entanglement for multipartite quantum systems remain invariant under local unitary transformations on the individual subsystems, the entanglement can be characterized in principle by all the invariants under local unitary transformations. For instance, the trace norms of realigned or partial transposed density matrices in entanglement measure [2] and separability criteria [3] are some of these invariants. Therefore a complete set of invariants gives rise to the classification of the quantum states under local unitary transformations. Two quantum states are locally equivalent if and only if all these invariants have equal values for these states.

There are many ways to construct such invariants of local unitary transformations. The method developed in [4, 5], in principle, allows one to compute all the invariants of local unitary transformations, though it is not generally operational. There have been some results on calculating invariants related to the equivalence of quantum states under local unitary transformations, e.g. for general two-qubit systems [6], three-qubit states [7, 8], some generic mixed states [9–11], tripartite pure and mixed states [12]. In particular, in terms of the Bloch representation of density matrices for general two-qubit systems, 18 invariants have been presented in [6]. It has been shown that these 18 invariants are sufficient to guarantee that two two-qubit states are equivalent under local unitary transformations, and lack of anyone of these 18 invariants would result in incompleteness of the set of invariants.

However generally we still have no operational criteria to judge the equivalence for two general bipartite mixed states under local unitary transformations. In this letter we investigate the equivalence of quantum states under local unitary transformations according to the spectral decompositions and the Schmidt expressions of the eigenvectors of bipartite density matrices. We give a general theorem on the local equivalence relations. From this theorem one can in principle construct the complete set of invariants under local unitary transformations, according to detailed cases. For comparison we calculate the invariants for two-qubit systems. Marvelously in our scheme we only need at most 12 invariants to characterize the local equivalence of two-qubit systems. As an example we also study in detail the invariants for qubit-qutrit systems.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex Hilbert spaces of dimension m and n respectively, $m \geq n \geq 2$. The tensor space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is a complex Hilbert space of dimension mn. Let ρ and $\tilde{\rho}$ be two bipartite density matrices defined on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. ρ and $\tilde{\rho}$ are said to be equivalent under local unitary transformations if there exist unitary operators U on $\mathcal{H}_1$ and V on $\mathcal{H}_2$ such that

$$\tilde{\rho} = (U \otimes V)\rho(U \otimes V)^\dagger, \tag{1}$$

where $\dagger$ stands for transpose and conjugation.

As a hermitian operator, a mixed state ρ with rank l has the spectral decomposition

$$\rho = \lambda_1 |e_1\rangle\langle e_1| + \cdots + \lambda_l |e_l\rangle\langle e_l|, \tag{2}$$

where λ_i, $i = 1, \cdots, l$, are the nonzero eigenvalues of ρ, $|e_i\rangle$ are the corresponding eigenvectors associated with λ_i, which can be chosen as orthonormal vectors. For convenience we set $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$.

Every eigenvector $|e_i\rangle$ with Schmidt rank k_i has Schmidt decomposition, namely there exist orthonormal vectors a^i_j and b^i_j of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, $j = 1, \cdots, k_i$, such that

$$|e_i\rangle = \mu_i^1 a_1^i \otimes b_1^i + \cdots + \mu_i^{k_i} a_{k_i}^i \otimes b_{k_i}^i, \quad k_i \leq n, \quad i = 1, \cdots, l,$$

where μ_i^j, $j = 1, \cdots, k_i$, are so called Schmidt coefficients satisfying $(\mu_i^1)^2 + \cdots + (\mu_i^{k_i})^2 = 1$. Without loss of generality we assume $\mu_i^1 \geq \mu_i^2 \geq \cdots \geq \mu_i^{k_i} > 0$.

We extend the set of k_1 orthonormal vectors $a_1^1, a_2^1, \cdots, a_{k_1}^1$ to be an orthonormal basis of $\mathcal{H}_1$, $\{a_1, a_2, \cdots, a_{k_1}, \cdots, a_m\}$, and $b_1^1, b_2^1, \cdots, b_{k_1}^1$ to an orthonormal basis of $\mathcal{H}_2$, $\{b_1, b_2, \cdots, b_{k_1}, \cdots, b_n\}$. Therefore the vectors a^i_j and b^i_j, $j = 1, \cdots, k_i$ can be represented according to the two bases respectively,

$$(a_1^i, a_2^i, \cdots, a_{k_i}^i) = (a_1, a_2, \cdots, a_m)X_i, \quad (b_1^i, b_2^i, \cdots, b_{k_i}^i) = (b_1, b_2, \cdots, b_n)Y_i, \tag{3}$$

for some $m \times k_i$ matrix X_i and $n \times k_i$ matrix Y_i. Denote $r(\rho)_i = (\lambda_i, \mu_i^1, \cdots, \mu_i^{k_i}, X_i, Y_i)$, $i = 1, ..., l$. We say that

$$r(\rho) = (r(\rho)_1, \cdots, r(\rho)_l) \tag{4}$$

is a representation of the mixed state ρ. We call the set of all the representations of ρ the representation class of ρ, denoted by $\mathscr{R}(\rho)$.

[Theorem] Two mixed states ρ and $\tilde{\rho}$ of bipartite quantum systems are equivalent under local unitary transformations if and only if they have the same representation class, i.e. $\mathscr{R}(\rho) = \mathscr{R}(\tilde{\rho})$.

[Proof] We firstly prove the sufficient part of the condition. Assume that the mixed states ρ and $\tilde{\rho}$ have the same representation class, $\mathscr{R}(\rho) = \mathscr{R}(\tilde{\rho})$. Hence there exists a representation $r(\rho) \in \mathscr{R}(\rho)$, $r(\tilde{\rho}) \in \mathscr{R}(\tilde{\rho})$ such that $r(\rho) = r(\tilde{\rho})$. Let us assume that $a_1, \cdots, a_m$

and $b_1, \cdots, b_n$ be the orthonormal basis of $\mathcal{H}_1$ and $\mathcal{H}_2$ with respect to the representation $r(\rho)$, $\tilde{a}_1, \cdots, \tilde{a}_m$ and $\tilde{b}_1, \cdots, \tilde{b}_n$ the orthonormal basis of $\mathcal{H}_1$ and $\mathcal{H}_2$ with respect to the representation $r(\tilde{\rho})$.

Then there exist unitary transformations U on $\mathcal{H}_1$ and V on $\mathcal{H}_2$ such that

$$(\tilde{a}_1, \cdots, \tilde{a}_m) = U(a_1, \cdots, a_m), \quad (\tilde{b}_1, \cdots, \tilde{b}_n) = V(b_1, \cdots, b_n).$$

From $r(\rho) = r(\tilde{\rho})$ we have $\lambda_i = \tilde{\lambda}_i$, $\mu_i^j = \tilde{\mu}_i^j$, $X_i = \tilde{X}_i$, $Y_i = \tilde{Y}_i$ for $i = 1, \cdots, l$, $j = 1, \cdots, k_i$. Therefore $\tilde{\rho} = (U \otimes V)\, \rho\, (U^\dagger \otimes V^\dagger)$.

For the necessary pat of the condition, we assume $\tilde{\rho} = (U \otimes V)\, \rho\, (U \otimes V)^\dagger$. If $r(\rho)$ is a representation of ρ, from the spectral decomposition of mixed states, and the properties of the unitary transformations, we have that $r(\rho)$ is also a representation of $\tilde{\rho}$, hence $\mathscr{R}(\rho) \subset \mathscr{R}(\tilde{\rho})$. Similarly, we have $\mathscr{R}(\tilde{\rho}) \subset \mathscr{R}(\rho)$. Therefore $\mathscr{R}(\rho) = \mathscr{R}(\tilde{\rho})$. $\square$

Remark The representations in the representation class of a mixed state are not independent. Actually if one representation is given, the others are also known. And the equivalence of two quantum states can be studied by calculating their representation classes.

We consider as an example the two-qubit systems, $m = n = 2$. Generally, a mixed state ρ has four different eigenvalues $\lambda_i, (i = 1, \cdots, 4)$. Here as the trace of ρ is one, only three eigenvalues are independent. Let $|e_i\rangle$, $i = 1, \cdots, 4$, be the corresponding orthonormal eigenvectors. Since $|e_4\rangle$ is determined by other three eigenvectors up to a scale $e^{i\theta}$, we only need to take into account three eigenvectors. Every eigenvector of these three $|e_i\rangle$, $i = 1, 2, 3$, can have at most Schmidt rank two. But only one of the Schmidt coefficients μ_i^1, μ_i^2 of $|e_i\rangle$, $i = 1, 2, 3$, is independent. Therefore only three eigenvalues and three Schmidt coefficients (all together 6 quantities) are free. The matrices X_1 and Y_1 are unit matrices of order 2. While X_2, Y_2, X_3, Y_3 are unitary matrices of order 2, taking the following form

$$\begin{pmatrix} re^{i\alpha_1} & -\sqrt{1-r^2}e^{-i\alpha_2}e^{i\alpha_3} \\ \sqrt{1-r^2}e^{i\alpha_2} & re^{-i\alpha_1}e^{i\alpha_3} \end{pmatrix},$$

where $r > 0$, $\alpha_1, \alpha_3, \alpha_3 \in \mathbb{R}$. That is, every matrix has four free quantities. Since $|e_i\rangle$, $i = 1, 2, 3$, are perpendicular to each other, and $|e_2\rangle$ and $|e_3\rangle$ are determined up to a phase factor $e^{i\theta}$, there are only 6 free parameters left. Therefore we only need at most 12 invariants to check the local equivalence for two-qubit bipartite quantum systems, which is different from [6] where 18 invariants are needed.

As an example we consider a two-qubit modified Werner state

$$\rho=\begin{pmatrix} (1-e-f)/3 & 0 & 0 & 0 \\ 0 & (1+2f)/6 & (1-4f)/6 & 0 \\ 0 & (1-4f)/6 & (1+2f)/6 & 0 \\ 0 & 0 & 0 & (1+e-f)/3 \end{pmatrix}, \tag{5}$$

where $0\leq f\leq 1-e$, $e\geq 0$. When $e=0$, ρ is just the usual two-qubit Werner state [13], which is separable for $f\leq 1/2$.

ρ has eigenvalues $\lambda_1=(1-f+e)/3$, $\lambda_2=(1-f)/3$, $\lambda_3=(1-f-e)/3$, $\lambda_4=f$, with the corresponding eigenvectors

$$|\nu_1>=\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\quad |\nu_2>=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix},\quad |\nu_3>=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad |\nu_4>=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\\0\end{pmatrix}.$$

Set $a_1=b_1=(0,1)^t$, $a_2=b_2=(1,0)^t$, we have $|\nu_1>=a_1\otimes b_1$. From (3), we have $X_1=Y_1=(1,0)^t$. Up to a global phase factor, the eigenvector $|\nu_2>$ can be expressed as

$$|\nu_2>=\frac{1}{\sqrt{2}}(a_1\otimes b_2+a_2\otimes b_1)\sim\frac{1}{\sqrt{2}}e^{i\theta}(e^{i\theta_2}a_1\otimes e^{-i\theta_2}b_2+e^{i\theta_1}a_2\otimes e^{-i\theta_1}b_1)=\frac{1}{\sqrt{2}}(a_1^2\otimes b_1^2+a_2^2\otimes b_2^2),$$

where $a_1^2=e^{i\theta}e^{i\theta_2}a_1$, $a_2^2=e^{i\theta}e^{i\theta_1}a_2$, $b_1^2=e^{-i\theta_2}b_2$, $b_2^2=e^{-i\theta_1}b_1$. From (3), we have

$$X_2=e^{i\theta}\begin{pmatrix}e^{i\theta_2} & 0\\ 0 & e^{i\theta_1}\end{pmatrix},\quad Y_2=\begin{pmatrix}0 & e^{-i\theta_1}\\ e^{-i\theta_2} & 0\end{pmatrix}.$$

Similarly one can obtain

$$X_3=\begin{pmatrix}0\\ e^{i(\beta+\beta_1)}\end{pmatrix},\quad Y_3=\begin{pmatrix}0\\ e^{-i\beta_1}\end{pmatrix},\quad X_4=e^{i\gamma}\begin{pmatrix}e^{i\gamma_2} & 0\\ 0 & -e^{i\gamma_1}\end{pmatrix},\quad Y_4=\begin{pmatrix}0 & e^{-i\gamma_1}\\ e^{-i\gamma_2} & 0\end{pmatrix}.$$

The representation class $\mathscr{R}(\rho)$ is given by $\mathscr{R}(\rho)=(r(\rho)_1,r(\rho)_2,r(\rho)_3,r(\rho)_4)$, where

$$\begin{aligned} r(\rho)_1&=(\frac{1-e-f}{3},1,X_1,Y_1),\quad r(\rho)_2=(\frac{1-f}{3},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},X_2,Y_2),\\ r(\rho)_3&=(\frac{1+e-f}{3},1,X_3,Y_3),\quad r(\rho)_4=(f,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},X_4,Y_4).\end{aligned} \tag{6}$$

This representation class is parameterized by 8 free parameters. Any states with representations of the form (6), for some given values $\theta,\theta_1,\theta_2,\gamma,\gamma_1,\gamma_2,\beta,\beta_1$, are equivalent to the state (5) under local unitary transformations.

The representation class can be analytically calculated in principle according to detailed situations: the rank of the density matrix, the property of the eigenvalues and the Schmidt rank of the eigenvectors. In the following we investigate as another example in detail the local equivalence of qubit-qutrit systems, $m = 2$, $n = 3$. We compute the representations for the cases that the mixed states have two different nonzero eigenvalues.

Let λ_1 and λ_2 be the two nonzero eigenvalues of ρ, $\lambda_1 > \lambda_2$, $\lambda_1 + \lambda_2 = 1$, with $|e_1\rangle$ and $|e_2\rangle$ the corresponding eigenvectors. Then mixed state ρ has the following spectral decomposition

$$\rho = \lambda_1 |e_1\rangle\langle e_1| + \lambda_2 |e_2\rangle\langle e_2|.$$

As the eigenvalues are different, for given ρ, $|e_j\rangle$ is determined up to a phase factor $e^{i\theta_j}$. We calculate the representation classes according to various cases.

Case 1 The eigenvectors $|e_1\rangle$ and $|e_2\rangle$ are all separable, i.e. the Schmidt rank $k_1 = k_2 = 1$. Their Schmidt decompositions are of the forms

$$|e_1\rangle = a_1^1 \otimes b_1^1, \qquad |e_2\rangle = a_1^2 \otimes b_1^2,$$

where for fixed ρ, a_i and b_i are determined up to a phase factor $e^{i\theta}$. To calculate the matrices X_i, Y_i in (3) we choose the orthonormal basis of $\mathcal{H}_1$ to be $\{a_1, a_2\}$ with $a_1 = a_1^1$, which is determined up to a rotation

$$\left(\begin{array}{cc} e^{i\alpha_1} & 0 \\ 0 & e^{i\alpha_2} \end{array}\right), \qquad \alpha_1,\ \alpha_2 \in \mathbb{R}$$

and the orthonormal basis of $\mathcal{H}_2$ to be $\{b_1, b_2, b_3\}$ with $b_1 = b_1^1$, up to a rotation

$$\left(\begin{array}{cc} e^{i\alpha} & 0 \\ 0 & u(2) \end{array}\right),$$

where $\alpha \in \mathbb{R}$, $u(2) \in U(2)$ is a 2×2 unitary matrix.

Therefore according to (3) if we set $a_1^1 = (a_1, a_2)X_1$, $b_1^1 = (b_1, b_2, b_3)Y_1$, and $a_1^2 = (a_1, a_2)X_2$, $b_1^2 = (b_1, b_2, b_3)Y_2$, we have

$$X_1 = \left(\begin{array}{c} 1 \\ 0 \end{array}\right), \quad Y_1 = \left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right); \qquad X_2 = \left(\begin{array}{cc} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{array}\right) X_2^0, \quad Y_2 = \left(\begin{array}{cc} e^{i\theta} & 0 \\ 0 & u(2) \end{array}\right) Y_2^0, \tag{7}$$

where

$$X_2^0 = \left(\begin{array}{c} x_1 \\ x_2 \end{array}\right), \quad Y_2^0 = \left(\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array}\right) \tag{8}$$

for some x_1, x_2 and $y_1, y_2, y_3 \in \mathbb{C}$. The representation of ρ is given by $r(\rho) = (r(\rho)_1, r(\rho)_2)$, where $r(\rho)_1 = (\lambda_1, 1, X_1, Y_1)$, $r(\rho)_2 = (\lambda_2, 1, X_2, Y_2)$. The representation class is

$$\mathscr{R}(\rho) = \{(r(\rho)_1, r(\rho)_2) \mid \theta_i \in \mathbb{R},\ u(2) \in U(2)\}.$$

Case 2 The eigenvector $|e_1\rangle$ is separable, while $|e_2\rangle$ is entangled, i.e. $k_1 = 1,\ k_2 = 2$. The Schmidt decompositions are of the forms

$$|e_1\rangle = a_1^1 \otimes b_1^1, \quad |e_2\rangle = \mu_2^1\, a_1^2 \otimes b_1^2 + \mu_2^2\, a_2^2 \otimes b_2^2.$$

We choose the orthonormal basis of $\mathcal{H}_1$ (resp. $\mathcal{H}_2$) to be $\{a_1^1, a_2\}$ (resp. $\{b_1^1, b_2, b_3\}$) as defined in the case 1. Then we have the same X_1 and Y_1 as in (7). Set

$$X_2^0 = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \quad Y_2^0 = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \\ y_5 & y_6 \end{pmatrix}. \tag{9}$$

If $\mu_2^1 \neq \mu_2^2$, then $a_1^2,\ b_1^2;\ a_2^2,\ b_2^2$ are determined up to a factor $e^{i\theta}$. We have

$$X_2 = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} X_2^0 \begin{pmatrix} e^{i\beta_1} & 0 \\ 0 & e^{i\beta_2} \end{pmatrix}, \quad Y_2 = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & u(2) \end{pmatrix} Y_2^0 \begin{pmatrix} e^{i\alpha_1} & 0 \\ 0 & e^{i\alpha_2} \end{pmatrix},$$

where $\beta_1 + \alpha_1 = \beta_2 + \alpha_2$. The representations of ρ are $r(\rho) = (r(\rho)_1, r(\rho)_2)$ with $r(\rho)_1 = (\lambda_1, 1, X_1, Y_1)$, $r(\rho)_2 = (\lambda_2, \mu_2^1, \mu_2^2, X_2, Y_2)$. The representation class is given by

$$\mathscr{R}(\rho) = \{(r(\rho)_1, r(\rho)_2) \mid \theta, \beta_i, \theta_i \in \mathbb{R}, \beta_1 + \alpha_1 = \beta_2 + \alpha_2,\ u(2) \in U(2)\}.$$

If $\mu_2^1 = \mu_2^2 = \frac{\sqrt{2}}{2}$, then $a_1^2,\ b_1^2;\ a_2^2,\ b_2^2$ are determined up to a rotation $u(2)$. The representations of ρ are $r(\rho) = (r(\rho)_1, r(\rho)_2)$ with $r(\rho)_1 = (\lambda_1, 1, X_1, Y_1)$, $r(\rho)_2 = (\lambda_2, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, X_2, Y_2)$, where

$$X_2 = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} X_2^0\ u_1(2), \quad Y_2 = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & u_2(2) \end{pmatrix} Y_2^0\ u_1(2)^\dagger, \quad u_1(2),\ u_2(2) \in U(2).$$

The representation class is given by

$$\mathscr{R}(\rho) = \{(r(\rho)_1, r(\rho)_2) \mid \theta, \theta_i \in \mathbb{R};\ u_1(2), u_2(2) \in U(2)\}.$$

Case 3 The eigenvector $|e_2\rangle$ is separable, while $|e_1\rangle$ is entangled, i.e. $k_1 = 2,\ k_2 = 1$. The Schmidt decompositions are of the forms

$$|e_1\rangle = \mu_1^1\, a_1^1 \otimes b_1^1 + \mu_1^2\, a_2^1 \otimes b_2^1, \quad |e_2\rangle = a_1^2 \otimes b_1^2.$$

In the bases $\{a_1^1, a_2^1\}$ (resp. $\{b_1^1, b_2^1, b_3\}$) of $\mathcal{H}_1$ (resp. $\mathcal{H}_2$), at first we have

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{10}$$

If $\mu_1^1 \neq \mu_1^2$, the representations of ρ are $r(\rho) = (r(\rho)_1, r(\rho)_2)$ with $r(\rho)_1 = (\lambda_1, \mu_1^1, \mu_1^2, X_1, Y_1)$, $r(\rho)_2 = (\lambda_2, 1, X_2, Y_2)$, where

$$X_2 = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} X_2^0, \quad Y_2 = \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{i\beta_3} \end{pmatrix} Y_2^0,$$

with $\theta_1 + \beta_1 = \theta_2 + \beta_2$ and X_2^0 and Y_2^0 given in (8). The representation class is

$$\mathscr{R}(\rho) = \{(r(\rho)_1, r(\rho)_2) \mid \beta_i, \theta_i \in \mathbb{R};\; \theta_1 + \beta_1 = \theta_2 + \beta_2\}.$$

If $\mu_1^1 = \mu_1^2 = \frac{\sqrt{2}}{2}$, the representations of ρ are given by $r(\rho) = (r(\rho)_1, r(\rho)_2)$ with $r(\rho)_1 = (\lambda_1, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, X_1, Y_1)$, $r(\rho)_2 = (\lambda_2, 1, X_2, Y_2)$, where

$$X_2 = u(2) X_2^0, \quad Y_2 = \begin{pmatrix} u(2)^\dagger & 0 \\ 0 & e^{i\beta} \end{pmatrix} Y_2^0\, e^{i\theta}.$$

The representation class is

$$\mathscr{R}(\rho) = \{(r(\rho)_1, r(\rho)_2) \mid \beta, \theta \in \mathbb{R};\; u(2) \in U(2)\}.$$

Case 4 Both eigenvectors $|e_1\rangle$ and $|e_2\rangle$ are entangled, i.e. $k_1 = k_2 = 2$. Their Schmidt decompositions are given by

$$|e_1\rangle = \mu_1^1\, a_1^1 \otimes b_1^1 + \mu_1^2\, a_2^1 \otimes b_2^1, \quad |e_2\rangle = \mu_2^1\, a_1^2 \otimes b_1^2 + \mu_2^2\, a_2^2 \otimes b_2^2.$$

In the basis $\{a_1^1, a_2^1\}$ (resp. $\{b_1^1, b_2^1, b_3\}$) of $\mathcal{H}_1$ (resp. $\mathcal{H}_2$), we have X_1 and Y_1 as given in (10). If $\mu_1^1 > \mu_1^2$, $\mu_2^1 > \mu_2^2$, we have

$$X_2 = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} X_2^0 \begin{pmatrix} e^{i\gamma_1} & 0 \\ 0 & e^{i\gamma_2} \end{pmatrix}, \quad Y_2 = \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{i\beta_3} \end{pmatrix} Y_2^0 \begin{pmatrix} e^{i\alpha_1} & 0 \\ 0 & e^{i\alpha_2} \end{pmatrix},$$

with $\theta_1+\beta_1 = \theta_2+\beta_2,\ \gamma_1+\alpha_1 = \gamma_2+\alpha_2$, where X_2^0 and Y_2^0 are given in (9). Hence $r(\rho)_1 = (\lambda_1, \mu_1^1, \mu_1^2, X_1, Y_1)$ and $r(\rho)_2 = (\lambda_2, \mu_2^1, \mu_2^2, X_2, Y_2)$. And the representation class is

$$\mathscr{R}(\rho) = \{(r(\rho)_1, r(\rho)_2) \mid \beta_i, \theta_i, \gamma_i, \alpha_i \in \mathbb{R};\ \theta_1+\beta_1 = \theta_2+\beta_2,\ \gamma_1+\alpha_1 = \gamma_2+\alpha_2\}.$$

If $\mu_1^1 > \mu_1^2,\ \mu_2^1 = \mu_2^2 = \frac{\sqrt{2}}{2}$, we have

$$X_2 = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} X_2^0\, u(2), \quad Y_2 = \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{i\beta_3} \end{pmatrix} Y_2^0\, u(2)^\dagger\, e^{i\theta},$$

with $\theta_1+\beta_1 = \theta_2+\beta_2$. We have $r(\rho)_1 = (\lambda_1, \mu_1^1, \mu_1^2, X_1, Y_1)$ and $r(\rho)_2 = (\lambda_2, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, X_2, Y_2)$. The representation class is

$$\mathscr{R}(\rho) = \{(r(\rho)_1, r(\rho)_2) \mid \theta, \theta_i, \beta_i \in \mathbb{R};\ \theta_1+\beta_1 = \theta_2+\beta_2,\ u(2) \in U(2)\}.$$

If $\mu_1^1 = \mu_1^2 = \frac{\sqrt{2}}{2},\ \mu_2^1 > \mu_2^2$, we have

$$X_2 = u(2) X_2^0 \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix}, \quad Y_2 = \begin{pmatrix} u(2)^\dagger & 0 \\ 0 & e^{i\theta} \end{pmatrix} Y_2^0 \begin{pmatrix} e^{i\beta_1} & 0 \\ 0 & e^{i\beta_2} \end{pmatrix},$$

with $\theta_1+\beta_1 = \theta_2+\beta_2$. In this case we have $r(\rho)_1 = (\lambda_1, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, X_1, Y_1)$ and $r(\rho)_2 = (\lambda_2, \mu_2^1, \mu_2^2, X_2, Y_2)$. And the representation class is given by

$$\mathscr{R}(\rho) = \{(r(\rho)_1, r(\rho)_2) \mid \theta, \theta_i, \beta_i \in \mathbb{R};\ \theta_1+\beta_1 = \theta_2+\beta_2,\ u(2) \in U(2)\}.$$

If $\mu_1^1 = \mu_1^2 = \frac{\sqrt{2}}{2},\ \mu_2^1 = \mu_2^2 = \frac{\sqrt{2}}{2}$, we have

$$X_2 = u_1(2)\, X_2^0\, u_2(2), \quad Y_2 = \begin{pmatrix} u_1(2)^\dagger & 0 \\ 0 & e^{i\theta} \end{pmatrix} Y_2^0\, u_2(2)^\dagger\, e^{i\gamma}$$

and $r(\rho)_1 = (\lambda_1, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, X_1, Y_1)$, $r(\rho)_2 = (\lambda_2, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, X_2, Y_2)$,

$$\mathscr{R}(\rho) = \{(r(\rho)_1, r(\rho)_2) \mid \theta, \gamma \in \mathbb{R};\ u_1(2), u_2(2) \in U(2)\}.$$

In stead of usual algebraic construction of invariants under local unitary transformations, we have presented a geometrical approach to the classification of quantum bipartite mixed states under local unitary transformations, which works for arbitrary $m \times n$ dimensional quantum systems. It has been shown that two bipartite mixed states are equivalent under local unitary transformations if and only if they have the same representation class. As shown in the examples, these representation classes can be calculated in detail according to the eigenvalues and the Schmidt decompositions of the eigenvectors of density matrices. Although general analysis could be rather complicated as one has to take into account many different cases, for a given density matrix the calculation would be quite direct.

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