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by

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Simultaneous Finite Time Blow-up in a Two-Species Model for Chemotaxis

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In honor of Erhard Heinz, on the occasion of his 85th birthday.

Abstract

A system of two classical chemotaxis equations, coupled with an elliptic equation for an attractive chemical, is analyzed. Depending on the parameter values for the three respective diffusion coefficients and the two chemotactic sensitivities in the radial symmetric setting, conditions are given for global existence of solutions and finite time blow-up. A question of interest is, whether there exist parameter regimes, where the two chemotactic species differ in their long time behavior. This questions arises in the context of differential chemotactic behavior in early aggregates of *Dictyostelium discoideum*

1 Introduction

In this paper we address the question if differential chemotaxis can serve as the main mechanism for cell sorting in *Dictyostelium discoideum* during self-organisation under starvation conditions. In laboratory experiments [10], cell sorting was observed in the early aggregation stages during mound formation. Cells moving in the top of the aggregates show stronger chemotactic abilities when being observed separately, than the cells moving at the bottom. As a first mathematical ansatz to approach this question on cell sorting, a model of two chemotactic species is considered, both of which react to the same chemical, cAMP, but with different chemotactic strength.

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So we are interested to understand the qualitative behavior of the following system

$$\begin{aligned}\partial_t u_1 &= \Delta u_1 - \chi_1 \nabla \cdot (u_1 \nabla v), \\ \partial_t u_2 &= \mu_2 \Delta u_2 - \chi_2 \nabla \cdot (u_2 \nabla v), \\ 0 &= \Delta v + u_1 + u_2 - 1, \qquad\qquad \text{in } B_R(0).\ t > 0,\end{aligned} \tag{1.1}$$

$$\text{with} \quad \mu_2, \chi_1, \chi_2 \;>\; 0 \tag{1.2}$$

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial v}{\partial \nu} \;=\; 0\ , \quad x \in \partial B_R(0), t > 0\ , \tag{1.3}$$

$$u_1(0,x) = u_{1,0}(x) \geq 0 \quad , \quad u_2(0,x) = u_{2,0}(x) \geq 0 \quad , \quad x \in B_R(0)\ , \tag{1.4}$$

where $B_R(0)$ is the two-dimensional ball of radius R, which is centered in the origin, $u_{1,0}, u_{2,0}$ are not identically zero, and u_1 and u_2 are the density variables for the two cell types with different chemotactic sensitivities, and v is the chemo-attractant.

Chemotaxis models of Keller-Segel type with several chemotactic populations reacting to several chemicals have be studied in [11]. By using energy methods sufficient conditions for global existence of solutions, especially for the case we are interested in, but with Dirichlet boundary conditions, were derived.

In [4] the dynamics of two chemotactic species are considered, which are modeled by two classical chemotaxis equations. They are interacting via two chemicals. One of the chemotactic populations is attracted by one chemical and repelled by the other. For the second population this is the other way around. The chemicals are modeled by elliptic equations. Under a certain symmetry condition for the ratio of the attractive and repulsive effects, the full model is numerically analyzed and solutions for the stationary system are described. The stationary system could be simplified by this symmetry assumption.

A question of interest in the context of our paper is, if the two chemotactic species in (1.1) can separate. One possibility could be that for one species blow-up happens, whereas the solution for the other species is still bounded. We will show that this is not the case. If blow-up happens, it is simultaneous for both chemotactic species. Specifically, it will be proved that blow-up for one chemotactic species implies blow-up for the other one. The difference between the species though is the following. For some parameter ranges of the chemotactic sensitivities and the diffusion coefficients, but probably not for all of them, dirac-mass formation in one of the chemotactic species can be observed, whereas the blow-up asymptotics for the other species grow like an integrable power law. This means that for the first species mass aggregation happens, whereas for the second species no accumulation of mass at the origin takes place, so we obtain blowup without mass aggregation.

The outline of this paper is as follows. First, in Section 2 we present a dimensional analysis of the model. Then, in Section 3, we recall several results on sufficient conditions for local and global existence of solutions in multi-component chemotaxis systems. Also we give sufficient conditions for finite time blow-up in the radial symmetric situation. In Section 4 we prove that blow-up for one of the chemotactic species implies also blow-up for the other one at the same time. In the last section we derive formal asymptotics of the blow-up profiles for a certain range of parameters of our system.

2 Dimensional analysis

The original model setup is

$$\partial_t u_1 = \mu_1 \Delta u_1 - \chi_1 \nabla(u_1 \nabla v) \tag{2.1}$$

$$\partial_t u_2 = \mu_2 \Delta u_2 - \chi_2 \nabla(u_2 \nabla v) \tag{2.2}$$

$$\partial_t v = D\Delta v + \alpha_1 u_1 + \alpha_2 u_2 - \beta v \quad \text{in} \quad B_R(0), t > 0 \tag{2.3}$$

with Neumann boundary conditions.
We will restrict our attention to the case $\mu_1 \approx \mu_2 \approx \mu$, $\chi_1 \approx \chi_2 \approx \chi$, $\alpha_1 \approx \alpha_2 \approx \alpha$. Moreover, we will assume that the concentrations u_1, u_2 are of the same order of magnitude. However, many other interesting limit cases would arise without posing these assumptions.
Suppose that $D >> \mu$, since molecular diffusion can be expected to be much faster than cell diffusion. The characteristic time scale for diffusion of the chemical in $B_R(0)$ then is $t_{chem} = \frac{R^2}{D}$, and the characteristic time scale for cell diffusion is given by $t_{cell} = \frac{R^2}{\mu}$. Thus $t_{chem} << t_{cell}$. We are interested in the regime in which the diffusive and chemotactic effects in (2.1), (2.2) are of the same order of magnitude. If v_δ denotes the typical spatial variation of the concentration of the chemical, and $\|u\|$ denotes a typical measure of the order of magnitude of the cell concentrations u_i, and if we assume that the initial cell concentrations change in scales of the order of magnitude of R, we then need

$$\mu \frac{\|u\|}{R^2} \approx \frac{\chi \|u\| v_\delta}{R^2} \quad , \quad \text{whence} \quad \mu \approx \chi v_\delta \ .$$

On the other hand (2.3) yields

$$\frac{v_\delta}{R^2} \approx \frac{\alpha \|u\|}{D} \quad , \quad \text{so} \quad \mu \approx \chi \frac{R^2 \alpha}{D} \|u\| \ .$$

Notice that $\|u\| \approx \frac{N_{cells}}{R^2}$, where N_{cells} is the number of cells. In the regime in which the Keller-Segel model yields interesting critical dynamics, if the diffusive and chemotactic terms are balanced, we must have that N_{cells} is of order one. Therefore, we will consider the limit

$$\frac{\chi\alpha}{D\mu} \approx 1 \ . \tag{2.4}$$

To non-dimensionalize the system we introduce the following new variables

$$t = t_{cell}t' \ , \ x = Rx' \ , \ u_i = \frac{\mu D}{\chi\alpha R^2}u_i' \ , \ v - v_{base} = \frac{\alpha R^2}{D}\left(\frac{\mu D}{\chi\alpha R^2}\right)v' = \frac{\mu}{\chi}v' \ ,$$

where $v_{base} = \frac{1}{\beta\pi R^2}\int(\alpha_1 u_1 + \alpha_2 u_2)dx$ is the base level of the chemo-attractor density. We further assume

$$\mu_i = \mu\mu_i' \ , \ \chi_i = \chi\chi_i' \ , \ \alpha_i = \alpha\alpha_i' \ , \quad \text{for } i = 1, 2 \ ,$$

with μ_i', χ_i', α_i' all being of order one. The original system of equations then becomes

$$\frac{\partial u_1'}{\partial t'} = \mu_1'\Delta_{x'}u_1' - \chi_1'\nabla_{x'}(u_1'\nabla_{x'}v') \ , \tag{2.5}$$

$$\frac{\partial u_2'}{\partial t'} = \mu_2'\Delta_{x'}u_2' - \chi_2'\nabla_{x'}(u_2'\nabla_{x'}v') \ , \tag{2.6}$$

$$\frac{1}{t_{cell}}\frac{\mu}{\chi}\frac{\partial v'}{\partial t'} = D\frac{\mu}{\chi R^2}\Delta_{x'}v' + \alpha(\alpha_1'u_1' + \alpha_2'u_2')\frac{\mu D}{\chi\alpha R^2} - \beta(v_{base} + \frac{\mu}{\chi}v') \ .$$

Therefore

$$\frac{t_{chem}}{t_{cell}}\frac{\partial v'}{\partial t'} = \Delta_{x'}v' + (\alpha_1'u_1' + \alpha_2'u_2') - \frac{\beta t_{chem}\chi}{\mu}v_{base} - \frac{t_{chem}}{t_{degrad}}v' \ ,$$

where $t_{degrad} = \frac{1}{\beta}$ is the characteristic time for the degradation of the chemical and

$$v_{base} = \frac{\mu}{\beta\chi}\frac{1}{t_{chem}}v_{base}' \ ,$$

with

$$v_{base}' \equiv \frac{1}{\pi}\int_{B_1(0)}(\alpha_1'u_1' + \alpha_2'u_2')dx' \ . \tag{2.7}$$

So

$$\frac{t_{chem}}{t_{cell}}\frac{\partial v'}{\partial t'} = \Delta_{x'}v' + (\alpha_1'u_1' + \alpha_2'u_2') - v_{base}' - \frac{t_{chem}}{t_{degrad}}v' \ . \tag{2.8}$$

By assumption all coefficients and functions in (2.5) and (2.6) are of order one. On the other hand, (2.8) might be simplified under the assumptions

$$t_{chem} << t_{cell} \quad \text{and} \quad t_{chem} << t_{degrad} \,. \tag{2.9}$$

If the conditions (2.4), (2.9) are satisfied, then the original system (2.1)-(2.3) reduces formally to (2.5), (2.6) and

$$0 = \Delta_{x'} v' + (\alpha_1' u_1' + \alpha_2' u_2') - v_{base}' \tag{2.10}$$

with v_{base} given as in (2.7).

Actually this limit is similar to the limit considered by Jäger and Luckhaus in [7]. Of course there are other regimes yielding interesting limits, for instance $t_{chem} << t_{cell}$, $t_{chem} \approx t_{degrad}$, but these will not be considered in this paper. Notice that now (2.5)-(2.7), (2.10) is solved for $x' \in \Omega' = B_1(0)\,,\ t' \in \mathbb{R}^+$.
It is possible to reduce (2.5)-(2.7), (2.10) to a problem containing a minimal number of parameters. To this end we define

$$\begin{aligned} U_i &= \alpha_i' u_i' \ \ , \ \ i = 1,2 \\ T &= \mu_1' t' \ \ , \ \ V = v' \ , \ X = x' \ . \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial U_1}{\partial T} &= \Delta_X U_1 - \frac{\chi_1'}{\mu_1'} \nabla_X (U_1 \nabla_X V) \ \ , \ \ X \in B_1(0) \ , \ T > 0 \\ \frac{\partial U_2}{\partial T} &= \frac{\mu_2'}{\mu_1'} \Delta_X U_2 - \frac{\chi_2'}{\mu_1'} \nabla_X (U_2 \nabla_X V) \ \ , \ \ X \in B_1(0) \ , \ T > 0 \\ 0 &= \Delta_X V + (U_1 + U_2) - \frac{1}{\pi} \int_{B_1(0)} (U_1 + U_2)\, dX \ \ , \ \ X \in B_1(0) \ , \ T > 0 \end{aligned}$$

Renaming $\frac{\chi_1'}{\mu_1'}$, $\frac{\chi_2'}{\mu_1'}$ and $\frac{\mu_2'}{\mu_1'}$ as χ_1, χ_2, μ_2 respectively, and T by t we finally obtain

$$\frac{\partial U_1}{\partial t} = \Delta_X U_1 - \chi_1 \nabla_X (U_1 \nabla_X V) \ \ , \ \ X \in B_1(0) \ , \ t > 0 \tag{2.11}$$

$$\frac{\partial U_2}{\partial t} = \mu_2 \Delta_X U_2 - \chi_2 \nabla_X (U_2 \nabla_X V) \ \ , \ \ X \in B_1(0) \ , \ t > 0 \tag{2.12}$$

$$0 = \Delta_X V + (U_1 + U_2) - \frac{1}{\pi} \int_{B_1(0)} (U_1 + U_2)\, dX \ \ , \ \ X \in B_1(0) \ , \ t > 0 \tag{2.13}$$

with boundary conditions:

$$\frac{\partial U_1}{\partial \nu_X} = \frac{\partial U_2}{\partial \nu_X} = \frac{\partial V}{\partial \nu_X} = 0 \ \ , \ \ X \in \partial B_1(0) \ , \ t > 0 \tag{2.14}$$

Later we will set the last term on the right hand side of (2.13) equal to 1.

In case of radially symmetric solutions it is more convenient to rewrite our problem in terms of the rescaled mass functions

$$M_1(t,r) = \int_0^r U_1(t,\xi)\,\xi d\xi \tag{2.15}$$

$$M_2(t,r) = \int_0^r U_2(t,\xi)\,\xi d\xi \ , \quad r = |X| \ . \tag{2.16}$$

As a consequence we obtain

$$\frac{\partial V}{\partial r} = \frac{M_1 + M_2}{r} - c\frac{r}{2} \ , \tag{2.17}$$

where $c = \frac{1}{\pi}\int_{B_1(0)} (U_1(0,X) + U_2(0,X))\,dX$. Now (2.11)-(2.14) become

$$\frac{\partial M_1}{\partial t} = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial M_1}{\partial r}\right) - \chi_1\left(\frac{cr}{2} - \frac{M_1 + M_2}{r}\right)\frac{\partial M_1}{\partial r} \ , \tag{2.18}$$

$$\frac{\partial M_2}{\partial t} = \mu_2 r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial M_2}{\partial r}\right) - \chi_2\left(\frac{cr}{2} - \frac{M_1 + M_2}{r}\right)\frac{\partial M_2}{\partial r} \ , \tag{2.19}$$

with boundary conditions

$$M_1(t,1) = \int_0^1 U_1(0,\xi)\,\xi d\xi =: m_1 \ , \quad M_2(t,1) = \int_0^1 U_2(0,\xi)\,\xi d\xi =: m_2 \tag{2.20}$$

In the following we will use u_1, u_2, v instead of U_1, U_2, V.

3 Global Solutions and Blow-up

Here we formulate some standard solvability conditions for system (1.1) together with (1.2), (1.4), and boundary conditions. First, notice that the function v is not uniquely defined by the given equations, since a solution (u_1, u_2, v) automatically implies a solution $(u_1, u_2, v + g(t))$ for any time dependent real function $g(t)$. The precise normalization of $v(t,x)$ though is not relevant for the problem we are considering, since all estimates are done in terms of ∇v, which is unique. More precisely, v does not represent the chemical attractor itself, but the difference of chemical concentration of the attractor with respect to its basal concentration. Let

$$\int_{B_1(0)} v(t,x)dx = \phi(t) \tag{3.1}$$

be a normalization condition for this difference, where $\phi(t)$ can be an arbitrary function of time. Integrating (2.8) with respect to x we obtain

$$\int_{B_1(0)} v dx = \left(\int_{B_1(0)} v'(0,x) dx \right) \exp\left(-\frac{t_{cell}}{t_{degrad}} t \right) \, .$$

Thus the relative size of t_{cell} and t_{degrad} provides several natural choices for ϕ.

For our system the following local existence result holds.

Theorem 1 *Let $u_{1,0}, u_{2,0} \in C^{2,\alpha}(B_1(0))$ for some $0 < \alpha < 1$, with $u_{1,0}, u_{2,0} \geq 0$ not being identically zero, and $\int_{B_1(0)} (u_{1,0}(x) + u_{2,0}(x)) dx = \pi$.*

Then there exists a unique classical solution of (1.1) with (1.2), (1.3), and (3.1) satisfying $u_1(0,x) = u_{1,0}(x), u_2(0,x) = u_{2,0}(x)$ for $0 \leq t < T$ for a $T > 0$.

Moreover, if $\|u_1(t,\cdot)\|_\infty + \|u_2(t,\dot{})\|_\infty \leq C$ in $0 \leq t \leq T$, it is possible to extend the solution (u_1, u_2, v) to an interval $0 \leq t \leq T + \delta$ for some $\delta > 0$.

Proof. The proof of this result can be done by standard arguments and was shown in the PhD-thesis by E.E. Espejo Arenas, [3]. ∎

Global existence results for more general multi-component chemotaxis systems were considered by Wolansky in [11]. Writing the notation by *Wolansky* on the *left* hand side of the following equations and *our* notation in the present paper on the *right* hand side we have, that $i = 1, 2$ is the index for the chemotactic species, and $k = 1$ is the index for the involved chemical.
Then $\nu_1 = 1$, $\frac{1}{\nu_2} = \mu_2$ and $\theta_{1,1} = \chi_1$, $\theta_{2,1} = \frac{\chi_2}{\mu_2}$.
For the chemo-attractant we have $\sigma_1 = \alpha = 0$, $\gamma_{1,1} = \gamma_{2,1} = 1$ and $f_1 = -1$.
Just to make the cross-check easier, we also mention further introduced notation in [11], namely: $\lambda_{i,j} = \theta_{i,1}\gamma_{j,1} = \theta_{i,1}$. Thus $\lambda_{1,1} = \chi_1$ and $\lambda_{2,1} = \frac{\chi_2}{\mu_2}$.
Additionally some constants $a_j > 0$ are defined in [11] by
$a_1\lambda_{1,2} = a_1\chi_1 = a_2\lambda_{2,1} = \frac{a_2\chi_2}{\mu_2}$. So for instance $a_1 = \frac{1}{\chi_1}$ and $a_2 = \frac{\mu_2}{\chi_2}$.
Then Theorem 5 in [11] applied to our context and our notation reads

Theorem 2 *Consider our system (1.1) on the two-dimensional disc of radius 1. Assume Dirichlet boundary conditions for the chemo-attractant. Further let*

$$8\pi \left(\frac{1}{\chi_1} N_1 + \frac{\mu_2}{\chi_2} N_2 \right) - (N_1^2 + 2N_1N_2 + N_2^2) > 0 \ ,$$

with $N_i = 2\pi m_i$ for $i = 1, 2$ are the respective total preserved masses of the two chemotactic species.

Then for $(u_1(0,\cdot),u_2(0,\cdot)) \in Y_N$ *with*
$Y_N := \left\{(u_1,u_2) : B_1(0) \to \mathbb{R}^+ : \int_{B_1(0)} u_i = N_i \; ; \int_{B_1(0)} u_i \log u_i \, dx < \infty\right\}$ *there exists a global in time classical solution.*

Remark 3 *For one species the above mentioned inequality is the classical condition for global solutions of the classical chemotaxis model.*

For system (1.1) with (1.2), (1.3), (1.4), so Neumann boundary conditions, existence of global solutions for the radial symmetric case, like in our situation, was proved in [3]. The result basically says the following:
If $T_{\max}$ *is the maximal time of existence for the classical radially symmetric solutions of our problem and if*

$$m_1 < \min\left\{\frac{\mu_2}{\chi_2}, \frac{2(2-\chi_1 m_2)}{\chi_1}\right\} \qquad \textit{and} \qquad m_2 < \min\left\{\frac{1}{\chi_1}, \frac{2(2\mu_2-\chi_2 m_1)}{\chi_2}\right\} \ , \tag{3.2}$$

then $T_{\max} = \infty$ *and* $\sup_{t\geq 0}\left\{\|u_1(t,\cdot)\|_{L^\infty} + \|u_2(t,\cdot)\|_{L^\infty} + \|v(t,\cdot)\|_{L^\infty}\right\} < \infty$.

Now we will derive sufficient conditions for blow-up in finite time for the solutions of our system.

Theorem 4 *For* $u_{1,0}, u_{2,0}$ *assume the conditions in Theorem 1.*
Let $\sigma_1(t) = \int_{B_1(0)} |x|^2 u_1(t,x)dx$ *and* $m_1(4-2\chi_1 m_1)+\chi_1\sigma_1(0) < 0$, *with* m_1, m_2 *being given as in (2.20). Then there exists a* $T < \infty$, *such that*

$$\overline{\lim}_{t\to T}\Big(\|u_1(t,\cdot)\|_{L^\infty} + \|u_2(t,\cdot)\|_{L^\infty} \Big) = \infty \ . \tag{3.3}$$

Remark 5 *From this result one can deduce that if* $m_1 > \frac{2}{\chi_1}$ *and* σ_1 *is "small" then blow-up happens in finite time.*

Proof. of Theorem 4
We will argue by contradiction and use techniques related to [8] . Suppose that (3.3) is bounded for any T. Then Theorem 1 implies that the solution is classical for any time. Now multiplying

$$\partial_t u_1 \;\; = \;\; \Delta u_1 - \chi_1 \nabla\cdot(u_1 \nabla v)$$

by $|x|^2$ and integrating the resulting relation over $B_1(0)$ we obtain

$$\partial_t \int_{B_1(0)} u_1 |x|^2 \, dx = \int_{B_1(0)} |x|^2 \Delta u_1 dx - \chi_1 \int_{B_1(0)} |x|^2 \nabla\cdot(u_1\nabla v)dx \ .$$

From Green's identity we get

$$\begin{aligned}
\partial_t \int_{B_1(0)} u_1 |x|^2 \, dx &= \int_{B_1(0)} \left(\Delta |x|^2\right) u_1 dx - \int_{\partial B_1(0)} u_1 \nabla |x|^2 \cdot dS \\
&\quad -\chi_1 \int_{B_1(0)} |x|^2 \, \nabla \cdot (u_1 \nabla v) dx \\
&= 4 \int_{B_1(0)} u_1 dx - \int_{\partial B_1(0)} u_1 \nabla |x|^2 \cdot dS + \chi_1 \int_{B_1(0)} \nabla |x|^2 \cdot (u_1 \nabla v) dx \\
&\leq 4 \int_{B_1(0)} u_1 dx + 2\chi_1 \int_{B_1(0)} u_1 (x \cdot \nabla v) dx \qquad (3.4)
\end{aligned}$$

From (2.17) and using the identity $x \cdot \nabla v = r \frac{\partial v}{\partial r}$ we derive

$$\begin{aligned}
\int_{B_1(0)} u_1 (x \cdot \nabla v) dx &= 2\pi \int_0^1 u_1 r \frac{\partial v}{\partial r} r dr \\
&= -2\pi \int_0^1 u_1 \left(M_1 + M_2 - \frac{r^2}{2} \right) r dr \\
&= -2\pi \int_0^1 M_1 \frac{\partial M_1}{\partial r} dr - 2\pi \int_0^1 u_1 M_2 dr + \frac{1}{2} \int_{B_1(0)} u_1 |x|^2 \, dx \\
&\leq -\pi \int_0^1 \frac{\partial M_1^2}{\partial r} dr + \frac{1}{2} \int_{B_1(0)} u_1 |x|^2 \, dx = -\pi m_1^2 + \frac{1}{2} \sigma_1(t)
\end{aligned}$$

From (3.4) it follows that

$$\frac{d}{dt} \sigma_1(t) \leq 4m_1 + 2\chi_1 \left(-\pi m_1^2 + \frac{1}{2} \sigma_1(t) \right) = m_1 (4 - 2\chi_1 m_1) + \chi_1 \sigma_1(t).$$

Since $m_1(4 - 2\chi_1 m_1) + \chi_1 \sigma_1(0) < 0$, it follows that $\sigma_1(t)$ is strictly decreasing with respect to t. Then for every t we have that

$$\frac{d}{dt} \sigma_1(t) < m_1 (4 - 2\chi_1 m_1) + \chi_1 \sigma_1(0) \ .$$

So

$$0 \leq \sigma_1(t) < \sigma_1(0) + \left[m_1 (4 - 2\chi_1 m_1) + \chi_1 \sigma_1(0) \right] t.$$

and $\sigma_1(t) \to 0$ as $t \to T_0^-$ for some finite T_0^-. But then (u_1, u_2, v) cannot be a classical solution. Hence due to Theorem 1 there exists $T \leq T_0$ such that

$$\overline{\lim}_{t \to T} \left(\|u_1(t, \cdot)\|_{L^\infty} + \|u_2(t, \cdot)\|_{L^\infty} \right) = \infty \ .$$

■

4 Simultaneous Blow-up

In this chapter we investigate if blow-up at two different times is possible in all densities variables of our system. Some of the techniques and tools that we use here are taken from [5] and [6]. The proof will be done by contradiction. To obtain this result we will prove that u_2 being bounded will imply a lower bound for M_1 near the blow-up point. This estimate will allow us to construct a sub-solution for (2.19) that behaves like $M_2 \geq Cr^{2\lambda}$ for small values of r at the blow-up time. This is a contradiction to $M_2 = \int_0^r u_2 \rho d\rho \leq Cr^2$ for small values of r.

To obtain our result we will first prove

Lemma 6 *Suppose that u_1 blows up at $t = T^*$ and that $\|u_2(t,\cdot)\|_{L^\infty} \leq C$ for $0 \leq t \leq T^*$. Then there exist $\varepsilon_0 > 0$ and $L > 0$, both depending on μ_2, χ_1, χ_2, and C such that*

$$\liminf_{t \to (T^*)^-} M_1(t, L\sqrt{T^* - t}) \geq \varepsilon_0.$$

Proof. We argue by contradiction. Suppose there exists a sequence $(t_n)_{n \geq 0}$ such that $t_n \to (T^*)^-$ and $M_1(t_n, L\sqrt{T^* - t_n}) < \varepsilon_0$. Define a sequence of functions

$$m_n(s,\xi) = M_1(t_n + L^2(T^* - t_n)s, L\sqrt{T^* - t_n}\,\xi) \tag{4.1}$$

for $0 \leq s < \frac{1}{L^2}$ and $0 \leq \xi \leq \frac{1}{L\sqrt{T^* - t_n}}$. Using (2.18) we obtain

$$m_{n,s} = m_{n,\xi\xi} - \frac{1}{\xi} m_{n,\xi} - \chi_1 \frac{c}{2} L^2 (T^* - t_n) \xi m_{n,\xi} + \chi_1 \frac{m_n + M_2}{\xi} m_{n,\xi}, \qquad n \geq 0\ . \tag{4.2}$$

By assumption the functions m_n blow up at $s = \frac{1}{L^2}$.

Our strategy is to construct a super-solution for (4.2), that shows that this is not the case. To obtain this contradiction the proof is divided into three steps

Step 1. Construction of an auxiliary super-solution
In order to obtain a super-solution for (4.2) we are looking for a function M which satisfies

$$M_{\xi\xi} - \frac{1}{\xi} M_\xi + \chi_1 \frac{M + M_2}{\xi} M_\xi - \chi_1 \frac{c}{2} L^2 (T^* - t_n) \xi M_\xi - M_s \leq 0 \tag{4.3}$$

in the sense of distributions. Since $u_2 \leq C$ it follows that $M_2 \leq C\frac{L^2}{2}(T^* - t_n)\xi^2$. Therefore, if $M(t,\xi)$ satisfies

$$M_{\xi\xi} - \frac{1}{\xi} M_\xi + \chi_1 \frac{M}{\xi} M_\xi + \chi_1 \frac{(C - c)}{2} L^2 (T^* - t_n) \xi M_\xi - M_s \leq 0 \tag{4.4}$$

in the sense of distributions, then M automatically satisfies (4.3).

First we obtain an auxiliary function

$$\overline{M}(\xi) = M_0 + \xi^\alpha \ , \ M_0, \alpha > 0 \ , \tag{4.5}$$

which satisfies

$$\overline{M}_{\xi\xi} - \frac{1}{\xi}\overline{M}_\xi + \chi_1 \frac{\overline{M}\ \overline{M}_\xi}{\xi} + \chi_1 \frac{(C-c)}{2} L^2(T^* - t_n)\xi \overline{M}_\xi \leq 0 \tag{4.6}$$

for $0 \leq \xi \leq \overline{\xi}$, with $\overline{M}(\overline{\xi}) < \frac{1}{\chi_1}$, and $0 \leq t \leq T^*$. This is possible, since (4.6) is equivalent to

$$\alpha(\alpha - 2) + \alpha\chi_1\{(M_0 + \xi^\alpha) + \chi_1 \frac{(C-c)}{2} L^2(T^* - t_n)\xi^2\} \leq 0 \ .$$

And the above inequality holds for any $\alpha \in (0,2)$ in $0 \leq \xi \leq \overline{\xi}$, $0 \leq t_n \leq T^*$ for $M_0, \overline{\xi}$ small enough.

Now we construct another auxiliary function $\widetilde{M}(s,\xi)$. Define $\widetilde{M}_0(\xi)$ for $\xi \geq \overline{\xi}$ as an increasing, strictly concave function, satisfying the following conditions

$$\begin{aligned}
\widetilde{M}_0(\overline{\xi}) = \overline{M}(\overline{\xi}) \quad &, \quad \frac{\partial \widetilde{M}_0}{\partial \xi}(\overline{\xi}) < \frac{\partial \overline{M}}{\partial \xi}(\overline{\xi}) \\
\frac{\partial^2 \widetilde{M}_0}{\partial \xi^2}(\xi) \quad &< \quad 0 \ \text{ for } \ \overline{\xi} \leq \xi < \infty \\
\widetilde{M}_0(\infty) \geq 2m_1 \quad &, \quad \widetilde{M}_0 \in C^\infty([\overline{\xi}, \infty)) \ .
\end{aligned} \tag{4.7}$$

We then define $\widetilde{M}(s,\xi)$ as the solution of

$$\widetilde{M}_s = \left(\chi_1 \frac{\widetilde{M}}{\xi} + \delta\xi\right) \frac{\partial \widetilde{M}}{\partial \xi} \quad \text{for } \hat{\xi}(s) \leq \xi < \infty \ , \quad \text{with } \widetilde{M}(0,\xi) = \widetilde{M}_0(\xi) \ , \tag{4.8}$$

where

$$\frac{d\hat{\xi}}{ds}(s) = \chi_1 \frac{\widetilde{M}(\overline{\xi})}{\hat{\xi}(s)} + \delta\hat{\xi}(s) \quad \text{for} \quad s \geq 0 \ , \quad \text{with } \hat{\xi}(0) = \overline{\xi} \tag{4.9}$$

and for $\delta > 0$ sufficiently small.

In the following we assume, that $\chi_1 \frac{(C-c)}{2} L^2(T - t_n) \leq \delta$. Problem (4.8) can be solved explicitly for short time, by using the method of characteristics. Moreover, for small values of s, i.e. $s \in [0, s_0]$, with $s_0 > 0$ sufficiently small and independent of L, the following estimates hold

$$
\begin{aligned}
&(i) \quad \widetilde{M}(x, \hat{\xi}(s)) = \overline{M}(\xi) && (4.10)\\
&(ii) \quad 0 < \frac{\partial \widetilde{M}}{\partial \xi}(s,\xi) < \frac{1}{\hat{\xi}(s)} \frac{\partial \overline{M}}{\partial \xi}(\xi) && (4.11)\\
&(iii) \quad \frac{\partial^2 \widetilde{M}}{\partial \xi}(s,\xi) < 0 \quad \text{for} \quad \overline{\xi} \le \xi < \infty && (4.12)\\
&(iv) \quad \widetilde{M}(s,\cdot) \in C^\infty([\overline{\xi}, \infty)) &&\\
&(v) \quad \widetilde{M}(s,\infty) > 2m_1 &&\\
&(vi) \quad \hat{\xi}(s) \ge \frac{\overline{\xi}}{w} \quad \text{for} \quad 0 \le s \le s_0 \ . &&
\end{aligned}
$$

These estimates can be derived by integration over characteristics. From (4.12) it follows that

$$\widetilde{M}_{\xi\xi} - \frac{1}{\xi}\widetilde{M}_\xi + \chi_1 \frac{\widetilde{M}\,\widetilde{M}_\xi}{\xi} + \chi_1 \frac{(C-c)}{2} L^2 (T - t_n) \xi \widetilde{M}_\xi - \widetilde{M}_s \le 0 \tag{4.13}$$

for $0 \le s \le s_0$, $\hat{\xi}(s) \le \xi < \infty$, if $\frac{\chi_1(C-c)}{2} L^2(T-t_n) \le \delta$.

We now define our super-solution as

$$\hat{M}(s,\xi) = \begin{cases} \overline{M}\left(\frac{\xi}{\hat{\xi}(s)}\right) \ , \ 0 \le \xi \le \hat{\xi}(s) \\ \\ \widetilde{M}(s,\xi) \ , \ \xi \ge \hat{\xi}(s) \end{cases} \tag{4.14}$$

with $0 \le s \le s_0$.

Due to (4.6),(4.10),(4.11), and (4.13), we have that $M(s,\xi)$ satisfies (4.3) in the sense of distributions for $0 \le s \le s_0$, $\xi \ge 0$. Moreover, $\hat{M}(s,\xi) \ge m_1$ for $\xi \ge \xi_\alpha$, with some $\xi_\alpha > 0$.

Finally define $M(s,\xi) = \hat{M}(\lambda s, \lambda \xi)$ for $\lambda > 0$ sufficiently large, then $M(s,\xi)$ satisfies (4.3) in the sense of distributions for $s \in (0, \frac{s_0}{\lambda})$, $\xi \ge 0$, $(T - t_n)$ small and

$$M(s,\xi) \ge m_1 \quad \text{for} \quad \xi \ge 1 \ . \tag{4.15}$$

Step 2. M_1 is "small" close to the origin for $t \in [t_n, T^*]$.

Suppose that $\varepsilon_0 < M_0$, with M_0 as in (4.5). Then from (4.1) it follows, that for L sufficiently large and $\frac{1}{L^2} \le \frac{s_0}{\lambda}$ that

$$m_n(0,\xi) \le M(0,\xi) \quad \text{for } \xi \ge 0\ . \tag{4.16}$$

By comparison we obtain

$$\begin{aligned} m_n(s,\xi) &\le M(s,\xi) \quad \text{for} \quad \xi \ge 0,\ 0 \le s \le \frac{1}{L^2} && (4.17)\\ m_n(s,\xi) &\le \overline{M}\,\overline{\xi} < \frac{1}{\chi_1} \quad \text{for} \quad 0 \le \xi \le \frac{1}{2}\overline{\xi},\ 0 \le s \le \frac{1}{L^2} && (4.18)\end{aligned}$$

respectively using (4.5),(4.7),(4.14)

$$M_1(t,r) \le \frac{1}{\chi_1}\ ,\ \text{for}\ \ 0 \le r \le \frac{\overline{\xi}}{2} L\sqrt{T^* - t_n}\ ,\ t_n \le t \le T^*\ . \tag{4.19}$$

Step 3. (4.19) implies that u_1 does not blow up at $t = T^*$.

We give a comparison argument. To do so, a super-solution for (2.18) is constructed. Since $M_2 \le \frac{C}{2} r^2$ due to the boundedness of u_2, we obtain this super-solution by finding $\hat{M}_1(t,r)$ such that

$$\frac{\partial \hat{M}_1}{\partial t} \ge \frac{\partial^2 \hat{M}_1}{\partial r^2} - \frac{1}{r}\frac{\partial \hat{M}_1}{\partial r} + \chi_1 \frac{(C-c)}{2} r \frac{\partial \hat{M}_1}{\partial r} + \chi_1 \frac{\hat{M}_1}{r}\frac{\partial \hat{M}_1}{\partial r}\ . \tag{4.20}$$

We can obtain a solution for (4.20) by setting the right hand side equal to zero. By simple ODE arguments one can solve the stationary system to get $\hat{M}_1 = \hat{M}_1(r)$. We can express

$$\hat{M}_1(r) = \varphi(\frac{r}{\varepsilon}) \text{ with } \varepsilon << 1 \text{ and } \varphi(\xi) = \frac{4}{\chi_1}\frac{\xi^2}{(\xi^2+1)} + O(\varepsilon \frac{\xi^2}{\xi^2+1}) \tag{4.21}$$

Choosing $\varepsilon > 0$ sufficiently small, we can then compare $M_1(t,r)$ and $\hat{M}_1(r)$ to obtain

$$M_1(t,r) \le K r^2 \ \ \text{for } 0 \le t \le T^*\ ,\ r \le 1\ , \tag{4.22}$$

and $K > 0$ fixed.

Finally we provide an estimate for u_1. We have that

$$u_1(t,r) = \frac{1}{r}\frac{\partial M_1}{\partial r} \ . \tag{4.23}$$

Therefore we have to estimate $\frac{\partial M_1}{\partial r}$. To do this, we define for small $R > 0$ the family of functions

$$M_{1,R}(\tau,\rho) = \frac{M_1(t_0 + \tau R^2, R\rho)}{R^2} \tag{4.24}$$

for $0 \leq \rho \leq 2$ and $t_0 \in (0, T^* - R^2)$. Now (4.23) in particular implies

$$|M_{1,R}(\tau,\rho)| \leq C \text{ for } \frac{1}{4} \leq \rho \leq 2 \ , \ \tau \in [0,1] \ .$$

On the other hand

$$\frac{\partial M_{1,R}}{\partial \tau} = \rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial M_{1,R}}{\partial \rho}\right) + f\left(\rho, M_{1,R}, \frac{\partial M_{1,R}}{\partial \rho}\right) \ , \tag{4.25}$$

where f is bounded for any $R \leq 1$, $\rho \in (\frac{1}{4}, 2)$, $\tau \in [0,1]$. Therefore, classical interior regularity theory for parabolic equations yields

$$\left|\frac{\partial M_{1,R}}{\partial \rho}\right| \leq C \ \text{ for } \rho \in \left(\frac{1}{2}, 1\right) \ , \ \tau \in \left[\frac{1}{2}, 1\right] \ . \tag{4.26}$$

Combining (4.24) and (4.26) it follows, that $|\frac{\partial M_1}{\partial r}| \leq Cr$ for $r \leq 1$, $t \in (\frac{T^*}{2}, T^*)$. Using (4.23) we obtain that $u_1(t,r)$ is bounded for $0 \leq r \leq 1$, $t \in (0, T^*)$, and our Lemma follows. ■

Now we focus on the main result of our paper, which is

Theorem 7 *Assume that $u_{1,0}, u_{2,0}$ fulfill the assumptions of Theorem 1. Also, suppose that*

$$\limsup_{t \to (T^*)^-} \|u_1(t,\cdot)\|_{L^\infty(B_1(0))} = \infty \ , \ \textit{for some } T^* > 0 \tag{4.27}$$

$$\textit{then} \qquad \limsup_{t \to (T^*)^-} \|u_2(t,\cdot)\|_{L^\infty(B_1(0))} = \infty \ . \tag{4.28}$$

Proof. We argue by contradiction. Suppose that (4.27) is fulfilled and (4.28) does not hold. Then there exists a constant $C > 0$ such that

$$0 \le u_2(t,x) \le C \quad \text{for } x \in B_1(0)\ ,\ 0 \le t \le T^*\ . \tag{4.29}$$

Thus

$$M_2(t,r) \le \frac{Cr^2}{2} \quad \text{for } 0 \le r \le 1\ ,\ 0 \le t \le T^*\ . \tag{4.30}$$

Our strategy is to construct a sub-solution for (2.19) behaving like $Kr^{2\lambda}$ with $K > 0$, for $r \to 0^+$ with $\lambda \in (0,1)$. This then contradicts (4.30).
Let $I_{\{r \ge a\}}$ be the characteristic function of the set $\{r \ge a\}$. Due to Lemma 6 and the monotonicity of M_1 with respect to r we have that $M_1(t,r) \ge \varepsilon_0 > 0$ for $L\sqrt{T^* - t} \le r \le 1$, with $(T^* - t)$ sufficiently small. Using (2.19) as well as the fact that $\frac{\partial M_2}{\partial r} \ge 0$, it follows that

$$\mu_2 \frac{\partial^2 M_2}{\partial r^2} - \mu_2 \frac{1}{r}\frac{\partial M_2}{\partial r} - \chi_2 \left(\frac{cr}{2} - \frac{\varepsilon_0 I_{\{r \ge L\sqrt{T^*-t}\}}}{r} \right) \frac{\partial M_2}{\partial r} - \frac{\partial M_2}{\partial t} \le 0\ .$$

Therefore, we can obtain a sub-solution for M_2 by finding a function M satisfying

$$\mu_2 \frac{\partial^2 M}{\partial r^2} - \mu_2 \frac{1}{r}\frac{\partial M}{\partial r} - \chi_2 \left(\frac{cr}{2} - \frac{\varepsilon_0 I_{\{r \ge L\sqrt{T^*-t}\}}}{r} \right) \frac{\partial M}{\partial r} - \frac{\partial M}{\partial t} \ge 0\ . \tag{4.31}$$

We are looking for M in the form

$$M(t,r) = \psi\left(\frac{r}{\sqrt{T^* - t}}, -\log(T^* - t) \right)\ . \tag{4.32}$$

Let $y = \frac{r}{\sqrt{T^*-t}}$ and $\tau = -\log(T^* - t)$, then

$$\mu_2 \frac{\partial^2 \psi}{\partial y^2} - \frac{y}{2}\frac{\partial \psi}{\partial y} - \mu_2 \frac{1}{y}\frac{\partial \psi}{\partial y} - \chi_2 \left(\frac{cy\exp(-\tau)}{2} - \frac{\varepsilon_0 I_{\{y \ge L\}}}{y} \right) \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial \tau} \ge 0\ . \tag{4.33}$$

The structure of (4.33) suggests to look for solutions of the form

$$\psi(\tau, y) = \exp(-\lambda\tau)Q(y) + \exp(-(\lambda+1)\tau)G(y)\ , \tag{4.34}$$

where

$$\mu_2 \frac{d^2 Q}{dy^2} - \left(\frac{y}{2} + \frac{\mu_2}{y} - \chi_2 \frac{\varepsilon_0 I_{\{y \ge L\}}}{y} \right) \frac{dQ}{dy} + \lambda Q = 0\ . \tag{4.35}$$

and

$$\mu_2 \frac{d^2G}{dy^2} - \left(\frac{y}{2} + \frac{\mu_2}{y} - \chi_2 \frac{\varepsilon_0 I_{\{y \geq L\}}}{y}\right) \frac{dG}{dy} + (\lambda+1)G \geq c\chi_2 y \frac{dG}{dy} \ . \tag{4.36}$$

It will later be checked if $\psi(\tau, y)$ in (4.34) yields a sub-solution for $0 \leq y \leq \delta_0 \exp(\tau)$ for $\delta_0 \geq 0$ small enough.

Before, we study the function $Q(y)$ as defined in (4.35).

Step 1. Analysis of $Q(y)$:
Equation (4.35) can be solved explicitly using Kummer functions. Let $z = \frac{y^2}{4\mu_2}$, then (4.35) transforms into

$$z \frac{d^2Q}{dz^2} - \left(z - \frac{\chi_2 \varepsilon_0}{2\mu_2} I_{\{z \geq \frac{L^2}{4\mu_2}\}}\right) \frac{dQ}{dz} + \lambda Q = 0 \ . \tag{4.37}$$

If $z \leq \frac{L^2}{4\mu_2}$, this equation reduces to

$$zQ_{zz} - zQ_z + \lambda Q = 0 \ , \tag{4.38}$$

and the unique solution of this equation, vanishing at the origin is (cf. [1])

$$Q(z) = AzM(-\lambda + 1; 2; z) \ , \tag{4.39}$$

with

$$M(a; b; z) = \sum_{l=0}^{\infty} \frac{(a_l)}{(b)_l} \frac{z^l}{l!} \ ,$$

where $(a)_l = a(a+1)(a+2)...(a+l-1)$ and $(a)_0 = 1$. On the other hand, for $z \geq \frac{L^2}{4\mu_2}$ equation (4.37) becomes

$$z \frac{d^2Q}{dz^2} - \left(z - \frac{\chi_2 \varepsilon_0}{2\mu_2}\right) \frac{dQ}{dz} + \lambda Q = 0 \ . \tag{4.40}$$

The only solution of this equation that does not grow exponentially for large z is

$$Q(z) = BU(-\lambda; \frac{\varepsilon_0 \chi_2}{2\mu_2}; z) \ , \tag{4.41}$$

with

$$U(a, b, z) = \frac{\pi}{\sin(\pi b)} \left(\frac{M(a; b; z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b} \frac{M(1+a-b; 2-b; z)}{\Gamma(a)\Gamma(2-b)}\right)$$

for $b \notin \mathbb{Z}$. In particular, using (4.39), (4.41) one obtains that the solution of (4.35) is given as

$$Q(y) = \frac{Ay^2}{4\mu_2} M(-\lambda+1;2;\frac{y^2}{4\mu_2}) \ \text{ for } y < L \ , \tag{4.42}$$

$$Q(y) = BU(-\lambda;\frac{\varepsilon_0\chi_2}{2\mu_2};\frac{y^2}{4\mu_2}) \ \text{ for } y > L \ . \tag{4.43}$$

Suppose that $A > 0$ is a given constant. To obtain a solution of (4.35) with the aid of (4.42), (4.43) we need to impose continuity for Q and $\frac{dQ}{dy}$ at $y = L$. Using 13.4.8 and 13.4.21 in [1] this requires

$$\frac{L^2}{4\mu_2} AM(-\lambda+1;2;\frac{L^2}{4\mu_2}) = BU\left(-\lambda;\frac{\varepsilon_0\chi_2}{2\mu_2};\frac{L^2}{4\mu_2}\right) \tag{4.44}$$

$$A\frac{(-\lambda+1)}{16\mu_2^2}L^3 M(-\lambda+2;3;\frac{L^2}{4\mu_2}) + \frac{AL}{2\mu_2}M(-\lambda+1;2;\frac{L^2}{4\mu_2})$$
$$= \frac{B\lambda L}{2\mu_2}U\left(-\lambda+1;1+\frac{\varepsilon_0\chi_2}{2\mu_2};\frac{L^2}{4\mu_2}\right) \ . \tag{4.45}$$

For the moment suppose

$$U(-\lambda;\frac{\varepsilon_0\chi_2}{2\mu_2};\frac{L^2}{4\mu_2}) \neq 0 \ \text{ for } \lambda \in (0,1) \ . \tag{4.46}$$

Without loss of generality we can always assume that L is sufficiently large. Then

$$B = \frac{L^2}{4\mu_2}\frac{M(-\lambda+1;2;\frac{L^2}{4\mu_2})}{U\left(-\lambda;\frac{\varepsilon_0\chi_2}{2\mu_2};\frac{L^2}{4\mu_2}\right)}A \ .$$

So (4.45) can be rewritten as

$$f(\lambda) := \frac{(1-\lambda)L^2}{2}M(-\lambda+2;3;\frac{L^2}{4\mu_2}) + 4\mu_2 M(-\lambda+1;2;\frac{L^2}{4\mu_2}) \tag{4.47}$$
$$-\lambda L^2 M(-\lambda+1;2;\frac{L^2}{4\mu_2})\frac{U\left(-\lambda+1;1+\frac{\varepsilon_0\chi_2}{2\mu_2};\frac{L^2}{4\mu_2}\right)}{U\left(-\lambda;\frac{\varepsilon_0\chi_2}{2\mu_2};\frac{L^2}{4\mu_2}\right)} = 0$$

It can be shown that there exists at least one solution of (4.47) for $\lambda \in (0,1)$ and L large enough, if (4.46) holds. Indeed, we have

$$f(0) = \frac{L^2}{2}M(2;3;\frac{L^2}{4\mu_2}) + 4\mu_2 M(1;2;\frac{L^2}{4\mu_2}) \ ,$$

and using

$$M(a;b;z) \approx \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} \quad \text{for} \quad z \to \infty$$

(cf. 13.1.4 in [1]), it follows that

$$f(0) \approx 4\mu_2 \exp(\frac{L^2}{4\mu_2}) > 0 \quad \text{for} \quad L \to \infty \ .$$

On the other hand

$$f(1) = 4\mu_2 M(0;2;\frac{L^2}{4\mu_2}) - L^2 M\left(0;2;\frac{L^2}{4\mu_2}\right) \frac{U(0;1+\frac{\varepsilon_0\chi_2}{2\mu_2};\frac{L^2}{4\mu_2})}{U(-1;\frac{\varepsilon_0\chi_2}{2\mu_2};\frac{L^2}{4\mu_2})} \ .$$

Since $M(0;b;z) = 1$, it follows that

$$f(1) = 4\mu_2 \left(1 - \frac{L^2}{4\mu_2} \frac{U(0;1+\frac{\varepsilon_0\chi_2}{2\mu_2};\frac{L^2}{4\mu_2})}{U(-1;\frac{\varepsilon_0\chi_2}{2\mu_2};\frac{L^2}{4\mu_2})}\right) \ .$$

So we need to study the sign of

$$g(b,x) := \left(1 - x\frac{U(0;b+1;x)}{U(-1;b;x)}\right) \ .$$

Using 13.4.21 in [1] we obtain $U'(a;b;x) = -aU(a+1;b+1;x)$. Thus

$$g(b,x) = 1 - x\frac{U'(-1;b;x)}{U(-1;b;x)} \ .$$

Using 13.5.2 in [1] we get

$$U(0;b;x) \approx x - b + O\left(\frac{1}{x}\right) \quad \text{for} \quad x \to \infty \ ,$$

and

$$g(b,x) \approx 1 - \frac{x}{(x-b)} \approx -\frac{b}{x} < 0 \quad \text{for} \quad x \to \infty \ .$$

With $b = \frac{\varepsilon_0\chi_2}{2\mu_2} > 0$ it follows, that $f(1) < 0$. Therefore there exists a root of (4.47) in the interval $\lambda \in (0,1)$. It only remains to prove (4.46). For large L this is a direct consequence of 13.5.2 in [1].

Step 2. Analysis of $G(y)$:
Now we study the properties of $G(y)$ in (4.34). A solution of the inequality (4.37) can be obtained by finding a solution of

$$\mu_2 \frac{\partial^2 G}{\partial y^2} - \left(\frac{y}{2} + \frac{\mu_2}{y}\right) \frac{\partial G}{\partial y} + (\lambda+1) G(y) = c\chi_2 y \frac{\partial Q}{\partial y} \tag{4.48}$$

for $y \geq 0$, satisfying $\frac{\partial G}{\partial y} > 0$ for $y \geq L$. The unique solution of (4.48), which is polynomially bounded for $y \to \infty$ and bounded quadratically for $y \to 0$, is given by

$$\begin{aligned} G(y) &= y^2 M(-\lambda; 2; \frac{y^2}{4\mu_2}) \int_y^\infty \frac{\chi_2 \eta Q_y(\eta) U(-\lambda; 2; \frac{\eta^2}{4\mu_2}) \eta^2}{W(\eta)} d\eta && (4.49) \\ &\quad + y^2 U(-\lambda; 2; \frac{y^2}{4\mu_2}) \int_0^y \frac{\chi_2 \eta Q_y(\eta) M(-\lambda; 2; \frac{\eta^2}{4\mu_2}) \eta 2}{W(\eta)} d\eta \ , && (4.50) \end{aligned}$$

where

$$W(y) = \left| \begin{array}{cc} y^2 M(-\lambda; 2; \frac{y^2}{4\mu_2}) & y^2 U(-\lambda; 2; \frac{y^2}{4\mu_2}) \\ \frac{d}{dy}\left(y^2 M(-\lambda; 2; \frac{y^2}{4\mu_2})\right) & \frac{d}{dy}\left(y^2 U(-\lambda; 2; \frac{y^2}{4\mu_2})\right) \end{array} \right| . \tag{4.51}$$

Using (4.41) we obtain with 13.1.8 in [1] that

$$Q(y) \approx B \left(\frac{y^2}{4\mu_2}\right)^\lambda \quad \text{for} \quad y \to \infty \ . \tag{4.52}$$

Combining this with (4.49), (4.51), and 13.1.4, 13.1.8 in [1] we get

$$\begin{aligned} G(y) &= O\left(y^{2(\lambda+1)}\right) \quad \text{for} \quad y \to \infty && (4.53) \\ G(y) &= O(y^2) \quad \text{for} \quad y \to 0 \ . && (4.54) \end{aligned}$$

Similar asymptotics and estimates for the derivatives can be obtained for Q and G.

Step 3. Comparison argument:
The function ψ defined in (4.34) satisfies (4.33) for values of y for which

$$\left|\frac{\partial G}{\partial y}\right| \exp(-(\lambda+1)\tau) \leq \frac{\partial Q}{\partial y} \exp(-\lambda\tau)$$

The asymptotics (4.52), (4.53) imply that this inequality is satisfied for $|y| \leq \delta \exp(\frac{\tau}{2})$ with $\delta > 0$ small enough.

A comparison argument using (2.19) with $M_2 = 0$ for $r = \frac{\delta}{2}$ implies that $M_2 \geq \nu > 0$ for $r = \delta$, $\frac{T^*}{2} \leq t \leq T^*$. Choosing $B > 0$ sufficiently small, we obtain that the subsolution $\psi(\tau, y)$ in (4.34) is smaller than $M_2(t, r)$ for $r = \delta$, $t \in (T^* - \theta, T^*)$ with $\theta > 0$ small. The same is true for $t = T^* - \theta$, $0 \leq r \leq \delta$. Overall, due to the self-similar structure of $\psi(\tau, y)$ and (4.52), (4.53) it follows that

$$\lim_{r \to 0} \frac{M_2((T^*)^-, r)}{r^{2\lambda}} = \infty$$

But since $\lambda \in (0, 1)$, this contradicts (4.30), thus the result follows. ∎

5 Formal derivation of the blowup profile

In this section we derive the asymptotics of the solutions of system (2.18), (2.19) near the blow-up time by formal asymptotic expansions Our analysis will be restricted to parameters χ_1, χ_2, μ_2 for which the resulting asymptotics yields $M_2 << M_1$ near the blow-up point. It seems possible to obtain other asymptotics yielding mass aggregation for both species at time $t = T^*$ when the chemotactic strength of both species is comparable. But this case will not be considered here. In this paper we will consider the situation where the chemotactic strength of the second species is much weaker than that of the first species. For $M_2 << M_1$ we can approximate (2.18), (2.19) by

$$\frac{\partial M_1}{\partial t} = r \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial M_1}{\partial r}\right) + \chi_1 \frac{M_1}{r}\frac{\partial M_1}{\partial r} + h.o.t. \tag{5.1}$$

$$\frac{\partial M_2}{\partial t} = \mu_2 r \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial M_2}{\partial r}\right) + \chi_2 \frac{M_1}{r}\frac{\partial M_2}{\partial r} + h.o.t.\ . \tag{5.2}$$

Therefore the dynamics of M_1 is decoupled from that of M_2 to the leading order. Thus the asymptotics of M_1 can be calculated like proved for the one species case in [9]. So one obtains

$$M_1(t, r) \approx M_{1,s}\left(\frac{r}{\sqrt{T - t}\,\delta(t)}\right) \tag{5.3}$$

$$\text{with} \quad M_{1,s} = \frac{4}{\chi_1}\frac{r^2}{(1 + r^2)} \tag{5.4}$$

$$\text{and} \quad \delta(t) \approx 2\exp\left(-\frac{(\gamma + 2)}{2}\right)\exp\left(-\sqrt{\frac{|\log(T^* - t)|}{2}}\right) \quad \text{for} \quad t \to T^* \ , \tag{5.5}$$

where $M_{1,s}$ is the stationary solution of the equation for M_1 and γ is the classical Euler constant.

So the problem reduces to the description of the asymptotics of M_2 in (5.2) with M_1 as in (5.3). Arguing like in the asymptotics for (5.3) it is convenient to reformulate (5.2) using self-similar variables

$$y = \frac{r}{\sqrt{T^* - t}} \ , \ \tau = -\log(T^* - t) \ , \ M_1 = \phi(\tau, y) \ , \ M_2 = \psi(\tau, y) \ . \tag{5.6}$$

Then (5.2) becomes to leading order

$$\psi_\tau \quad = \quad \mu_2 y \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial \psi}{\partial y}\right) - \frac{y}{2}\psi_y + \chi_1 \frac{\phi}{y}\psi_y \ . \tag{5.7}$$

Additionally we will need the equation for M_2 in the so-called inner region $y \approx \delta$. For this we introduce new variables

$$\xi = \frac{y}{\delta(\tau)} \ , \ \psi(\tau, y) = Q(\tau, \xi) \ , \ \phi(\tau, y) = G(\tau, \xi) \ , \tag{5.8}$$

where δ is given as in (5.5) with some slight abuse of notation. Therefore

$$\frac{\partial Q}{\partial \tau} = \mu_2 \xi \frac{\partial}{\partial \xi}\left(\frac{1}{\xi}\frac{\partial Q}{\partial \xi}\right) + \frac{\chi_2}{\xi} G \frac{\partial Q}{\partial \xi} - \left(\frac{\delta^2}{2} - \delta \delta_\tau\right) \xi \frac{\partial Q}{\partial \xi} \ . \tag{5.9}$$

Due to (5.3) we have $G(\tau, \xi) \to M_{1,s}(\xi)$ for $\tau \to \infty$. Thus to the leading order $Q(\tau, \xi) \to Q_0(\tau, \xi)$ for $\tau \to \infty$, where

$$\mu_2 \xi \frac{\partial}{\partial \xi}\left(\frac{1}{\xi}\frac{\partial Q_0}{\partial \xi}\right) + \frac{\chi_2}{\xi} M_{1,s}(\xi) \frac{\partial Q_0}{\partial \xi} = 0 \ . \tag{5.10}$$

Then

$$Q_0(\tau, \xi) = K(\tau) L(\xi) \ , \tag{5.11}$$

with

$$L(\xi) = \int_0^\xi z \exp\left(-\frac{\chi_2}{\mu_2}\int_0^z \frac{M_{1,s}(\eta)}{\eta} d\eta\right) dz \ .$$

Using (5.4) it follows that

$$L(\xi) \approx A_1 \xi^\beta + A_2 \ \text{ for } \ \xi \to \infty \ . \tag{5.12}$$

Here $\beta = 2 - \frac{4\chi_2}{\mu_2\chi_1}$ and

$$A_1 \;=\; \exp\left(-\frac{\chi_2}{\mu_2}\int_0^1 \frac{M_{1,s}(\eta)}{\eta}d\eta - \frac{\chi_2}{\mu_2}\int_1^\infty \frac{(M_{1,s}(\eta) - \frac{4}{\chi_1})}{\eta}d\eta \right) \;, \tag{5.13}$$

$$A_2 \;=\; A_1\int_0^\infty z^{1-\frac{4\chi_2}{\mu_2\chi_1}}\left[\exp\left(\frac{\chi_2}{\mu_2}\int_z^\infty \frac{(M_{1,s}(\eta) - \frac{4}{\chi_1})}{\eta}d\eta \right) - 1\right] dz \;. \tag{5.14}$$

Combining (5.11) and (5.12) we obtain the following matching condition for $\psi(\tau, y)$ in (5.7)

$$\psi(\tau, y) \approx A_1\frac{K(\tau)}{(\delta(\tau))^\beta}y^\beta + A_2K(\tau) \;\;\text{for}\;\; \tau \to \infty \;, \tag{5.15}$$

with $\delta(\tau) << y << 1$.
In order to compute the asymptotics of $\psi(\tau, y)$ we introduce a new variable

$$\Psi(\tau, y) = \frac{\psi(\tau, y)}{y^\beta} \;.$$

Then

$$\Psi_\tau = \mu_2\left(\Psi_{yy} - (2\beta + \frac{4\chi_2}{\chi_1\mu_2} - 1)\frac{\Psi_y}{y} \right) - \frac{y}{2}\Psi_y - \frac{\beta}{2}\Psi \tag{5.16}$$

Writing $N = 2\beta + \frac{4\chi_2}{\chi_1\mu_2} = 2 + \beta$, then the right-hand side of (5.16) is self-adjoint in $L^2(\mathbb{R}^+; y^{N-1}\exp(-\frac{y^2}{4\mu_2}))$. On the other hand (5.15) implies

$$\Psi(\tau, y) \approx \frac{K(\tau)A_1}{\delta^\beta} + \frac{K(\tau)A_2}{y^{N-2}} \;. \tag{5.17}$$

To compute the asymptotics of $\Psi(\tau, y)$ we argue as in [9] and write

$$\Psi(\tau, y) = a_0(\tau) + Q(\tau, y) \;, \tag{5.18}$$

with

$$\int_0^\infty y^{N-1}Q(\tau, y)\exp(-\frac{y^2}{4\mu_2})dy = 0 \;. \tag{5.19}$$

Plugging (5.18) into (5.16), multiplying by $y^{N-1}\exp(-\frac{y^2}{4\mu_2})$, integrating in $[\varepsilon, \infty)$ for $\varepsilon > 0$, we obtain, after some computations

$$\frac{d}{d\tau}\left(\int_\varepsilon^\infty y^{(N-1)}\exp\left(-\frac{y^2}{4\mu_2}\right)\varphi(\tau, y)dy \right)$$
$$= -\mu_2\varepsilon^{N-1}\varphi_y(\tau, \varepsilon)\exp(-\frac{\varepsilon^2}{4\mu_2}) - \frac{\beta}{2}\int_\varepsilon^\infty \varphi(\tau, y)y^{N-1}\exp(-\frac{y^2}{4\mu_2})dy \;. \tag{5.20}$$

Using (5.17) and taking the limit $\varepsilon \to 0$ we get

$$\frac{d}{d\tau}\left(\int_0^\infty y^{N-1}\exp(-\frac{y^2}{4\mu_2})\varphi(\tau,y)dy\right)$$
$$= -\frac{\beta}{2}\int_0^\infty \varphi(\tau,y)y^{N-1}\exp(-\frac{y^2}{4\mu_2})dy + \mu_2 K(\tau)(N-2)A_2\ , \quad (5.21)$$

and using (5.19) we obtain

$$a_{0,\tau} = -\frac{\beta}{2}a_0 + \mu_2(N-2)A_2K(\tau)\ . \quad (5.22)$$

Here $K(\tau)$ could be expected to behave like $\exp(-\frac{\beta}{2}\tau)$ up to logarithmic corrections, since $\delta(\tau) >> \exp(-\sigma\tau)$ for any $\sigma > 0$. Due to (5.17) we can assume

$$Q(\tau,y) = K(\tau)W(y)\ ,$$

where to the leading order

$$\begin{aligned}
\mu_2\left(W_{yy} + \frac{(N-1)}{2}W_y\right) - \frac{y}{2}W_y &= 0 \ \text{ for } \ y > 0\ ,\\
W(y) &\approx \frac{A_2}{y^{N-2}} \ \text{ for } \ y \to 0\ ,\\
\int_0^\infty y^{N-1}\exp\left(-\frac{y^2}{4\mu_2}\right)W(y)dy &= 0\ .
\end{aligned}$$

On the other hand, (5.17) and (5.18) yield

$$a_0(\tau) = \frac{K(\tau)}{\delta^\beta}A_1\ . \quad (5.23)$$

So (5.22) reads

$$a_{0,\tau} = -\frac{\beta}{2}a_0 + \mu_2(N-2)A_2\frac{\delta^\beta}{A_1}a_0\ ,$$

and since $\int^{+\infty}(\delta(\tau))d\tau < \infty$ it follows that $a_0(\tau) \approx C\exp(-\frac{\beta}{2}\tau)$ for $\tau \to \infty$ and some $C > 0$. Using (5.23) we get

$$K(\tau) \approx \frac{C}{A_1}(\delta(\tau))^\beta \exp(-\frac{\beta}{2}\tau) \ \text{ for } \ \tau \to \infty\ .$$

This concludes the computation for $K(\tau)$ in (5.11) to the leading order. Combining this with (5.8), (5.15) gives the asymptotics for M_2 in the region, where $|y|$ is large. Then

$$M_2(t,x) \approx Cr^{\beta} \quad , \quad \text{for } (T-t)^{\frac{1}{2}} << r << 1 \quad , \quad \text{where } t \to T \ ,$$

where $\beta = 2 - \frac{4\chi_2}{\mu_2\chi_1}$. This implies in particular that

$$u_2(T,r) = \frac{1}{r}\frac{\partial M_2}{\partial r} \approx \frac{C\beta}{r^{\frac{4\chi_2}{\mu_2\chi_1}}} \quad , \quad \text{as } r \to 0^+ \ .$$

This shows that there is a singularity for u_2 but no mass aggregation.

6 Conclusions

We analyzed a two chemotactic species model, where the chemotactic species are attracted by the same chemical. In the radial symmetric situation we wanted to see, if chemotaxis can separate the two species. From our results so far this does not seem to be the case. There is simultaneous blow-up for both chemotactic species. Specifically it is proved that if there is blow-up in one chemotactic species, then there is also blow-up in the other one. We give conditions for local and global existence. Also the existence of blow-up in this system is proved. The blow-up asymptotics of the solutions are formally calculated for certain parameter regimes. The two species can be different in this respect, since one of them shows mass concentration in the blow-up regime, whereas the other one does not.

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