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by

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Abstract

At first we express the higher order Riccati equation or Faá di Bruno polynomial in terms of the modified Ramanujan differential equations in analogy to the relation of the Chazy III equation and the well known Ramanujan equations for the Eisenstein series of the modular group. We relate Ramanujan's series connected with the pentagonal numbers, introduced by Ramanujan in his Lost Notebook, to the Faá di Bruno polynomials and the Riccati chain determined by the Eisenstein series of weight two for the modular group. As a first step to get an explicit expression for the general term in Ramanujan's polynomial of degree k we derive a formula for the n-th order differential equations this Eisenstein series fulfill.

Keywords and phrases . Eisenstein series, Ramanujan differential equation, Riccati equation, Faá di Bruno polynomial.

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1 Introduction

The ordinary Riccati equation plays a very important role in the solution of nonlinear integrable partial differential equations. Higher order Riccati equations can be obtained by reduction of the Matrix Riccati equation. All the higher order Riccati equations can be linearlized via a Cole-Hopf transformation to linear differential equations. It is known, that the higher order Riccati equations play the role of Bäcklund transformations for integrable partial differential equations of higher order than the KdV equation. The Riccati chain without potential is naturally associated to Faá di Bruno polynomials. The Faá di Bruno polynomials appear in several branches of mathematics and physics and can be introduced in several ways. The generalized Riccati equation can be written in a compact form in terms of Faá di Bruno polynomials. It plays a very important role in integrable systems.

The geometrical meaning of the Faá di Bruno polynomials in the context of KP theory was studied in [10]. They provide a basis in a subspace W of the universal Grassmannian associated to the KP hierarchy. When W comes from geometrical data via the Krichever map, the Faá di Bruno recursion relation turns out to be the cocycle condition for the deformations of the dynamical line bundle on the spectral curve together with the meromorphic sections which give rise to the Krichever map.

Applications of modular forms in integrable systems is a less explored subject in integrable systems. But this issue always draws attention to integrable specialists and number theorists.

In a remarkable paper Takhtajan [22] considers integrable reductions of the self-dual Yang-Mills equations [9]. As is shown by the author, these reductions with infinite-dimensional gauge groups lead to integrable equations with well-known modular forms as τ-functions. For example, the famous cusp form Δ appears as the τ-function of the Chazy equation. The Chazy equation is deeply connected to special automorphic functions. In 1909, Chazy [8] in his study of Painlevé type equations of third order, considered the following nonlinear differential equation

$$Y''' = 2YY'' - 3Y'^2.$$

It is known that a particular solution of the Chazy equation is given in terms of weight 2 Eisenstein series of the full modular group $SL(2, \mathbf{Z})$.

Harnad and McKay [13] studied solutions to a class of differential systems that generalize the Halphen system and which are determined in terms of automorphic functions whose groups are commensurable with the modular group. These functions uniformize Riemann surfaces of genus zero and have q-series with integral coefficients.

Inspired by the work of Ablowitz et. al [1] in this article we study further connections between modified Ramanujan Eisenstein series and Riccati type differential equations. In particular, we express the third order Riccati equation with modified Ramanujan differential equations. We also show that a class of infinite series connected with pentagonal numbers can be expressed in terms of the Faá di Bruno polynomials.

Result In this article we derive two different results:

1. In 1916, Ramanujan [18, 19] introduced the functions $P(q), Q(q)$ and $R(q)$ defined for $|q| < 1$ by
$$P(q) := 1 - 24\sum_{i=1}^{\infty}\sigma_1(n)q^n, \qquad Q(q) := 1 + 240\sum_{i=1}^{\infty}\sigma_3(n)q^n,$$
$$R(q) := 1 - 504\sum_{i=1}^{\infty}\sigma_5(n)q^n.$$
We relate the third order Riccati equation in terms of modified Ramanujan differential equations for certain functions $\tilde{P}, \tilde{Q}$ and $\tilde{R}$ which should be closely related to Ramanujan's Eisenstein series P, Q and R.
2. Ramanujan examines a class of infinite series $\frac{T_{2k}(q)}{(q;q)_\infty}$ (defined later) connected with Euler's pentagonal numbers and expressed the first few of them as polynomials in the functions P , Q and R. In an interesting paper Berndt and Yee [7] found a new general recurrence formula for these poynomials. They have implicitly assumed the Faá di Bruno formula [14, 17], which is hidden in their proof. In this paper we unveil this relationship and prepare a first step in getting explicit expressions for the general term in these polynomials..

2 Eisenstein series and Ramanujan differential equations

Let τ be a complex number with strictly positive imaginary part. In contemporary notation, the Eisenstein series $G_{2k}(\tau)$ of weight $2k$ on the full modular group $\Gamma(1)$, where $k \geq 1$, are defined by
$$G_{2k}(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^{2k}} \qquad m, n \in \mathbf{Z}.$$
Suppose $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$ then for $k > 1$
$$G_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} G_{2k}(\tau)$$
and G_{2k} is therefore a modular form of weight $2k$ for $k \geq 2$.

Let us define $q = e^{2\pi i\tau}$. Then the Fourier series of the Eisenstein series is
$$G_{2k}(\tau) = 2\zeta(2k)\left(1 + c_{2k}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n\right),$$
where the Fourier coefficients c_{2k} are given as
$$c_{2k} = \frac{(2\pi i)^{2k}}{(2k-1)!\zeta(2k)} = \frac{-4k}{B_{2k}},$$

with $\sigma_\nu(n)$ the sum of the ν-th powers of the divisors of n.

This allows us to define the *normalized Eisenstein series* $E_{2k}(\tau)$, with $E_{2k}(i\infty) = 1$ as

$$E_{2k}(\tau) := \frac{G_{2k}(\tau)}{2\zeta(2k)}. \tag{1}$$

In Ramanujan's notation, the three most relevant Eisenstein series are defined for $|q| < 1$ by

$$P(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},\ Q(q) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n},\ R(q) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}.$$

In contemporary notation we have

$$P = E_2, \qquad Q = E_4 \qquad R = E_6.$$

Then Ramanujan's famous differential equations are given as

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12} \qquad q\frac{dQ}{dq} = \frac{PQ - R}{3} \qquad q\frac{dR}{dq} = \frac{PR - Q^2}{2}. \tag{2}$$

Let us introduce the variable y as $q = e^{2y}$. Then Ramanujan's differential equations become

$$\frac{dP}{dy} = \frac{1}{6}(P^2 - Q) \qquad \frac{dQ}{dy} = \frac{2}{3}(PQ - R) \qquad \frac{dR}{dy} = (PR - Q^2),$$

which lead to a third order differential equation for $P(y) = E_2(e^{2y})$:

$$P_{yyy} - 2PP_{yy} + 3P_y^2 = 0. \tag{3}$$

This equation is called the Chazy III equation. It arises in the study of third order ordinary differential equations having the " Painleve property", that means, their solutions have only poles as moveable singularities.

2.1 The Hecke congruence subgroup $\Gamma_0(2)$ and differential equations

The Hecke congruence subgroup $\Gamma_0(2)$ is defined as

$$\Gamma_0(2) := \left\{\gamma \in SL(2,\mathbf{Z}) \middle| \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \, mod\, 2\right\}. \tag{4}$$

Since $\Gamma_0(2)$ has two inequivalent cusps, there are also two independent Eisenstein series. Namely, if E_{2k} is the normalized Eisenstein series of weight $2k$ for $SL(2,\mathbf{Z})$ then it is well known, that $E_{2k}(\tau)$ and $E_{2k}(2\tau)$ are Eisenstein series of the same weight for $\Gamma_0(2)$, also called ' old ' Eisenstein series in Atkin-Lehner's terminology.

Special normalized Eisenstein series of weight $2k$ for $\Gamma_0(2)$ are defined, for integer $k \geq 2$, by

$$\mathcal{E}_{2k}(\tau) = 1 - \frac{4k}{(1-2^{2k})B_{2k}} \sum_{n=1}^{\infty} \frac{(-1)^n n^{2k-1} q^n}{1-q^n}. \tag{5}$$

This Eisenstein series $\mathcal{E}_{2k}(\tau)$ can be simply expressed in terms of the old ones as follows:

$$\mathcal{E}_{2k}(\tau) = \frac{1}{4^k - 1}\big(4^k E_{2k}(2\tau) - E_{2k}(\tau)\big) \tag{6}$$

Ramamani studied Ramanujan-type differential equations for certain Eisenstein series associated with $\Gamma_0(2)$. She introduced three modular functions, given for $|q| < 1$ by

$$\mathcal{P}(q) := 1 - 8\sum_{n=1}^{\infty} \frac{(-1)^n n q^n}{1-q^n}, \qquad \tilde{\mathcal{P}}(q) := 1 + 24\sum_{n=1}^{\infty} \frac{n q^n}{1+q^n}$$

$$\mathcal{Q}(q) := 1 + 16\sum_{n=1}^{\infty} \frac{(-1)^n n^3 q^n}{1-q^n}.$$

By a simple computation it is easy to check that

$$\mathcal{P}(q) = \mathcal{E}_2(\tau) = \frac{4}{3}E_2(2\tau) - \frac{1}{3}E_2(\tau) \tag{7}$$

$$\mathcal{Q}(q) = \mathcal{E}_4(\tau) = \frac{16}{15}E_4(2\tau) - \frac{1}{15}E_4(\tau) \tag{8}$$

$$\tilde{\mathcal{P}}(q) = \frac{1}{2}\big(3\mathcal{E}_2(\tau) - E_2(\tau)\big) = 2E_2(2\tau) - E_2(\tau). \tag{9}$$

Thus all $\mathcal{P}(q)$, $\tilde{\mathcal{P}}(q)$ and $\mathcal{Q}(q)$ are linear combination of $E_{2k}(2\tau)$ and $E_{2k}(\tau)$. One noticable feature compared to the case of the group $SL(2,\mathbb{Z})$ is the absence of the $E_6(\tau)$ term.

Ramamani showed that the functions $\mathcal{P}(q)$, $\tilde{\mathcal{P}}(q)$ and $\mathcal{Q}(q)$ satisfy the following differential equations

$$q\frac{d\mathcal{P}}{dq} = \frac{\mathcal{P}^2 - \mathcal{Q}}{4} \qquad q\frac{d\tilde{\mathcal{P}}}{dq} = \frac{\tilde{\mathcal{P}}\mathcal{P} - \mathcal{Q}}{2} \qquad q\frac{d\mathcal{Q}}{dq} = \mathcal{P}\mathcal{Q} - \tilde{\mathcal{P}}\mathcal{Q} \tag{10}$$

In a recent paper Ablowitz et al. [1] showed that Ramamani's system of differential equations is equivalent to a third order scalar nonlinear ODE found by Bureau [5], whose solutions are given implicitly by a Schwarz triangle function.

3 Riccati equation

The Riccati equation

$$v' = f(x) + g(x)v + h(x)v^2, \tag{11}$$

is a simple first order nonlinear differential equation with quadratic nonlinearity. The solutions of the Riccati equation are free from movable branch points and can have only movable poles. This is a simplest nonlinear differential equation which can be completely solved. The general fractional change of variables

$$v' = \frac{a(x)v + b(x)}{c(x)v + d(x)} \tag{12}$$

transforms one Riccati equation into another one.

In order to solve a Riccati equation, it is enough to know one particular solution: let v_1 be a particular solution of the Riccati equation (11). Consider a new function w defined by

$$w = \frac{1}{v - v_1}.$$

This transforms the Riccati equation into a linear equation satisfied by the new function w

$$\frac{dw}{dx} = -(2v_1 h(x) + g(x))w - h(x). \tag{13}$$

Once it is solved, we go back to v via the relation

$$v = v_1 + \frac{1}{w}.$$

When two solutions $v_1(z)$ and $v_2(z)$ are known, the general solution can be found by means of only one quadrature. In fact the change of variable

$$\tilde{v} = \frac{v - v_1}{v - v_2}$$

transforms the original equation into a homogeneous linear differential equation in the new variable $\tilde{v}$

$$\tilde{v}' = h(z)(v_1(x) - v_2(x))\tilde{v}, \tag{14}$$

whose most general solution is

$$\tilde{v}(x) = \tilde{v}(0)exp(\int_0^x h(s)(v_1(s) - v_2(s))ds.$$

It must be noticed that if for the Riccati equation three particular solutions v_1, v_2, v_3 are known, we can construct all other solutions v without use of any further quadrature. This is obtained using a very simple formula called cross-ratio:

$$\frac{v - v_1}{v - v_2} = k\frac{v_3 - v_1}{v_3 - v_2} \tag{15}$$

where k is an arbitrary constant characterizing each particular solution.

In order to verify the cross-ratio formula consider the following system of equations

$$v' = f(x) + g(x)v + h(x)v^2,$$

$$v_1' = f(x) + g(x)v_1 + h(x)v_1^2$$
$$v_2' = f(x) + g(x)v_2 + h(x)v_2^2$$
$$v_3' = f(x) + g(x)v_3 + h(x)v_3^2$$

Then from the consistency condition we obtain

$$\begin{vmatrix} v' & v^2 & v & 1 \\ v_1' & v_1^2 & v_1 & 1 \\ v_2' & v_2^2 & v_2 & 1 \\ v_3' & v_3^2 & v_3 & 1 \end{vmatrix} = 0.$$

and direct calculation shows that this condition is equivalent to

$$\frac{d}{dt}\left(\frac{v - v_1}{v - v_2} \cdot \frac{v_3 - v_1}{v_3 - v_2}\right) = 0. \tag{16}$$

It was shown by Lie and Scheffers [16] that the Riccati equation is the only ordinary nonlinear differential equation of first order which possesses a nonlinear superposition formula

$$\hat{v}(x) = \frac{v_1(v_3 - v_2) + kv_2(v_1 - v_3)}{v_3 - v_2 + k(v_1 - v_3)}.$$

Therefore the solution $\hat{v}(x)$ is nonlinearly expressed in terms of v_1, v_2, v_3 and k is a constant parameter.

3.1 The Riccati chain and modified Ramanujan differential equation

It is known that by using the Cole-Hopf transformation one obtains a whole class of nonlinear ODEs, which possesses the same kind of properties as the Riccati equation, known as Riccati chain ([11], references therein).

Let L be the following differential operator

$$L = \frac{d}{dx} + v(x). \tag{17}$$

For n any integer the nth–order equation of the Riccati chain is given by the following formula

$$L^n v(x) + \sum_{j=1}^{n-1} \alpha_j(x)\big(L^{j-1}v(x)\big) + \alpha_0(x) = 0, \tag{18}$$

where $\alpha_j(x)$, $j = 0, 1, \cdots N$, are arbitrary functions.

The lowest-order equations for $n = 2$ and $n = 3$ in this chain after the ordinary Riccati equation have the explicit form:

$$v_{xx} + 3v(x)v_x + v^3(x) + \alpha_1(x)v(x) + \alpha_0(x) = 0 \tag{19}$$

$$\begin{aligned} v_{xxx} + 4vv_{xx} + 3v_x^2 + 6v^2v_x + \alpha_2(x)v_x + \\ +v^4(x) + \alpha_2v^2(x) + \alpha_1(x)v(x) + \alpha_0(x) = 0. \end{aligned} \tag{20}$$

Let us consider now the case $\alpha_i = 0$ for all i. Then the third order Riccati equation (20) becomes

$$v_{xxx} + 3v_x^2 + 4vv_{xx} + 6v^2v_x + v^4 = 0. \tag{21}$$

Lemma 1 *The third order Riccati equation (20) is equivalent to*

$$v_{xxx} - 2vv_{xx} + 3v_x^2 = \frac{1}{8}(6v_x - v^2)^2. \tag{22}$$

Proof: The proof follows from rescaling $v \to v/2$ and $x \to -x$.
□

Proposition 1 *The third order Riccati equation*

$$P_{yyy} - 2PP_{yy} + 3P_y^2 = \frac{1}{8}(6P_y - P^2)^2$$

is equivalent to some modified Ramanujan's equations

$$\frac{dP}{dy} = \frac{1}{6}(P^2 - Q) \qquad \frac{dQ}{dy} = \frac{2}{3}(PQ - R)$$

$$\frac{dR}{dy} = (PR + \frac{1}{8}Q^2),$$

which differ from the ordinary equations of Ramanujan only by the factor $1/8$ *in the equation for* R.

Proof: From the first equation we obtain $Q = P^2 - 6P_y$. This yields

$$R = 9P_{yy} - 9PP_y + P^3$$

from the second equation. The last equation can also be rewritten as $R_y = (PR - Q^2(1 - 9/8))$ Substituting the values of R and Q in the last equation we obtain our desired result. If on the other hand P solves equation (22) then Q and R are determined by the modified Ramanujan equations. □

In fact if P solves the second Riccati equation, then Q and R are given by the Faá di Bruno polynomials determined by P. These are the underlying structures also of the ordinary Ramanujan differential equations. It would be interesting to clarify the nature of these functions P, Q and R and especially their relation to the functions of Ramanujan.

3.2 The Restricted Riccati chain and the Faá di Bruno polynomials

In this section we will study the restricted Riccati chain, that is the Riccati chain with all the coefficients α_i vanishing and its relation to the Faá di Bruno formula.

We consider the sequence of derivatives $f^{(j)} = (L^j v)f,\ j = 0, 1, 2, 3, \cdots$ with the operator L in (17) and $f = f(x)$ an arbitrary function. Thereby one arrives at the following functions depending on an increasing number of arguments

$$f^{(0)} \equiv f^{(0)}(x, 0) = v(x)f(x)$$

$$f^{(1)} \equiv f^{(1)}(x, v, v_x) = (\frac{d}{dx} + v)f^{(0)} = (v_x + v^2)f$$

$$f^{(2)} \equiv f^{(2)}(x, v, v_x, v_{xx}) = (\frac{d}{dx} + v)f^{(1)} = (v^3 + 3vv_x + v_{xx})f$$

and so on.

Then the restricted Riccati chain is closely related to the famous *Faá di Bruno polynomials* defined by

$$f^{(j+1)} = (\partial_x + v)f^{(j)} \tag{23}$$

The ordinary Riccati equation $v_x + v^2 + u = 0$ for instance can be written also in the form

$$f^{(1)}(x, v, v_x) + uf = 0, \tag{24}$$

where u is a potential and f an arbitrary function.

3.3 Faá di Bruno's formula and the Riccati chain

In this section we discuss the Faá di Bruno formula and the nature of the differential polynomials appearing in this formula.

If g and h are arbitrary functions with a sufficient number of derivatives, then

$$\frac{d^n}{dt^n} g(h(t)) = \sum \frac{n!}{b_1!b_2! \cdots b_n!} g^{(k)}(h(t)) (\frac{h'(t)}{1!})^{b_1} \cdots (\frac{h^{(n)}(t)}{n!})^{b_n}$$

where the sum is over all different solutions in nonnegative integers $b_1, \cdots, b_n$ of $b_1 + 2b_2 + \cdots + nb_n = n$ and $k := b_1 + \cdots + b_n$. For example, when $n = 3$, Faá di Bruno's formula reads

$$\frac{d^3}{dt^3} g(h(t)) = g'(h(t))h''' + 3g''h'(t)h''(t) + g'''((h(t))\big(h'(t)\big)^3.$$

$$\Longrightarrow \qquad \sum \frac{n!}{b_1!b_2!b_3!} g^{(k)}(h(t)) (\frac{h'(t)}{1!})^{b_1} (\frac{h''(t)}{2!})^{b_2} (\frac{h'''(t)}{3!})^{b_3}$$
$$= g'(h(t))h''' + 3g''h'(t)h''(t) + g'''((h(t))\big(h'(t)\big)^3,$$

where $b_1 + 2b_2 + 3b_3 = 3$ and $k = b_1 + b_2 + b_3$.

One can associate partitions of the sets $\{1, 2, 3, .., n\}$ to the n-th derivative of composite functions $g(h(t))$ in the following way: for $n = 1$ we associate $\{1\}$ with the term $g'(h(t))h'(t)$, for $n = 2$ there are two partitions, namely $\{1, 2\}$ and $\{1\}$, $\{2\}$. To them we associate the term $g'(h(t))h''(t)$ respectively the term $g''(h(t))(h'(t))^2$. Di Bruno's formula can then be reformulated as follows [14]:

Proposition 2 *(Faá di Bruno) If g and h are functions with a sufficient number of derivatives, then*

$$\frac{d^n}{dt^n} g(h(t)) = \sum g^{(k)}(h(t)\left(h'(t)\right)^{b_1}\left(h''(t)\right)^{b_2} \cdots \left(h^{(n)}(t)\right)^{b_n} \tag{25}$$

where the sum is over all partitions of $\{1, 2, \cdots, n\}$, and, for each partition, k is its number of blocks and b_i is the number of blocks with exactly i elements.

There is a direct relation of this formula with the n-th order equation $L^n v = 0$ of the Riccati chain (18) for vanishing α's:

Corollary 1 *For L the operator $L = \frac{d}{dx} + v$, the following formula holds*

$$L^n v(x) = \sum_{\substack{\mathcal{A}=\{A:\, A\subset\{1,..,n+1\},\, \cup\, A=\{1,..,n+1\},\\ b_i=\sharp\{A:\,|A|=i\}\}}} (v)^{b_1}(v')^{b_2}\cdots(v^{(n)})^{b_{n+1}}.$$

where the sum is over all partitions $\mathcal{A}$ of the set $\{1, .., n+1\}$ into non empty different subsets A.

Proof The proof is by induction on n and follows similar arguments in [14]. For $n = 0$ the formula is trivial. Let's assume the formula holds for n. Then one gets

$$L^{n+1}v = LL^n v = L \sum_{\substack{\mathcal{A}=\{A:\, \cup A=\{1,..,n+1\},\\ b_i=\sharp\{A:|A|=i\}\}}} (v)^{b_1}(v')^{b_2}\cdots(v^{(n)})^{b_{n+1}}.$$

Hence inserting expression (17) we get

$$\begin{aligned} L^{n+1}v = & \sum_{\substack{\mathcal{A}=\{A:\, \cup A=\{1,..,n+1\},\\ b_i=\sharp\{A:|A|=i\}\}}} (v)^{b_1+1}(v')^{b_2}\cdots(v^{(n)})^{b_{n+1}} + \\ + & \sum_{i=1}^{n} \sum_{\substack{\mathcal{A}=\{A:\, \cup A=\{1,..,n+1\},\\ b_i=\sharp\{A:|A|=i\}\}}} (v)^{b_1}(v')^{b_2}\cdots b_i(v^{(i-1)})^{b_i-1}(v^{(i)})^{b_{i+1}+1}(v^{(n)})^{b_{n+1}} + \\ + & \sum_{\substack{\mathcal{A}=\{A:\, \cup A=\{1,..,n+1\},\\ b_i=\sharp\{A:|A|=i\}\}}} (v)^{b_1}(v')^{b_2}\cdots(v^{(n)})^{b_{n+1}-1} b_{n+1} v^{(n+1)}. \end{aligned} \tag{26}$$

But given a partition $\mathcal{A} = \{A\}$ of the set of numbers $\{1, ..., n+1\}$ then $\mathcal{A}' = \mathcal{A} \cup \{n+2\}$ defines a partition of the set of numbers $\{1, .., n+2\}$ with $b'_1 = b_1 + 1$ and $b'_i = b_i$, $i =$

$2,..,n+1$ respectively $b_{n+2}=0$. This corresponds just to the first term on the right hand side of eq. (26). Furthermore, for any $1 \leq i \leq n$ and any element $B \in \mathcal{A}$ with $|B| = i$ the set $\mathcal{A}' = \mathcal{A}_B = \{A'\}$, where $A' = \{B \cup \{n+2\}\}$ and $A' = A$ for all $A \in \mathcal{A}, A \neq B$, defines a partition $\mathcal{A}'$ of the set $\{1,...,n+2\}$ with $b'_i = b_i - 1$ and $b'_{i+1} = b_{i+1}+1$ respectively $b'_j = b_j$ for $j \neq i, i+1$. There are exactly b_i such partitions of the set of numbers $\{1,..,n+2\}$. Obviously they give rise to the middle term in eq. (26). The last term in this equation however corresponds to the partition $\mathcal{A}' = \{1,...,n+1\}$ with $b'_i = 0$, $1 \leq i \leq n+1$ and $b_{n+2} = 1$. This concludes the proof of the Corollary.
Hence the n-th order Ricatti equation can be directly expressed with Faá di Bruno's formula.

3.4 Ramanujan's Eisenstein series and Riccati chain

On page 188 of his lost notebook, Ramanujan examines the series

$$\begin{aligned} T_{2k} := T_{2k}(q) := 1 + \sum_{n=1}^{\infty} (-1)^n \{ (6n-1)^{2k} q^{n(3n-1)/2} \\ + (6n+1)^{2k} q^{n(3n+1)/2} \}, \qquad |q| < 1. \end{aligned} \tag{27}$$

One must notice that the exponents $n(3n \pm 1)/2$ are the generalized pentagonal numbers. Ramanujan recorded formulas for $T_{2k}, k = 1, 2, \cdots 6$ in terms of the Eisenstein series P, Q and R. The first three are given by

$$\frac{T_2}{(q,q)_\infty} = P,$$

$$\frac{T_4}{(q,q)_\infty} = 3P^2 - 2Q,$$

$$\frac{T_6}{(q,q)_\infty} = 15P^3 - 30PQ + 16R,$$

$$\frac{T_8}{(q,q)_\infty} = 105P^4 - 420P^2Q + 448PR - 132Q^2$$

etc., where

$$(1-q)(1-q^2)(1-q^3)\cdots := (q,q)_\infty = 1 + \sum_{n=1}^{\infty} (-1)^n [q^{n(3n-1)/2} + q^{n(3n+1)/2}],$$

for $|q| < 1$.

Remark The first formula has an interesting arithmetical interpretation in terms of $\sigma(n) = \sum_{d|n} d$ for $\sigma(0) = -\frac{1}{24}$. Then

$$-24 \sum_{j+k(3k\pm1)/2=n} (-1)^k \sigma(j) = \begin{cases} (-1)^r (6r-1)^2 & \text{if } n = r(3r-1)/2 \\ (-1)^r (6r+1)^2 & \text{if } n = r(3r+1)/2 \\ 0 & \text{otherwise} \end{cases},$$

where the sum is over all nonnegative pairs of integers (j,k) such that $j+k(3k\pm1)/2 = n$.

Ramanujan's work on this page [3] can be considered a continuation of his study of representing certain kinds of series as polynomials in Eisenstein series. It has been shown by Berndt and Yee [7] that the R.H.S. of $T_{2k}/(q;q)_\infty$ contain all products $P^aQ^bR^c$, such that $2a+4b+6c=2k$. This is clearly related to Faá di Bruno polynomials as described in the previous section. Their precise result is the following

Theorem 1 (Berndt-Yee) *Define the polynomials $f_{2k}(P,Q,R)$ (for $k \geq 1$), by*

$$f_{2k}(P,Q,R) := \frac{T_{2k}}{(q;q)_\infty}, \tag{28}$$

then

$$f_{2k}(P,Q,R) + 1.3.\cdots(2k-1)\{P^k - \frac{k(k-1)}{3}P^{k-2}Q + \frac{8k(k-1)(k-2)}{45}P^{k-3}R$$
$$-\frac{11k(k-1)(k-2)(k-3}{210}P^{k-4}Q^2 + \cdots\},$$

where the missing terms represented by $+\cdots$ contain all products $P^aQ^bR^c$, such that $2a+4b+6c=2k$.

Remark Up to now there is no explicit formula known for the coefficients in front of the general product $P^aQ^bR^c$.

The proof of this theorem is given in [7]. The identity for $k=1$ is proved by direct computation:

$$(q;q)_\infty P(q) = (1+24q\frac{d}{dq})(q;q)_\infty = T_2(q).$$

The remaining identities are proved using the formulas

$$24q\frac{d}{dq}T_{2k}(q) = T_{2k+2} - T_{2k}. \tag{29}$$

respectively

$$f_{2k}(P,Q,R) := \frac{T_{2k}}{(q;q)_\infty} = (P+24\frac{d}{dq})^{k-1}P(q) = (\frac{d}{dy}+\tilde{P})^{k-1}\tilde{P} \tag{30}$$

where $\tilde{P}(y) = P(e^{24y})$. Obviously, this formula can be easily related to the Faá di Bruno polynomial in section 3.

Corollary 2 *Suppose P,Q, R satisfy the Ramanujan differential equations. Then Ramanujan's polynomials f_{2k} can be directly related to the Riccati chain $L^k\tilde{P}$ with $L = \frac{d}{dy}+\tilde{P}$.*

By Faá di Bruno's formula the right hand side of f_{2k} can be expressed in terms of the derivatives of the function $\tilde{P}$. Since this function fulfills the differential equation

$$\tilde{P}''' + 36(\tilde{P}')^2 - 24\tilde{P}\tilde{P}'' = 0. \tag{31}$$

all its higher derivatives fulfill the following k-th order equations: if k is even, then

$$\tilde{P}^{(k)} + \sum_{l=0}^{\frac{k-1}{2}} c_l^{(k)} \tilde{P}^{(l)} \tilde{P}^{(k_1-l)} = 0 \tag{32}$$

and if k is odd

$$\tilde{P}^{(k)} + \sum_{l=0}^{\frac{k}{2}-1} c_l^{(k)} \tilde{P}^{(l)} \tilde{P}^{(k_1-l)} = 0. \tag{33}$$

The coefficients $c^{(k)}$ thereby fulfill the following recursion relations:

$$\begin{aligned} c_0^{(k+1)} &= c_0^{(k)} = -2 \times 12 \\ c_{\frac{k}{2}}^{(k+1)} &= c_{\frac{k}{2}-1}^{(k)}, \\ c_l^{(k+1)} &= c_{l-1}^{(k)} + c_l^{(k)},\ 1 \le l \le \frac{k}{2} - 1, \end{aligned} \tag{34}$$

for k even , respectively

$$\begin{aligned} c_0^{(k+1)} &= c_0^{(k)} = -2 \times 12 \\ c_{\frac{k-1}{2}}^{(k+1)} &= 2c_{\frac{k-1}{2}}^{(k)} + c_{\frac{k-1}{2}-1}^{(k)}, \\ c_l^{(k+1)} &= c_{l-1}^{(k)} + c_l^{(k)},\ 1 \le l \le \frac{k-3}{2}, \end{aligned} \tag{35}$$

for k odd. These coefficients can be determined recursively as follows:

For even k,$k = 4 + 2n$, $n \ge 1$ and $k \ge 6$, respectively $2 \le l \le k/2 - 1$ one finds

$$c_l^{(k+1)} = \sum_{r=2l}^{k} c_{l-1}^{(r)} + c_{l-1}^{(2l)}.$$

For $l = k/2$ on the otherhand we get

$$c_{k/2}^{(k+1)} = \sum_{r=1}^{n} 2^{r-1} c_{k/2-r-1}^{(k+1-2r)} + 2^n c_1^{(4)}.$$

For k odd,$k = 2n + 3$, $n \ge 1$, $k \ge 7$ and $2 \le l \le \frac{k-3}{2}$ one also gets $c_l^{(k+1)} = \sum_{r=2l}^{k} c_{l-1}^{(r)} + c_{l-1}^{(2l)}$.

For $l = k - 1/2$ we find

$$c_{\frac{k-1}{2}}^{(k+1)} = \sum_{r=1}^{n} 2^{r-1} c_{\frac{k-1-2r}{2}}^{(k+2-2r)} + 2^n c_1^{(4)}$$

Together with readily obtained formulas

$$c_0^{(k)} = -2.12 \qquad \text{and} \qquad c_1^{(k)} = -2(k-6)\times 12$$

the above relations determine the coefficients $c_l^{(k)}$ completely.

It would be nice to have explicit expressions for the $c_l^{(k)}$. For $l=2$ and $l=3$ they read

$$c_2^{(k+1)} = -12(k^2-11k+20)$$

and

$$c_3^{(k+2)} = -4k^3+60k^2-176k+120.$$

It turns out that $c_3^{(k+2)}$ is always a multiple of 12.

By using these recursion relations, formula (30) and Ramanujan's equations for the functions P, Q and R we hope to give another proof of the result of Berndt et al. which also includes an explicit expression for the coefficient in front of $P^aQ^bR^c$.

□

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