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Virasoro Action on Pseudo-differential Symbols and (Noncommutative) Supersymmetric Peakon Type Integrable Systems

by

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Abstract

Using Grozman's formalism of invariant differential operators we demonstrate the derivation of $N = 2$ Camassa-Holm equation from the action of $Vect(S^{1|2})$ on the space of pseudo-differential symbols. We also use generalized logarithmic 2-cocycles to derive $N = 2$ super KdV equations. We show this method is equally effective to derive Camassa-Holm family of equations and these system of equations can also be interpreted as geodesic flows on the Bott-Virasoro group with respect to right invariant H^1- metric. In the second half of the paper we focus on the derivations of the fermionic extension of a new peakon type systems. This new one-parameter family of $N = 1$ super peakon type equations, known as $N = 1$ super b- field equations, are derived from the action of $Vect(S^{1|1})$ on tensor densities of arbitrary weights. Finally, using the formal Moyal deformed action of $Vect(S^{1|1})$ on the space of Pseudo-differential symbols to derive the noncommutative analogues of $N = 1$ super b- field equations.

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1 Introduction

Noncommutative geometry [5] extends the notions of classical differential geometry from differential manifold to discrete spaces, like finite sets and fractals, and noncommutative spaces which are given by noncommutative associative algebras. It was an idea of Descartes that we can study a space by means of functions on the space, in other words, the algebra of functions determines the space. Quantum physics suggests that some physical systems should be modeled by spaces on which functions are not commutative. In fact, C^*- algebras are natural models for the function algebras. In recent years it is appreciated that such noncommutative spaces retain a rich topology and geometry expressed in terms of K-theory and K-homology, as well as in finer aspects of the theory. The subject has also been approached from a more algebraic side with the advent of quantum groups and quantum homogeneous spaces [31].

Noncommutative geometry has recently been involved in a noncommutaive gauge theory related to strings. Noncommutative spaces are characterized by the noncommutative coordinates

$$[x_i, x_j] = i\theta^{ij}, \tag{1}$$

where θ^{ij} are real constants. During the last few years there has been a steady growth in the interest in noncommutative geometry, which appears in string theory in several ways. Much attention has been paid also to field theories on noncommutative spaces and especially Moyal deformed [33] space-time. This theory appears as certain limits of strings, D-branes and M-theory.

Noncommutative gauge theories are naively realized from ordinary commutative theories just by replacing all products of the fields with $\star$ product. String theory proposed a new non-commutative gauge theories that describe the dynamics of branes. The case of a general non-Abelian gauge group is dealt with a construction of a Seiberg-Witten map [39], i.e. a map which connects the gauge theory on the noncommutative space with gauge theory on a commutative space and for this purpose the $\star$-product formalism is used.

Several classical integrable models have been generalized to noncommutative spaces [10, 20]. Also, under the Moyal deformation, the self-dual Yang-Mills equation is considered to preserve the integrability in the same sense as in commutative cases. Noncommutative KdV and nonlinear Schrödinger equations are derived from the reduction of self dual Yang-Mills equation [21, 28] and other methods [13-16]. There exist a method, namely the bicomplex method [10], to yield noncommutative integrable equations which have many conserved quantities. Certainly all these equations are derived formally from the Lax representation by replacing ordinary product by $\star$ product.

Noncommutative extension of integrable systems such as the KdV equation, the super KdV equation are also one of the hot topics in noncommutative geometry and physics. In fact, some time ago Kupershmidt [26] considered a generalization of the Moyal approach to the problem of quantization of classical integrable dynamical systems. The nature of these noncommutative integrable equations are strange. They do not have good integrable property and on top of that Noncommutative extension of

$(1+1)$-dimensional equations introduces infinite number of time derivatives. But they do possess the existence of infinite number of conserved quantities which are widely accepted as definition of complete integrability of underlying equations.

Roughly there are *two different methods* to construct such noncommutative extension of integrable systems. By deforming the Lax equation (for example, [20]) one can derive such equations. This is a bit adhoc formalism and does not incorporate geometry. Recently a deformation of the algebra of diffeomorphism is constructed by Wess and coworkers [2] for canonically deformed spaces with constant deformation parameter θ. Using this method we constructed the noncommuative version of periodic KdV and the Burgers equation [17], and this gives us the second method for the construction of Moyal deformed integrable systems. Although this method is far more geometrical than Lax equation method but certainly we faced problems for not having a proper (noncommutative) Hamiltonian formalism, since infinite time derivatives hidden inside the $\star$–product. One must remember that the deformation theory is a homotopy theory and does not incorporate dynamics easily.

The **motivation** of this present article to offer another interesting method of construction of noncommutative integrable systems. This method has been sketched in our earlier papers [15, 16, 18]. We embed the vector field and its dual to pseudodifferential symbols on S^1 [35, 38]. These are functions on cotangent bundle. We use logarithmic 2-cocycle [25, 27] to derive the dispersion term. Therefore we lift the systems on the space of pseudodifferential symbols where the natural action of vector field on its dual is given by Poisson action. Thus coadjoint action [16] of Virasoro algebra on its dual can be manifested in a simple manner. In this paper we apply Grozman [14] programme of invariant bilinear differential operators on tensor fields to compute this action. It is shown that this scheme can be applied to the $N = 2$ supersymmetry theory [41] provided one must take care of anticommutative properties of fermions. In this work we reexamine the proof of $N = 2$ supersymmetric KdV equation using Grozman prescription. We also derive the $N = 2$ supersymmetric Camassa-Holm equation from our method. In both cases we derive explicit representation of the second Hamiltonian structures.

By deforming the Poisson action to Moyal action we obtain the quantum integrable systems. In fact Kupershmidt [26] also proposed such method to quantize integrable systems. We argue that this method is deeply rooted inside the Moyal-Weyl-Wigner formalism. The Moyal algebra $(C^\infty(M), \{.,.\}_{Moyal})$ is an algebra of quantum observables and it can be continously reduced the Poisson algebra $(C^\infty(M), \{.,.\}_{PB})$ of classical observables. Using this new approach we derive various integrable and superintegrable systems. Using this approach we derive various integrable and superintegrable systems. In this method we are able to quantize a completely new and exotic super b-field equation. This new class of partial differential equations recently obtained by Degasperis Holm and Hone [6, 7] using the asymptotic integrability method. The second member of this one-parameter family of pdes is called Degasperis-Procesi equation [8]. Degasperis et al. proved the exact integrability of the new equation by constructing its Lax pair and explain its relation to a negative flow in the Kaup-Kupershmidt hierarchy via a reciprocal transformation.

Devchand and Schiff [9] showed that the fermionic extension of the Camassa-Holm equation arises as a geodesic flows of an invariant H^1-metric on the group of superconformal transformations. In this paper we focus on to fermionic extension of the b-field equations. We derive this equation from the action of $Vect(S^{1|1})$ on tensor densities of weight b form embedded in $\psi D(S^1)$. We also derive the noncommutative version of these system of equations. Therefore we also able to quantize a completely new and exotic super b-field equation. The *advantage* of this method is that we can avoid the the quantization of coadjoint orbit to derive quantum (or noncommutative) integrable systems.

This paper is **organized** as follows: In Section 2 we give a brief description of pseudodifferential symbols on S^1 and the construction of KdV equations. We introduce generalized Souriau cocycle in Section 3. In this section we demonstrate the construction of the $N = 2$ super KdV equation. Section 4 is devoted to the construction of the $N = 2$ supersymmetric Camassa-Holm equations. We also show its geodesic connection to superconformal group. Section 5 is devoted to the construction of the b field and super b-field equations. Finally, the noncommutative (or Moyal deformed) analogue of the super peakon type equations are given in Section 6.

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2 Background: Pseudodifferential Symbols on S^1 and KdV equation

The ring of pseudodifferential symbols on S^1, $\Psi D(S^1)$, is defined to be the ring of formal Laurent series $\sum_{k\geq k_0} f_k(x)\,\xi^k$ over $C^\infty(S^1)$ with finite number of positive powers. There are two differentiations defined in this ring $\Psi D(S^1)$:

$$\partial_\xi : \sum_k f_k\xi^k \longmapsto \sum_k k f_k \xi^{k-1} \qquad \partial : \sum_k f_k\xi^k \longmapsto \sum_k f_k'\xi^k. \tag{2}$$

These differentiations may be used to define the symbolic multiplication on the ring by setting

$$(F\circ G) = \sum_{k=0}^{\infty} \frac{1}{k!} : \frac{\partial^k F}{\partial \xi^k}(x,\xi)\frac{\partial^k G}{\partial x^k}(x,\xi) : .$$

Here : : is called the Normal Ordering; it is defined by

$$: f(x)\xi^k g(x)\xi^l := f(x)g(x)\xi^{k+l}.$$

This multiplication rules yields an associative and and a Lie algebra operation on the ring. The commutator and the residue map are defined, respectively, by setting $[F,G] = F \circ G - G \circ F$ and $\text{res} \; : \psi D(S^1) \to C^\infty(S^1)$. The "trace operation" is defined by

$$Tr(F) = \int_{S^1} res \; F dx = \int_{S^1} f_{-1} dx. \tag{3}$$

The main property of the residue (or trace) is for $F, G \in \psi D(S^1)$,

$$Tr([F,G]) = \int res \, [F,G] \, dx = 0 \, .$$

The embedding of the vector field $\pi(f(x)\partial) = f(x)\xi$ enables one to pass from the Virasoro algebra to the algebra of groups of area preserving diffeomorphisms. The dual of the vector field is identified with $u(x)\xi^{-2} \in \mathcal{F}_2$.

This algebra of embedded vector fields can be extended via logathimic cocycle.

Theorem 1 (Kravchenko and Khesin) *Let F and G be pseudo differential symbols on the circle The nontrivial central extension $\psi D(S^1)$ is given by the cocycle*

$$c(F,G) = \int \, res \, ([ln \, \xi, F], G) \, . \tag{4}$$

The restriction of Kravchenko-Khesin cocycle [25] to the subalgebra of vector fields is the Gelfand–Fuchs cocycle of the Virasoro algebra. This follows from a simple calculation

$$\begin{aligned} & c(f(x)\xi, g(x)\xi) = \int res \, ([ln\xi, f(x)\xi] \circ g(x)\xi) \\ = \; & \int res(f'(x) - \tfrac{f''(x)}{2}\xi^{-1} + \tfrac{f'''(x)}{3}\xi^{-2} + \cdots)g(x)\xi \\ = \; & \tfrac{1}{6} \int f'''(x) g(x) \, dx. \end{aligned}$$

The term containing the $\xi^{-2} \in \mathcal{F}_2$ in the expansion

$$\mathbb{S}(f(x)\frac{d}{dx}) = f'''(x)\xi^{-2}$$

is called the *Souriau cocycle* of the Virasoro algebra. One should note that the space of tensor density of degree 2 is the regular dual of $Vect(S^1)$.

2.1 Construction of full KdV equation

Our goal is to give a geometric formulation of noncommutative integrable systems. Let us reformulate the Euler-Poincaré flow in a following form:

Definition 1 *Let dH be the gradient of the Hamiltonian function $H(u)$. The Euler-Poincaré equation induced by the action of $Vect(S^1)$ on its dual is:*

$$u_t = ad^*_{(dH)\xi} u\xi^{-2} \equiv \{(dH)\xi, u\xi^{-2}\}. \tag{5}$$

Theorem 2 *The Euler-Poincaré flow which is induced by the action of $Vect(S^1)$ on the extended space of $\psi D(S^1)$ is defined as*

$$u_t = \{\frac{\delta H}{\delta u}\xi, u\xi^{-2}\} + (\frac{\delta H}{\delta u})'''\xi^{-2}\,. \tag{6}$$

Let $H = \frac{1}{2}\int_{S^1} u^2\, dx$. This yields the KdV equation

$$u_t + u_{xxx} + 3uu_x = 0\,. \tag{7}$$

Proof: The action of $Vect(S^1)$ on the extended $\Psi D(S^1)$ comes from two sources; (a) the orginal action of $Vect(S^1)$ on $\Psi D(S^1)$ and (b) Souraiu term. The term $f'''(x)\xi^{-2}$ in the following equation is the Souraiu term.

The coadjoint action of $Vect(S^1)$ on its dual $u(x)\xi^{-2}$ is given by

$$\begin{aligned} & ad^*_{f(x)\xi}u(x)\xi^{-2} = \{f(x)\xi, u(x)\xi^{-2}\} \\ = \;& -(\frac{\partial}{\partial\xi}(f(x)\xi)\frac{\partial}{\partial x}(u\xi^{-2} - \frac{\partial}{\partial x}(f(x)\xi)\frac{\partial}{\partial\xi}(u\xi^{-2}) \\ = \;& -(fu' + 2f'u)\xi^{-2}. \end{aligned}$$

Therefore the action of $Vect(S^1)$ on the extended $\Psi D(S^1)$ is given by

$$\begin{aligned} \widetilde{a}d^*_{f(x)\xi}(u(x)\xi^{-2}, f'''(x)\xi^{-2}) &= \{f(x)\xi, u(x)\xi^{-2}\} + f'''\xi^{-2} \\ &= (fu' + 2f'u + f''')\xi^{-2}\,. \end{aligned} \tag{8}$$

One can easily use equation (8) to compute the second Hamiltonian operator $\mathcal{O}_{KdV}$ of the KdV equation which corresponds to the action of $Vect(S^1)$ on the extended $\psi D(S^1)$ given by:

$$\mathcal{O}_{KdV} = \partial u + u\partial + \partial^3. \tag{9}$$

□

3 The $N = 2$ Neveu-Schwarz super algebra and construction of the $N = 2$ Super KdV equation

Let us introduce the Neveu-Schwarz superalgebra. Consider the space of $-1/2$-tensor densities $\mathcal{F}_{1/2}$ on S^1. There exists a natural Lie superalgebra structure on the space of $Vect(S^1) \oplus \mathcal{F}_{1/2}$. The anticommutator

$$[,]_+ : \mathcal{F}_{1/2} \otimes \mathcal{F}_{1/2} \longrightarrow Vect(S^1) \in \mathcal{F}_1$$

is just the product of tensor densities:

$$[\psi(x)(dx)^{-1/2}, \phi(x)(dx)^{-1/2}]_+ := \psi(x)\,\phi(x)\frac{d}{dx}.$$

The space of $-1/2$ densities on S^1 can be periodic or anti-periodic

$$\phi(x)(dx)^{-1/2}, \qquad \phi(x+2\pi) = \pm\phi(x),$$

known as Ramond space and Neveu-Schwarz space respectively and these spaces of $-1/2$-tensor densities are $Vect(S^1)$-module. The Neveu-Schwarz space is customarily assigned by $\mathcal{F}^-_{1/2}$.

Definition 2 *Let $\mathcal{F}^-_{-3/2}$ be the space of antiperiodic $3/2$ densities. The space*

$$\mathfrak{g} = Vect(S^1) \oplus \mathcal{F}^-_{1/2} \oplus \mathcal{F}^-_{1/2} \oplus C^\infty(S^1)$$

defines a $N=2$ Lie superalgebra structure, it is known as $N=2$ Neveu-Schwarz algebra $\mathfrak{g}$. The (regular) dual space of the Neveu-Schwarz algebra is given as

$$\mathfrak{g}^* = \mathcal{F}_{-2} \oplus \mathcal{F}^-_{-3/2} \oplus \mathcal{F}^-_{-3/2} \oplus \mathcal{F}^-_{-1}$$

Since we consider only the Neveu-Schwarz algebra so we drop the '$-$' sign from $\mathcal{F}$.

Let us now embed the Neveu-Schwarz algebra into space of pseudodifferential symbols. We obtain following mappings for algebra

$$(f(x)\frac{d}{dx} + \phi_i(x)(dx)^{-1/2} + g(x)) \longmapsto (f(x)\xi + \phi_i(x)\xi^{1/2} + g(x)),$$

and the corresponding dual is given by

$$(u(x)dx^2 + \eta_i(x)(dx)^{3/2}) + w(x)(dx) \longmapsto (u(x)\xi^{-2} + \eta_i(x)\xi^{-3/2} + w(x)\xi^{-1}),$$

where sum over repeated index is implied. One must note that $N=2$ superconformal algebra has two fermionic fields, denoted by ϕ_1 and ϕ_2, and two bosonic fields $u(x)$ and $w(x)$.

Proposition 1 (Grozman) *The classification of invariant bilinear differential operators on tensor fields is due to P. Grozman. Let us recall the zeroth-order and the first order cases:*

1. *There exists a suitable constant so that a zeroth-order operator $\mathcal{F}_\nu \otimes \mathcal{F}_\mu \longrightarrow \mathcal{F}_{\nu+\mu}$ has the form*
$$\phi(x)(dx)^\nu \otimes u(x)(dx)^\mu \longmapsto k\phi(x)u(x)(dx)^{\nu+\mu}\,. \tag{10}$$

2. *Every first order operator $\mathcal{F}_\nu \otimes \mathcal{F}_\mu \longrightarrow \mathcal{F}_{\nu+\mu+1}$ is given by*
$$\{\phi(x)(dx)^\nu, u(x)(dx)^\mu\} = (\nu\phi(x)u'(x) - \lambda\phi'(x)u(x))(dx)^{\nu+\mu+1}. \tag{11}$$
For every $(\nu,\mu) \neq (0,0)$, the operator (11) is the only $Diff(S^1)$ operator, otherwise there are two linearly independent operators $\phi d(u)$ and $d(\phi)u$, where d is the de Rham differential operator.

Let us study the action of the Neveu-Schwarz algebra on $u(x)\xi^{-2} + \eta(x)\xi^{-3/2}$. The computation of this action is based on the Grozman formalism [14] of bilinear invariant differential operators on tensor densities.

Lemma 1 *The Hamiltonian operator $\mathcal{O}$ corresponding to the action of the $N = 2$ Neveu-Schwarz algebra on $u(x)\xi^{-2} + \eta_i(x)\xi^{-3/2} + w(x)\xi^{-1} \in \Psi D(S^1)$ yields*

$$\mathcal{O} = \begin{pmatrix} u\partial + \partial u & \frac{1}{2}\partial\eta_1 + \eta_1\partial & \frac{1}{2}\partial\eta_2 + \eta_2\partial & w\partial_x \\ \partial\eta_1 + \frac{1}{2}\eta_1\partial & -\frac{1}{2}u & \frac{1}{2}\partial w + \frac{1}{2}w\partial & \frac{k_4}{2}\eta_2 \\ \partial\eta_2 + \frac{1}{2}\eta_2\partial & -\frac{1}{2}w\partial - \frac{1}{2}\partial w & -\frac{1}{2}u & \frac{k_3}{2}\eta_1 \\ \partial w & \frac{k_1}{2}\eta_2 & \frac{k_2}{2}\eta_1 & 0 \end{pmatrix}. \tag{12}$$

Proof: Let $\mathcal{F}_\lambda$ be the space of $-\lambda$-densities on S^1. Any zeroth-order differential operator is the operator of multiplication by a $(\mu - \lambda)$ density:

$$\phi(x)(dx)^{\mu-\lambda} \ : \ u(x))(dx)^\lambda \longmapsto \phi(x)u(x)(dx)^\mu.$$

In standard Darboux coordinates this can be written as:

$$\phi(x)\xi^{-\mu+\lambda} \ : \ u(x))\xi^{-\lambda} \longmapsto \phi(x)u(x)\xi^{-\mu}.$$

Moreover, there exists a pair of duals $(\eta_1(dx)^{3/2}, \eta_2(dx)^{3/2})$ corresponding to a pair $-1/2$ densities functions $(\phi_1(x)(dx)^{-1/2}, \phi_2(x)(dx)^{-1/2})$. It is clear that $\phi_1(x)$ acts on its dual η_1 in an obvious way and η_2 by the principle of zeroth-order differential operator.

Hence, the action of $f(x)\xi + \phi_i(x)\xi^{1/2} + g(x) \in sVect(S^1)$ on its dual is:

$$\begin{aligned} & ad^*_{f(x)\xi+\phi_i(x)\xi^{1/2}+g(x)}(u(x)\xi^{-2} + \eta_i(x)\xi^{-3/2} + w(x)\xi^{-1}) \\ = \ & \{f(x)\xi, u(x)\xi^{-2} + \eta_i(x)\xi^{-3/2} + w(x)\xi^{-1}\} + \{\phi_1(x)\xi^{1/2}, \eta_1(x)\xi^{-3/2} + w(x)\xi^{-1}\} \\ & +\{\phi_2(x)\xi^{1/2}, \eta_2(x)\xi^{-3/2}\} - \{\phi_2(x)\xi^{1/2}, w(x)\xi^{-1}\} + \{g(x), w(x)\xi^{-1}\} + \frac{k}{2}\phi_i(x)u(x)\xi^{-3/2} \\ & \quad +\frac{k_1}{2}\phi_1(x)\eta_2\xi^{-1} + \frac{k_2}{2}\phi_2(x)\eta_1(x)\xi^{-1} + \frac{k_3}{2}g(x)\eta_1\xi^{-3/2} + \frac{k_4}{2}g(x)\eta_2\xi^{-3/2} \end{aligned}$$

where the last expression follows from the definition of zeroth operator and we have chosen $k = -\frac{1}{2}$. Therefore, we obtain the Hamiltonian operator from this expression, thereby completing the proof:

$$\begin{aligned} = & \Big(fu' + 2f'u + \tfrac{1}{2}\phi_i\eta_i' + \tfrac{3}{2}\phi_i'\eta_i + fw' + f'w\Big)\xi^{-2} + \\ & \Big(f(x)\eta' + \tfrac{3}{2}f'\eta_i(x) - \tfrac{1}{2}\phi_i(x)u(x) + (-1)^{i-1}\tfrac{1}{2}(\phi_i w' + \phi_i w') + \tfrac{k_3}{2}g(x)\eta_1 + \tfrac{k_4}{2}g(x)\eta_2\Big)\xi^{-3/2} \\ & +\Big(g(x)w'(x) + \tfrac{k_1}{2}\phi_1\eta_2 + \tfrac{k_2}{2}\phi_2(x)\eta_1(x)\Big)\xi^{-1}. \qquad \square \end{aligned}$$

Comment One must careful to apply Grozman prescription for extended supersymmetry case. In this situation we encounter multiple fermions. Thus the action should respect the ordering of fermions. This does not appear in $N = 1$ case, since we just play with a single fermion.

Remark All these constants k and k_is are free parameters. These play an important role in integrability. For $N = 1$ case we have just one parameter k and and its value directly connected to nature of the system. It can be shown that the supersymmetric may be bihamiltonian for particular values ks. Following Oevel and Popowicz [37] we assume $k = -1 = k_4 = -k_3 = -k_2 = k_1$. Many of these k_is are chosen in such a way so that one can perform cancellation of certain terms.

Our goal is to construct the noncommutative analogue of the super-KdV equation. Thus, once again we use the modified definition of the Euler-Poincaré equation to obtain the dispersionless super-KdV equation.

Definition 3 *The Euler-Poincaré equation induced by the action of supersymmetric vector field $sVect(S^1)$ on its dual $\mathcal{F}_{-2} + \mathcal{F}_{-3/2} + \mathcal{F}_{-3/2} + \mathcal{F}_{-1}$ is defined as*

$$\begin{aligned} & (u\xi^{-2} + \eta_1\xi^{-3/2} + \eta_2\xi^{-3/2} + w\xi^{-1})_t \\ = \; & ad^*_{(\frac{\delta H}{\delta u}\xi + \frac{\delta H}{\delta \eta_1}\xi^{1/2} + \frac{\delta H}{\delta \eta_2}\xi^{1/2} + \frac{\delta H}{\delta w})}(u\xi^{-2} + \eta_1\xi^{-3/2} + \eta_2\xi^{-3/2} + w\xi^{-1}) \end{aligned} \tag{13}$$

3.1 Generalization of Souriau cocycle to $N = 2$ superconformal algebras

Let Φ_i and Ψ_i be the odd parts of the super $\psi D(S^1)$. The nontrivial central extension of the Fermionic part of the super $\psi D(S^1)$ is:

$$c_{fermionic}(\Phi_i, \Psi_i) = \int res([ln\xi, \Phi_i(x)]\Psi_i)\, dx\,. \tag{14}$$

Let us compute the pair of cocycles connected to the fermions.

$$\begin{aligned} & c_{fermionic}(\phi_i(x)\xi^{1/2}, \psi_i(x)\xi^{1/2}) = \int res([ln\xi, \phi_i(x)\xi^{1/2}]\psi_i\xi^{1/2})\, dx \\ = \; & \int res((\phi_i'\xi^{-1/2} - \tfrac{1}{2}\phi_i''\xi^{-3/2} + \cdots)\psi_i\xi^{1/2}) \\ = \; & \int res(\cdots - -\tfrac{1}{2}\phi_i''\psi_i\xi^{-1} + \cdots) = \tfrac{1}{2}\int \phi_i''\psi_i\, dx. \end{aligned}$$

There exists another bosonic cocycle in $N = 2$ superconformal algebra, given by

$$\begin{aligned} & cboson(v(x), w(x)) = \int res\,([ln\xi, v(x)] \circ w(x)) \\ = \; & \int v'(x)w(x)\, dx. \end{aligned}$$

Thus using the definition of logrithmic cocycle we derive the two cocycle of $N = 2$ Neveu-Schwarz algebra.

Proposition 2 *The expression of $N = 2$ superalgebra 2 cocycle in component form is given as*

$$\Omega\Big((f_1\frac{d}{dx}, \phi_1(dx)^{-1/2}, \phi_2(dx)^{-1/2}, f_2(x)), (g_1\frac{d}{dx}, \psi_1(dx)^{-1/2}, \psi_2(dx)^{-1/2}g_2(x))\Big)$$

$$= \int_{S^1} (f_1''g_1' + \phi_1\psi_1'' + \phi_2\psi_2'' - f_2g_2')\, dx.$$

Thus $N=2$ Lie superalgebra can be extended by the two cycle Ω. This allows us to compute the super-Hamiltonian operator for $N=2$ super KdV equation.

Proposition 3 *The super-Hamiltonian operator $\mathcal{O}_{skdv_2}$ corresponding to the action of the centrally extended $N=2$ Neveu-Schwarz algebra on its extended dual $\Big(u(x)\xi^{-2}+\eta_i(x)\xi^{-3/2}+w(x)\xi^{-1},\mathbf{c}\Big)$ yields*

$$\mathcal{O}_{skdv_2}=\begin{pmatrix} -\partial^3+2u\partial+2\partial u & \partial\eta_1+2\eta_1\partial & \partial\eta_2+2\eta_2\partial & 2w\partial_x \\ 2\partial\eta_1+\eta_1\partial & \partial^2-u & \partial w+w\partial & -\eta_2 \\ 2\partial\eta_2+\eta_2\partial & -w\partial-\partial w & \partial^2-u & \eta_1 \\ \partial w & \eta_2 & -\eta_1 & \partial \end{pmatrix}, \tag{15}$$

where we have normalized the operator $\mathcal{O}$.

Thus we give alternative derivation of super-Hamiltonian operator, given by [37]. We consider the following Laberge-Mathieu [29] Hamiltonian function

$$H=\frac{1}{2}\int(u^2-ww'-\eta_i\eta_i'+auw^2-2aw\eta_1\eta_2)\,dx \tag{16}$$

to study flow on the orbits of $N=2$ superconformal algebra. Using bosonic superfield

$$\Phi=\theta_2\theta_1u(x)+\theta_1\eta_1+\theta_2\eta_2+w(x)$$

Hamiltonian H can be rewritten as

$$H=\frac{1}{2}\int dxd\theta_1d\theta_2(\Phi D_1D_2\Phi+\frac{a}{3}\Phi^3).$$

Using the Hamiltonian operator $\mathcal{O}_{skdv_2}$ and Hamiltonian H we obtain $N=2$ super KdV equation, given by Labelle and Mathieu [see [29], eqn. (2.4), also citeop].

4 Geodesic Flows on group of area preserving diffeomorphisms on cyclider

Groups of area-preserving diffeomorphisms and their Lie algebras play an important role in modern physics literature. It is known that in a suitable basis, the Lie algebra of the group $SDiff(2)$ tends to that of $SU(N)$ as $N\to\infty$.

Using Adler's trace formula [1] we fix the ad-invariant quantity

$$TrL=\int_C Ldxd\xi. \tag{17}$$

Thus, we can define a weakly nondegenerate invariant inner product [3] on $sdiff(2)$ by

$$<L,M> \;=\; Tr(LM) \;=\; \int_C LM\,dxd\xi \qquad L,M\in sdiff(2). \tag{18}$$

The Lie-Poisson bracket on $sdiff(2)$ is given by

$$\{\{f,g\}\}(\alpha)=<\alpha,\{\frac{\delta f}{\delta\alpha},\frac{\delta g}{\delta\alpha}\}>, \tag{19}$$

where $\frac{\delta f}{\delta\alpha}$ denotes the Frechét derivative. Here we have used the double curly bracket notations from Bloch et. al [3].

4.1 Geodesic flow with respect to H^1-norm

We study geodesic flow on the area preserving diffeomorphism group with respect to H^1-Sobolev norm on the $sdiff(\mathcal{A})$ algebra. It is defined by

$$< L, M >= \int_C LM \; dxd\xi + \int_C L'M'dxd\xi \qquad L, M \in sdiff(2), \tag{20}$$

where L' denotes derivatives with respect to both x and ξ. But since all L and M are polynomial ξ. Hence for all practical purposes it boils down to

$$< L, M >= \int_C LM \; dxd\xi + \nu \int_C L_x M_x dxd\xi \qquad \nu \in \mathbb{R}. \tag{21}$$

In other words $'$ always means the derivative with respect to x.

Let us compute again the coadjoint action:

Lemma 2 *The coadjoint action with respect to H^1 metric is given by*

$$ad^*_F(G)|_{H^1} = (1 - \nu\partial^2)^{-1}\{(F, (1 - \nu\partial^2)G\}. \tag{22}$$

Proof: We start from

$$\begin{aligned} < F, \{G, H >_{H^1} &= \int_C F'\{G, H\}'dxd\xi + \int_C F\{G, H\}dxd\xi \\ &= \int_C \{F', G'\}Hdxd\xi + \int_C \{F', G\}H'dxd\xi \\ &= \int_C \{F, (1 - \nu\partial^2)G\}Hdxd\xi. \end{aligned}$$

Let us compute now the L.H.S. of equation (22)

$$\begin{aligned} L.H.S. &= \int_{\mathcal{A}} (ad^*_G F)Hdxd\xi + \int_{\mathcal{A}} (ad^*_G F)'H'dxd\xi \\ &= \int_{\mathcal{A}} [(1 - \nu\partial^2)ad^*_G F]Hdxd\xi. \end{aligned}$$

Thus by equating the R.H.S. and L.H.S. we obtain the above formula.
□

Therefore, we conclude:

Proposition 4 *The Euler-Poincaré equation with respect to right invariant H^1 metric on the dual space of $sdiff(2)^*$ yields*

$$\frac{\partial m}{\partial t} = -ad^*_{\frac{\delta H}{\delta u}} m, \tag{23}$$

where $m = (1 - \partial^2)u$ and H is Hamiltonian.

Thus we justify the replacement of u by $m = u - u_{xx}$ for the computation of peakon type equations, and it is not at all a computational trick.

4.2 Computation of $N = 2$ Supersymmetric Camassa-Holm equation

In this Section we derive $N = 2$ super Camassa-Holm equation by replacing all variables by their Helmholtz counterparts. In our previous section we justify the replacement for the computation of peakon type equations.

Lemma 3 *The Hamiltonian operator $\mathcal{O}_{super}$ corresponding to the H^1- action of $f(x)\xi + \phi_i(x)\xi^{-1/2} + g(x) \in Vect(S^{1|2})$ on the dual space of the Neveu-Schwarz algebra on $u(x)\xi^{-2} + \eta_i(x)\xi^{-3/2} + w(x)\xi^{-1}$ yields*

$$\mathcal{O}_{super} = (1 - \nu\partial_x^2)^{-1} \begin{pmatrix} 2m\partial + 2\partial m & \partial\beta_1 + 2\beta_1\partial & \partial\beta_2 + 2\beta_2\partial & 2n\partial_x \\ 2\partial\beta_1 + \beta_1\partial & -m & \partial n + n\partial & -\beta_2 \\ 2\partial\beta_2 + \beta_2\partial & -n\partial - \partial n & -m & \beta_1 \\ \partial w & \beta_2 & -\beta_1 & 0 \end{pmatrix}, \tag{24}$$

where $m = u - \nu u_{xx}, \beta_i = \eta_i - \nu\eta_{ixx}$ and $n = w - \nu w_{xx}$.

Proof: This follows directly from our previous results

$$ad^*_{f(x)\xi+\phi_i(x)\xi^{1/2}+g(x)}(u(x)\xi^{-2} + \eta_i(x)\xi^{-3/2} + w(x)\xi^{-1})|_{H^1}$$
$$= (1 - \nu\partial^2)^{-1}\{f(x)\xi + \phi_i(x)\xi^{1/2} + g(x), (1 - \nu\partial^2)(u(x)\xi^{-2} + \eta_i(x)\xi^{-3/2} + w(x)\xi^{-1})\}$$

and the computation of the left hand side $\{.,.\}$ is similar to previous section.
□

Once we transfer the Helmholtz operator to left hand side we express EP equation in the following framework.

Definition 4 *The Euler-Poincaré equation induced by the action of $N = 2$ supersymmetric $Vect(S^1)$ on its dual $\mathcal{F}_{-2} + \mathcal{F}_{-3/2} + \mathcal{F}_{-3/2} + \mathcal{F}_{-1}$ with respect to H^1 norm is defined as*

$$(m\xi^{-2} + \beta_i\xi^{-3/2} + w\xi^{-1})_t = ad^*_{(\frac{\delta H}{\delta u}\xi + \frac{\delta H}{\delta \eta_i}\xi^{1/2} + \frac{\delta H}{\delta g})}(m\xi^{-2} + \beta_i\xi^{-3/2} + w(x)\xi^{-1}), \tag{25}$$

where $m = u - \nu u_{xx}, \beta_i = \eta_i - \nu\eta_{ixx}$ and $n = w - \nu w_{xx}$ are the Helmholtz counter parts of u, η_i and w respectively.

Proposition 5 *The Euler-Poincaré flow with respect to H^1-metric on the dual space of $N = 2$ Neveu-Schwarz algebra yields the $N = 2$ super Camassa-Holm equation*

$$\begin{aligned} m_t = 4mu_x + 2m_x u + 4amww_x + am_x w^2 - 3\eta_i\eta_i'' + \nu\eta_i'\eta_i''' + 3a(w\eta_2\beta_1 + w\beta_2\eta_1)_x \\ +2aw(\beta_1'\eta_2 - \eta_1\beta_2') - 2n(x)w''' + 2an(x)(uw)_x - 2an(x)(\eta_1\eta_2)_x \end{aligned} \tag{26}$$

$$\begin{aligned} \beta_{1t} = 3(\beta_1 u + \frac{1}{2}a\beta_1 w^2)_x - \beta_1'(u + \frac{1}{2}aw^2) + m\eta_1' + amw\eta_2 - 2(n(x)\eta_2')_x \\ +2(nw\eta_1)_x + n'(\eta_2' - aw\eta_1) + \beta_2(w'' - auw + a\eta_1\eta_2) \end{aligned} \tag{27}$$

$$\beta_{2t} = 3(\beta_2 u + \frac{1}{2}a\beta_2 w^2)_x - \beta_2'(u + \frac{1}{2}aw^2) + m\eta_1' - amw\eta_1 + 2(n(x)\eta_1')_x$$
$$+2a(n(x)w\eta_2)_x - n'(\eta_1' + aw\eta_2) + \beta_1(-w'' + auw - a\eta_1\eta_2) \qquad (28)$$

$$n_t(x) = (nu + \frac{1}{2}nw^2)_x - \beta_2(\eta_1' + aw\eta_2) + \beta_1(\eta_2' - aw\eta_1). \qquad (29)$$

Proof: We use the above definition for our proof. □

Unfortunately there are not so much cancellation of terms due the existence Helmholtz functions are absence of cocycle terms. These two play very important role to write the $N = 2$ super KdV equation in a compact form.

4.3 $N = 2$ supersymmetric Camassa-Holm equation in covariant form

Clearly one can see that the $N = 2$ is bit cumbersome in component form. One easily express this in superfield form. Here we express the

We define the supercircle $S^{1|2}$ in terms of its superalgebra of functions denoted by $C^\infty_{\mathbb{C}}(S^{1|2})$ consisting of elements of the form

$$F(X) = f(x) + \theta_1\phi_1(x) + \theta_2\phi_2(x) + \theta_1\theta_2 g(x), \qquad (30)$$

where f, g and ϕ_i are smooth functions on S^1. Here X stands for the triplet (x, θ_1, θ_2), we assume x is an arbitrary parameter on S^1 and θ_1 and θ_2 are formal Grassmann coordinates. These anticommuting variables satisfy

$$\theta_1\theta_2 = -\theta_2\theta_1, \qquad \theta_1^2 = \theta_2^2 = 0.$$

Identifying the element $(w, \eta_2, \eta_1, u) \in Vect(S^{1|2})^*$ with the odd (parity) dual super element

$$U = w + \theta_1\eta_2 + \theta_2\eta_1 + \theta_1\theta_2 u.$$

In the superfield form the (second) super Hamiltonian operator of $N = 2$ super Camassa-Holm equation can be rewritten as

$$\mathcal{O}'_{super} = (1 - \nu\partial_x^2)^{-1}(2\partial_x M + 2M\partial_x - (D_i M)D_i), \qquad (31)$$

where the superfield $M = U - \nu U_{xx}$. This allows us to express the $N = 2$ supersymmetric Camassa-Holm equation in a more compact form.

Proposition 6 *The Euler-Poincaré flow on the dual of $N = 2$ superconformal algebra with respect to H^1 norm yields the $N = 2$ super Camassa-Holm equation*

$$M_t + 2(MD_1D_2U)_x - \epsilon_{ij}D_i M D_j U_x + a(MU^2)_x + aM(U^2)_x - aD_i M D_i U = 0. \qquad (32)$$

Moreover this supersymmetric equation is equivalent to equations (26-29).

Proof: We use the Hamiltonian equation

$$U_t = \mathcal{O}'_{super}(\frac{\delta H}{\delta U}),$$

where $H = \frac{1}{2}\int dX(UD_1D_2U + \frac{a}{3}U^3)$. Thus we obtain

$$M_t = (2\partial_x M + 2M\partial_x - (D_iM)D_i)(D_1D_2U + \frac{a^2}{2}U^2)$$

to obtain our result.

□

5 Euler-Poincaré flow and (super) b–field equation

In this section we derive of the Degasperis–Procesi equation and b-field equation. The DP equation is considered to be the second member of the one parameter b-field family of partial differential equations, which is given by

$$m_t = 3mu_x + m_x u, \qquad m = u - u_{xx}. \tag{33}$$

At first our goal is to derive this equation from the action of $Vect(S^1)$ on tensor densities. It is clear that

$$\{f\xi, g\xi^{-(b-1)}\} = (fg' - (b-1)f'g)\xi^{(b-1)}.$$

Thus we consider the deformation of the algebra of vector fields

$$\begin{aligned}
[v,w]_b \quad &:= \quad \frac{b}{2}[v,w] - \frac{b-2}{2}(vw)_x \\
&= \quad vw_x - (b-1)v_x w\ .
\end{aligned} \tag{34}$$

We note that the deformation is symmetric and a total divergence, reminiscent of the Dorfman bracket. This b–bracket allows interpretation as an action of $Vect(S^1)$ on $\mathcal{F}_{(b-1)}(S^1)$. For $b = 2$ this is merely the vector field action corresponding to the Lie bracket. The b–bracket is clearly not skewsymmetric. However, it has several interesting properties.

There exists a pairing [13]

$$\mathcal{F}_\lambda \otimes \mathcal{F}_{1-\lambda} \to \mathbb{R}$$

given by

$$\langle a(x)(dx)^\lambda, b(x)(dx)^{1-\lambda}\rangle = \int a(x)b(x)\,dx.$$

Therefore the above pairing allows us to identify the dual of $\mathcal{F}_{(b-1)}(S^1)$ with $\mathcal{F}_{-b}(S^1)$.

Definition 5 *The generalized EPDiff flow induced by the action $Vect(S^1)$ on $\Psi D(S^1)$ space is defined as*

$$m_t = \{\frac{\delta H}{\delta u}\xi, m\xi^{-b}\}, \tag{35}$$

where $m = u - \nu u_{xx}$ is the standard Helmholtz operator acting on u.

It is clear that equation (37) is equivalent to Hamiltonian flow on the tensor densities $\mathcal{F}_{-b}$ of weight b (or b-forms) given by

$$u_t = \mathcal{O}_b^{H^1}\frac{\delta H}{\delta u}.$$

Theorem 3 *Suppose we define Hamiltonian $H = \frac{1}{2}\int_{S^1} u^2\, dx$. The Euler–Poincaré equation for the right invariant H^1 metric on tensor densities $\mathcal{F}_{-b}$ (the dual space of b algebra) yields the b–field equation*

$$m_t = m_x u + bmu_x, \qquad m = u - \nu u_{xx}. \tag{36}$$

Proof: The action of $\frac{\delta H}{\delta u}$ on $m\xi^{-b}$ is given by

$$\begin{aligned}
& ad_{\frac{\delta H}{\delta u}(x)\xi} m(x)\xi^{-b} \\
= \quad & -(\frac{\partial}{\partial \xi}(\frac{\delta H}{\delta u}\xi)\frac{\partial}{\partial x}(m\xi^{-b}) - \frac{\partial}{\partial x}(\frac{\delta H}{\delta u}\xi)\frac{\partial}{\partial \xi}(m\xi^{-b}) \\
= \quad & -(\frac{\delta H}{\delta u}m' + b(\frac{\delta H}{\delta u})' m)\xi^{-b-1}.
\end{aligned}$$

Substituting $\frac{\delta H}{\delta u} = u$ we obtain our result. □

This equation was introduced in Degasperis, Holm and Hone [6, 7] based on Degasperis and Procesi [8] who singled out the cases $b = 2$ Camassa-Holm equation and $b = 3$ Degasperis-Procesi (DP) equation. Hamiltonian structure from the Euler-Poincaré formalism is given in [19].

5.1 Supersymmetric b–field equation

Let us study the fermionic (i.e. $N = 1$ supersymmetry) extension of the b–field equation. Consider the space of $-(2b-1)/2$–tensor densities $\mathcal{F}_{-2b-1/2}$. There exists a natural action of Lie superalgebra $Vect(S^1) \oplus \mathcal{F}_{1/2}$ on $\mathcal{F}_b \oplus \mathcal{F}_{-2b-1/2}$.

In our case, the super b bracket is the deformation of the Neveu-Schwarz superconformal algebra, consisting of pairs $(u(x), \phi(x))$, where u is a bosonic field and $\phi(x)$ is a fermionic field. The bracket is defined by

$$[(u,\phi),(v,\xi)]_b = (uv_x - (b-1)u_x v, u\xi_x - (b-\frac{3}{2})u_x\xi - (b-1)v\phi_x + \frac{1}{2}v_x\phi). \tag{37}$$

Consideration of the supersymmetrisation of this algebra then opens the door to the construction of supersymmetric extensions of the b-field equations.

Remark A vector field X_f on $S^{1|1}$, for any $f \in C^\infty_{\mathbb{C}}(S^{1|1})$, is said to be contact if it preserves the contact distributions. The contact bracket is defined by $[X_f, X_g] = X_{\{f,g\}}$, where the space $C^\infty_{\mathbb{C}}(S^{1|1})$ is thus equipped with a Lie superalgebra structure given by

$$\{f, g\} = fg' - f'g + (-1)^{p(f)p(g)+1}\frac{1}{2}D(f)D(g), \tag{38}$$

where $D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$. The super b bracket that we are considering belongs to a class of generalized contact bracket [12] which extends to densities of arbitrary weight:

$$\{.,.\} : \mathcal{F}_\lambda(S^{1|1}) \otimes \mathcal{F}_\mu(S^{1|1}) \to \mathcal{F}_{\lambda+\mu+1}(S^{1|1}),$$

explicitly given by

$$\{f, g\} = \lambda fg' - \mu f'g + (-1)^{p(f)p(g)+1}\frac{1}{2}D(f)D(g). \tag{39}$$

Lemma 4 *The Hamiltonian operator $\mathcal{O}_b$ corresponding to the action of the Neveu-Schwarz algebra on $m(x)\xi^{-b} + \beta(x)\xi^{-2b-1/2} \in \Psi D(S^1)$ yields*

$$\mathcal{O} = -\left(\begin{array}{c|c} \partial m + (b-1)m\partial & \frac{1}{2}\partial\beta + (b-1)\beta\partial \\ \hline \partial\beta + \frac{2b-3}{2}\beta\partial & \frac{1}{2}m \end{array}\right) \tag{40}$$

Proof: It follows straight away from

$$\begin{aligned} & ad^*_{f(x)\xi+\phi(x)\xi^{1/2}}(m(x)\xi^{-b} + \beta(x)\xi^{-2b-1/2}) \\ = \; & \{f(x)\xi, m(x)\xi^{-b} + \beta(x)\xi^{-2b-1/2}\} + \{\phi(x)\xi^{1/2}, \beta(x)\xi^{-2b-1/2}\} \\ & -\frac{1}{2}\phi(x)m(x)\xi^{-3/2}, \end{aligned}$$

where the last expression follows from the definition of zeroth operator given by Grozman and we have chosen $k = -\frac{1}{2}$.

□

Proposition 7 *Let $\frac{\delta H}{\delta u} = 2u$ and $\frac{\delta H}{\delta \eta} = 8\eta_x$. The Euler-Poincaré flow on $\mathcal{F}_{-b} + \mathcal{F}_{-2b-1/2}$, yields the supersymmetric b–field equation*

$$\begin{aligned} m_t &= 2m_x u + 2bmu_x + 4(2b-1)\beta\eta_{xx} + 4\beta_x\eta_x \\ \beta_t &= 2\beta_x u + 4m\eta_x + (2b-1)\beta u_x\,. \end{aligned} \tag{41}$$

Proof: It is clear that $(\frac{\delta H}{\delta u}, \frac{\delta H}{\delta \eta}) \in Vect(S^{1|1})$. Thus computing the (coadjoint) action $ad^*_{\frac{\delta H}{\delta u}(x)\xi + \frac{\delta H}{\delta \eta}\xi^{1/2}}(m(x)\xi^{-b} + \beta(x)\xi^{-2b-1/2})$ for $\frac{\delta H}{\delta u} = 2u$ and $\frac{\delta H}{\delta \eta} = 8\eta_x$ we obtain our desired result.

□

6 Noncommutative analogue of the Kuper Camassa-Holm and supersymmetric b–field equations

Finally in this section we propose another construction of noncommutative (or Moyal deformed) integrable systems. In particular, we demonstrate the derivation of noncommutative super peakon type systems. The main idea of this construction is to replace Poisson action by Moyal action of vector fields $Vect(S^1)$ (or $Vect(S^{1|1}$) on the space of pseudodifferential symbols,

6.1 Construction of Noncommutative bosonic systems

All the equations described in this article can be $\star$ deformed using following rules:
(A) All Poisson brackets should be replaced by Moyal brackets [33] defined by:

$$\{F, G\}_{Moyal} := \frac{F \star G - G \star F}{\kappa}. \tag{42}$$

(B) The derivatives act on the $\star$-deformed space in a usual way.

We will study two types of systems here - one is purely bosonic (or ordinary) system and other one is $N = 1$ supersymmetric system.

The bosonic part can be quantized by using the Moyal product

$$f \star_M g = f exp\Big[\frac{i\hbar}{2}(\overleftarrow{\partial_q}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_q})\Big] g. \tag{43}$$

The Moyal star product replaces the ordinary product between functions on the phase space.

Definition-Proposition 1 *Let dH be the gradient of the Hamiltonian. In the noncommutative case $Vect(S^1)$ acts on the $\star$- deformed dual embedded in $\Psi D(S^1)$ is given by*

$$\begin{aligned} u_t &= \{dH(x)\xi, u(x)\xi^{-2}\}_{Moyal} + \tfrac{1}{6}(dH)'''\xi^{-2} \\ &= (dH \star u' + 2dH' \star u + \tfrac{1}{6}dH''')\xi^{-2}. \end{aligned} \tag{44}$$

Corollary 1 *Suppose $H = \frac{1}{2}\int_{S^1} u^2\, dx$. The Moyal deformed Euler-Poincaré flow on $\mathcal{F}_{-2}$ yields the noncommutative KdV (ncKdV) equation*

$$u_t = u \star u_x + 2u_x \star u + u_{xxx}. \tag{45}$$

Proposition 8 *Let dH be the gradient of the Hamiltonian with respect to Helmholtz function m. Let $Vect(S^1)$ acts on the $\star$- deformed space of tensor densities $\mathcal{F}_{-b}$ embedded in $\Psi D(S^1)$. The noncommutative analogue of the EPDiff flow is given by*

$$\begin{aligned} m_t &= \{dH(x)\xi, m(x)\xi^{-b}\}_{Moyal} \\ &= dH \star m' + b\, dH' \star m. \end{aligned} \tag{46}$$

Suppose $H = \int_{S^1} mu\, dx$, we obtain the noncommutative b–field equation

$$m_t + u \star m_x + bu_x \star m = 0. \tag{47}$$

Proof: Straightforward. □

6.2 Construction of Noncommutative $N = 1$ super b-field systems

Non

Noncommutativity in superspace naturally arise in string theory in several contexts. Imposing the worldsheet supersymmetry to the noncommutativity relation of the space-time creates $\star$ products between the boson-boson, boson-fermion and fermion-fermion fields. It is known that the $\star$ product of two superfields is a superfield.

In this study the Moyal deformed super b-field equation using results of noncommutative superspaces [11, 22].

Here we invoke a generalization of Moyal-Weyl deformation to functions on super-phase space. This algebraic structure corresponds to the quantization of systems with both, bosonic and fermionic degrees of freedom. The fermionic variables involves a $\star$-product that is given by

$$\phi \star_C \psi = \phi exp\Big[\frac{i\hbar}{2}(\overleftarrow{\partial_{\theta_i}}\overrightarrow{\partial_{\theta_i}})\Big]\psi. \tag{48}$$

This star product is called the Clifford star product because it leads to a cliffordization of the Grassmann algebra of the odd coordinates θ_i. Moreover, the star anticommutator is given by

$$\{\theta_i, \theta_j\}_C = \theta_i \star_C \theta_j + \theta_j \star_C \theta_i = \hbar\delta_{ij}.$$

Proposition 9 *The Euler-Poincaré flow with respect to H^1-metric on the Moyal deformed dual space of Neveu-Schwarz algebra yields the noncommutative Kupershmidt-Camassa-Holm equation*

$$\begin{aligned} m_t + 4u_x \star m + 2u \star m_x + 4\eta_x \star \beta_x + 12\eta_{xx} \star \beta = 0 \\ \beta_t + 2u \star \beta_x + 3u_x \star \beta + +4\eta_x \star m = 0 \end{aligned} \tag{49}$$

for (super) Hamiltonian $\frac{\delta H}{\delta u} = 2u$ and $\frac{\delta H}{\delta \eta} = 8\eta_x$.

Proof: The Euler-Poincaré equation induced by the action of Moyal deformed supersymmetric $Vect(S^1)$ on its dual $\mathcal{F}_{-2} + \mathcal{F}_{-3/2}$ is given by

$$\begin{aligned} & (m\xi^{-2} + \tilde{\eta}\xi^{-3/2})_t \\ = \; & \{\frac{\delta H}{\delta u}\xi, u(x)\xi^{-2} + \eta\xi^{-3/2}\}_{Moyal} + \{\frac{\delta H}{\delta \eta}\xi^{1/2}, \eta\xi^{-3/2}\}_{Moyal} + \frac{1}{2}\frac{\delta H}{\delta \eta}m(x) \star \xi^{-3/2} \end{aligned} \tag{50}$$

Therefore, using Hamiltonian $\frac{\delta H}{\delta u} = 2u$ and $\frac{\delta H}{\delta \eta} = 8\eta_x$ we obtain our desired result. $\square$

Similarly one can compute the noncommutative version of supersymmetric b–field equation.

Proposition 10 *The Euler-Poincaré flow with respect to H^1-metric on $\star$- deformed $\mathcal{F}_{-b} + \mathcal{F}_{-\frac{2b-1}{2}}$ yields the Moyal deformed supersymmetric b–field equation*

$$\begin{aligned} m_t + 2bu_x \star m + 2u \star m_x + 4\eta_x \star \beta_x + 4(2b-1)\eta_{xx} \star \beta = 0 \\ \beta_t + 2u \star \beta_x + (2b-1)u_x \star \beta + +4\eta_x \star m = 0 \end{aligned} \tag{51}$$

Obviously, the best way to consider such deformations through the introduction of the (super) Poisson bracket between two superfields and consider the Moyal-Weyl star product of superfields.

7 Outlook

In the present paper, we have constructed various noncommutative integrable and superintegrable systems in $(1 + 1)$ through embedding of vector fields and its dual on the space of pseudodifferential symbols on S^1. In this process we have tacitly moved the coadjoint action to Poisson action. We have used Grozman's method coupled with the anticommutativity properties of fermions to compute these actions. Then replacing the Poisson action by Moyal action we have constructed the noncommutative or Moyal deformed integrable systems. We claim that this method is much more elegant and geometrical than previously known methods. It would be nice to formulate the solutions of these equations - hope one should be able implement the methods developed by Takasaki [40] to study the geometry of the corresponding Riemann-Hilbert problem in some Moyal algebra valued loop group.

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