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by

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Abstract

We study the coarsening rates for attachment-limited kinetics which is modeled by nonlocal mean-curvature flow. Attachment-limited kinetics is observed during solidification processes, in which the system is divided into two domains of the two pure phases, more precisely islands of a solid phase surrounded by an undercooled liquid phase, and the relaxation process is due to material redistribution form high to low interfacial curvature regions. The interfacial area between the phases decreases in time while the volume of each phase is preserved. Consequently, the domain morphology coarsens. Experiments, heuristics and numerics suggest that the typical domain size ℓ of the solid islands grows according to the power law $\ell \sim t^{1/2}$, when t denotes time.

In this paper, we prove a weak one-sided version of this coarsening rate, namely we prove that $\ell \lesssim t^{1/2}$ in time average. The bound on the coarsening rate is uniform in the initial configuration but requires some control on collisions of different domains. Our approach is based on a method, introduced by Kohn and Otto [18], relying on the gradient flow structure of the dynamics.

1 Introduction

For more than one century, coarsening phenomena in physics and material science have been the object of extensive theoretical, experimental, and numerical research. Coarsening is observed in two-phase (or multi-phase) systems far from equilibrium, where thermodynamics favors the separation of the different phases and drives thus the formation of microstructure. Eventually after an initial relaxation stage, the system is essentially divided into domains

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of the pure phases and the system energy is concentrated along the interface between these domains. In the subsequent evolution, the systems tends to minimize the interfacial area while the volume of each phase is preserved. Consequently, the domain morphology coarsens. The coarsening behavior can be quantified in terms of a characteristic length scale ℓ, the average size of the domains of the pure phases, that grows as a function of time t, typically as power law.

In many situations the decrease of interfacial area is realized by mass transport from regions with high interfacial curvature to regions with low interfacial curvature. In this paper we consider the simplest volume-preserving evolution in a two-phase system that implements this idea: the nonlocal mean-curvature flow. Mathematically this flow relates the normal velocity V at any point of the interface to its mean curvature κ. More precisely, if $\Omega(t)$ denotes the volume that is occupied by one of the two phases at time t, the evolution of its boundary $\partial\Omega(t)$ can be expressed via

$$V = \langle\kappa\rangle - \kappa \quad \text{on } \partial\Omega(t), \tag{1}$$

where $\langle\kappa\rangle$ denotes the average of the mean curvature over the interface, i.e., $\langle\kappa\rangle = \frac{1}{|\partial\Omega(t)|}\int_{\partial\Omega(t)} \kappa \, d\mathcal{H}^{d-1}$.

Nonlocal mean-curvature flow models attachment-limited kinetics. This latter can be observed during the growth process of a solid phase that is surrounded by an undercooled liquid phase of the same substance, in which diffusion is so fast that the chemical potential can be considered as constant and excess mass of the solid in the substrate is negligible [29, 8, 28]. In this case, solid particles move with "infinite" velocity from regions with large interfacial curvature to regions with small interfacial curvature. More general solidification processes can be modeled by the Mullins–Sekerka equations, where the Gibbs–Thomson relation is modified by a kinetic drag term [20, 16]:

$$\begin{aligned} -\Delta\mu &= 0 && \text{in } \Omega\cup\Omega^c \\ -[\nabla\mu\cdot\nu] &= V && \text{on } \partial\Omega \\ V &= \mu-\kappa && \text{on } \partial\Omega. \end{aligned} \tag{2}$$

Here, μ is the chemical potential, ν is the outward normal on $\partial\Omega$, and the brackets $[\cdot]$ denote the jump of a quantity over the boundary. Furthermore, Ω^c is the (open) complement set of Ω. This model allows for attachment-limited *and* bulk-diffusion-limited kinetics. It is observed that the early stages of the phase separation process are dominated by attachment-limited kinetics, cf. [11], so that nonlocal mean-curvature flow (1) can be considered as a singular limit of (2).

Nonlocal mean-curvature flow can also be considered as a toy model for grain growth, where grain boundaries move according to mean-curvature flow with a nonlocal condition at the grain vertices, see *e.g.*, [24]. Aside from their application in phase separation models, variants of nonlocal mean-curvature flow have been applied to shape recovery in image processing [7].

We also recall that nonlocal mean-curvature flow has a phase field formulation, the so-called nonlocal Allen–Cahn equation. In this formulation a smooth order-parameter ϕ evolves in time by

$$\partial_t \phi - \Delta\phi - \phi(1-\phi^2) + \fint \phi(1-\phi^2)\,dx \;=\; 0. \tag{3}$$

The order-parameter ϕ varies smoothly between the regions of the pure phases, *e.g.*, $\{\phi = 1\}$ and $\{\phi = -1\}$, with a characteristic interfacial profile. The integral in (3) is averaged over the system size, so that the volume fraction of each phase is preserved. The connection between nonlocal mean-curvature flow (1) and nonlocal Allen–Cahn equation (3) is investigated in [26, 4, 9]. In particular, in [9] the authors prove that, in the so called *sharp-interface limit*, solutions of (3) converge (in an appropriate sense) to solutions of (1), as long as a smooth solution of (1) exists.

Opposed to the situation of the nonlocal Allen–Cahn equation (3), for which well-posedness is relatively easy, the mathematical treatment of nonlocal mean-curvature flow (1) is delicate. Local in time existence and uniqueness of a smooth solution of (1) follow from the results obtained in [15, 17, 13]. Moreover, existence and uniqueness of a global (in time) classical solution, as well as its asymptotic convergence to a sphere, has been proved for a smooth convex initial datum Ω_0 (see [15, 17]) and for an initial datum Ω_0 whose boundary $\partial\Omega_0$ is smooth and "near" to a sphere (see [13, 21]). For generic initial data singularities occur in finite time, due, for example, to topological changes of the domain morphology: small domains shrink and disappear; domains collide and merge; parts of domains are pinched off. A few notions of weak-solutions have been proposed, *e.g.*, level set solutions [25] and diffusion generated solutions [27]. However, to the best of our knowledge, no rigorous results are available in the literature concerning the global existence and the uniqueness of such weak solutions. Despite these analytical difficulties, in this paper, we work with equation (1) under the assumption that piecewise smooth solutions exist and that the evolution of the perimeter of $\partial\Omega$ is continuous for all times. Considering (1) instead of (3) has the advantage that geometric information on the domain morphology is directly accessible.

In this paper we study the coarsening behavior for attachment-limited kinetics as it is modelled by nonlocal mean-curvature flow (1). Systems evolving by attachment kinetics are supposed to coarsen according to

$$\ell \sim t^{1/2}, \tag{4}$$

where ℓ denotes the average size of the islands of the minority phase (i.e., the solid phase), and t denotes time. This coarsening rate has first been predicted by Wagner [29] using a mean-field theory for the evolution of droplet radii (Lifshitz–Slyozov–Wagner theory). Real experiments, *e.g.*, [2], and numerical simulations, *e.g.*, [19], support Wagner's predictions. Evidence for this growth law is given by scale invariance: As the equation (1) is invariant under the scaling $t = \lambda^2 \hat{t}$, $x = \lambda \hat{x}$, the only possible growth law for a characteristic length scale ℓ is (4).

Our main result is a time-averaged one-sided version of (4). More precisely we prove an upper bound on the coarsening rate which is *universal* in the sense that it is uniform in the particular choice of the initial data. However, to carry on our analysis we have to impose two restrictions on the evolution: First, we assume that *collisions of domains are rare events*, in a sense that we will make precise and discuss later on pages 7f.; second we assume that the area of the boundary of the domain-phases is continuous in time — a fact that we expect to be true for generic evolutions. Consequences of the latter restriction will be discussed on page 8. Our analysis follows a method proposed by Kohn and Otto in [18].

The paper is organized as follows. In Section 2 we recall the (formal) gradient flow structure of the dynamics and introduce the mathematical framework we apply for our studies: the Kohn–Otto method; in Section 3 we present and discuss our main results, the proofs of which we collect in Section 4. Finally, in Section 5 we state and prove a technical lemma that we need in the proof of Proposition 2.

2 Gradient flow structure and method

Our mathematical investigation follows a method introduced by Kohn and Otto [18] in order to study the coarsening rates for two variants of the Cahn–Hilliard equation modeling bulk and surface diffusion. The method is based on the gradient flow structure of the dynamics and analyzes the evolution of the configuration in the energy landscape. Before introducing the Kohn–Otto method, we first discuss the gradient flow structure of (1).

The nonlocal mean-curvature flow has a formal gradient flow interpretation, that is, the dynamics follows the steepest descend in an energy landscape:

$$V \;=\; -\nabla E.$$

The energy E is the surface area functional, which we normalize by the total volume:

$$E := \frac{|\partial\Omega|}{|\Omega|}. \tag{5}$$

(Here and in the following, we will often write $|\cdot|$ both for the $(d-1)$-dimensional Hausdorff measure and the d-dimensional Lebesgue measure.) The energy gradient ∇E in (5) is defined implicitly via the Riesz Representation Theorem

$$\mathrm{diff}E.W \;=\; g_{\partial\Omega}(\nabla E, W) \quad \text{for any } W \text{ s. t. } \int_{\partial\Omega} W(y)\, d\mathcal{H}^{d-1}(y) = 0,$$

where the metric tensor $g_{\partial\Omega}$ is the (normalized) L^2 scalar product on $\partial\Omega$,

$$g_{\partial\Omega}(V,W) := \frac{1}{|\Omega|}\int_{\partial\Omega} V(y)W(y)\, d\mathcal{H}^{d-1}(y)$$
$$\text{for any } V, W \text{ s. t. } \int_{\partial\Omega} V(y)\, d\mathcal{H}^{d-1}(y) = \int_{\partial\Omega} W(y)\, d\mathcal{H}^{d-1}(y) = 0.$$

The constraints on the admissible normal velocity fields V and W ensure that the volume of Ω is preserved and cause thereby the nonlocal correction in (1). Indeed, if $\Omega(t)$ is evolving with normal velocity $V(t,y)$ for $y \in \partial\Omega(t)$, then

$$\frac{d}{dt}|\Omega(t)| \;=\; \int_{\partial\Omega(t)} V(t,y)\, d\mathcal{H}^{d-1}(y) \;=\; 0. \tag{6}$$

Notice also the dissipative structure of the gradient flow:

$$\frac{d}{dt}E(t) \;=\; g_{\partial\Omega(t)}(V(t,\,\cdot\,), \nabla E(t)) \;=\; -g_{\partial\Omega(t)}(\nabla E(t), \nabla E(t)) \;\leq\; 0. \tag{7}$$

We now come back to the description of the strategy we will adopt to prove our main result. The Kohn–Otto method translates information on the energy landscape, i.e., information on how fast the energy decreases as a function of the distance to some reference configuration, into information on the coarsening rates, more precisely, a lower bound on how fast the energy decreases as a function of time. Since the energy (in the original paper an appropriate Ginzburg–Landau regularization of surface energy normalized by system volume) scales like an inverse length,

$$E \;\sim\; \frac{1}{\text{length}},$$

assuming there is only one length scale present in the dynamics, lower bounds on the energy can be interpreted as upper bounds on the coarsening rates. The energy bounds come in time averages.

The key idea of [18] is to compare the geometric length scale E to a second intrinsic quantity which is rather of physical origin: a distance L in configuration space to some reference configuration. In the simplest application, this distance is just the distance which is induced by the metric tensor in the gradient flow formulation of the dynamics. Since in a gradient flow formulation the metric tensor encodes the limiting dissipation mechanisms, cf. (7), the induced distance measures the minimal amount of energy which is dissipated along trajectories between two points in configuration space. Kohn and Otto establish two basic relations between E and L: Both quantities are dual in the sense that they satisfy an isoperimetric (or interpolation) inequality

$$EL \;\gtrsim\; 1;$$

and the rate of change of L is controlled via a dissipation inequality

$$\frac{dL}{dt} \;\lesssim\; L^{1/2}\left(-\frac{dE}{dt}\right)^{1/2}.$$

Notice that above, we present the dissipation inequality precisely in the version that we will adopt in our analysis below. If the induced distance function is not known explicitly, a proxy has to be introduced for L that satisfies both inequalities. Finally, the core of the Kohn–Otto-method is an abstract ODE argument that is based on these two relations and produces a time-averaged version of the coarsening rate

$$E \gtrsim \frac{1}{t^{1/2}}.$$

We also recall that the Kohn–Otto method does not provide pointwise bounds on the energy. A counterexample is stated in [18, Remark 4].

The method is quite robust and it has been applied to a variety of models, see [3, Sec. 1.2] for a review. Within the Kohn–Otto method, the challenge is that of identifying a distance function L that satisfies the above inequalities. Before discussing our choice of L, we comment on previous (partial) results on coarsening rates for nonlocal mean-curvature flow.

- In [12], Dai and Pego consider a mean-field approximation of a Mullins–Sekerka model with kinetic undercooling, cf. (2), which allows for attachment kinetics *and* bulk diffusion. The model comes as an evolution equation for the radii of a collection of spherical islands, i.e., $\Omega = \bigcup B_{R_i}(x_i)$, (Lifshitz–Slyozov–Wagner theory) and coincides in the regime of dominant attachment kinetics with nonlocal mean-curvature flow under the assumption that each island remains spherical during the evolution. The authors derive an upper bound on the $t^{1/2}$ coarsening rate using the method from [18] with distance
$$L := \frac{\sum R_i^{d+1}}{\sum R_i^d}.$$
A simplified proof can be found in [11].
- In [10], Dai considers nonlocal mean-curvature flow in two space dimensions assuming that Ω is a finite collection of convex domains Ω_i. Using the Kohn–Otto method with distance
$$L := \frac{\sum |\Omega_i|^{3/2}}{\sum |\Omega_i|},$$
the author obtains, as desired, a $t^{1/2}$ bound on the energy decay. However, this bound is not universal in the sense that the coarsening rate depends on the geometry of the initial data. More precisely, the bound becomes trivial if the isoperimetric ratio of the initial data becomes large.

In contrast with the strategy of Kohn and Otto described above, in our setting it seems hopeless to look for a distance function L that can be considered as a proxy for the induced geodesic distance. In fact, it turns out that the distance induced by an L^2 metric tensor on the (suitably defined) manifold of *geometric curves* in the plane is degenerate, in that the distance between two well separated configurations on the manifold is zero (see [23]). This insight suggests that we need a more subtle choice of L. In fact, our choice of L is rather geometrically motivated and generalizes the one of [12] and its interpretation as a *volume-weighted average of radii* [12, page 5], cf. (8) below. Moreover, in order to apply the Kohn–Otto method, besides the gradient flow structure, in the proof of the dissipation inequality we need to use some characteristic properties of nonlocal mean-curvature flow.

3 Main results

We are now in the position to state our main results. We limit ourselves to the two-dimensional setting, but we comment on the higher dimensional case in Remark 1 below. By $\Omega \subset \mathbb{R}^2$ we denote an open, bounded subset of finite topological genus, that is

$$\Omega = \bigcup_{i=1}^{N} \Omega_i \setminus \Big(\bigcup_{j=1}^{M} \omega_j \Big),$$

where Ω_i, ω_j are open, simply connected subsets and $N, M \in \mathbb{N}$. We suppose that Ω evolves according to nonlocal mean-curvature flow (1). More precisely, we consider a time-parametrized family of subsets $\{\Omega(t)\}_{t\in[0,T)}$ verifying

$$V(t,x) = \langle \kappa(t,\,\cdot\,) \rangle - \kappa(t,x) \quad \text{for } x \in \partial\Omega(t) \text{ and } t \in [0,T) \setminus \{T_1, \ldots, T_K\},$$

where $V(t,x)$, $\kappa(t,x)$ denote respectively the (outward) normal velocity and the curvature of $\partial\Omega(t)$ at $x \in \partial\Omega(t)$ (with the sign convention that the curvature is non-negative at x if $\Omega(t)$ is locally convex around x). Moreover, the brackets $\langle\cdot\rangle$ denote the average over the boundary of $\Omega(t)$. We suppose that $\partial\Omega(t)$ is smooth except for at most finitely many times $\{T_1, \ldots, T_K\}$, corresponding to possible topological changes of $\Omega(t)$ (collisions, pinch-offs, shrinkage to points) in the time interval $[0, T)$. Finally, we suppose that the perimeter of $\Omega(t)$ evolves continuously in the whole time interval $[0, T]$. For a more detailed description of solutions to the nonlocal mean-curvature flow (1), we refer to Definitions 1 & 2 in Section 4.

As generalized average domain radius, we consider

$$L := \frac{1}{|\Omega|} \sum_{i=1}^{N} \int_{\widehat{\Omega}_i} \operatorname{dist}(x, \partial\widehat{\Omega}_i)\, dx, \tag{8}$$

where $\widehat{\Omega}_i$ denotes the convex hull of Ω_i. For given initial data and with the evolution described above, L is a function of time: $L = L(t)$. Notice that we consider the convex hulls componentwise. In particular, the convex hulls of two (or more) distinct domains possibly overlap. Furthermore, due to the discontinuous evolution of the convex hulls, L has a positive jump when two domains collide, while, in general, L develops a negative jump in case a domain pinches-off. To measure the height of the jumps of L we introduce a total variation measure at the topological changes in the time interval $[0, T)$. We define

$$\mathrm{TV}^{1/2}(L,T) := \sum_{k=1}^{K} \Big(L(T_k+)^{1/2} - L(T_k-)^{1/2} \Big) .$$

As we already said above, our result applies to a restricted class of solutions of the nonlocal mean-curvature flow satisfying the two following assumptions. The first assumption states that *collisions of different domains are rare events.* More precisely, we suppose that

$$L(T)^{1/2} \;\geq\; 4\,\mathrm{TV}^{1/2}(L,T). \tag{9}$$

When domains only either shrink to a point or pinch-off, L is either continuous or has negative jumps. Hence, in these cases assumption (9) does not represent a restriction on the evolution. However, it does in the case of collisions. Condition (9) rules out series of many collisions all happening in short time intervals, and collisions of large domains, i.e., domain sizes comparable to the system size. We notice that in the case where the typical length scale is small compared to the system size, that is, in the case of a large number of domains, the height of the jump of L at the time two domains collide is relatively small compared to L. Therefore, in generic large-size, low-volume fraction configurations, that is, when the domains are well separated from each other, and collisions do indeed only happen rarely, we expect condition (9) to be generically satisfied.

The second assumption we make is that the perimeter of $\Omega(t)$ evolves continuously in the whole time interval. This second restriction on the admissible flows is rather of mathematical nature, and we expect it to be generically true. The reason why we have to state this condition as an assumption is due to the lack of results ensuring the continuity of the energy of (suitably defined) weak solutions, with smooth initial data, when topological changes occur during the flow.

Under these hypotheses we prove:

Theorem 1. *Suppose that $L(T)^{1/2} \geq 4\,\mathrm{TV}^{1/2}(L,T)$. Then, for any $1 < \sigma < 2$ we have*

$$\int_0^T E^\sigma \, dt \;\gtrsim\; \int_0^T \left(\frac{1}{t^{1/2}}\right)^\sigma dt,$$

provided that $T \gg L(0)^2$.

We have to comment on our sloppy notation. The result has to be read as follows: For every given σ, if T is sufficiently large compared to $L(0)^2$ then the constant in the statement can be chosen *uniformly* (only dependent on σ).

Notice that the above result is in agreement with the coarsening rate (4). Indeed, since the energy scales like an inverse length and assuming that there is only one length scale present in the dynamics, a lower bound on the energy translates into an upper bound on the coarsening rate.

We also recall that proving upper bounds on coarsening rates is substantially different from proving lower bounds. While upper bounds can be universally true, lower bounds depend strongly on the initial data: There are (infinitely many) configurations that do not coarsen at all, *e.g.*, a collection of spheres with the same radii. Therefore, lower bounds can only be "generically" true.

For proving Theorem 1, we use the method proposed by Kohn and Otto in [18], discussed in the previous section. As a first step, we derive the isoperimetric inequality:

Proposition 1.

$$EL \;\gtrsim\; 1.$$

Our main contribution is the dissipation inequality:

Proposition 2. *For all but a finite number of times* $\{t_1, \ldots, t_K\}$ *we have*

$$\frac{dL}{dt} \lesssim L^{1/2} \left(-\frac{dE}{dt}\right)^{1/2}.$$

The core of the Kohn–Otto method is the ODE argument that, in our context, has the following form:

Proposition 3 ([18, 12])**.** *Let* E *and* L *be continuous on* $[0, T) \setminus \{T_1, \ldots, T_K\}$*. Suppose that* (9) *holds and*

$$EL \gtrsim 1 \quad \text{and} \quad \frac{dL}{dt} \lesssim L^{1/2} \left(-\frac{dE}{dt}\right)^{1/2}.$$

Then, for any $1 < \sigma < 2$ *we have*

$$\int_0^T E^\sigma \, dt \gtrsim \int_0^T \left(\frac{1}{t^{1/2}}\right)^\sigma dt,$$

provided that $T \gg L(0)^2$*.*

Remark 1. *Let us also mention that our main result can be easily generalized to any dimension* $d \geq 2$*, under the additional assumption that the initial datum is given by a finite collection of convex domains and that the domains remain convex during the evolution. In fact, since in this situation, apart from the gradient flow structure, no additional information on the flow is needed, the techniques developed in Proposition 1 and 2 above can be used to generalize the results from [12] (and from [11] for the evolution* (2)*) to collections of convex domains.*

Before turning to the proofs of our results, we motivate our choice of L. A natural generalization of the length scale used by Dai and Pego in [12] to configurations with nonspherical islands leads to the choice $\tilde{L} = \frac{1}{|\Omega|} \int_\Omega \operatorname{dist}(x, \partial\Omega)\, dx$. Indeed, this quantity measures the average diameter of the configuration and reduces to the average radius in the case of spherical islands. The reason for our modification (i.e., to consider the convex hull of the connected components of Ω) is the following: On the one hand, infinitesimally small "holes" ω_j inside the components may decrease the average distance to $\partial\Omega$ while keeping the perimeter of Ω constant. In particular, $\tilde{L}$ can become arbitrary small and the interpolation/isoperimetric inequality stated in Proposition 1 may fail. Therefore it is convenient to "fill these holes". On the other hand, small negative-curvature regions on the outer boundaries $\partial\Omega_i$ result in a steep ascend of $\tilde{L}$ during the evolution which might not be controlled by the energy dissipation. In order to rule out this inconvenience, we convexify the connected components — with the negative effect, that L jumps up when two (or more) components collide.

4 Proofs

Before presenting the proof of our main result we introduce some notation. In particular we give a rigorous definition of what we mean by open set with smooth boundary, and of the admissible initial data and solutions of (1) to which our result apply.

Definition 1 (Smooth domains)**.** *Let $\Omega \subset \mathbb{R}^2$ be open. We say that Ω has smooth boundary if*

$$\text{for every } x \in \partial\Omega \text{ there exists } r(x) > 0 \text{ such that (up to rigid motions) } \Omega \cap B_r(x) \text{ can be written as the subgraph of a } C^\infty \text{ function.} \tag{10}$$

Definition 2 (Admissible Flows)**.** *Let $\{\Omega(t)\}_{t\in[0,T)}$ be a time-parametrized family of open subsets. We say that $\Omega(t)$ is is an admissible flow with initial condition $\Omega(0)$ if*

- *for every $t \in [0,T)$ but possibly a finite set J of times the set $\Omega(t)$ is open, bounded, with smooth boundary and of finite topological genus. That is, for every $t \in [0,T) \setminus J$ the set $\Omega(t)$ has smooth boundary, and we have*

$$\Omega(t) \; = \; \bigcup_{i=1}^{N(t)} \Omega_i(t) \setminus \Big(\bigcup_{j=1}^{M(t)} \omega_j(t) \Big); \tag{11}$$

  where $\Omega_i(t)$, $\omega_j(t)$ are open, simply connected subsets with smooth boundary, $N(t), M(t) \in \mathbb{N}$. Moreover, for $t \in J$ we require that $\Omega(t)$ has piecewise smooth boundary (i.e., (10) *holds for all but (possibly) a finite number of points of $\partial\Omega(t)$) and that* (11) *still holds, but now $\Omega_i(t)$, $\omega_j(t)$ are only piecewise smooth;*

- *for every $\psi \in C_c^\infty(\mathbb{R}^2)$ the function*

$$t \mapsto \int_{\partial\Omega(t)} \psi \, d\mathcal{H}^1,$$

  is continuous in $[0,T]$. Moreover $\{\Omega(t)\}_{t\in[0,T)}$ is a smooth solution of (1) *up to a finite set of singular times $\mathrm{Sing}_{\Omega_T} = \{0 < t_1 < t_2 < \cdots < t_K < T\} \supseteq J$ with $K \in \mathbb{N}$. That is, letting $T_0 = 0$ and $T_{K+1} = T$, for every $i = \{0, \ldots, K\}$ the functions $M(t)$, $N(t)$ are constant for every $t \in [t_i, t_{i+1})$, and*

$$V(t,x) \; = \; \langle \kappa(t,\,\cdot\,) \rangle - \kappa(t,x) \quad \text{for any } x \in \Omega(t),$$

  where $V(t,x)$ and $\kappa(t,x)$ respectively denote the velocity of the boundary at x in direction of the outer normal to $\partial\Omega(t)$ and the curvature of $\partial\Omega(t)$ at x, with the sign convention that $\kappa(t,x)$ is non-negative if $\Omega(t)$ is locally convex around x.

In our definition above, the set J consists of the points of time at which different domains *collide* or *pinch-off*, while Sing_{Ω_T} additionally contains the points of time at which connected components *shrink to a point*.

We notice that if $\{\Omega(t)\}_{t\in[0,T)}$ is an admissible flow, the function $L(t)$, defined in (8), is smooth on $[0,T]\setminus \mathrm{Sing}_{\Omega_T}$, and Lipschitz continuous on $[0,T]\setminus J$. Moreover we have: $L \in L^\infty(0,T)\cap BV(0,T)$, and in particular,

$$\mathrm{TV}^{1/2}(L,T) \;=\; \sum_{k=1}^{K} \Big(L(T_k+)^{1/2} - L(T_k-)^{1/2} \Big) \;<\; \infty.$$

Hence the singular part of distributional derivative of L can only concentrate on the (possibly not empty) set J.

Finally we also remark that the continuity assumption of $|\partial\Omega(t)|$ on $[0,T]$, together with the assumption that $\{\Omega(t)\}_{t\in[0,T)}$ is a smooth solution of the nonlocal mean-curvature flow on $[0,T]\setminus \mathrm{Sing}_{\Omega_T}$, ensure that $E \in W^{1,1}(0,T)\cap C^0([0,T])$.

Next we observe that the volume of $\Omega(t)$ is preserved during the evolution, cf. (6), so that, by rescaling length, w.l.o.g. from here on we may assume that

$$|\Omega(t)| \;=\; |\Omega(0)| \;=\; 1.$$

We then have

$$E(t) \;=\; \sum_{i=1}^{N(t)} |\partial\Omega_i(t)| + \sum_{j=1}^{M(t)} |\partial\omega_j(t)|$$

and

$$L(t) \;=\; \sum_{i=1}^{N(t)} \int_{\widehat{\Omega}_i(t)} \mathrm{dist}(x, \partial\widehat{\Omega}_i(t))\, dx.$$

We are now in a position to prove Proposition 1.

Proof of Proposition 1. We first observe that the convexification reduces the length of the boundary,

$$E(t) \;\geq\; \sum_{i=1}^{N(t)} |\partial\Omega_i(t)| \;\geq\; \sum_{i=1}^{N(t)} |\partial\widehat{\Omega}_i(t)|, \tag{12}$$

while it enlarges the area

$$1 \;=\; |\Omega(t)| \;\leq\; \sum_{i=1}^{N(t)} |\widehat{\Omega}_i(t)|. \tag{13}$$

We split each set $\widehat{\Omega}_i$ into a boundary and a bulk part. More precisely, for $\ell \geq 0$ we have

$$|\widehat{\Omega}_i(t)| \;=\; \int_{\widehat{\Omega}_i(t)\cap\{\mathrm{dist}(\cdot,\partial\widehat{\Omega}_i(t))\leq \ell\}} dx + \int_{\widehat{\Omega}_i(t)\cap\{\mathrm{dist}(\cdot,\partial\widehat{\Omega}_i(t))> \ell\}} dx.$$

The bulk part is easily estimated:

$$\int_{\widehat{\Omega}_i(t)\cap\{\mathrm{dist}(\cdot,\partial\widehat{\Omega}_i(t))> \ell\}} dx \;\leq\; \frac{1}{\ell} \int_{\widehat{\Omega}_i(t)} \mathrm{dist}(x, \partial\widehat{\Omega}_i(t))\, dx.$$

For the boundary part, we invoke the co-area formula[14, Theorem 1.3.4.2] and find

$$\begin{aligned}\int_{\widehat{\Omega}_i(t)\cap\{\mathrm{dist}(\cdot,\partial\widehat{\Omega}_i(t))\le\ell\}} dx &\le \int_0^\ell \mathcal{H}^1\left(\widehat{\Omega}_i(t)\cap\{\mathrm{dist}(\cdot,\partial\widehat{\Omega}_i(t))=s\}\right)ds\\ &\le \int_0^\ell \mathcal{H}^1\left(\partial\widehat{\Omega}_i(t)\right)ds\\ &= \ell|\partial\widehat{\Omega}_i(t)|,\end{aligned}$$

where the second inequality holds because $\widehat{\Omega}_i$ is simply connected.

Combining the estimates for the boundary and the bulk part with (12) and (13), we have

$$\begin{aligned}1 &\le \ell\sum_{i=1}^{N(t)}|\partial\widehat{\Omega}_i(t)| + \frac{1}{\ell}\int_{\widehat{\Omega}_i(t)}\mathrm{dist}(x,\partial\widehat{\Omega}_i(t))\,dx\\ &\le \ell E(t) + \frac{1}{\ell}L(t).\end{aligned}$$

We optimize in $\ell>0$ by choosing $\ell\sim E(t)^{-1/2}L(t)^{1/2}$ and obtain

$$1 \lesssim E(t)\,L(t).$$

□

Before proceeding with the proof of Proposition 2, we collect some easy consequence of basic properties of convex hulls and of the definition of admissible flows in a technical lemma we will need later on.

Lemma 1. *Suppose $\{\Omega(t)\}_{t\in[0,T)}$ is an admissible flow with initial condition $\Omega(0)=(\cup_{i=1}^{N(0)}\Omega_i(0))\setminus(\cup_{j=1}^{M(0)}\omega_j(0))$. Let $T_l\in\mathrm{Sing}_{\Omega_T}$. For every $t\in[T_l,T_{l+1}]$ and $i\in\{1,\dots,N(T_l)\}$ we define the following subsets of $\partial\widehat{\Omega}_i(t)$:*

$$\Sigma_i^-(t) := \partial\widehat{\Omega}_i(t)\cap\{x:\ \mathrm{dist}(x,\partial\Omega_i(t))>0\},\quad \Sigma_i^+(t):=\partial\widehat{\Omega}_i(t)\setminus\Sigma_i^-(t).$$

The following properties hold for every $t\in[0,T_i)$:

(i) *$\Sigma_i^+(t)=\partial\Omega_i(t)\cap\partial\widehat{\Omega}_i(t)$, and for every $y\in\Sigma_i^+(t)$ we have $\kappa(t,y)\ge 0$;*

(ii) *$\Sigma_i^-(t)=\cup_{k\in I(t)}\alpha_{k,t}$, where: for every $t\in[0,T_i)$ the index set $I(t)$ is at most countable; for every $k\in I(t)$ and $t\in[0,T_i)$ every $\alpha_{k,t}$ is an open, connected flat segment of finite length whose extremes are points of $\Sigma_i^+(t)$. That is, for every $y\in\Sigma_i^-(t)$ we can find $p_1(t,y),p_2(t,y)\in\Sigma_i^+(t)$ and $s\in(0,1)$, such that $y=s\,p_1(t,y)+(1-s)\,p_2(t,y)$;*

(iii) *the (outer) normal velocity $\widehat{V}$ of $\partial\widehat{\Omega}_i(t)$ is given by*

$$\widehat{V}(t,y) = \begin{cases} V(t,y) & \text{if } y\in\Sigma_i^+(t)\\ s\,V(t,p_1(t,y))+(1-s)\,V(t,p_2(t,y)) & \text{if } y\in\Sigma_i^-(t).\end{cases}$$

Proof of Lemma 1. (ii) follows from the fact that $\Sigma_i^-(t)$ is an open subset of $\partial\widehat{\Omega}_i(t)$ and basic properties of the convex hull. (i) follows from the fact that $\Sigma_i^+(t)$ is a closed subset of $\partial\widehat{\Omega}_i(t)$, the smoothness of $\partial\Omega_i(t)$ and again basic properties of the convex hull. Eventually (iii) is a consequence of (i), (ii), and the definition of admissible flow. $\square$

Let us now proceed with the

Proof of Proposition 2. Let $t \in (0,T) \setminus \mathrm{Sing}_{\Omega_T}$. Then $M(t+h)$, $N(t+h)$ are constant for $|h| << 1$. Moreover, for every $1 \le i \le N(t)$ we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\widehat{\Omega}_i(t)} \mathrm{dist}(x, \partial\widehat{\Omega}_i(t))\, dx \\
&\quad = \int_{\widehat{\Omega}_i(t)} \partial_t \mathrm{dist}(x, \partial\widehat{\Omega}_i(t))\, dx + \int_{\partial\widehat{\Omega}_i(t)} \widehat{V}(t,y)\mathrm{dist}(x, \partial\widehat{\Omega}_i(t))\, d\mathcal{H}^1(y) \\
&\quad = \int_{\widehat{\Omega}_i(t)} \widehat{V}(t, \pi_{\partial\widehat{\Omega}_i(t)}(x))\, dx.
\end{aligned}
$$

where $\pi_{\partial\widehat{\Omega}_i(t)}(x)$ denotes the projection of x onto $\partial\widehat{\Omega}_i(t)$.

Let $R(t,y)$ denote the distance along normals, i.e., $-\nu_{\partial\widehat{\Omega}(t)}(y)$, from a point y on the boundary $\partial\widehat{\Omega}(t)$ to the singular set $\mathcal{S}(t)$, i.e.,

$$
R(t,y) := \sup\{s \ge 0 : \ [y,\, y + s\nu_{\partial\widehat{\Omega}(t)}(y)[\subset \widehat{\Omega}(t) \setminus \mathcal{S}(t)\}
$$

with

$$
\mathcal{S}(t) := \{x \in \widehat{\Omega}(t) : \ \pi_{\partial\widehat{\Omega}(t)}(x) \text{ is not a singleton}\},
$$

and $[y,\, y+s\nu_{\partial\widehat{\Omega}(t)}(y)[:= \{z = \lambda y + (1-\lambda)s\nu_{\partial\widehat{\Omega}(t)}(y) : \lambda \in [0,1)\}$. Since $\partial\Omega_i(t)$ is smooth, we have that $\partial\widehat{\Omega}_i(t)$ is of class $C^{1,1}$ and thus, applying Lemma 2 at fixed time t yields

$$
\begin{aligned}
&\frac{d}{dt}\int_{\widehat{\Omega}_i(t)} \mathrm{dist}(x, \partial\widehat{\Omega}_i(t))\, dx \\
&\quad = \int_{\partial\widehat{\Omega}_i(t)} \widehat{V}(t,y) \left(\int_0^{R(t,y)} |1 - s\widehat{\kappa}(t,y)|\, ds \right) d\mathcal{H}^1(y),
\end{aligned}
$$

where $\widehat{\kappa}(t,y)$ is the curvature at $y \in \partial\widehat{\Omega}(t)$, which is well defined $\mathcal{H}^1$-a.e. since $\partial\widehat{\Omega}(t)$ is of class $C^{1,1}$. By the convexity of $\widehat{\Omega}_i(t)$ we can conclude that

$$
0 \le \widehat{\kappa}(t,y) \le \frac{1}{R(t,y)} \quad \text{for } \mathcal{H}^1 - \text{a.e. } y \in \partial\widehat{\Omega}_i(t). \tag{14}
$$

Moreover, as a consequence of Lemma 1-(i)&(ii),

$$
\widehat{\kappa}(t,y) \;=\; \begin{cases} \kappa(t,y) & \text{for } y \in \Sigma_i^+(t) \\ 0 & \text{for } y \in \Sigma_i^-(t). \end{cases} \tag{15}
$$

so that

$$\frac{d}{dt}\int_{\widehat{\Omega}_i(t)} \mathrm{dist}(x,\partial\widehat{\Omega}_i(t))\,dx$$

$$\overset{(14)\&(15)}{=} \int_{\Sigma_i^+(t)} \widehat{V}(t,y)\left(\int_0^{R(t,y)}(1-s\kappa(t,y))\,ds\right)d\mathcal{H}^1(y) \tag{16}$$

$$+\int_{\Sigma_i^-(t)} \widehat{V}(t,y)R(t,y)\,dsd\mathcal{H}^1(y). \tag{17}$$

We first observe that

$$\int_{\partial\widehat{\Omega}_i(t)} R(t,y)^2\,d\mathcal{H}^1(y) \;\lesssim\; \int_{\widehat{\Omega}_i(t)} \mathrm{dist}(x,\widehat{\Omega}_i(t))\,dx. \tag{18}$$

Indeed, again by Lemma 2, we have

$$\begin{aligned}
&\int_{\widehat{\Omega}_i(t)} \mathrm{dist}(x,\partial\widehat{\Omega}_i(t))\,dx\\
&= \int_{\partial\widehat{\Omega}_i(t)}\int_0^{R(t,y)} s|1-s\widehat{\kappa}(t,y)|\,dsd\mathcal{H}^1(y)\\
&\overset{(14)\&(15)}{=} \int_{\Sigma_i^+(t)}\int_0^{R(t,y)} s(1-s\widehat{\kappa}(t,y))\,dsd\mathcal{H}^1(y)\\
&\qquad + \int_{\Sigma_i^-(t)}\int_0^{R(t,y)} s\,dsd\mathcal{H}^1(y)\\
&= \int_{\Sigma_i^+(t)}\left(\frac{1}{2}R(t,y)^2-\frac{1}{3}R(t,y)^3\widehat{\kappa}(t,y)\right)\,d\mathcal{H}^1(y)\\
&\qquad + \int_{\Sigma_i^-(t)}\frac{1}{2}R(t,y)^2\,dsd\mathcal{H}^1(y)\\
&\overset{(14)}{\geq} \frac{1}{6}\int_{\Sigma_i^+(t)} R(t,y)^2\,d\mathcal{H}^1(y)+\frac{1}{2}\int_{\Sigma_i^-(t)} R(t,y)^2\,d\mathcal{H}^1(y)\\
&\gtrsim \int_{\partial\widehat{\Omega}_i(t)} R(t,y)^2\,d\mathcal{H}^1(y).
\end{aligned}$$

We turn to the estimate of (16). In view of Lemma 1-(iii) and the definition of $\Sigma_i^+(t)$ we have

$$\begin{aligned}
&\int_{\Sigma_i^+(t)} \widehat{V}(t,y)\left(\int_0^{R(t,y)}(1-s\widehat{\kappa}(t,y))\,ds\right)d\mathcal{H}^1(y)\\
&\overset{(14)}{\leq} \left(\int_{\Sigma_i^+(t)} V(t,y)^2\,d\mathcal{H}^1(y)\right)^{1/2}\left(\int_{\Sigma_i^+(t)} R(t,y)^2\,d\mathcal{H}^1(y)\right)^{1/2}\\
&\overset{(18)}{\lesssim} \left(\int_{\partial\Omega_i(t)} V(t,y)^2\,d\mathcal{H}^1(y)\right)^{1/2}\left(\int_{\widehat{\Omega}_i(t)} \mathrm{dist}(x,\partial\widehat{\Omega}_i(t))\,dx\right)^{1/2}.
\end{aligned} \tag{19}$$

To estimate the term (17), we use the fact that $\{\Omega(t)\}_{t\in[0,T)}$ evolves with nonlocal mean-curvature flow (1). Let $y \in \Sigma_i^-(t)$. By Lemma 1 we have

$$\begin{aligned}\widehat{V}(t,y) &= s\,V(t,p_1(t,y)) + (1-s)\,V(t,p_2(t,y)) \\ &\overset{(1)}{=} \langle\kappa(t,\cdot)\rangle - (s\,\kappa(t,p_1(t,y)) + (1-s)\kappa(t,p_1(t,y))) \\ &\overset{(14)}{\leq} \langle\kappa(t,\cdot)\rangle.\end{aligned}$$

and hence

$$\widehat{V}(t,y) \;\leq\; \max\{0,\langle\kappa(t,\cdot)\rangle\} \quad \text{for all } y \in \Sigma_i^-(t).$$

It is enough to consider the situation where $\langle\kappa(t,\cdot)\rangle \geq 0$. In this case, we conclude that

$$\begin{aligned}&\int_{\Sigma_i^-(t)} \widehat{V}(t,y)R(t,y)d\mathcal{H}^1(y) \\ &\leq \left(|\Sigma_i^-(t)|\langle\kappa(t,\cdot)\rangle^2\right)^{1/2}\left(\int_{\Sigma_i^-(t)} R(t,y)^2\,d\mathcal{H}^1(y)\right)^{1/2} \\ &\overset{(18)}{\lesssim} \left(|\partial\Omega_i(t)\setminus\Sigma_i^+(t)|\langle\kappa(t,\cdot)\rangle^2\right)^{1/2}\left(\int_{\widehat{\Omega}_i(t)} \operatorname{dist}(x,\partial\widehat{\Omega}_i(t))\,dx\right)^{1/2},\end{aligned}$$

where in the last estimate we used that $|\Sigma_i^-(t)| \leq |\partial\Omega_i(t)\setminus\Sigma_i^+(t)|$, as convexification reduces the length of the boundary. However, since $\partial\widehat{\Omega}_i(t)$ is of class $C^{1,1}$, the oriented tangents at the two endpoints of each connected component of $\partial\Omega_i(t)\setminus\Sigma_i^+(t)$ are parallel and therefore we have

$$\int_{\partial\Omega_i(t)\setminus\Sigma_i^+(t)} \kappa(t,y)\,d\mathcal{H}^1(y) \;=\; 0,$$

and thus

$$\begin{aligned}|\partial\Omega_i(t)\setminus\Sigma_i^+(t)|\langle\kappa(t,\cdot)\rangle^2 &\leq \int_{\partial\Omega_i(t)\setminus\Sigma_i^+(t)} \langle\kappa(t\,\cdot)\rangle^2 + \kappa(t,y)^2\,d\mathcal{H}^1(y) \\ &= \int_{\partial\Omega_i(t)\setminus\Sigma_i^+(t)} \left(\langle\kappa(t,\cdot)\rangle - \kappa(t,y)\right)^2\,d\mathcal{H}^1(y) \\ &= \int_{\partial\Omega_i(t)\setminus\Sigma_i^+(t)} V(t,y)^2\,d\mathcal{H}^1(y).\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{\Sigma_i^-(t)} \widehat{V}(t,y)R(t,y)d\mathcal{H}^1(y) \\ &\leq \left(\int_{\partial\Omega_i(t)} V(t,y)^2\,d\mathcal{H}^1(y)\right)^{1/2}\left(\int_{\widehat{\Omega}_i(t)} \operatorname{dist}(x,\partial\widehat{\Omega}_i(t))\,dx\right)^{1/2}. \qquad (20)\end{aligned}$$

Putting together (16), (17), (19), and (20), for every $t \in [0,T)\setminus \mathrm{Sing}_{\Omega_T}$, we get

$$\begin{aligned}&\frac{d}{dt}\int_{\widehat{\Omega}_i(t)} \operatorname{dist}(x,\partial\widehat{\Omega}_i(t))\,dx \\ &\lesssim \left(\int_{\partial\Omega_i(t)} V(t,y)^2\,d\mathcal{H}^1(y)\right)^{1/2}\left(\int_{\widehat{\Omega}_i(t)} \operatorname{dist}(x,\partial\widehat{\Omega}_i(t))\,dx\right)^{1/2}.\end{aligned}$$

Since

$$\frac{dE}{dt} = -\int_{\partial\Omega(t)} V(t,y)^2 \, d\mathcal{H}^1(y),$$

cf. (7), the above inequality turns into

$$\frac{d}{dt}L(t) \lesssim L(t)^{1/2}\left(-\frac{d}{dt}E(t)\right)^{1/2},$$

for every $t \in (0,T) \setminus \mathrm{Sing}_{\Omega_T}$. □

Finally, in order to complete the proof of our main result, we prove Proposition 3.

Proof of Proposition 3. The proof of this Proposition closely follows the one of [18, Lemma 3] (see also [12, lemma 4.2]). Our modifications are due to the fact that in our case $L(t)$ may jump at times $t \in \mathrm{Sing}_{\Omega_T}$.

We consider separately the two possible cases $L(T) \geq 4L(0)$ and $L(T) < 4L(0)$. Let us begin supposing that $L(T) \geq 4L(0)$. By the dissipation inequality of our assumptions we have

$$\frac{d}{dt}L(t)^{1/2} \lesssim \left(-\frac{d}{dt}E(t)\right)^{1/2} \quad \text{for } t \in [0,T) \setminus \mathrm{Sing}_{\Omega_T},$$

and thus, after integration,

$$L(T_k-)^{1/2} - L(T_{k-1}+)^{1/2} \lesssim \int_{T_{k-1}}^{T_k} \left(-\frac{d}{dt}E(t)\right)^{1/2} dt$$

for any $k \in \{1,\dots,K+1\}$, recalling that $T_0 = 0$ and $T_{K+1} = T$. Summing over all k yields

$$\begin{aligned} L(T)^{1/2} - L(0)^{1/2} - \mathrm{TV}^{1/2}(L,T) &= \sum_{k=1}^{K+1}\left(L(T_k-)^{1/2} - L(T_{k-1}+)^{1/2}\right) \\ &\lesssim \sum_{k=1}^{K+1}\int_{T_{k-1}}^{T_k}\left(-\frac{d}{dt}E(t)\right)^{1/2} dt. \end{aligned}$$

Making use of $L(T) \geq 4L(0)$ and (9) we observe that

$$\frac{1}{4}L(T)^{1/2} \leq L(T)^{1/2} - L(0)^{1/2} - \mathrm{TV}^{1/2}(L,T),$$

and thus, thanks to the continuity of $E(t)$ guaranteed by the choice of admissible flows in Definition 2, we obtain

$$L(T)^{1/2} \lesssim \int_0^T \left(-\frac{d}{dt}E(t)\right)^{1/2} dt.$$

Estimating the integral on the right hand side with the help of the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} L(T)^{1/2} &\lesssim \left(\int_0^T \left(-\frac{d}{dt}E(t)\right) E(t)^{-\sigma}\,dt\right)^{1/2} \left(\int_0^T E(t)^\sigma\,dt\right)^{1/2} \\ &= \left(\frac{1}{\sigma-1}\left(E(T)^{1-\sigma} - E(0)^{1-\sigma}\right)\right)^{1/2} \left(\int_0^T E(t)^\sigma\,dt\right)^{1/2} \\ &\lesssim \left(E(T)^{1-\sigma}\right)^{1/2} \left(\int_0^T E(t)^\sigma\,dt\right)^{1/2}, \end{aligned}$$

where in the last inequality we use the fact that $\sigma > 1$. Taking the square on both sides, and applying the isoperimetric inequality of our assumptions, we obtain

$$\left(E(T)^{-\sigma}\right)^{\frac{2-\sigma}{\sigma}} \lesssim E(T)^{\sigma-1}L(T) \lesssim \int_0^T E(t)^\sigma\,dt.$$

Setting $h(T) = \int_0^T E^\sigma\,dt$ and using $\sigma < 2$, this reads

$$\frac{d}{dT}\left(h(T)^{\frac{2}{2-\sigma}}\right) \gtrsim 1 \quad \text{provided } L(T) \geq 4L(0). \tag{21}$$

Now we consider the case $L(T) < 4L(0)$. Under this assumption, the isoperimetric inequality yields

$$L(0)^\sigma \gtrsim \frac{1}{E(T)^\sigma},$$

which in terms of $h(T)$ reads

$$\frac{d}{dT}\left(h(T)L(0)^\sigma\right) \gtrsim 1 \quad \text{provided } 4L(0) \geq L(T). \tag{22}$$

Combining (21) and (22) we get

$$\frac{d}{dT}\left(h(T)^{\frac{2}{2-\sigma}} + h(T)\,L(0)^\sigma\right) \gtrsim 1.$$

Integrating in T and using Young's inequality we then have

$$T \lesssim h(T)^{\frac{2}{2-\sigma}} + h(T)\,L(0)^\sigma \lesssim h(T)^{\frac{2}{2-\sigma}} + L(0)^{1/2}.$$

Finally, since $T \gg L(0)^2$, we conclude that

$$h(T) \gtrsim T^{1-\frac{\sigma}{2}} \sim \int_0^T t^{-\sigma/2}\,dt.$$

□

5 Appendix

In the present section we state and prove a technical lemma needed in the proof of Proposition 2.

Lemma 2. *Let $\Omega \subset \mathbb{R}^2$ be open, bounded, connected and with $C^{1,1}$-smooth boundary. Let $\mathcal{S}$ denote the singular set of the distance function, that is*

$$\mathcal{S} := \{x \in \Omega : \ \pi_{\partial\Omega}(x) \text{ is not a singleton}\},$$

and define

$$R : \partial\Omega \to [0, +\infty), \quad y \mapsto \sup\{s \geq 0 : \ [y, y + s\nu_{\partial\Omega}(y)[\subset \Omega \setminus \mathcal{S}\},$$

where $[y, y + s\nu_{\partial\Omega}(y)[:= \{z = \lambda y + (1-\lambda)s\nu_{\partial\Omega}(y) : \lambda \in [0,1)\}$. For every $f \in L^1(\Omega)$ we have

$$\int_\Omega f(x)\, dx = \int_{\partial\Omega} \int_0^{R(y)} f(y + s\nu_{\partial\Omega}(y))|1 - s\kappa_{\partial\Omega}(y)|\, ds d\mathcal{H}^1(y). \tag{23}$$

Remark 2. *The proof of (23) in case Ω has C^2-boundary is straightforward. In fact in this case the closure of the singular set $\overline{\mathcal{S}}$ has zero Lebesgue measure (see [5, 6]), and the function $R(y)$ defined above coincides with the continuous function $\tau(y) := \min\{s \geq 0 : \ y + s\nu_{\partial\Omega}(y) \in \overline{\mathcal{S}}\}$. However, as shown in [22], there exist open (convex) subsets of $\mathbb{R}^2$ with $C^{1,1}$-smooth boundary such that $\overline{\mathcal{S}}$ has positive Lebesgue measure. In this case the function $R(\cdot)$ is only upper semi-continuous, it verifies $R(y) > \tau(y)$, and the proof of (23) requires a bit more of work.*
Finally we also remark that, up to minor changes, the proof of Lemma 2 works also when Ω is a $C^{1,1}$ open bounded connected subset of $\mathbb{R}^d$, ($d \geq 2$). In this case (23) reads:

$$\int_\Omega f(x)\, dx = \int_{\partial\Omega} \int_0^{R(y)} f(y + s\nu_{\partial\Omega}(y)) \prod_{i=1}^{d-1} |1 - s\kappa^{(i)}_{\partial\Omega}(y)|\, ds d\mathcal{H}^{d-1}(y),$$

where $\kappa^{(1)}_{\partial\Omega}(y), \ldots, \kappa^{(d-1)}_{\partial\Omega}(y)$ denote the principal curvatures of $\partial\Omega$ at $y \in \partial\Omega$.

Proof of Lemma 2. As Ω has $C^{1,1}$-smooth boundary by [22, Proposition 4.2, Remark 4.3] we obtain $\inf_{y \in \partial\Omega} R(y) > 0$. Moreover by [5, Proposition 2.2] (see also [1, Theorem 5.9, Remark 5.10]) we have that for every $x \in \Omega$ and $y \in \pi_{\partial\Omega}(x)$ the function $\mathrm{dist}(\cdot, \partial\Omega)$ is differentiable in every point z belonging to the open segment $]y, x[:= \{z = \lambda x + (1-\lambda)y : \lambda \in (0,1)\}$, and $\nabla \mathrm{dist}(z, \partial\Omega) = (x-y)/|x-y|$. As a consequence,

$$]y, x[\cap \mathcal{S} = \emptyset \quad \text{for every } x \in \Omega, \text{ and } y \in \pi_{\partial\Omega}(x). \tag{24}$$

Hence $y + R(y)\nu_{\partial\Omega}(y) \in \Omega$. Moreover, again due to the $C^{1,1}$-regularity hypothesis on $\partial\Omega$, the curvature $\kappa_{\partial\Omega}$ is well defined $\mathcal{H}^1$-a.e. on $\partial\Omega$, and it verifies $1 > R(y)\kappa_{\partial\Omega}(y)$ for $\mathcal{H}^1$-a.e. $y \in \partial\Omega$.

We claim that the function R is a bounded upper semi-continuous function (and hence measurable). To prove that, we argue by contradiction: Suppose that we can find a sequence $\{y_n\}_n \subset \partial\Omega$ such that, as $n \to \infty$, $y_n \to y \in \partial\Omega$ and $R(y_n) \to R^* > R(y)$. Let $\delta > 0$ be such that $R(y) + \delta < R^*$. By the definition of $R(y)$ we can find a $\bar{s} \in [R(y), R(y)+\delta)$ such that $y + \bar{s}\nu_{\partial\Omega}(y) \in \mathcal{S}$. Furthermore, fixed $\tilde{s} \in (R(y) + \delta, R^*)$, for every $n \in \mathbb{N}$ large enough we have $\tilde{s} < R(y_n)$, so that $z_n := y_n + \tilde{s}\nu_{\partial\Omega}(y_n) \in \Omega \setminus \mathcal{S}$, and $\pi_{\partial\Omega}(z_n) = \{y_n\}$ and $\mathrm{dist}(z_n, \partial\Omega) = \tilde{s}$. Since $\lim_{n\to\infty} z_n = y + \tilde{s}\nu_{\partial\Omega}(y) =: z$ and, by the continuity of the distance function, we have $\mathrm{dist}(z, \partial\Omega) = \tilde{s}$, we conclude that $y \in \pi_{\partial\Omega}(z)$. However, we also have using $\bar{s} < \tilde{s}$ that

$$y + \bar{s}\nu_{\partial\Omega}(y) \in \mathcal{S} \cap]y, z[,$$

which is in contradiction with (24).

Since Ω is bounded, connected and with $C^{1,1}$-boundary, we have

$$\Omega = \Omega_0 \setminus \Big(\bigcup_{j=1}^{M} \omega_j \Big),$$

where $\Omega_0, \omega_1, \dots, \omega_M$ are open, simply connected subsets with $C^{1,1}$-boundary. We fix $I_k = [a_k, b_k)$ ($k = 0, \dots, M$) such that: $\cap_{k=0}^M I_k = \emptyset$; $|b_0 - a_0| = \mathcal{H}^1(\Omega_0)$ and $|b_k - a_k| = \mathcal{H}^1(\omega_k)$ for $k = 1, \dots, M$. We then consider $\gamma_0 \in C^{1,1}([a_0, b_0), \mathbb{R}^2)$ (respectively $\gamma_j \in C^{1,1}([a_j, b_j), \mathbb{R}^2)$ for every $j = 0, 1, \dots, M$) be the arc length parametrization of $\partial\Omega_0$ (respectively of $\partial\omega_j$) such that the vector $\dot{\gamma}_0^{\perp}(\sigma)$, obtained rotating counter-clockwise (respectively clockwise) by $\pi/2$ the tangent vector $\dot{\gamma}_0(\sigma)$ (respectively $\dot{\gamma}_j(\sigma)$), verifies $\dot{\gamma}_0^{\perp}(\sigma) = -\nu_{\partial\Omega}(\gamma_0(\sigma))$ (respectively $\dot{\gamma}_j^{\perp}(\sigma) = -\nu_{\partial\Omega}(\gamma_j(\sigma))$). We consider the map

$$\Psi : \Big(\bigcup_{k=0}^{M} I_k \Big) \times [0, +\infty) \to \mathbb{R}^2, \qquad (\sigma, s) \in I_k \times [0, +\infty) \mapsto \gamma_k(\sigma) + s\dot{\gamma}_k^{\perp}(\sigma).$$

By the regularity assumptions on Ω, we can conclude that the map Ψ is locally Lipschitz on $(\cup_{k=0}^M I_k) \times [0, +\infty)$, and uniformly Lipschitz on every subset of the form $(\cup_{k=0}^M I_k) \times [0, M)$ with a Lipschitz constant that depends on M and the Lipschitz constant of $\nu_{\partial\Omega}$.

We define

$$U := \{(\sigma, s) : \ \sigma \in (0, \mathcal{H}^1(\partial\Omega)), \, 0 < s < R(\gamma(\sigma))\},$$

and notice that: by the upper semi-continuity of $R(\cdot)$, U is a measurable subset of $\mathbb{R}^2$; by construction $\Psi(U) \subset \Omega$, and for every $x \in \Psi(U)$ we have that $\Psi^{-1}(x) \cap U$ is a singleton. Thus applying the area formula (see [14, Theorem 3.3.3.2]) we obtain

$$\begin{aligned}
&\int_{\partial\Omega}\int_0^{R(y)} f(y+s\nu_{\partial\Omega}(y))|1-s\kappa_{\partial\Omega}(y)|\, dsd\mathcal{H}^1(y) \\
&= \int_0^{\mathcal{H}^1(\partial\Omega)}\int_0^{R(y)} f(\Psi(\sigma,s))|\det J\Psi(\sigma,s)|\, dsd\sigma \\
&= \int_{\Psi(U)}\Big[\sum_{(\sigma,s)\in\Psi^{-1}(x)\cap U} f(\Psi(\sigma,s))\Big]\, dx = \int_{\Psi(U)} f(x)\, dx.
\end{aligned}$$

Since, by (24), for every $x\in\Omega\setminus\mathcal{S}$ we have $]\pi_{\partial\Omega}(x),x[\subset\Omega\setminus\mathcal{S}$, we then have $|\pi_{\partial\Omega}(x)-x|<R(\pi_{\partial\Omega}(x))$ and hence $x=\pi_{\partial\Omega}(x)+|\pi_{\partial\Omega}(x)-x|\nu_{\partial\Omega(x)}\in\Psi(U)$. Therefore $\Psi(U)\supset(\Omega\setminus\mathcal{S})$. Since by [22] $\mathcal{S}$ has null Lebesgue measure, we can conclude that (23) holds. □

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