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Abstract

We construct explicit examples of Dirac-harmonic maps (ϕ, ψ) between Riemannian manifolds (M, g) and (N, g') which are non-trivial in the sense that ϕ is not harmonic. When $\dim M = 2$, we also produce examples where ϕ is harmonic, but not conformal, and ψ is non-trivial.

Key words and phrases: Dirac-harmonic map, twistor spinor, totally umbilical
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1 Introduction

A Dirac-harmonic map is a pair that couples a map between Riemannian manifolds with a nonlinear spinor field along that map [10]. Dirac-harmonic maps arise from the supersymmetric nonlinear sigma model of quantum field theory [12]. They are a generalization and combination of harmonic maps and harmonic spinors while preserving the essential properties of the former.

Both harmonic maps and harmonic spinors have been extensively studied. See, for instance [13, 15]. In particular, many non-trivial examples of harmonic maps and harmonic spinors

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are known [2, 3, 4, 13]. A harmonic map and a vanishing spinor, or conversely a constant map and a harmonic spinor constitute an example of a Dirac-harmonic map. A natural question then is whether there exist other examples that couple a map and a spinor in a non-trivial manner. The purpose of this paper therefore is to manufacture non-trivial examples of Dirac-harmonic maps between Riemannian manifolds. For hypersurfaces in a Riemannian manifold of constant sectional curvature, we prove the following:

Theorem 1 *Let M be an n-dimensional manifold which is immersed in an $(n+1)$-dimensional Riemannian manifold $N(c)$ of constant sectional curvature c. Assume Φ is a harmonic spinor on M, and $\Psi \in \Gamma(\Sigma M)$ satisfies*

$$-2cRe\langle\Phi, \Psi\rangle\nu = H. \tag{1}$$

where H is the mean curvature vector field of ϕ, ν is the unit normal field of ϕ and ΣM is the spinor bundle of M. We define a spinor field ψ along the immersion ϕ by

$$\psi = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha) + \Phi \otimes \nu$$

where ϵ_α is a local orthonormal basis of M.

(i) If $n = 2$, ϕ is minimal and Ψ satisfies

$$\epsilon_1 \cdot \nabla_{\epsilon_1}\Psi - \epsilon_2 \cdot \nabla_{\epsilon_2}\Psi = \lambda_1 \Phi \tag{2}$$

where λ_1 is the principal curvature in the principal direction ϵ_1, then (ϕ, ψ) is Dirac-harmonic.

(ii) If $n \geq 3$, ϕ is totally umbilical and Ψ is a twistor spinor satisfying

$$\partial\!\!\!/ \Psi = -\frac{n\langle H, \nu\rangle}{n-2}\Phi \tag{3}$$

then (ϕ, ψ) is Dirac-harmonic.

Using Theorem 1, we can construct many Dirac-harmonic maps (ϕ, ψ) from $\mathbb{R}^n$ into $\mathbb{H}^{n+1}(-1)$ where $n \geq 3$ and $\phi : \mathbb{R}^n \to \mathbb{H}^{n+1}(-1)$ is not harmonic (see Section 5, Example 3).

Finding explicit non-trivial explicit solutions of (2) and (3) turns out to be difficult. However in some special cases, we are able to get the non-trivial solutions, as in Example 3.

Let us take a look at the following special case of (i) of Theorem 1: when $\Phi = 0$, then (ϕ, ψ) is Dirac-harmonic if $\phi : M \to N(c)$ is minimal and Ψ is a twistor spinor. In fact, in the general case, we have the following:

Theorem 2 *Let M be a Riemann surface and N a Riemannian manifold. Assume $\phi : M \to N$ is a harmonic map and $\Psi \in \Gamma(\Sigma M)$ is a twistor spinor. We define a spinor field $\psi_{\phi,\,\Psi}$ along map ϕ by*

$$\psi_{\phi,\,\Psi} := \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha) \tag{4}$$

where ϵ_α ($\alpha = 1,\, 2$) is a local orthonormal basis of M. Then $(\phi,\, \psi_{\phi,\,\Psi})$ is a Dirac-harmonic map.

By using Theorem 2, we can manufacture Dirac-harmonic maps $(\phi,\, \psi_{\phi,\,\Psi})$ from a surface for a (not necessarily conformal) map ϕ (see Section 4). Theorem 2 generalizes the result of [10] that was derived for the special case when both source and target manifolds are two-dimensional spheres.

Finally, by investigating spinor fields along a hypersurface with two constant principal curvatures in a Riemannian manifold of constant curvature, we get Dirac-harmonic maps $(\phi,\, \psi)$ from surfaces for which ϕ is not harmonic (see Section 6 and Section 7).

Let us describe our construction. Let $M := S^1(r) \times H^1(\sqrt{R^2 + r^2})$ be a hyperbolic surface of revolution (see Section 6 for definitions). Let a and b be arbitrary complex constants and m be an arbitrary non-negative integer. For each $k \in \{0, \pm 1 \cdots, \pm m\}$, let c_k and d_k be complex constants satisfying

$$Re(a\bar{d}_0 + \bar{b}c_0) = \frac{\sqrt{R^2 + r^2}(R^2 + 2r^2)}{2rR} \tag{5}$$

and

$$a\bar{d}_k + \bar{b}c_{-k} = 0. \tag{6}$$

We obtain the following result (see Section 7):

Theorem 3 *Let $\phi : M \hookrightarrow H^3(R)$ be an isometric immersion from M into a hyperbolic space and $\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TH^3(R))$ defined by*

$$\psi = \epsilon_1 \cdot \Psi \otimes \phi_*(\epsilon_1) - \frac{r^2}{R^2 + r^2}\epsilon_2 \cdot \Psi \otimes \phi_*(\epsilon_2) + \chi \otimes \nu$$

where

$$\nu(\theta,\, t) = -\left(\frac{\sqrt{R^2 + r^2}}{R} \cos\frac{\theta}{r},\, \frac{\sqrt{R^2 + r^2}}{R} \sin\frac{\theta}{r},\, \frac{r}{R} \sinh\frac{t}{\sqrt{R^2 + r^2}},\, \frac{r}{R} \cosh\frac{t}{\sqrt{R^2 + r^2}} \right)$$

is a unit normal vector of M, and $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\Psi(\theta,\, t) = i\frac{\sqrt{R^2 + r^2}}{rR} t \begin{pmatrix} b \\ a \end{pmatrix} + \sum_{k=-m}^{m} e^{i\frac{k}{r}\theta} \begin{pmatrix} d_k e^{-\frac{k}{r}t} \\ c_k e^{\frac{k}{r}t} \end{pmatrix}$$

are the spinors on M with respect to the "untwisted" spinor bundle on M satisfying (5) and (6). $\{\epsilon_1, \epsilon_2\}$ is a local orthonormal basis of M such that

$$\epsilon_1(\theta, t) = \left(-\sin\frac{\theta}{r}, \cos\frac{\theta}{r}, 0, 0\right)$$

is a principal curvature $\frac{\sqrt{R^2+r^2}}{rR}$ direction and

$$\epsilon_2(\theta, t) = \left(0, 0, \cosh\frac{t}{\sqrt{R^2+r^2}}, \sinh\frac{t}{\sqrt{R^2+r^2}}\right)$$

is a principal curvature $\frac{r}{R\sqrt{R^2+r^2}}$ direction. Then (ϕ, ψ) is a Dirac-harmonic map from M into $H^3(R)$ for which ϕ is not harmonic.

The proofs of our results are essentially of an algebraic nature. They carefully match the algebraic structure of the curvature term in the Dirac-harmonic map equation, as displayed in the next section, with the special properties of twistor spinors or those of particular submanifolds defined in terms of ambient curvature properties in spaces of constant curvature.

2 Dirac-harmonic maps

Let (N, h) be a Riemannian manifold of dimension n', (M, g) be an n-dimensional Riemannian manifold with fixed spin structure, ΣM its spinor bundle, on which we have a Hermitian metric $\langle\cdot, \cdot\rangle$ induced by the Riemannian metric $g(\cdot, \cdot)$ of M. Let ϕ be a smooth map from (M, g) to (N, h) and $\phi^{-1}TN$ the pull-back bundle of TN by ϕ. On the twisted bundle $\Sigma M \otimes \phi^{-1}TN$ there is a metric (still denoted by $\langle\cdot, \cdot\rangle$) induced from the metrics on ΣM and $\phi^{-1}TN$. There is also a natural connection $\tilde{\nabla}$ on $\Sigma M \otimes \phi^{-1}TN$ induced from those on ΣM and $\phi^{-1}TN$ (which in turn come from the Levi-Civita connections of (M, g) and (N, h), resp.).

For $X \in \Gamma(TM)$, $\xi \in \Gamma(\Sigma M)$, denote by $X \cdot \xi$ their Clifford product, which satisfies the skew-symmetry relation

$$\langle X \cdot \xi, \eta\rangle = -\langle \xi, X \cdot \eta\rangle \tag{7}$$

as well as the Clifford relations

$$X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2g(X, Y)\psi$$

for $X, Y \in \Gamma(TM)$, $\xi, \eta \in \Gamma(\Sigma M)$.

Let ψ be a section of the bundle $\Sigma M \otimes \phi^{-1}TN$. The *Dirac operator along the map ϕ* is defined as

$$\not{D}\psi := \epsilon_\alpha \cdot \tilde{\nabla}_{\epsilon_\alpha}\psi$$

where ϵ_α is a local orthonormal basis of M. For more details about the spin bundle and Dirac operator, we refer to [17, 21].

Set

$$\chi := \{(\phi,\, \psi) \mid \phi \in C^\infty(M,\, N) \text{ and } \psi \in C^\infty(\Sigma M \otimes \phi^{-1}TN)\}.$$

On χ, we consider the following functional

$$L(\phi,\, \psi) := \frac{1}{2}\int_M \left[|d\phi|^2 + \langle \psi,\, \not{D}\psi\rangle\right] {}^*1_M.$$

This functional couples the two fields ϕ and ψ because the operator $\not{D}$ depends on the map ϕ. The Euler-Lagrange equations of $L(\phi,\, \psi)$ then also couple the two fields; they are:

$$\tau(\phi) = \mathcal{R}(\phi,\, \psi) \tag{8}$$

and

$$\not{D}\psi = 0 \tag{9}$$

where $\tau(\phi) := \mathrm{trace}\nabla d\phi$ is the tension field of the map ϕ and $\mathcal{R}(\phi,\, \psi)$ is defined by

$$\mathcal{R}(\phi,\, \psi) = \frac{1}{2} R^i{}_{jkl}\langle \psi^k,\, \nabla\phi^j \cdot \psi^l\rangle \frac{\partial}{\partial y^i},$$

where

$$\psi = \psi^i \otimes \frac{\partial}{\partial y^i},$$

$$(d\phi)^\sharp = \nabla\phi^i \otimes \frac{\partial}{\partial y^i},$$

$$R^{\phi^{-1}TN}\left(\frac{\partial}{\partial y^k},\, \frac{\partial}{\partial y^l}\right)\frac{\partial}{\partial y^j} = R^i{}_{jkl}\frac{\partial}{\partial y^i}$$

where $\sharp : T^*M \otimes \phi^{-1}TN \to TM \otimes \phi^{-1}TN$ is the standard ("musical") isomorphism obtained from the Riemannian metric g.

Solutions $(\phi,\, \psi)$ to (8) and (9) are called *Dirac-harmonic maps* from M into N [9].

We now start with some differential geometric identities: Let ϵ_α be a local orthonormal basis of M. By using the Clifford relations we have

$$\epsilon_\alpha \cdot \epsilon_\beta \cdot \psi = (-1)^{\delta_{\alpha\beta}+1}\epsilon_\beta \cdot \epsilon_\alpha \cdot \psi = \begin{cases} -\psi, & \alpha = \beta \\ -\epsilon_\beta \cdot \epsilon_\alpha \cdot \psi, & \alpha \neq \beta \end{cases} \tag{10}$$

for $\psi \in \Gamma(\Sigma M)$.

Lemma 2.1 $\mathcal{R}(\phi,\, \psi) \in \Gamma(\phi^{-1}TN)$; in particular, it is real.

Proof For any (not necessarily orthonormal) frame $\{\epsilon_i\}$ on $\phi^{-1}TN$, we put

$$\psi = \psi^a \otimes \epsilon_a, \tag{11}$$

$$(d\phi)^\sharp = \nabla\phi^a \otimes \epsilon_a, \tag{12}$$

$$R^{\phi^{-1}TN}(\epsilon_a,\, \epsilon_b)\epsilon_c = R^d{}_{abc}\epsilon_d$$

where $^\sharp : T^*M \otimes \phi^{-1}TN \to TM \otimes \phi^{-1}TN$ is the musical isomorphism as before. Take

$$\epsilon_a = u_a^i \frac{\partial}{\partial y^i},$$

then

$$\psi^i = u_a^i \psi^a, \qquad \nabla\phi^i = u_a^i \nabla\phi^a, \qquad u_a^j u_b^k a_c^l R^i{}_{jkl} = R^d{}_{abc} u_d^i.$$

A simple calculation gives following

$$R^i{}_{jkl}\langle\psi^k,\, \nabla\phi^j \cdot \psi^l\rangle \frac{\partial}{\partial y^i} = R^a{}_{bcd}\left(\phi(x)\right)\langle\psi^c,\, \nabla\phi^b \cdot \psi^d\rangle \epsilon_a\left(\phi(x)\right). \tag{13}$$

It follows that the definition of $\mathcal{R}(\phi,\, \psi)$ is independent of the choice of frame. Moreover, from the skew-symmetry of $R^i{}_{jkl}$ with respect to the induces k and l, we have

$$\begin{aligned}
\overline{\tfrac{1}{2}R^i{}_{jkl}\langle\psi^k,\, \nabla\phi^j \cdot \psi^l\rangle} &= \tfrac{1}{2}R^i{}_{jkl}\langle\nabla\phi^j \cdot \psi^l,\, \psi^k\rangle \\
&= \tfrac{1}{2}R^i{}_{jlk}\langle\nabla\phi^j \cdot \psi^k,\, \psi^l\rangle \\
&= -\tfrac{1}{2}R^i{}_{jkl}\langle\nabla\phi^j \cdot \psi^k,\, \psi^l\rangle = \tfrac{1}{2}R^i{}_{jkl}\langle\psi^k,\, \nabla\phi^j \cdot \psi^l\rangle.
\end{aligned}$$

It follows that $\mathcal{R}(\phi,\, \psi)$ is well-defined vector field on $\phi^{-1}TN$, i.e., $\mathcal{R}(\phi,\, \psi) \in \Gamma(\phi^{-1}TN)$. $\square$

A spinor (field) $\Psi \in \Gamma(\Sigma M)$ is called a *twistor spinor* if Ψ belongs to the kernel of the twistor operator, equivalently,

$$\nabla_X\Psi + \frac{1}{n}X \cdot \partial\!\!\!/\Psi = 0 \qquad \forall X \in \Gamma(TM)$$

where n is the dimension of Riemannian manifold M, ΣM is the associated spinor bundle of M and $\partial\!\!\!/$ is the usual Dirac operator (cf. [1, 14, 20, 23]).

In fact the concept of a twistor spinor (in particular, a Killing spinor) is motivated by theories from physics, like general relativity, 11-dimensional (resp. 10-dimensional) supergravity theory, supersymmetry (see, for example [5, 8, 11]).

3 Dirac-harmonic maps from surfaces I

In this section, we consider *two*-dimensional Riemannian manifolds $(M,\, g)$. Since a metric on a two-dimensional Riemannian manifold defines a conformal structure, we then also have the structure of a Riemann surface. In fact, since the functional L and its critical points, the Dirac-harmonic maps are conformally invariant (see [10]), in our subsequent considerations, we only need the conformal structure in place of the full Riemannian metric g.

Lemma 3.1 *Let Ψ be a section of ΣM. Then $\langle \epsilon_\alpha \cdot \Psi,\, \epsilon_\beta \cdot \epsilon_\gamma \cdot \Psi\rangle$ is purely imaginary for any α, β, γ where ϵ_α ($\alpha = 1, 2$) is a local orthonormal basis of M.*

Proof: For the Hermitian product $\langle\cdot, \cdot\rangle$ on the spinor bundle ΣM, we have

$$\begin{aligned}
\overline{\langle \epsilon_\alpha \cdot \Psi,\, \epsilon_\beta \cdot \epsilon_\gamma \cdot \Psi\rangle} &= \langle \epsilon_\beta \cdot \epsilon_\gamma \cdot \Psi,\, \epsilon_\alpha \cdot \Psi\rangle \\
&= -\langle \epsilon_\gamma \cdot \Psi,\, \epsilon_\beta \cdot \epsilon_\alpha \cdot \Psi\rangle \\
&= -(-1)^{\delta_{\alpha\beta}+1}\langle \epsilon_\gamma \cdot \Psi,\, \epsilon_\alpha \cdot \epsilon_\beta \cdot \Psi\rangle \\
&= (-1)^{\delta_{\alpha\beta}+1}\langle \epsilon_\alpha \cdot \epsilon_\gamma \cdot \Psi,\, \epsilon_\beta \cdot \Psi\rangle \\
&= (-1)^{\delta_{\alpha\beta}+1}(-1)^{\delta_{\gamma\alpha}+1}\langle \epsilon_\gamma \cdot \epsilon_\alpha \cdot \Psi,\, \epsilon_\beta \cdot \Psi\rangle \\
&= (-1)^{\delta_{\alpha\beta}+\delta_{\gamma\alpha}}\langle \epsilon_\gamma \cdot \epsilon_\alpha \cdot \Psi,\, \epsilon_\beta \cdot \Psi\rangle \\
&= -(-1)^{\delta_{\alpha\beta}+\delta_{\gamma\alpha}}\langle \epsilon_\alpha \cdot \Psi,\, \epsilon_\gamma \cdot \epsilon_\beta \cdot \Psi\rangle \\
&= -(-1)^{\delta_{\beta\gamma}+1}(-1)^{\delta_{\alpha\beta}+\delta_{\gamma\alpha}}\langle \epsilon_\alpha \cdot \Psi,\, \epsilon_\beta \cdot \epsilon_\gamma \cdot \Psi\rangle \\
&= (-1)^{\delta_{\alpha\beta}+\delta_{\beta\gamma}+\delta_{\gamma\alpha}}\langle \epsilon_\alpha \cdot \Psi,\, \epsilon_\beta \cdot \epsilon_\gamma \cdot \Psi\rangle = -\langle \epsilon_\alpha \cdot \Psi,\, \epsilon_\beta \cdot \epsilon_\gamma \cdot \Psi\rangle
\end{aligned}$$

where we have used (10) and (7). It follows that

$$Re\langle \epsilon_\alpha \cdot \Psi,\, \epsilon_\beta \cdot \epsilon_\gamma \cdot \Psi\rangle = 0.$$

□

Proposition 3.2 *For a map $\phi : (M, g) \to (N, h)$ and a spinor $\Psi \in \Gamma(\Sigma M)$, we define a spinor field $\psi_{\phi,\Psi}$ along the map by (4). Then*

(i) $\mathcal{R}(\phi, \psi_{\phi,\Psi}) \equiv 0$;

(ii) $\not{D}\psi_{\phi,\Psi} = -\Psi \otimes \tau(\phi) - 2(\nabla_{\epsilon_\alpha}\Psi + \frac{1}{2}\epsilon_\alpha \cdot \not{\partial}\Psi) \otimes \phi_*(\epsilon_\alpha)$ *where ϵ_α ($\alpha = 1, 2$), as always, is a local orthonormal basis of M.*

Remark (a) The Dirac-harmonicity of $(\phi, \psi_{\phi,\Psi})$ implies the harmonicity of ϕ by (i) and (8).

(b) $(\nabla_{\epsilon_\alpha}\Psi + \frac{1}{2}\epsilon_\alpha \cdot \not{\partial}\Psi) \otimes \phi_*(\epsilon_\alpha)$ is globally defined.

Proof of Proposition 3.2: (i) Define local vector fields $\nabla\phi^i$ on M by

$$\nabla\phi^i := (d\phi)^\sharp(dy^i)$$

where $\{dy^i\}$ is the natural local dual basis on N. By using (4), we have

$$\psi^i := \psi_{\phi,\,\Psi}(dy^i) = \nabla\phi^i \cdot \Psi$$

Set $d\phi = \phi^i_\alpha \theta^\alpha \otimes \frac{\partial}{\partial y^i}$ where θ^α is the dual basis for ϵ_α. Then $\nabla\phi^i = \sum \phi^i_\alpha \epsilon_\alpha$ and

$$\langle \psi^k,\, \nabla\phi^j \cdot \psi^l \rangle = \phi^k_\alpha \phi^j_\beta \phi^l_\gamma \langle \epsilon_\alpha \cdot \Psi,\, \epsilon_\beta \cdot \epsilon_\gamma \cdot \Psi \rangle.$$

Together with Lemma 3.1, we conclude that $R^i{}_{jkl}\langle \psi^k,\, \nabla\phi^j \cdot \psi^l \rangle$ is purely imaginary. On the other hand, from the proof of Lemma 2.1, $R^i{}_{jkl}\langle \psi^k,\, \nabla\phi^j \cdot \psi^l \rangle$ must be real, and hence

$$\mathcal{R}(\phi,\, \psi_{\phi,\,\Psi}) \equiv \frac{1}{2} R^i{}_{jkl} \langle \psi^k,\, \nabla\phi^j \cdot \psi^l \rangle \frac{\partial}{\partial y^i} \equiv 0.$$

(ii) By using (10) we have

$$\nabla_{\epsilon_\alpha}\Psi + \frac{1}{2}\epsilon_\alpha \cdot \partial\!\!\!/ \Psi = \nabla_{\epsilon_\alpha}\Psi + \frac{1}{2}\epsilon_\alpha \cdot \left[\Sigma \epsilon_\beta \cdot \nabla_{\epsilon_\beta}\Psi\right] = \begin{cases} \frac{1}{2}(\nabla_{\epsilon_1}\Psi + \epsilon_1 \cdot \epsilon_2 \cdot \nabla_{\epsilon_2}\Psi), & \alpha = 1 \\ \frac{1}{2}(\nabla_{\epsilon_2}\Psi - \epsilon_1 \cdot \epsilon_2 \cdot \nabla_{\epsilon_1}\Psi), & \alpha = 2 \end{cases}. \quad (14)$$

We choose a local orthonormal frame field ϵ_α such that $\nabla_{\epsilon_\alpha}\epsilon_\beta = 0$ at $x \in M$. Then

$$\begin{aligned} D\!\!\!\!/\, \psi_{\phi,\,\Psi} &= \epsilon_\beta \cdot \tilde{\nabla}_{\epsilon_\beta} \psi_{\phi,\,\Psi} \\ &= \epsilon_\beta \cdot \tilde{\nabla}_{\epsilon_\beta} \left(\epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)\right) \\ &= \epsilon_\beta \cdot \left[\nabla_{\epsilon_\beta}(\epsilon_\alpha \cdot \Psi) \otimes \phi_*(\epsilon_\alpha) + \epsilon_\alpha \cdot \Psi \otimes \nabla_{\epsilon_\beta}(\phi_*(\epsilon_\alpha))\right] \\ &= \epsilon_\beta \cdot \left[((\nabla_{\epsilon_\beta}(\epsilon_\alpha) \cdot \Psi + \epsilon_\alpha \cdot \nabla_{\epsilon_\beta}\Psi) \otimes \phi_*(\epsilon_\alpha) + \epsilon_\alpha \cdot \Psi \otimes \nabla_{\epsilon_\beta}(\phi_*(\epsilon_\alpha))\right] \\ &= \epsilon_\beta \cdot \epsilon_\alpha \cdot \left\{\nabla_{\epsilon_\beta}\Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes \nabla_{\epsilon_\beta}(\phi_*(\epsilon_\alpha))\right\} \\ &= (\Sigma_{\alpha=\beta} + \Sigma_{\alpha\neq\beta})\epsilon_\beta \cdot \epsilon_\alpha \cdot \left\{\nabla_{\epsilon_\beta}\Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes \nabla_{\epsilon_\beta}(\phi_*(\epsilon_\alpha))\right\} \\ &= (I) + (II). \end{aligned} \quad (15)$$

where

$$\begin{aligned} (I) &= \epsilon_\alpha \cdot \epsilon_\alpha \cdot \{\nabla_{\epsilon_\alpha}\Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes \nabla_{\epsilon_\alpha}(\phi_*(\epsilon_\alpha))\} \\ &= -\{\nabla_{\epsilon_\alpha}\Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes [\nabla_{\epsilon_\alpha}(\phi_*(\epsilon_\alpha)) - \phi_*(\nabla_{\epsilon_\alpha}(\phi_*(\epsilon_\alpha))]\} \\ &= -\{\nabla_{\epsilon_\alpha}\Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes \tau(\phi)\} \end{aligned} \quad (16)$$

and

$$\begin{aligned} (II) &= \epsilon_1 \cdot \epsilon_2 \cdot \{\nabla_{\epsilon_1}\Psi \otimes \phi_*(\epsilon_2) + \Psi \otimes \nabla_{\epsilon_1}(\phi_*(\epsilon_2))\} \\ &\quad + \epsilon_2 \cdot \epsilon_1 \cdot \{\nabla_{\epsilon_2}\Psi \otimes \phi_*(\epsilon_1) + \Psi \otimes \nabla_{\epsilon_2}(\phi_*(\epsilon_1))\} \\ &= \epsilon_1 \cdot \epsilon_2 \cdot \{\nabla_{\epsilon_1}\Psi \otimes \phi_*(\epsilon_2) - \nabla_{\epsilon_2}\Psi \otimes \phi_*(\epsilon_1) + \Psi \otimes \nabla_{\epsilon_1}(\phi_*(\epsilon_2)) - \Psi \otimes \nabla_{\epsilon_2}(\phi_*(\epsilon_1)\} \\ &= \epsilon_1 \cdot \epsilon_2 \cdot \{\nabla_{\epsilon_1}\Psi \otimes \phi_*(\epsilon_2) - \nabla_{\epsilon_2}\Psi \otimes \phi_*(\epsilon_1)\} \end{aligned} \quad (17)$$

here we have used the following

$$\nabla_{\epsilon_1}(\phi_*(\epsilon_2)) = (\nabla_{\epsilon_1}\phi_*)(\epsilon_2) = (\nabla_{\epsilon_2}\phi_*)(\epsilon_1) = \nabla_{\epsilon_2}(\phi_*(\epsilon_1))$$

Substituting (16) and (17) into (15) yields

$$
\begin{aligned}
\not{D}\psi_{\phi,\,\Psi} &= -\left\{\nabla_{\epsilon_\alpha}\Psi\otimes\phi_*(\epsilon_\alpha)+\Psi\otimes\tau(\phi)\right\}+\epsilon_1\cdot\epsilon_2\cdot\left\{\nabla_{\epsilon_1}\Psi\otimes\phi_*(\epsilon_2)-\nabla_{\epsilon_2}\Psi\otimes\phi_*(\epsilon_1)\right\}\\
&= -\Psi\otimes\tau(\phi)-(\nabla_{\epsilon_1}\Psi+\epsilon_1\cdot\epsilon_2\cdot\nabla_{\epsilon_2}\Psi)\otimes\phi_*(\epsilon_1)\\
&\qquad +(\epsilon_1\cdot\epsilon_2\cdot\nabla_{\epsilon_1}\Psi-\nabla_{\epsilon_2}\Psi)\otimes\phi_*(\epsilon_2)
\end{aligned}
\tag{18}
$$

Plugging (14) into (18) yields (ii). □

4 Proof of Theorem 2 and Examples

Proof of Theorem 2 By using (i) of Proposition 3.2 and the harmonicity of ϕ we have

$$\mathcal{R}(\phi,\,\psi_{\phi,\,\Psi})\equiv 0\equiv\tau(\phi).$$

Thus, $(\phi,\,\psi_{\phi,\,\Psi})$ satisfies (8). On the other hand, since Ψ is a twistor spinor and $n=2$ we get

$$\nabla_{\epsilon_\alpha}\Psi+\frac{1}{2}\epsilon_\alpha\cdot\not{\partial}\Psi=0.$$

Plugging this into the equation in (ii) of Proposition 3.2 yields $\not{D}\psi_{\phi,\,\Psi}=0$. It follows that $(\phi,\,\psi_{\phi,\,\Psi})$ satisfies (9), and hence $(\phi,\,\psi_{\phi,\,\Psi})$ is a Dirac-harmonic map. □

Corollary 4.1 *Let $\psi_{\phi,\,\Psi}$ be defined by (4) from a branched minimal conformal immersion $\phi : M \hookrightarrow N$ and a twistor spinor $\Psi\in\Gamma(\Sigma M)$. Then $(\phi,\,\psi_{\phi,\,\Psi})$ is a Dirac-harmonic map.*

This corollary comes from the fact that a conformal map from a Riemann surface is harmonic if and only if it is a branched minimal immersion [6]. Say that an almost Hermitian manifold $(N,\,h,\,J)$ is $(1,\,2)$-*symplectic* if

$$\nabla^N_{\bar{Z}}W\in\Gamma(T^{1,\,0}N)\quad\text{for every}\quad Z,\,W\in\Gamma(T^{1,\,0}N).$$

Lichnerowicz proved in [22] that any holomorphic map from a cosymplectic manifold to a (1, 2)-symplectic manifold is harmonic. Since a Riemann surface is automatically cosymplectic, we have the following:

Corollary 4.2 *Let $\psi_{\phi,\,\Psi}$ be defined by (4) from a holomorphic map $\phi : M\to N$ and a twistor spinor $\Psi\in\Gamma(\Sigma M)$ where N is a $(1,\,2)$-symplectic manifold. Then $(\phi,\,\psi_{\phi,\,\Psi})$ is a Dirac-harmonic map.*

Example 1 (non-conformal Dirac-harmonic maps) Suppose that $\mathbb{R}^2$ is given the metric $ds^2 = 2dzd\bar{z}$, where $z = x + iy$ is the standard complex coordinate, and let $\mathbf{e}_0, \cdots, \mathbf{e}_n$ be a unitary basis of $\mathbb{C}^{n+1}$. Define $\phi : \mathbb{R}^2 \to \mathbb{C}\mathrm{P}^n$ by

$$\phi(z) = \left[\sum_{j=0}^{n} r_j \exp(\mu_j z - \overline{\mu_j z})\mathbf{e}_j\right]$$

where $r_0, \cdots, r_n$ are strictly positive real numbers and $\mu_0, \cdots, \mu_n$ are complex numbers of unit modulus satisfying

$$\sum_{j=0}^{n} r_j^2 = 1, \qquad \sum_{j=0}^{n} r_j \mu_j = 0.$$

Then ϕ is a harmonic map [6, 19]. In particular, ϕ is totally real, and it is conformal if and only if

$$\sum_{j=0}^{n} r_j \mu_j^2 = 0.$$

Let us consider a twistor spinor $\Psi : \mathbb{R}^2 \to \Delta_2 = \mathbb{C}^2$ on $\mathbb{R}^2$ (cf [18]). According to Example 1 of [1] the set of all twistor spinors on $\mathbb{R}^2$ is given by

$$\Psi(z) = \Psi_0 - \frac{1}{2} z \cdot \Psi_1$$

with $\Psi_0, \Psi_1 \in \Delta_2$. From Theorem 2, we obtain that $(\phi, \psi_{\phi, \Psi})$ is a Dirac-harmonic map from $\mathbb{R}^2$ into $\mathbb{C}\mathrm{P}^n$ where

$$\psi_{\phi, \Psi} := \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$$

where ϵ_α $(\alpha = 1, 2)$ is a local orthonormal basis of M. Furthermore, ϕ is non-conformal if $\sum_{j=0}^{n} r_j \mu_j^2 \neq 0$.

Example 2 (Dirac-harmonic sequence) For each $p = 0, \cdots, n$, let $\phi_p : S^2 \to \mathbb{C}\mathrm{P}^n$ be given by

$$\phi_p[z_0, z_1] = [f_{p,0}(z_0/z_1), \cdots, f_{p,n}(z_0/z_1)]$$

where $[z_0, z_1] \in \mathbb{C}\mathrm{P}^1 = S^2$, and for $r = 0, \cdots, n$, $f_{p,r}(z)$ is given by

$$f_{p,r}(z) = \frac{p\,!}{(1 + z\bar{z})^p} \sqrt{C_r^n} z^{r-p} \sum_k (-1)^k C_{p-k}^r C_k^{n-r} (z\bar{z})^k$$

where

$$C_r^n = \frac{n(n-1)\cdots(n-r+1)}{r!}.$$

Then ϕ_p is a conformal minimal immersion (therefore it is a harmonic map) with induced metric

$$ds_p^2 = \frac{n + 2p(n-p)}{(1 + z\bar{z})^2} dzd\bar{z}.$$

According to Theorem 7 of [1] the twistor spinors on $(S^2,\, ds_p^2)$ are given by

$$\Psi(z) = \frac{\Psi_0 + z \cdot \Psi_1}{\sqrt{1 + z\bar{z}}}$$

where $\Psi_0,\, \Psi_1 \in \Delta_2$ are constants and where we identify the new and old spin bundles as in [1]. Thus we obtain a Dirac-harmonic sequence $(\phi_p,\, \psi_{\phi_p,\,\Psi})$ from S^2 into $\mathbb{C}\mathrm{P}^n$ (cf. [7]) where

$$\psi_{\phi_p,\,\Psi} := \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_{p*}(\epsilon_\alpha).$$

5 Dirac-harmonic maps from Riemannian manifolds

In this section, we are going to construct Dirac-harmonic maps $(\phi,\, \psi)$ for which ϕ is not harmonic.

Let $(N,\, h)$ be a Riemannian manifold of dimension n', $(M,\, g)$ be an n-dimensional Riemannian manifold with fixed spin structure, ΣM its spinor bundle, with induced Hermitian metric $\langle \cdot,\, \cdot \rangle$. Let $\phi : M \hookrightarrow N$ be an isometric immersion which means that the natural induced Riemannian metric on M from the ambient space N coincides with the original one on M. We identify M with its immersed image in N. For each $x \in M$ the tangent space T_xN can be decomposed into a direct sum of T_xM and its orthogonal complement $T_x^\perp M$. Such a decomposition is differentiable. Thus, we have an orthogonal decomposition of the tangent bundle TN along M

$$TN|_M = \phi^{-1}TN = TM \oplus T^\perp M.$$

For a global section $\mathcal{R}(\phi,\, \psi)$ on $\phi^{-1}TN$ (see Section 2), we have

$$\mathcal{R}(\phi,\, \psi) = \mathcal{R}^T(\phi,\, \psi) + \mathcal{R}^N(\phi,\, \psi)$$

where

$$\mathcal{R}^T(\phi,\, \psi) \in \Gamma(TM), \qquad \mathcal{R}^N(\phi,\, \psi) \in \Gamma(T^\perp M).$$

Similarly, for $\not{D}\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$, we have

$$\not{D}\psi = \not{D}^T\psi + \not{D}^N\psi$$

where

$$\not{D}^T\psi \in \Gamma(\Sigma M \otimes TM), \qquad \not{D}^N\psi \in \Gamma(\Sigma M \otimes T^\perp M).$$

The mean curvature vector of M in N is

$$H = \frac{1}{n}\tau(\phi) \in \Gamma(T^\perp M)$$

where $\tau(\phi)$ is the tension field of the map ϕ. Hence we have the following:

Lemma 5.1 *Let $\phi : M \hookrightarrow N$ be an isometric immersion with the mean curvature vector H and $\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$. Then (ϕ, ψ) is a Dirac-harmonic map from M into N if and only if*

(i) $\mathcal{R}^T(\phi, \psi) = 0$;
(ii) $\mathcal{R}^N(\phi, \psi) = nH$ where $n = \dim M$;
(iii) $\not{D}^T \psi = 0$;
(iv) $\not{D}^N \psi = 0$.

In this section we shall be using the following ranges of indices:

$$1 \leq \alpha, \beta, \cdots \leq n, \qquad n+1 \leq s, t, \cdots \leq n', \qquad 1 \leq i, j, \cdots \leq n'.$$

Choose a local frame field $\{\epsilon_i\}$ of $\phi^{-1}TN$ such that $\{\epsilon_\alpha\}$ lies in the tangent bundle TM and $\{\epsilon_s\}$ in the normal bundle $T^\perp M$ of M. By using (12) we have

$$\nabla \phi^j = \sum_{\alpha=1}^{n} \delta^j_\alpha \epsilon_\alpha. \tag{19}$$

Plugging (19) into (13) yields

$$\mathcal{R}(\phi, \psi) = \frac{1}{2} R^i{}_{\alpha kl}\,(x)\,\langle \psi^k, \epsilon_\alpha \cdot \psi^l \rangle \epsilon_i\,(x)\,. \tag{20}$$

Choose a local orthonormal frame field $\{\epsilon_\alpha\}$ near $x \in M$ with $\nabla_{\epsilon_\alpha}\epsilon_\beta|_x = 0$. By (11) we have

$$\begin{aligned} \not{D}\psi &= \not{D}(\psi^i \otimes \epsilon_i) \\ &= \epsilon_\alpha \cdot \tilde{\nabla}_{\epsilon_\alpha}(\psi^i \otimes \epsilon_i) \\ &= \epsilon_\alpha \cdot \left[(\nabla_{\epsilon_\alpha}\psi^i) \otimes \epsilon_i + \psi^i \otimes \nabla_{\epsilon_\alpha}\epsilon_i\right] \\ &= (\epsilon_\alpha \cdot \nabla_{\epsilon_\alpha}\psi^i) \otimes \epsilon_i + \epsilon_\alpha \cdot \left[\psi^\beta \otimes \nabla_{\epsilon_\alpha}\epsilon_\beta + \psi^s \otimes \nabla_{\epsilon_\alpha}\epsilon_s\right] \\ &= \not{\partial}\psi^i \otimes \epsilon_i + \epsilon_\alpha \cdot \psi^s \otimes \nabla_{\epsilon_\alpha}\epsilon_s \end{aligned} \tag{21}$$

at x.

Let A_ν be the shape operator and $\nabla^\perp_X$ the normal connection of M in N where X denotes a tangent vector of M and ν a normal vector to M. Then

$$\nabla_{\epsilon_\alpha}\epsilon_s = -A_{\epsilon_s}\epsilon_\alpha + \nabla^\perp_{\epsilon_\alpha}\epsilon_s. \tag{22}$$

Let B be the second fundamental form of M in N. Then B satisfies the Weingarten equation

$$\langle B(X, Y), \nu \rangle = \langle A_\nu(X), Y \rangle \tag{23}$$

where $X,\, Y \in \Gamma(TM)$. By using (22) and (23) we have

$$\nabla_{\epsilon_\alpha}\epsilon_s = -\langle B(\epsilon_\alpha,\, \epsilon_\beta),\, \epsilon_s\rangle \epsilon_\beta + \nabla^\perp_{\epsilon_\alpha}\epsilon_s. \tag{24}$$

By plugging (24) into (21) we obtain

$$\not{D}\psi = \not{\partial}\psi^i \otimes \epsilon_i - \langle B(\epsilon_\alpha,\, \epsilon_\beta),\, \epsilon_s\rangle \epsilon_\alpha \cdot \psi^s \otimes \epsilon_\beta + \epsilon_\alpha \cdot \psi^s \otimes \nabla^\perp_{\epsilon_\alpha}\epsilon_s. \tag{25}$$

Let $(\cdots)^T$ and $(\cdots)^N$ denote the orthogonal projection into the tangent bundle $\Sigma M \otimes TM$ and the normal bundle $\Sigma M \otimes T^\perp M$ respectively.

Lemma 5.2 *Let ψ^T be defined by*

$$\psi^T = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$$

from an isometric immersion $\phi : M \hookrightarrow N$ and a spinor $\Psi \in \Gamma(\Sigma M)$ where ϵ_α is a local orthonormal basis on M. Then

$$\not{D}^T\psi = -\left[2\nabla_{\epsilon_\beta}\Psi + \epsilon_\beta \cdot \not{\partial}\Psi + \langle B(\epsilon_\alpha,\, \epsilon_\beta),\, \epsilon_s\rangle \epsilon_\alpha \cdot \psi^s\right] \otimes \epsilon_\beta \tag{26}$$

where $\psi^N = \Sigma_s \psi^s \otimes \epsilon_s$. In particular, if $N = N(c)$ is a Riemannian manifold of constant curvature c, then

$$\mathcal{R}^T(\phi,\, \psi) = 0,$$

$$\mathcal{R}^N(\phi,\, \psi) = -2ncRe\langle \psi^s,\, \Psi\rangle \epsilon_s$$

where $n = \mathrm{dim}M$.

Proof Choose a local orthonormal frame field $\{\epsilon_\alpha\}$ near $x \in M$ with $\nabla_{\epsilon_\alpha}\epsilon_\beta|_x = 0$.

$$\begin{aligned}
\not{\partial}\psi^\alpha &= \not{\partial}(\epsilon_\alpha \cdot \Psi) \\
&= \epsilon_\beta \cdot \nabla_{\epsilon_\beta}(\epsilon_\alpha \cdot \Psi) \\
&= \epsilon_\beta\left[(\nabla_{\epsilon_\beta}\epsilon_\alpha) \cdot \Psi + \epsilon_\alpha \cdot \nabla_{\epsilon_\beta}\Psi\right] \\
&= \epsilon_\beta \cdot \epsilon_\alpha \cdot \nabla_{\epsilon_\beta}\Psi \\
&= -\nabla_{\epsilon_\alpha}\Psi - \textstyle\sum_{\beta \neq \alpha} \epsilon_\alpha \cdot \epsilon_\beta \cdot \nabla_{\epsilon_\beta}\Psi \\
&= -2\nabla_{\epsilon_\alpha}\Psi - \epsilon_\alpha \cdot \not{\partial}\Psi.
\end{aligned} \tag{27}$$

Substituting (27) into (25) and taking the tangent projection yield (26). Now we assume that $N := N(c)$ is of constant curvature c. Then the components of the Riemannian curvature tensor of N satisfy

$$R^i{}_{jkl} = c(\delta^i{}_k\delta_{jl} - \delta^i{}_l\delta_{jk}).$$

From which together with (20) we obtain

$$\begin{aligned}\mathcal{R}(\phi,\,\psi) &= c(\delta^i{}_k\delta_{\alpha l}-\delta^i{}_l\delta_{\alpha k})Re\langle\psi^k,\,\epsilon_\alpha\cdot\psi^l\rangle\epsilon_i\\ &= c\left[Re\langle\psi^i,\,\epsilon_\alpha\cdot\psi^\alpha\rangle - Re\langle\psi^\alpha,\,\epsilon_\alpha\cdot\psi^i\rangle\right]\epsilon_i\\ &= 2cRe\langle\psi^i,\,\epsilon_\alpha\cdot\psi^\alpha\rangle\epsilon_i.\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{R}^T(\phi,\,\psi) &= 2cRe\langle\psi^\beta,\,\epsilon_\alpha\cdot\psi^\alpha\rangle\epsilon_\beta\\ &= 2cRe\langle\epsilon_\beta\cdot\Psi,\,\epsilon_\alpha\cdot\epsilon_\alpha\cdot\Psi\rangle\epsilon_\beta\\ &= -2cRe\langle\epsilon_\beta\cdot\Psi,\,\Psi\rangle\epsilon_\beta=0\end{aligned}\tag{28}$$

and

$$\begin{aligned}\mathcal{R}^N(\phi,\,\psi) &= 2cRe\langle\psi^s,\,\epsilon_\alpha\cdot\psi^\alpha\rangle\epsilon_s\\ &= 2cRe\langle\psi^s,\,\epsilon_\alpha\cdot\epsilon_\alpha\cdot\Psi\rangle\epsilon_s\\ &= -2cRe\langle\psi^s,\,\Psi\rangle\epsilon_s.\end{aligned}$$

Here we have used

$$\overline{\langle\epsilon_\beta\cdot\Psi,\,\Psi\rangle}=-\langle\epsilon_\beta\cdot\Psi,\,\Psi\rangle.$$

□

We call a spinor Φ *harmonic* if it satisfies the Dirac equation without potential [3],

$$\partial\!\!\!/\Phi=0$$

where $\partial\!\!\!/$ is the usual Dirac operator [14].

In the rest of this section, we discuss hypersurfaces in a Riemannian manifold.

Lemma 5.3 *Let $\phi: M\hookrightarrow N$ be an isometric immersion with codimension 1 and $\psi\in\Gamma(\Sigma M\otimes\phi^{-1}TN)$ defined by*

$$\psi=\Sigma_\alpha\epsilon_\alpha\cdot\Psi\otimes\phi_*(\epsilon_\alpha)+\Phi\otimes\nu$$

where ν is unit normal vector of M, $\Psi,\,\Phi\in\Gamma(\Sigma M)$ and ϵ_α is a local orthonormal basis of M. Then

(i)

$$D\!\!\!\!/^{\,T}\psi=0$$

if and only if for each β

$$2\epsilon_\beta\cdot\nabla_{\epsilon_\beta}\Psi-\partial\!\!\!/\Psi=\lambda_\beta\Phi\tag{29}$$

where λ_β is the principal curvature of M in the direction ϵ_β;

(ii)

$$\not{D}^N \psi = 0$$

if and only if Φ *is a harmonic spinor.*

Proof It is easy to see that

$$\langle B(\epsilon_\alpha,\, \epsilon_\beta),\, \nu\rangle \epsilon_\alpha \cdot \Phi \otimes \epsilon_\beta$$

is globally defined. Choose an adapted orthonormal frame of M such that

$$\langle B(\epsilon_\alpha,\, \epsilon_\beta),\, \nu\rangle = \lambda_\alpha \delta_{\alpha\beta}$$

where λ_α is the principal curvature of ϕ. Plugging this into (26) yields

$$\not{D}^T \psi = -(2\nabla_{\epsilon_\beta}\Psi + \epsilon_\beta \cdot \not{\partial}\Psi + \lambda_\beta \epsilon_\beta \cdot \Phi) \otimes \epsilon_\beta.$$

It follow that $\not{D}^T \psi = 0$ if and only if

$$2\nabla_{\epsilon_\beta}\Psi + \epsilon_\beta \cdot \not{\partial}\Psi = -\lambda_\beta \epsilon_\beta \cdot \Phi \tag{30}$$

for each β. From (10), we see that (30) holds if and only if (29) holds for each β.

(ii) Note that M is a hypersurface. It follows that $\nabla^\perp \nu = 0$. Plugging this into (25) yields

$$\not{D}^N \psi = \not{\partial}\Phi \otimes \nu + \epsilon_\alpha \cdot \Phi \otimes \nabla^\perp_{\epsilon_\alpha} \nu = \not{\partial}\Phi \otimes \nu$$

which immediately implies (ii). $\square$

Corollary 5.4 *Let* $\phi : M \hookrightarrow N$ *be an isometric immersion with codimension* 1. *If* $(\phi,\, \psi)$ *is a Dirac-harmonic map then* Φ *is a harmonic spinor where*

$$\psi = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha) + \Phi \otimes \nu$$

where ν *is unit normal vector of* M*,* $\Psi,\, \Phi \in \Gamma(\Sigma M)$ *and* ϵ_α *is a local orthonormal basis of* M*.*

Proof of Theorem 1 (ii) For a totally umbilical hypersurface M, we can assume that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = \langle H,\, \nu\rangle \tag{31}$$

where λ_α is the principal curvature of M. Note that Ψ is a twistor spinor. Hence from [1, page 23, Theorem 2] the spinor field $X \cdot \nabla_X \psi$ does not depend on the unit vector field X. Together with (3), we obtain

$$\epsilon_1 \cdot \nabla_{\epsilon_1} \Psi = \cdots = \epsilon_n \cdot \nabla_{\epsilon_n} \Psi = \frac{1}{n} \partial\!\!\!/ \Psi = -\frac{\langle H, \nu \rangle}{n-2} \Phi$$

where $n = \dim M$. It follows that

$$2\epsilon_\beta \cdot \nabla_{\epsilon_\beta} \Psi - \partial\!\!\!/ \Psi = -\frac{2\langle H, \nu \rangle}{n-2} \Phi + \frac{n\langle H, \nu \rangle}{n-2} \Phi = \langle H, \nu \rangle \Phi.$$

Now (ii) can be obtained from (31), Lemma 5.1, Lemma 5.2 and Lemma 5.3 immediately.

(i) For a minimal immersion ϕ, we can assume that

$$\lambda_1 = -\lambda_2. \tag{32}$$

On the other hand,

$$2\epsilon_1 \cdot \nabla_{\epsilon_1} \Psi - \partial\!\!\!/ \Psi = -[2\epsilon_2 \cdot \nabla_{\epsilon_2} \Psi - \partial\!\!\!/ \Psi].$$

Together with (2) and (32) we get (29) for $\beta = 1, 2$. Now (i) can be obtained from Lemma 5.1, Lemma 5.2 and Lemma 5.3 immediately. $\square$

Example 3 We consider a totally umbilical hypersurface $\mathbb{R}^n$ in a hyperbolic space form $\mathbb{H}^{n+1}(-1)$ where $n \geq 3$. We recall the corresponding construction: For any two vectors X and Y in $\mathbb{R}^{n+2}$, we set

$$g(X, Y) = \sum_{i=1}^{n+1} X^i Y^i - X^{n+2} Y^{n+2}.$$

We define

$$\mathbb{H}^{n+1}(-1) = \{x \in \mathbb{R}^{n+2} \mid x_{n+2} > 0, g(x, x) = -1\}.$$

Then $\mathbb{H}^{n+1}(-1)$ is a connected simply-connected hypersurface of $\mathbb{R}^{n+2}$ and the restriction of g to the tangent space of $\mathbb{H}^{n+1}(-1)$ yields a complete Riemannian metric of constant curvature -1.

Consider the following small spheres [26]

$$\mathbb{R}^n := \{x \in \mathbb{H}^{n+1}(-1) \,|\, x_{n+2} = x_{n+1} + 1\}.$$

Then the inclusion map $\phi : \mathbb{R}^n \hookrightarrow \mathbb{H}^{n+1}(-1)$ is a totally umbilical isometric immersion with respect to the induced metric. Furthermore its sharp operator is $A = Id$ [16], that is, its principal curvatures satisfy that

$$\lambda_1 = \cdots = \lambda_n = 1.$$

It follows that $H = \nu$. We take a constant $\Phi \in \Delta_n$ where

$$\Delta_n = \mathbb{C}^{2^k} \qquad \text{for} \qquad n = 2k,\ 2k+1$$

is the vector space of complex n spinors (cf. [14]). Then Φ is a harmonic spinor on $\mathbb{R}^n$. Let us consider a twistor spinor $\Psi : \mathbb{R}^n \to \Delta_n$ on $\mathbb{R}^n$ satisfying

$$\partial\!\!\!/ \Psi = -\frac{n}{n-2}\Phi$$

where $n \geq 3$. Now we integrate the twistor equation

$$\begin{aligned} 0 &= \nabla_X \Psi + \tfrac{1}{n} X \cdot \partial\!\!\!/ \Psi \\ &= \nabla_X \Psi - \tfrac{1}{n} X \cdot \left(\tfrac{n}{n-2} \Phi \right) \\ &= \nabla_X \Psi + \tfrac{1}{2-n} X \cdot \Phi \end{aligned}$$

along the line $\{sX \,|\, 0 \leq s \leq 1\}$, i.e.

$$\begin{aligned} \Psi(X) - \Psi(0) &= (\Psi \circ \sigma)(1) - (\Psi \circ \sigma)(0) \\ &= \int_0^1 \tfrac{d(\Psi\circ\sigma)}{ds} ds \\ &= \int_0^1 (\nabla_X \Psi) ds \\ &= \int_0^1 \tfrac{1}{n-2} X \cdot \Phi ds = \tfrac{1}{n-2} X \cdot \Phi \end{aligned}$$

where $\sigma(s) := sX$ and $\Psi(0) \in \Delta_n$ is constant (cf. [1]). It is easy to see that the solutions of the equation $\partial\!\!\!/ \Psi = -\frac{n}{n-2}\Phi$ are given by $\Psi(X) = \Psi(0) + \frac{1}{n-2} X \cdot \Phi$ (cf.[1,P29, Example 1]). Now we will find $\Psi_0 := \Psi(0)$ such that (1) holds. Note that $\langle \Phi,\, X \cdot \Phi \rangle$ is purely imaginary. Hence

$$\begin{aligned} \langle \Phi,\, \Psi \rangle &= \langle \Phi,\, \Psi_0 + \tfrac{1}{n-2} X \cdot \Phi \rangle \\ &= \langle \Phi,\, \Psi_0 \rangle + \tfrac{1}{n-2} \langle \Phi,\, X \cdot \Phi \rangle = \langle \Phi,\, \Psi_0 \rangle + \tfrac{1}{n-2} Im \langle \Phi,\, X \cdot \Phi \rangle. \end{aligned}$$

It is easy to see that (1) holds when $\Phi,\, \Psi_0 \in \Delta_n$ satisfy

$$Re\langle \Phi,\, \Psi_0 \rangle = \frac{1}{2}. \tag{33}$$

Thus we obtain that $(\phi,\, \psi)$ is a Dirac-harmonic map from $\mathbb{R}^n$ into $\mathbb{H}^{n+1}(-1)$ where

$$\psi(X) = \epsilon_\alpha \cdot \left(\Psi_0 + \frac{1}{n-2} X \cdot \Phi \right) \otimes \phi_* \epsilon_\alpha + \Phi \otimes \nu$$

and Φ, Ψ_0 satisfy (33).

Remark It is easy to prove that if $\psi^T = \sum \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$ and (ϕ, ψ) is Dirac-harmonic then $n = 2$ implies that $H = 0$. Hence when $\dim M = 2$, $\Phi = 0$, (1) automatically holds, and (2) holds if and only if Ψ is a twistor spinor.

6 Hypersurfaces with constant principal curvatures in a Riemannian manifold of constant curvature

In this section, we consider first the following example. Equipped with the pseudo-Riemannian metric

$$ds^2 = dx_1^2 + \cdots + dx_{n+1}^2 - dx_{n+2}^2,$$

$\mathbb{R}^{n+2}$ becomes *Minkowski space* $\mathbb{R}_1^{n+2}$. We define *(real) hyperbolic space*

$$H^{n+1}(R) := \big\{x \in \mathbb{R}^{n+2} \,|\, q(x) = -R^2,\, x_{n+2} > 0\big\}$$

where $q(x) := x_1^2 + \cdots + x_{n+1}^2 - x_{n+2}^2$. $H^{n+1}(R)$ is a connected, simply-connected hypersurface of $\mathbb{R}_1^{n+2}$ and the restriction of ds^2 to tangent vectors yields a (positive-definite) complete Riemannian metric in $H^{n+1}(R)$ of constant sectional curvature $c = -\frac{1}{R^2}$. We now define a family of product hypersurfaces

$$M := \big\{x \in H^{n+1}(R) \,|\, x_1^2 + \cdots + x_{k+1}^2 = r^2\big\} = S^k(r) \times H^{n-k}(\sqrt{R^2 + r^2}) \tag{34}$$

for $r > 0$ and $k = 1, \cdots, n-1$. The induced metric on M is

$$ds^2_{S^k(r)} + ds^2_{H^{n-k}(\sqrt{R^2+r^2})} = r^2 ds^2_{S^k(1)} + (R^2 + r^2) ds^2_{H^{n-k}(1)}. \tag{35}$$

M has principal curvatures $\frac{\sqrt{R^2+r^2}}{rR}$ with multiplicity k and $\frac{r}{R\sqrt{R^2+r^2}}$ with multiplicity $n-k$ [25]. Therefore, the trace of the shape operator of M in $H^{n+1}(R)$ is $\frac{kR^2+nr^2}{Rr\sqrt{R^2+r^2}}$. We have the following:

Lemma 6.1 *Let $M := S^k(r) \times H^{n-k}(\sqrt{R^2 + r^2})$ be a hypersurface in $H^{n+1}(R) \subset \mathbb{R}_1^{n+2}$. Then*

(i) M is non-minimal, therefore, $\phi : M \looparrowright H^{n+1}(R)$ is not harmonic;

(ii) M has two constant principal curvatures, with constant multiplicities.

In order to getting new non-trivial Dirac-harmonic maps, we construct a spinor field along a hypersurface with two constant principal curvatures in a Riemannian manifold of constant curvature. We shall be using the following ranges of indices:

$$1 \le i,\, j,\, \cdots \le k, \qquad k+1 \le r,\, s,\, \cdots \le n, \qquad 1 \le \alpha,\, \beta,\, \cdots \le n.$$

Lemma 6.2 *Let $\phi : M^n \looparrowright N^{n+1}(c)$ be a hypersurface with two principal curvatures λ and μ in a Riemannian manifold of constant curvature c, where λ has the multiplicity k and*

μ has the multiplicity $n-k$, and $\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$ is defined by

$$\psi = \Sigma_i \epsilon_i \cdot \Psi \otimes \phi_*(\epsilon_i) + \Sigma_r \epsilon_r \cdot \Phi \otimes \phi_*(\epsilon_r) + \chi \otimes \nu$$

where ν is the unit normal vector of M, Ψ, Φ, $\chi \in \Gamma(\Sigma M)$ and ϵ_α is a local orthonormal basis of M such that ϵ_i is the eigenvector of λ and ϵ_r is the eigenvector of μ. Then

$$\mathcal{R}^T(\phi, \psi) = 2c\left[Re\langle \epsilon_i \cdot \Phi, \Psi\rangle \epsilon_i - Re\langle \epsilon_r \cdot \Phi, \Psi\rangle \epsilon_r\right]; \tag{36}$$

$$\mathcal{R}^N(\phi, \psi) = -2cRe\langle \chi, k\Psi + (n-k)\Phi\rangle \nu; \tag{37}$$

$$\not{D}^T \psi = -(2\nabla_{\epsilon_i}\Psi + \epsilon_i \cdot \not{\partial}\Psi + \lambda_\beta \epsilon_i \cdot \chi) \otimes \epsilon_i - (2\nabla_{\epsilon_r}\Phi + \epsilon_r \cdot \not{\partial}\Phi + \mu_\beta \epsilon_r \cdot \chi) \otimes \epsilon_r; \tag{38}$$

$$\not{D}^N \psi = (\not{\partial}\chi) \otimes \nu. \tag{39}$$

Proof: Denote the distributions of the spaces of principal vectors corresponding to λ and μ by D_λ and D_μ, i.e.

$$D_\lambda := \{v \in TM \mid Av = \lambda v\}, \qquad D_\mu := \{v \in TM \mid Av = \mu v\}$$

where A is the shape operator of ϕ. Then

$$\epsilon_i \in D_\lambda, \qquad \epsilon_r \in D_\mu \tag{40}$$

and ψ is well-defined. Note that the multiplicities of the two principal curvatures are constant. Thus D_λ and D_μ are completely integrable [24]. In particular, we may choose a local orthonormal frame field $\{\epsilon_i, \epsilon_r\}$ near x with $\nabla_{\epsilon_\alpha}\epsilon_\beta|_x = 0$ and satisfying (40).

Denote ψ^T by

$$\psi^T = \psi^\alpha \otimes \phi_*(\epsilon_\alpha).$$

Then

$$\begin{array}{rcl} \not{\partial}\psi^i & = & \not{\partial}(\epsilon_i \cdot \Psi) \\ & = & \epsilon_\beta \cdot \nabla_{\epsilon_\beta}(\epsilon_i \cdot \Psi) \\ & = & \epsilon_\beta\left[(\nabla_{\epsilon_\beta}\epsilon_i) \cdot \Psi + \epsilon_i \cdot \nabla_{\epsilon_\beta}\Psi\right] \\ & = & \epsilon_\beta \cdot \epsilon_i \cdot \nabla_{\epsilon_\beta}\Psi \\ & = & -\nabla_{\epsilon_i}\Psi - \sum_{\beta \neq i} \epsilon_i \cdot \epsilon_\beta \cdot \nabla_{\epsilon_\beta}\Psi \\ & = & -2\nabla_{\epsilon_i}\Psi - \epsilon_i \cdot \not{\partial}\Psi. \end{array} \tag{41}$$

Similarly we have

$$\not{\partial}\psi^r = -2\nabla_{\epsilon_r}\Phi - \epsilon_r \cdot \not{\partial}\Phi. \tag{42}$$

By using (25), we have

$$\not{D}\psi = \not{\partial}\psi^\alpha \otimes \epsilon_\alpha + (\not{\partial}\chi) \otimes \nu - \langle B(\epsilon_\alpha, \epsilon_\beta), \nu\rangle \epsilon_\alpha \cdot \chi \otimes \epsilon_\beta. \tag{43}$$

Here we have used $\nabla^\perp \nu = 0$. Plugging (41) and (42) into (43) and using the Weingarten equation yield

$$\begin{aligned}
\not{D}\psi &= -\left[2\nabla_{\epsilon_i}\Psi + \epsilon_i \cdot \not{\partial}\Psi + B(\epsilon_\alpha, \epsilon_i), \nu\rangle\epsilon_\alpha \cdot \chi\right] \otimes \epsilon_i \\
&\quad - \left[2\nabla_{\epsilon_r}\Phi + \epsilon_r \cdot \not{\partial}\Phi + B(\epsilon_\alpha, \epsilon_r), \nu\rangle\epsilon_\alpha \cdot \chi\right] \otimes \epsilon_r + (\not{\partial}\chi) \otimes \nu \\
&= -(2\nabla_{\epsilon_i}\Psi + \epsilon_i \cdot \not{\partial}\Psi + \lambda_\beta \epsilon_i \cdot \chi) \otimes \epsilon_i \\
&\quad -(2\nabla_{\epsilon_r}\Phi + \epsilon_r \cdot \not{\partial}\Phi + \mu_\beta \epsilon_r \cdot \chi) \otimes \epsilon_r + (\not{\partial}\chi) \otimes \nu.
\end{aligned}$$

Thus we obtain (38) and (39).

Note that $N^{n+1}(c)$ has constant sectional curvature c. Consider ϵ_α, ν as a local orthonormal frame field of $\phi^{-1}TN$. By simple calculations, we have

$$\mathcal{R}^T(\phi, \psi) = 2cRe\langle\psi^\beta, \epsilon_\alpha \cdot \psi^\alpha\rangle\epsilon_\beta, \tag{44}$$

$$\mathcal{R}^N(\phi, \psi) = 2cRe\langle\chi, \epsilon_\alpha \cdot \psi^\alpha\rangle\nu. \tag{45}$$

By using the skew-symmetry relation of the Clifford product and the property of Hermitian metric we have

$$Re\langle\psi^i, \epsilon_j \cdot \psi^j\rangle = Re\langle\epsilon_i \cdot \Psi, \epsilon_j \cdot \epsilon_j \cdot \Psi\rangle = -Re\langle\epsilon_i \cdot \Psi, \Psi\rangle = 0, \tag{46}$$

$$\begin{aligned}
Re\langle\psi^i, \epsilon_r \cdot \psi^r\rangle &= -Re\langle\epsilon_i \cdot \Psi, \Phi\rangle \\
&= Re\langle\Psi, \epsilon_i \cdot \Phi\rangle \\
&= Re\overline{\langle\epsilon_i \cdot \Phi, \Psi\rangle} = Re\langle\epsilon_i \cdot \Phi, \Psi\rangle.
\end{aligned} \tag{47}$$

Similarly, we have

$$Re\langle\psi^r, \epsilon_i \cdot \psi^i\rangle = -Re\langle\epsilon_r \cdot \Phi, \Psi\rangle, \tag{48}$$

$$Re\langle\psi^r, \epsilon_s \cdot \psi^s\rangle = 0. \tag{49}$$

Substituting (46), (47), (48) and (49) into (44) yields

$$\begin{aligned}
\mathcal{R}^T(\phi, \psi) &= 2cRe\langle\psi^i, \epsilon_j \cdot \psi^j\rangle\epsilon_i + 2cRe\langle\psi^i, \epsilon_r \cdot \psi^r\rangle\epsilon_i \\
&\quad +2cRe\langle\psi^r, \epsilon_i \cdot \psi^i\rangle\epsilon_r + 2cRe\langle\psi^r, \epsilon_s \cdot \psi^s\rangle\epsilon_r \\
&= 2c\left(Re\langle\epsilon_i \cdot \Phi, \Psi\rangle\epsilon_i - Re\langle\epsilon_r \cdot \Phi, \Psi\rangle\epsilon_r\right).
\end{aligned}$$

Finally, using (10) and (45) we obtain (37). □

7 Dirac-harmonic maps from surfaces II

In this section, we give first a useful criterion for a class of maps from surfaces into a three-dimensional Riemannian manifold of constant curvature to be Dirac-harmonic. By using this criterion we manufacture Dirac-harmonic maps $(\phi,\, \psi)$ from surfaces for which ϕ is not harmonic.

Theorem 7.1 *Let $\phi : M^2 \looparrowright N^3(c)$ be a surface with two principal curvatures λ and μ in a Riemannian manifold of constant curvature c, where $\lambda \neq \mu$, and let $\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$ be defined by*

$$\psi = \epsilon_1 \cdot \Psi \otimes \phi_*(\epsilon_1) - \frac{\mu}{\lambda}\epsilon_2 \cdot \Psi \otimes \phi_*(\epsilon_2) + \chi \otimes \nu$$

where ν is unit normal vector of M, $\Psi,\, \chi \in \Gamma(\Sigma M)$ and $\{\epsilon_1,\, \epsilon_2\}$ is a local orthonormal basis of M such that ϵ_1 is the eigenvector of λ and ϵ_2 is the eigenvector of μ. Then $(\phi,\, \psi)$ is a Dirac-harmonic map from M into N if and only if

(i) χ is a harmonic spinor;

(ii) $c(\frac{\mu}{\lambda} - 1)Re\langle\chi,\, \Psi\rangle\nu = H$;

(iii) $\epsilon_1 \cdot \nabla_{\epsilon_1}\Psi - \epsilon_2 \cdot \nabla_{\epsilon_2}\Psi = \lambda\chi$.

Proof Take $\Phi = -\frac{\mu}{\lambda}\Psi$. Substituting this into (36) we get

$$\begin{aligned}\mathcal{R}^T(\phi,\, \psi) &= 2c\left[Re\langle\epsilon_1 \cdot (-\tfrac{\mu}{\lambda}\Psi),\, \Psi\rangle\epsilon_1 - Re\langle\epsilon_2 \cdot (-\tfrac{\mu}{\lambda}\Psi),\, \Psi\rangle\epsilon_2\right] \\ &= 2c\tfrac{\mu}{\lambda}\left[Re\langle\epsilon_2 \cdot \Psi,\, \Psi\rangle\epsilon_2 - Re\langle\epsilon_1 \cdot \Psi,\, \Psi\rangle\epsilon_1\right] = 0.\end{aligned}$$

Let us assume that (i) (ii) and (iii) hold. From (37) we have

$$\mathcal{R}^N(\phi,\, \psi) = -2cRe\langle\chi,\, \Psi - \frac{\mu}{\lambda}\Psi\rangle\nu = 2c(\frac{\mu}{\lambda} - 1)Re\langle\chi,\, \Psi\rangle\nu = 2H.$$

By using (39) we obtain

$$\not{D}^N\psi = (\not{\partial}\chi) \otimes \nu = 0.$$

From (iii) we have

$$\epsilon_2 \cdot \nabla_{\epsilon_2}(-\frac{\mu}{\lambda}\Psi) - \epsilon_1 \cdot \nabla_{\epsilon_1}(-\frac{\mu}{\lambda}\Psi) = \frac{\mu}{\lambda}\left[\epsilon_1 \cdot \nabla_{\epsilon_1}\Psi - \epsilon_2 \cdot \nabla_{\epsilon_2}\Psi\right] = \mu\chi.$$

Together with (38) and (iii) we have

$$\begin{aligned}\not{D}^T\psi &= -(2\nabla_{\epsilon_1}\Psi + \epsilon_1 \cdot \not{\partial}\Psi + \lambda\epsilon_1 \cdot \chi) \otimes \epsilon_1 \\ &\quad - \big(2\nabla_{\epsilon_2}(-\tfrac{\mu}{\lambda}\Psi) + \epsilon_2 \cdot \not{\partial}(-\tfrac{\mu}{\lambda}\Psi) + \mu\epsilon_2 \cdot \chi\big) \otimes \epsilon_2 \\ &= \epsilon_1 \cdot (2\epsilon_1 \cdot \nabla_{\epsilon_1}\Psi - \not{\partial}\Psi - \lambda\chi) \otimes \epsilon_1 \\ &\quad + \epsilon_2 \cdot \big(2\epsilon_2\nabla_{\epsilon_2}(-\tfrac{\mu}{\lambda}\Psi) - \not{\partial}(-\tfrac{\mu}{\lambda}\Psi) - \mu\chi\big) \otimes \epsilon_2 \\ &= \epsilon_1 \cdot (\epsilon_1 \cdot \nabla_{\epsilon_1}\Psi - \epsilon_2 \cdot \nabla_{\epsilon_2}\Psi - \lambda\chi) \otimes \epsilon_1 \\ &\quad + \epsilon_2 \cdot \big(\epsilon_2 \cdot \nabla_{\epsilon_2}(-\tfrac{\mu}{\lambda}\Psi) - \epsilon_1 \cdot \nabla_{\epsilon_1}(-\tfrac{\mu}{\lambda}\Psi) - \mu\chi\big) \otimes \epsilon_2 \\ &= (\epsilon_1 \cdot 0) \otimes \epsilon_1 + \tfrac{\mu}{\lambda}(\epsilon_2 \cdot 0) \otimes \epsilon_2 = 0.\end{aligned}$$

From Lemma 5.1 we see that $(\phi,\, \psi)$ is a Dirac-harmonic map.

Conversely, if (ϕ, ψ) is a Dirac-harmonic map, then it is easy to see from Lemma 5.1 that (i) (ii) and (iii) hold. $\square$

Remark In fact $\phi : S^1(r) \times H^1(\sqrt{R^2+r^2}) \hookrightarrow H^3(R)$ has two constant principal curvatures λ and μ, where $\lambda \neq \mu$ (see proof of Theorem 3 below).

Proof of Theorem 3 Let

$$M := S^1(r)\times H^1(\sqrt{R^2+r^2}) = \{(x_1,\, y_1;\, x_2,\, y_2)\,|\, x_1^2+y_1^2 = r^2,\, x_2^2-y_2^2 = -(R^2+r^2),\, y_2>0\}$$

be parameterized as

$$M = (\mathbb{R}/2\pi r\mathbb{Z})\times\mathbb{R} = \left\{(r\cos\tfrac{\theta}{r},\, r\sin\tfrac{\theta}{r},\, \sqrt{R^2+r^2}\sinh\tfrac{t}{\sqrt{R^2+r^2}},\, \sqrt{R^2+r^2}\cosh\tfrac{t}{\sqrt{R^2+r^2}})\,|\,(\theta,\, t)\in[0,\, 2\pi r)\times\mathbb{R}\right\}. \tag{50}$$

The induced metric on M is the flat metric

$$d\theta^2 + dt^2. \tag{51}$$

Since $2^{[\frac{dim M}{2}]} = 2$, we use "two-component" spinors. We identify the "untwisted" spinor bundle on M with $[(\mathbb{R}/2\pi r\mathbb{Z})\times\mathbb{R}]\times\mathbb{C}^2$, that is to say, the spinor on M is a single periodic spinor on $\mathbb{R}^2$ [1, 21]. Let $\epsilon_1 = \frac{\partial}{\partial\theta}$ and $\epsilon_2 = \frac{\partial}{\partial t}$. Then ϵ_1 and ϵ_2 acting on spinor fields can be identified by multiplication with matrices [9, 10]

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad i = \sqrt{-1}. \tag{52}$$

Since the metric (51) is flat, $\nabla = d$ is the Levi-Civita connection on 1-forms. Hence we have

$$\nabla_{\epsilon_\alpha} = \epsilon_\alpha \tag{53}$$

on ΣM [17, 22].

We take a constant $\chi = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$. Then χ is a harmonic spinor on M. Let us consider a spinor

$$\Psi = \begin{pmatrix} f \\ g \end{pmatrix} : (\mathbb{R}/2\pi r\mathbb{Z})\times\mathbb{R}\to\mathbb{C}^2 \tag{54}$$

on M satisfying

$$\epsilon_1\cdot\nabla_{\epsilon_1}\Psi - \epsilon_2\cdot\nabla_{\epsilon_2}\Psi = \lambda\chi \tag{55}$$

where λ is a real constant. By using (52) and (53), (55) is equivalent to

$$\left(\frac{\partial}{\partial\theta} - i\frac{\partial}{\partial t}\right) g = \lambda a, \tag{56}$$

$$-\left(\frac{\partial}{\partial\theta}+i\frac{\partial}{\partial t}\right)f=\lambda b. \tag{57}$$

For arbitrary $m\in\{0,\,1,\,\cdots\}$, we consider $g:(\mathbb{R}/2\pi r\mathbb{Z})\times\mathbb{R}\to\mathbb{C}$ defined by

$$g(\theta,\,t)=\sum_{k=-m}^{m}e^{i\frac{k}{r}\theta}g_k(t). \tag{58}$$

From (56) and (58) we have

$$\lambda a=\frac{\partial g}{\partial\theta}-i\frac{\partial g}{\partial t}=\frac{i}{r}\sum_{k=-m}^{m}ke^{i\frac{k}{r}\theta}g_k(t)-i\sum_{k=-m}^{m}e^{i\frac{k}{r}\theta}g_k'(t)=\frac{i}{r}\sum_{k=-m}^{m}e^{i\frac{k}{r}\theta}\left[kg_k(t)-rg_k'(t)\right].$$

Hence we take g_k satisfying

$$\begin{cases}-ig_k'(t)=\lambda a & \text{for} \quad k=0\\ kg_k(t)-rg_k'(t)=0 & \text{for} \quad k\neq 0\end{cases}. \tag{59}$$

One can verify that

$$g_k(t):=\begin{cases}i\lambda at+c_0 & \text{for} \quad k=0\\ c_ke^{\frac{k}{r}t} & \text{for} \quad k\neq 0\end{cases}$$

satisfies (59). Plugging this into (58) yields

$$g(\theta,\,t)=\sum_{k=-m}^{m}c_ke^{\frac{k}{r}(t+i\theta)}+i\lambda at. \tag{60}$$

Similarly, the following function

$$f:(\mathbb{R}/2\pi r\mathbb{Z})\times\mathbb{R}\to\mathbb{C}$$

defined by

$$f(\theta,\,t)=\sum_{k=-m}^{m}d_ke^{\frac{k}{r}(-t+i\theta)}+i\lambda bt \tag{61}$$

satisfies (57). Plugging (60) and (61) into (54) yields

$$\Psi=i\lambda t\begin{pmatrix}b\\a\end{pmatrix}+\sum_{k=-m}^{m}e^{i\frac{k}{r}\theta}\begin{pmatrix}d_ke^{-\frac{k}{r}t}\\c_ke^{\frac{k}{r}t}\end{pmatrix}.$$

Consider $\phi:M=S^1(r)\times H^1(\sqrt{R^2+r^2})\hookrightarrow H^3(R)$. Then $H^3(R)$ has constant sectional curvature $c=-\frac{1}{R^2}$. The principal curvatures of ϕ are (cf. Section 6)

$$\lambda=\frac{\sqrt{R^2+r^2}}{rR},\qquad \mu=\frac{r}{R\sqrt{R^2+r^2}}$$

and therefore the mean curvature of ϕ is

$$\xi=\frac{R^2+2r^2}{2Rr\sqrt{R^2+r^2}}.$$

By a straightforward computation one obtains

$$\left[c\left(\frac{\lambda}{\mu}-1\right)\right]^{-1}\xi = \frac{\sqrt{R^2+r^2}(R^2+2r^2)}{2rR}.$$

Now we will find c_k and d_k such that (ii) in Theorem 7.1 holds.

$$\langle\chi,\,\Psi\rangle = a\overline{\left(i\lambda bt+\sum_{k=-m}^{m} d_k e^{\frac{k}{r}(-t+i\theta)}\right)} + b\overline{\left(i\lambda at+\sum_{k=-m}^{m} c_k e^{\frac{k}{r}(t+i\theta)}\right)} = (I)+(II)$$

where

$$(I) = a\overline{i\lambda bt} + b\overline{i\lambda at} = -i(a\bar{b}+b\bar{a})\lambda = -2\lambda i Re(a\bar{b}),$$

$$(II) = a\sum_{k=-m}^{m}\overline{d_k e^{\frac{k}{r}(-t+i\theta)}} + b\sum_{k=-m}^{m}\overline{c_k e^{\frac{k}{r}(t+i\theta)}}.$$

Note that (I) is purely imaginary. Hence

$$\begin{aligned}
Re\langle\chi,\,\Psi\rangle &= Re(II) \\
&= \textstyle\sum_{k=-m}^{m} Re\left[a\bar{d}_k e^{-\frac{k}{r}(t+i\theta)}\right] + \sum_{k=-m}^{m} Re\left[b\bar{c}_k e^{\frac{k}{r}(t-i\theta)}\right] \\
&= \textstyle\sum_{k=-m}^{m} Re\left[a\bar{d}_k e^{-\frac{k}{r}(t+i\theta)}\right] + \sum_{k=-m}^{m} Re\left[b\bar{c}_{-k} e^{-\frac{k}{r}(t-i\theta)}\right] \\
&= \textstyle\sum_{k=-m}^{m} Re\left[a\bar{d}_k e^{-i\frac{k}{r}\theta} + b\bar{c}_{-k} e^{i\frac{k}{r}\theta}\right] e^{-\frac{k}{r}t} \\
&= \textstyle\sum_{k=-m}^{m} Re\left[(a\bar{d}_k + \bar{b}c_{-k})e^{-i\frac{k}{r}\theta}\right] e^{-\frac{k}{r}t}.
\end{aligned}$$

It follows that the sufficient conditions on c_k and d_k for (ii) in Theorem 7.1 to hold are

$$Re(a\bar{d}_0+\bar{b}c_0) = \frac{\sqrt{R^2+r^2}(R^2+2r^2)}{2rR}$$

and

$$a\bar{d}_k + \bar{b}c_{-k} = 0$$

for $k=\pm1,\cdots,\pm m$. $\square$

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