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Regularity of source-type solutions to the
thin-film equation with zero contact angle and
mobility exponent between 3/2 and 3

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by

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Regularity of source-type solutions to the thin-film equation with zero contact angle and mobility exponent between 3/2 and 3[‡]

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Abstract

In one space dimension, we consider source-type (self-similar) solutions to the thin-film equation with vanishing slope at the edge of its support (zero contact-angle condition) in the range of mobility exponents $n \in \left(\frac{3}{2}, 3\right)$. This range contains the physically relevant case $n = 2$ (Navier slip). The existence and (up to a spatial scaling) uniqueness of these solutions has been established in [F. Bernis, L.A. Peletier & S.M. Williams, Nonlinear Anal. 18 (1992), 217-234]. There, it is also shown that the leading order expansion near the edge of the support coincides with that of a travelling-wave solution. In this paper we substantially sharpen this result, proving that the higher order correction is analytic with respect to *two* variables: the first one is just the spatial variable, whereas the second one is a (generically irrational, in particular for $n = 2$) power of it, which naturally emerges from a linearisation of the operator around the travelling-wave solution.

This result shows that — as opposed to the case of $n = 1$ (Darcy) or to the case of the porous medium equation (the second-order analogue of the thin-film equation) — in this range of mobility exponents,

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source-type solutions are *not* smooth at the edge of their support, even when the behaviour of the travelling wave is factored off. We expect the same singular behaviour for a generic solution to the thin-film equation near its moving contact line. As a consequence, we expect a (short-time or "small" data) well-posedness theory for classical solutions — of which this paper is a natural prerequisite — to be more involved than in the case $n = 1$.

1 Introduction

In this paper we study the regularity of special solutions to the following free boundary problem for the thin-film equation (tfe):

$$\partial_t h + \partial_z \left(h^n \partial_z^3 h \right) = 0 \quad \text{for} \quad t > 0, \, z \in (z_-(t), z_+(t)), \quad (1a)$$

$$h = \partial_z h = 0 \quad \text{for} \quad t > 0, \, z = z_\pm(t), \quad (1b)$$

$$\frac{\mathrm{d}z_\pm}{\mathrm{d}t}(t) = \lim_{z \to z_\pm(t)\mp} \left(h^{n-1} \partial_z^3 h \right) \quad \text{for} \quad t > 0, \quad (1c)$$

with $\frac{3}{2} < n < 3$. Here $h = h(t, z)$ describes the height of a liquid droplet as a function of time $t \geq 0$ and base point $z \in \mathbb{R}$ on a one-dimensional surface. It is known that one can derive equation (1a) in a lubrication approximation from the Navier-Stokes equations of a liquid droplet on a solid driven by capillarity [11, 21]. The case $n = 2$ then corresponds to the Navier-slip condition at the liquid-solid interface for film heights below the slippage length and is contained in our setting. The functions $z_\pm(t)$ describe the contact points between the liquid, the solid, and the surrounding gas, and will be referred to as contact lines due to their analog for droplets on two-dimensional surfaces. The boundary condition $h = 0$ for $z = z_\pm(t)$ stated in equation (1b) therefore merely defines $z_\pm(t)$. Condition (1c) is of kinematic character: It states that the (vertically averaged) horizontal velocity $v := h^{n-1} \partial_z^3 h$ of the liquid equals the velocity of the contact line. This implies in particular that the volume is conserved in time:

$$\int_{z_-(t)}^{z_+(t)} h(t, z) \mathrm{d}z = \int_{z_-(0)}^{z_+(0)} h(0, z) \mathrm{d}z. \quad (2)$$

As the fourth order problem would be under-determined in this setting, a third boundary condition is needed. The usual choice is to prescribe the contact angle between the liquid-gas and liquid-solid interfaces: Here we consider a zero contact angle condition, $\partial_z h = 0$ at $z = z_\pm(t)$, corresponding to the so-called "complete wetting regime".

Equation (1a) is invariant under the two-parameter transformation

$$(h, z, t) \mapsto \left(H_* h, Z_* z, H_*^{-n} Z_*^4 t \right),$$

with $H_*, Z_* > 0$. Enforcing conservation of volume (i.e. $H_* Z_* = 1$, cf. (2)), this two-parameter scaling becomes a one-parameter scaling

$$(h, z, t) \mapsto \left(Z_*^{-1} h, Z_* z, Z_*^{n+4} t \right). \tag{3}$$

This motivates to look for selfsimilar solutions that conserve volume/mass, i.e. solutions of the form

$$h(t, z) = t^{-\frac{1}{n+4}} H(Z), \quad Z := z t^{-\frac{1}{n+4}}. \tag{4}$$

It is elementary to see that solutions to (1) of the form (4) converge to $M\delta_0$ for $t \searrow 0$ in $\mathcal{D}'(\mathbb{R})$ (where M is the volume of the droplet and δ_0 the Dirac distribution centered at 0). This is why one commonly calls them source-type selfsimilar solutions.

On substituting (4) into equation (1a), one finds

$$-\frac{1}{n+4} \frac{\mathrm{d}}{\mathrm{d}Z}(ZH) + \frac{\mathrm{d}}{\mathrm{d}Z}\left(H^n \frac{\mathrm{d}^3 H}{\mathrm{d}Z^3} \right) = 0 \tag{5}$$

in the interior of the droplet. Motivated by the analysis of [3], we assume that $H(Z)$ is even. Then we can integrate equation (5), using boundary conditions (1b) and (1c), and obtain the following problem for the unknowns H and Z_0:

$$H^{n-1} \frac{\mathrm{d}^3 H}{\mathrm{d}Z^3} = \frac{1}{n+4} Z \quad \text{for} \quad Z \in (-Z_0, 0), \tag{6a}$$

$$H = \frac{\mathrm{d}H}{\mathrm{d}Z} = 0 \quad \text{at} \quad Z = -Z_0, \tag{6b}$$

$$\frac{\mathrm{d}H}{\mathrm{d}Z} = 0 \quad \text{at} \quad Z = 0. \tag{6c}$$

We note that since

$$\int_{z_-(t)}^{z_+(t)} h(t, z)\mathrm{d}z = 2 \int_{-Z_0}^{0} H(Z)\mathrm{d}Z,$$

conservation of mass — and also boundary condition (1c) — is automatically fulfilled by ansatz (4).

By scaling both H and Z with dimensionless quantities, we can assume w.l.o.g. that $Z_0 = 1$ and that the pre-factor on the right hand side of equation

(6a) disappears. Furthermore, we can shift the problem by $Z_0 = 1$ to the right:
$$x = Z + 1.$$
Then problem (6) reduces to finding an $H \in C^1([0,1]) \cap C^3((0,1])$ s.t.

$$H^{n-1} \frac{\mathrm{d}^3 H}{\mathrm{d}x^3} = -1 + x \qquad \text{for} \quad x \in (0,1], \tag{7a}$$

$$H = \frac{\mathrm{d}H}{\mathrm{d}x} = 0 \qquad \text{at} \quad x = 0, \tag{7b}$$

$$\frac{\mathrm{d}H}{\mathrm{d}x} = 0 \qquad \text{at} \quad x = 1. \tag{7c}$$

In the past, some efforts have been made to characterise existence, uniqueness, and properties of various types of selfsimilar solutions to the thin-film equation [3, 4, 6, 12, 13, 22]. We mention in particular the work of Bernis, Peletier, and Williams [3]: They prove existence and uniqueness of nonnegative solutions to (7) in the class $C^1([0,1]) \cap C^\infty((0,1])$, and they show that the solution is to first approximation given by

$$H(x) = A^{-\frac{\nu}{3}} x^\nu (1 + o(1)) \quad \text{as} \quad x \searrow 0, \quad \text{with} \quad \nu := \tfrac{3}{n}, \quad A := \nu(\nu-1)(2-\nu). \tag{8}$$

Our aim, which is stated in the following theorem, is to go beyond the regularity of (8):

Theorem 1. *There exist $\varepsilon > 0$ and an analytic function*

$$v(x,y) : [0, \varepsilon^2] \times [0, \varepsilon] \to \mathbb{R}, \quad \text{with} \quad v(0,0) = 0 \quad \text{and} \quad \partial_y v(0,0) < 0,$$

such that the unique solution $H \in C^1([0,1]) \cap C^\infty((0,1])$ of problem (7) satisfies:

$$H(x) = A^{-\frac{\nu}{3}} x^\nu \left(1 + v\left(x, x^\beta\right)\right) \quad \text{for} \quad 0 \le x \le \varepsilon^2, \tag{9}$$

where ν and A are given by (8) and

$$\beta = \frac{\sqrt{-3\nu^2 + 12\nu - 8} - 3\nu + 4}{2}.$$

The theorem implies in particular that H has an expansion of the form

$$H(x) = A^{-\frac{\nu}{3}} x^\nu \left(1 - bx^\beta + O\left(x^{\min\{1, 2\beta\}}\right)\right) \quad \text{as} \quad x \searrow 0,$$

with $b > 0$ since $\partial_y v(0,0) < 0$. We also note that $\tfrac{3}{2} < n < 3$ corresponds to $1 < \nu < 2$ and $0 < \beta < 1$ with the limits $\beta \searrow 0$ and $\nu \nearrow 2$ for $n \searrow \tfrac{3}{2}$, $\beta \nearrow 1$ and $\nu \searrow 1$ for $n \nearrow 3$.

Theorem 1 shows in particular that, at least on the level of the source-type solution, the thin-film equation in the range of $n \in \left(\tfrac{3}{2}, 3\right)$ is qualitatively different from

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- the porous medium equation (pme) [23, p. 59–65]

$$\partial_t h - \partial_z \left(h^n \partial_z h \right) = 0$$

for all $n > 0$, and

- the thin-film equation for $n = 1$ [22], that is, the lubrication approximation of the Hele-Shaw flow [16].

Indeed, in both cases the (explicitly known) source-type solutions are of the form

$$H(x) = \text{const} \cdot x^\mu (1 + v(x))$$

with $\mu = \frac{1}{n}$ (for the pme), $\mu = 2$ (for the tfe with $n = 1$), and $v(x)$ an analytic function of a *single* variable. The reason for problem (7) to have an analytic expansion with respect to x *and* the fractional power x^β may be formally understood by writing, according to (8),

$$H = A^{-\frac{\nu}{3}} x^\nu (1 + u).$$

The linearisation of (7a) around $u = 0$ is given by

$$p\left(x \frac{\mathrm{d}}{\mathrm{d}x} \right) u = Ax, \quad u(0) = 0, \tag{10}$$

where

$$p(\zeta) := (\zeta + 1)(\zeta - \alpha)(\zeta - \beta) \quad \text{and} \quad \alpha < 0$$

(see § 2). Out of the homogeneous solutions of (10), the ones corresponding to negative zeros of $p(\zeta)$, i.e. x^{-1} and x^α for $\alpha \neq -1$ and x^{-1} and $x^{-1} \ln x$ for $\alpha = -1$, are ruled out by the boundary condition; however, x^β, i.e. the one corresponding to the sole positive zero, is not. Hence the solutions of (10) are of the form $u(x) = -bx^\beta + ax$ (with a such that $p(1)a = A$). The nonlinear operator then mixes the two powers into, at best, a power series in x, x^β: that is indeed what we prove to happen. The possibility of such a singular behaviour was in fact suggested already by Angenent [1, p. 467] in the context of parabolic equations of a more general form than the porous medium equation (nonlinear in $\partial_z h$, possibly in non-divergence form).

It should be noted that the leading order term in (8), i.e.

$$H_{\mathrm{TW}}(x) := A^{-\frac{\nu}{3}} x^\nu, \quad x > 0, \tag{11}$$

corresponds to a travelling-wave profile for (1), i.e. to a solution of

$$H_{\mathrm{TW}}^{n-1} \frac{\mathrm{d}^3 H_{\mathrm{TW}}}{\mathrm{d}x^3} = -1 \qquad \text{for} \quad x > 0, \tag{12a}$$

$$H_{\mathrm{TW}} = 0 \qquad \text{at} \quad x = 0, \tag{12b}$$

$$\frac{\mathrm{d}H_{\mathrm{TW}}}{\mathrm{d}x} = 0 \qquad \text{at} \quad x = 0. \tag{12c}$$

Then, using the transformation

$$h_{\mathrm{TW}}(t, z) = v^{\frac{1}{n+3}} H_{\mathrm{TW}}\left(v^{\frac{1}{n+3}}(z + vt)\right),$$

the function $h_{\mathrm{TW}}(t, z)$ describes a profile travelling with constant speed $v > 0$. Now, it is commonly believed that the local behaviour near the free boundary of generic solutions to (1) is the same as that of travelling-wave or source-type selfsimilar solutions: As far as the leading order expansion (8) is concerned, this has been proved to be true for a.e. t in [2, 5] by means of so-called entropy-estimates (see also [9, 17] for the higher-dimensional case). For this reason, we believe that the understanding of the fractional expansion in this particular case is a good indication of the regularity theory to be expected for the full parabolic free boundary value problem (1). Here, we think of a small-data theory, i.e. global existence for initial data that are small perturbations of a stationary solution, a travelling-wave solution, or a source-type solution. In case of the porous medium equation, this program was first carried out by Angenent [1] in one space dimension (linearising around the travelling-wave solution and using Hölder estimates and semi-group theory), then by Daskalopoulos and Hamilton [10] in two dimensions (linearising around the one-dimensional travelling-wave solution and using weighted Hölder estimates and contraction arguments), and finally by Koch in arbitrary space dimension [20] (linearising around the source-type solution and using singular integral methods in weighted L^p-spaces). In [15], Knüpfer and two of the authors considered the thin-film equation for $n = 1$ with zero-contact angle; a global existence result close to the equilibrium solution

$$H_{\mathrm{eq}}(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ 0 & \text{for } x \leq 0 \end{cases} \tag{13}$$

was obtained in weighted L^2-spaces (see also [14] for local existence in weighted Hölder spaces). Notably, no singular behaviour occurs there. In [19], Knüpfer considered the case of $n = 2$ and a non-zero contact angle (i.e. $\partial_x h = \pm 1$ at the boundary) and obtained a global existence result close to the stationary solution. In this situation the travelling wave is itself singular:

It is a smooth function in x and $\ln x$, and the full parabolic solution inherits this singular behaviour. In a work in progress with Knüpfer, the authors develop a parabolic theory for the case considered in this paper: $n = 2$ and zero contact angle, in which case one has to perturb around the travelling-wave solution (cf. (11)).

Obviously the constant A in equation (8) vanishes in the cases $n = \frac{3}{2}$ and $n = 3$. The case $n = \frac{3}{2}$ is borderline, i.e. it distinguishes between the parameter regions $n \in \left(0, \frac{3}{2}\right)$ and $n \in \left(\frac{3}{2}, 3\right)$ with its prominent members $n = 1$ (Darcy) and $n = 2$ (Navier slip), respectively. For $n \in \left(0, \frac{3}{2}\right)$ the situation is indeed completely different from the one we analyse here: The travelling-wave solution $(-A)^{-\frac{\nu}{3}} x^\nu$ solves (12) with reversed speed (+1 on the right hand side of (12a)) and is in fact non-generic. Both generic travelling waves and source-type solutions, that are expected to capture the behaviour of generic solutions to (1) near the contact line, behave as the equilibrium solution (13) (cf. [3]). Note also that, in contrast to $n \in \left(\frac{3}{2}, 3\right)$, where the droplet can only spread, for $n \in \left(0, \frac{3}{2}\right)$ the droplet can also recede (as the non-generic travelling wave does). The case $n = 3$ instead corresponds to the no-slip condition at the liquid-solid interface. It is critical in the sense that neither solutions of (12a)-(12b) nor compactly supported source-type solutions exist for $n \geq 3$ (cf. [3]), an indication that the support of generic solutions to (1) is constant in time (this is known for $n \geq \frac{7}{2}$, see [2, 5]).

We further note that (1a) is itself a leading order approximation, as $h \to 0$, of a slightly different pde:

$$\partial_t h + \partial_z \left(\left(h^3 + \lambda h^2 \right) \partial_z^3 h \right) = 0, \tag{14}$$

where $\lambda > 0$ is a slippage length-scale accounting for the frictional forces at the liquid-solid interface. Formal asymptotics suggest that, for a given contact-line speed $v > 0$, (14) has a unique advancing travelling-wave profile $H_{\mathrm{TW}}^{(\lambda)}(x)$ such that at leading order in x and up to a multiplicative constant depending on λ,

$$H_{\mathrm{TW}}^{(\lambda)}(x) = H_{\mathrm{TW}}(x)(1 + o(1)) \quad \text{as} \quad x \searrow 0$$

and

$$H_{\mathrm{TW}}^{(\lambda)}(x) = x(3 \ln x)^{\frac{1}{3}}(1 + o(1)) \quad \text{as} \quad x \nearrow \infty$$

(see e.g. [7] for a discussion). We believe that a similar approach may be applied to this case, too. More precisely we expect that $H_{\mathrm{TW}}^{(\lambda)}$ is analytic for $x \ll 1$ with respect to x^β and $x^{\frac{3}{2}}$.

A final, even more delicate issue concerns the existence of an analytic expansion (in fractional powers) for travelling-wave solutions (with zero contact-angle) of the two-dimensional Stokes equation with slippage, of which (14) is an approximation. Our hope is that the present note will serve as a first step in this direction, as well as in the ones just mentioned.

The rest of the paper is devoted to the proof of Theorem 1. In fact, our argument does *not* use the existence result of [3]. Incidentally, the existence result in [3] is based on a shooting argument emanating from the line of symmetry, $x = 1$; ours is based on a shooting argument starting from the free boundary, $x = 0$. As indicated in the next section, we could instead have used arguments from the theory of dynamical systems (existence and characterisation of unstable manifolds) to show that the solution constructed in [3] satisfies the expansion (9) with analytic $v(x, y)$. However, we still need the shooting argument to show that $v(x, y)$ *does* depend on y, i.e. $\partial_y v(0, 0) < 0$. Hence we opted for a self-contained argument that does not rely on dynamical systems theory.

Section 2 mainly deals with the unfolding of the singularity in the two variables x and x^β as stated in Theorem 1, resulting in a nonlinear pde. In Section 3 we treat the associated linearised problem (cf. (28)), proving maximal regularity estimates (cf. Proposition 1): The proof mainly relies on an explicit representation of the solution operator as a product of three singular integral operators. Section 4 is devoted to the nonlinear case (cf. Proposition 2), which is treated by a fixed point argument: We obtain contractivity by using the previous estimates and the sub-multiplicativity of the employed norm (cf. Lemma 3). In this way, we construct a family of solutions which has one free real parameter: the derivative $\partial_y v(0, 0) = -b$. Finally, in Section 5 we apply the above mentioned shooting method to show that b can be chosen such that it matches the boundary condition $\frac{dH}{dx}(1) = 0$ (cf. (7c)). Here we use a comparison with the travelling-wave solution $H_{\mathrm{TW}}(x)$ and nonlinear ode monotonicity arguments, and we show that $\partial_y v(0, 0) < 0$.

2 Unfolding of the singularity

2.1 The linearisation and the root β

Since to leading order the solution of problem (7) is given by equation (8), we can split off the travelling-wave behaviour and set

$$H(x) =: A^{-\frac{\nu}{3}} x^\nu F(x). \tag{15}$$

Motivated by (8), we impose

$$F(0) = 1, \qquad (16a)$$

which automatically implies (7b) if $F(x)$ is sufficiently smooth at $x = 0$ (see (57) below). Disregarding for the time being (7c), equation (7a) translates into

$$(F(x))^{n-1} q(D) F(x) = A(-1 + x) \quad \text{for} \quad x \in (0, 1], \qquad (16b)$$

with the logarithmic (scaling-invariant) derivative $D := x \frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}(\ln x)}$ and the polynomial $q(\zeta)$ given by

$$
\begin{aligned}
q(\zeta) &= (\zeta + \nu)(\zeta + \nu - 1)(\zeta + \nu - 2) \\
&= \zeta^3 + 3(\nu - 1)\zeta^2 + (3\nu^2 - 6\nu + 2)\zeta - \nu(\nu - 1)(2 - \nu). \quad (17)
\end{aligned}
$$

Indeed, by the definition of ν (cf. (8)), the differential operator $x^{\nu(n-1)} \frac{\mathrm{d}^3}{\mathrm{d}x^3} x^\nu$ that appears by inserting ansatz (15) into (7a), is of degree zero and thus necessarily is of the form $q(D)$ with a third order polynomial $q(\zeta)$ of degree 3. Moreover, $x^{\nu(n-1)} \frac{\mathrm{d}^3}{\mathrm{d}x^3} x^\nu$ vanishes on the monomials $x^{-\nu}$, $x^{1-\nu}$, and $x^{2-\nu}$, so that $q(\zeta)$ has the zeros $-\nu$, $1 - \nu$, and $2 - \nu$. Altogether we obtain (17).

Since $F(0) = 1$, it is convenient to set $F(x) =: 1 + u(x)$. Since $q(D)1 = q(0) = -A$ (cf. (8)&(17)), we have by expansion of $(1 + u)^{n-1}$

$$
\begin{aligned}
(1 + u)^{n-1} q(D)(1 + u) &= q(D)1 + q(D)u + (n - 1)u q(D)1 \\
&\quad + \left((1 + u)^{n-1} - 1\right) q(D)u \\
&\quad + \left((1 + u)^{n-1} - 1 - (n - 1)u\right) q(D)1 \\
&= -A + p(D)u + \left((1 + u)^{n-1} - 1\right) q(D)u \\
&\quad - A\left((1 + u)^{n-1} - 1 - (n - 1)u\right),
\end{aligned}
$$

with the polynomial

$$p(\zeta) := q(\zeta) + (n-1)q(0) = \zeta^3 + 3(\nu - 1)\zeta^2 + (3\nu^2 - 6\nu + 2)\zeta - 3(\nu - 1)(2 - \nu).$$

Hence problem (16) turns into

$$
\begin{aligned}
p(D)u &= Ax - \left((1 + u)^{n-1} - 1\right) q(D)u \\
&\quad + A\left((1 + u)^{n-1} - 1 - (n - 1)u\right) \quad \text{for} \quad x > 0, \qquad (18a)
\end{aligned}
$$

$$u(0) = 0. \qquad (18b)$$

It is easy to guess the roots of $p(\zeta)$: Because of translational invariance, $\zeta = -1$ must be a zero of $p(\zeta)$. Indeed, by definition of the travelling-wave solution $H_{\mathrm{TW}} = A^{-\frac{\nu}{3}} x^\nu$ (cf. (11)), we have for any translation $\delta \in \mathbb{R}$ that

$$\left(A^{-\frac{\nu}{3}}(x - \delta)^\nu\right)^{n-1} \frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(A^{-\frac{\nu}{3}}(x - \delta)^\nu\right) = -1. \qquad (19)$$

To first order in δ we have

$$A^{-\frac{\nu}{3}}(x - \delta)^\nu = A^{-\frac{\nu}{3}}x^\nu\left(1 - \delta\nu x^{-1}\right) + O(\delta^2),$$

so that (19), together with $x^{\nu(n-1)}\frac{\mathrm{d}^3}{\mathrm{d}x^3}x^\nu = q(D)$ and

$$(1 - \delta x^{-1})^{n-1} = 1 - (n-1)\delta\nu x^{-1} + O(\delta^2),$$

turns into

$$\left(1 - (n-1)\delta\nu x^{-1}\right)q(D)(1 - \delta\nu x^{-1}) = -A + O(\delta^2),$$

which yields as claimed

$$p(-1)x^{-1} = p(D)x^{-1} = (q(D) + (n-1)q(0))\,x^{-1} = 0.$$

Hence we find

$$p(\zeta) = (\zeta + 1)(\zeta - \alpha)(\zeta - \beta), \tag{20}$$

with roots α and β given by

$$\alpha := \frac{-\sqrt{-3\nu^2 + 12\nu - 8} - 3\nu + 4}{2} \in (-2, 0), \tag{21a}$$

$$\beta := \frac{\sqrt{-3\nu^2 + 12\nu - 8} - 3\nu + 4}{2} \in (0, 1). \tag{21b}$$

The limiting behaviour of α and β is given by

$$\lim_{n\nearrow 3}\alpha = 0, \quad \lim_{n\searrow\frac{3}{2}}\alpha = -2, \quad \lim_{n\nearrow 3}\beta = 1, \quad \text{and} \quad \lim_{n\searrow\frac{3}{2}}\beta = 0.$$

2.2   The dynamical systems argument

Let us now sketch the dynamical systems argument: We can rewrite (16b) as a four-dimensional autonomous system by introducing the independent variable $s = \ln x$ (note that $D = \frac{\mathrm{d}}{\mathrm{d}s}$) and the dependent variables x, F, $F' = \frac{\mathrm{d}F}{\mathrm{d}s}$, and $F'' = \frac{\mathrm{d}^2 F}{\mathrm{d}s^2}$. In view of (17) we so obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} x \\ F \\ F' \\ F'' \end{pmatrix} = \begin{pmatrix} x \\ F' \\ F'' \\ F''' \end{pmatrix} \quad \text{for} \ -s \gg 1, \tag{22}$$

where

$$F''' = -3(\nu - 1)F'' - (3\nu^2 - 6\nu + 2)F' + \nu(\nu - 1)(2 - \nu)F - \frac{A}{F^{n-1}}(1 - x).$$

By construction $(0, 1, 0, 0)$ is a stationary point of (22), cf. the definition (8) of A. At this stationary point, the linearisation is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ A & nA & -(3\nu^2 - 6\nu + 2) & -3(\nu - 1) \end{pmatrix}. \tag{23}$$

Its characteristic polynomial is given by $(\zeta - 1)p(\zeta)$ with $p(\zeta)$ given by (20). In particular, it has two positive zeros, namely 1 and β, and two negative zeros, namely -1 and α. Hence $(0, 1, 0, 0)$ is a hyperbolic stationary point and the unstable manifold $\mathcal{M}$ of (22) at $(0, 1, 0, 0)$, corresponding to the eigenvalues 1 and β, is two dimensional (cf. [18]). The unstable manifold $\mathcal{M}$ is characterised dynamically (locally near the stationary point) as the set of points (x, F, F', F'') in phase space such that the solution of (22) that passes through (x, F, F', F'') converges to $(0, 1, 0, 0)$ for $s \searrow -\infty$. This is the case for the solution $(x_{\mathrm{BPW}}, F_{\mathrm{BPW}}, F'_{\mathrm{BPW}}, F''_{\mathrm{BPW}})$ constructed by [3]. It thus remains to argue that all the solutions in the unstable manifold $\mathcal{M}$ have the desired asymptotic property for $s \searrow -\infty$, that is, $x \searrow 0$.

For this, we use that the unstable manifold is also characterised geometrically, namely as the invariant manifold in phase space that is tangent to the linear space $T\mathcal{M}$ spanned by the eigenvectors of the positive eigenvalues of (23) (and that locally is a graph over the corresponding affine subspace). Let us endow $T\mathcal{M}$ with coordinates (F_1, F_2) such that $\frac{\partial}{\partial F_1}$ and $\frac{\partial}{\partial F_2}$ correspond to the eigenvectors of (23) w.r.t. 1 and β, respectively. In a small neighbourhood of $(0, 1, 0, 0)$, we may then lift the restriction of the dynamical system (22) to $\mathcal{M}$ onto the (F_1, F_2) plane, yielding a two-dimensional system of the form

$$\frac{\mathrm{d}}{\mathrm{d}s} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} + \begin{pmatrix} r_1(F_1, F_2) \\ r_2(F_1, F_2) \end{pmatrix}, \tag{24}$$

where $r = (r_1, r_2)$ is analytic in (F_1, F_2) with $r = o(|F_1| + |F_2|)$ (because the dynamical system (22) is analytic in a neighbourhood of the hyperbolic stationary point, cf. [8]). We also note that trivially on $\mathcal{M}$, we have

$$F = r_0(F_1, F_2), \tag{25}$$

where r_0 is analytic in (F_1, F_2). Obviously, (24) can be reformulated as a fixed point equation

$$\begin{aligned} F_1(s) &= e^s F_1(0) - \int_s^0 e^{s-s'} r_1(F_1(s'), F_2(s')) \mathrm{d}s', \\ F_2(s) &= e^{\beta s} F_2(0) - \int_s^0 e^{\beta(s-s')} r_2(F_1(s'), F_2(s')) \mathrm{d}s'. \end{aligned}$$

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The uniqueness result in the contraction mapping theorem (provided $F_1(s)$ and $F_2(s)$ are small enough for $s \leq 0$, which we may assume w.l.o.g. since otherwise we shift s) yields the form

$$\left. \begin{array}{rcl} F_1(s) & = & e^s(1 + v_1(e^s, e^{\beta s})) \\ F_2(s) & = & e^{\beta s}(1 + v_2(e^s, e^{\beta s})) \end{array} \right\} \quad \text{for } -s \gg 1,$$

where $v_1(F_1, F_2)$, $v_2(F_1, F_2)$ are analytic functions in (F_1, F_2) that vanish in $(0,0)$ (and depend on $(F_1(0), F_2(0))$). In view of (25) and because of $F(-\infty) = 1$, this yields

$$F(s) = 1 + v(e^s, e^{\beta s}) \quad \text{for } -s \gg 1,$$

where $v(F_1, F_2)$ is an analytic function in (F_1, F_2) that vanishes in $(0,0)$. In terms of the original variables $x = e^s$ and $H = A^{-\frac{\nu}{3}} x^\nu$, this turns into the desired form (formula (9) in Theorem 1):

$$H(x) = A^{-\frac{\nu}{3}} x^\nu (1 + v(x, x^\beta)) \quad \text{for } 0 < x \ll 1.$$

2.3 The unfolding

As mentioned at the end of the introduction, rather than using the dynamical system argument sketched above, we prefer to give a self-contained proof that on one hand does not rely on the existence result in [3], and on the other hand shows that the solution is genuinely two-variable analytic (in the sense that $\partial_y v(x, y)|_{(x,y)=(0,0)} < 0$). However, as the dynamical systems argument, our proof relies on studying the linearisation of (18), namely

$$p(D)u = f \quad \text{for} \quad x > 0, \tag{26a}$$
$$u(0) = 0. \tag{26b}$$

This ordinary initial value problem can be solved by finding a particular and, as $x \searrow 0$, regular solution and then adding a linear combination of solutions to the homogeneous equation to it. A linear independent set of solutions to $p(D)u = 0$ is given by x^{-1}, x^α, and x^β for $\alpha \neq -1$ and x^{-1}, $x^{-1} \ln x$, and x^β for $\alpha = -1$. Among these, the solution x^β is obviously the only one, which does not violate the boundary behaviour (8). In fact, we have to use one free parameter to match condition (7c) and we will prove that this contribution does not vanish. Since $\frac{d^k}{dx^k} x^\beta$ is singular in $x = 0$ for $k \geq 1$, we introduce a second variable $y := bx^\beta$ for some $b \in \mathbb{R}$ to be fixed later, to unfold the singularity according to

$$u(x) = \bar{u}\left(x, bx^\beta\right).$$

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We note that if smooth functions $v(x)$ and $\bar{v}(x, y)$ are related via $v(x) = \bar{v}(x, bx^\beta)$, we have $Dv(x) = \bar{D}\bar{v}(x, bx^\beta)$, where

$$\bar{D} := x\partial_x + \beta y\partial_y.$$

The conditions $u(0) = 0$, $u(x) = -bx^\beta(1 + o(1))$ as $x \searrow 0$, and equation (26a) thus translate into

$$p\left(\bar{D}\right)\bar{u} = \bar{f} \qquad \text{for} \quad x, y > 0, \tag{27a}$$
$$(\bar{u}, \partial_y\bar{u})(0, 0) = (0, -1), \tag{27b}$$

where $\bar{f}(x, y)$ is smooth and obeys the compatibility conditions

$$\left(\bar{f}, \partial_y\bar{f}\right)(0, 0) = (0, 0)$$

that follow from

$$p(\bar{D})\bar{u}\big|_{(x,y)=(0,0)} = p(0)\bar{u}(0, 0) = 0$$

and

$$\partial_y p(\bar{D})\bar{u}\big|_{(x,y)=(0,0)} = p(\bar{D} + \beta)\partial_y\bar{u}\big|_{(x,y)=(0,0)} = p(\beta)\partial_y\bar{u}(0, 0) = 0$$

for sufficiently smooth $\bar{u}(x, y)$. The boundary conditions (27b) imply that $u(x) = \bar{u}(x, bx^\beta) = -bx^\beta(1 + o(1))$ for $x \ll 1$. The parameter b will later be selected so that H matches the symmetry condition (7c) at $x = 1$. It turns out that we can restrict our considerations to nonnegative b, since for $b < 0$ we will always have $\partial_x H_b > \partial_x H_{\text{TW}} = A^{-\frac{\nu}{3}}\nu x^{\nu-1} > 0$ for $x > 0$ (cf. § 5).

Throughout the paper, we will write $f \lesssim g$, resp. $f \ll g$, whenever a constant $C \geq 1$, only depending on n, exists such that $f \leq Cg$, resp. $Cf \leq g$. We write $f \sim g$ if $f \lesssim g$ and $g \lesssim f$. If C depends on parameters S, we write $f \lesssim_S g$ instead.

3   Linear theory

By replacing $\bar{u} = -y + \bar{u}_0$, it suffices to consider the linear problem (27a) with homogeneous boundary data:

$$p\left(\bar{D}\right)\bar{u} = \bar{f} \qquad \text{for} \quad x, y > 0, \tag{28a}$$
$$(\bar{u}, \partial_y\bar{u})(0, 0) = (0, 0). \tag{28b}$$

The aim of this section is to construct a solution to (28), or more precisely, to construct a linear solution operator T.

Proposition 1. *For all smooth $\bar{f}(x, y)$ with $\left(\bar{f}, \partial_y \bar{f}\right)(0, 0) = (0, 0)$, there exists $\bar{u}(x, y) = \left(T\bar{f}\right)(x, y)$ smooth satisfying (28). Furthermore $\bar{u}(x, y)$ satisfies the maximal regularity estimates*

$$\sum_{m=0}^{3} \left\| \partial_x^k \partial_y^l \bar{D}^m \bar{u} \right\| \lesssim \left\| \partial_x^k \partial_y^l \bar{f} \right\| \quad \text{for all} \quad (k, l) \in \mathbb{N}_0^2 - \{(0, 0), (0, 1)\}. \quad (29)$$

In this section, $\|\cdot\|$ denotes the sup-norm on an arbitrary rectangle $[0, \ell_x] \times [0, \ell_y]$.

Our strategy for solving the third order problem (28) is to split up the solution operator as $T = T_\beta T_\alpha T_{-1}$, where $\bar{u}(x, y) = T_\gamma \bar{f}(x, y)$ is a solution of

$$\left(\bar{D} - \gamma\right) \bar{u} = \bar{f}, \quad (30)$$

with $(\bar{u}, \partial_y \bar{u})(0, 0) = (0, 0)$, $\gamma \in \{-1, \alpha, \beta\}$. For negative roots we have a simple result:

Lemma 1. *Suppose $-\gamma \gtrsim 1$. Then, for all smooth $\bar{f}(x, y)$, there exists*

$$\bar{u}(x, y) = \left(T_\gamma \bar{f}\right)(x, y)$$

smooth such that (30) holds with the maximal regularity estimates

$$\sum_{m=0}^{1} \left\| \partial_x^k \partial_y^l \bar{D}^m \bar{u} \right\| \lesssim \left\| \partial_x^k \partial_y^l \bar{f} \right\| \quad \text{for all} \quad (k, l) \in \mathbb{N}_0^2. \quad (31)$$

Moreover $\left(\bar{f}, \partial_y \bar{f}\right)(0, 0) = (0, 0)$ implies $(\bar{u}, \partial_y \bar{u})(0, 0) = (0, 0)$ and the commutation relation $T_\gamma \bar{D} = \bar{D} T_\gamma$ holds.

Proof. We define

$$\bar{u}(x, y) = \left(T_\gamma \bar{f}\right)(x, y) := \int_0^1 r^{-\gamma} \bar{f}\left(rx, r^\beta y\right) \frac{\mathrm{d}r}{r} \quad (32)$$

and note that $\bar{f}(0, 0) = 0$ obviously implies $\bar{u}(0, 0) = 0$. Furthermore

$$|\bar{u}(x, y)| \leq \int_0^1 r^{-\gamma} \frac{\mathrm{d}r}{r} \left\| \bar{f} \right\| = \frac{1}{-\gamma} \left\| \bar{f} \right\|,$$

so that

$$|\gamma| \left\| \bar{u} \right\| \leq \left\| \bar{f} \right\|. \quad (33)$$

For the derivatives, we observe that

$$\partial_x^k \partial_y^l \bar{u}(x, y) = \int_0^1 r^{-\gamma + k + \beta l} \partial_x^k \partial_y^l \bar{f}\left(rx, r^\beta y\right) \frac{\mathrm{d}r}{r} = \left(T_{\gamma - k - \beta l} \partial_x^k \partial_y^l \bar{f}\right)(x, y)$$

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for $(k, l) \in \mathbb{N}_0^2$, so that $(\partial_x^k \partial_y^l \bar{f})(0,0) = 0$ implies $\partial_x^k \partial_y^l \bar{u}(0,0) = 0$ and (33) upgrades to

$$(|\gamma| + k + \beta l) \left\| \partial_x^k \partial_y^l \bar{u} \right\| \le \left\| \partial_x^k \partial_y^l \bar{f} \right\|. \tag{34}$$

We now prove that such defined $\bar{u}(x, y)$ obeys (30) by integrating by parts:

$$
\begin{aligned}
\left(\bar{D} \bar{u} \right)(x, y) &= \int_0^1 r^{-\gamma} \left(rx \partial_x \bar{f} \left(rx, r^\beta y \right) + \beta r^\beta y \partial_y \bar{f} \left(rx, r^\beta y \right) \right) \frac{\mathrm{d}r}{r} \\
&= \int_0^1 r^{-\gamma} \frac{\mathrm{d}}{\mathrm{d}r} \left[\bar{f} \left(rx, r^\beta y \right) \right] \mathrm{d}r \\
&= \int_0^1 \gamma r^{-\gamma - 1} \bar{f} \left(rx, r^\beta y \right) \mathrm{d}r + \left[r^{-\gamma} \bar{f} \left(rx, r^\beta y \right) \right]_{r=0}^1 \\
&\overset{(32)}{=} \gamma \bar{u}(x, y) + \bar{f}(x, y).
\end{aligned} \tag{35}
$$

Applying $\partial_x^k \partial_y^l$ to (35), this also implies

$$
\left\| \partial_x^k \partial_y^l \bar{D} \bar{u} \right\| + (|\gamma| + k + \beta l) \left\| \partial_x^k \partial_y^l \bar{u} \right\| \overset{(35)}{\le} (2|\gamma| + k + \beta l) \left\| \partial_x^k \partial_y^l \bar{u} \right\| + \left\| \partial_x^k \partial_y^l \bar{f} \right\|
$$
$$
\overset{(34)}{\le} 3 \left\| \partial_x^k \partial_y^l \bar{f} \right\|,
$$

which completes the proof of (31).

Finally note that by (32) and the chain rule

$$
\begin{aligned}
\bar{D} \left(T_\gamma \bar{f} \right)(x, y) &= \int_0^1 r^{-\gamma} \left(rx \partial_x \bar{f} \left(rx, r^\beta y \right) + \beta r^\beta y \partial_y \bar{f} \left(rx, r^\beta y \right) \right) \frac{\mathrm{d}r}{r} \\
&= \int_0^1 r^{-\gamma} \left(\bar{D} \bar{f} \right) \left(rx, r^\beta y \right) \frac{\mathrm{d}r}{r} = T_\gamma \left(\bar{D} \bar{f} \right)(x, y),
\end{aligned}
$$

i.e. $T_\gamma \bar{D} = \bar{D} T_\gamma$. $\qquad \square$

For the positive root $\beta \in (0, 1)$, the situation is slightly more involved:

Lemma 2. *For all smooth $\bar{f}(x, y)$ with $\left(\bar{f}, \partial_y \bar{f} \right)(0, 0) = (0, 0)$, there exists a smooth function $\bar{u}(x, y) = \left(T_\beta \bar{f} \right)(x, y)$ with $(\bar{u}, \partial_y \bar{u})(0, 0) = (0, 0)$ such that*

$$\left(\bar{D} - \beta \right) \bar{u} = \bar{f}$$

and

$$\sum_{m=0}^1 \left\| \partial_x^k \partial_x^l \bar{D}^m \bar{u} \right\| \lesssim \left\| \partial_x^k \partial_y^l \bar{f} \right\| \quad \textit{for all} \quad (k, l) \in \mathbb{N}_0^2 - \{(0,0), (0,1)\}. \tag{36}$$

Furthermore, the commutation relation $T_\beta \bar{D} = \bar{D} T_\beta$ holds true.

Proof. We again define

$$\bar{u}(x,y) = (T_\beta \bar{f})(x,y) := \int_0^1 r^{-\beta} \bar{f}\left(rx, r^\beta y\right) \frac{\mathrm{d}r}{r}.$$

This expression is well-defined, since $\left(\bar{f}, \partial_y \bar{f}\right)(0,0) = (0,0)$ implies

$$r^{-\beta} \bar{f}\left(rx, r^\beta y\right) = O\left(r^{1-\beta}x + rxy + r^\beta y^2\right)$$

and $\beta \in (0,1)$. This is also sufficient to see that $\bar{u}(0,0) = 0$ and to show that $(\bar{D} - \beta)\bar{u} = \bar{f}$ as in the proof of Lemma 1. Furthermore

$$\partial_y \bar{u}(x,y) = \int_0^1 \partial_y \bar{f}\left(rx, r^\beta y\right) \frac{\mathrm{d}r}{r}$$

holds, since $\partial_y \bar{f}(0,0) = 0$ implies

$$\partial_y \bar{f}\left(rx, r^\beta y\right) = O\left(rx + r^\beta y\right).$$

For $(k,l) \in \mathbb{N}_0^2 - \{(0,0), (0,1)\}$ we have as before

$$\partial_x^k \partial_y^l \bar{u}(x,y) = \left(T_{-k+\beta(1-l)} \partial_x^k \partial_y^l \bar{f}\right)(x,y).$$

Since $-k + \beta(1-l) < 0$, Lemma 1 is applicable, showing that

$$\left\|\partial_x^k \partial_y^l \bar{D}\bar{u}\right\| + (k + \beta(l-1)) \left\|\partial_x^k \partial_y^l \bar{u}\right\| \leq 3 \left\|\partial_x^k \partial_y^l \bar{f}\right\|.$$

The commutation relation $T_\beta \bar{D} = \bar{D} T_\beta$ can be obtained as in the proof of Lemma 1. $\qquad\square$

We can now conclude:

Proof of Proposition 1. We set $T := T_\beta T_{-1} T_\alpha$ and note that since

$$\left(\bar{f}, \partial_y \bar{f}\right)(0,0) = (0,0),$$

also

$$\left(T_{-1} T_\alpha \bar{f}, \partial_y T_{-1} T_\alpha \bar{f}\right)(0,0) = (0,0)$$

due to Lemma 1. Hence $\bar{u}(x,y) = \left(T\bar{f}\right)(x,y)$ is well-defined. By Lemma 2

$$\left(T\bar{f}, \partial_y T\bar{f}\right)(0,0) = \left(T_\beta \left(T_{-1} T_\alpha \bar{f}\right), \partial_y T_\beta \left(T_{-1} T_\alpha \bar{f}\right)\right)(0,0) = (0,0)$$

and since

$$\begin{aligned}
p\left(\bar{D}\right) T\bar{f} &= \left(\bar{D} - \alpha\right)\left(\bar{D} + 1\right)\left(\bar{D} - \beta\right) T_\beta T_{-1} T_\alpha \bar{f} \\
&= \left(\bar{D} - \alpha\right)\left(\bar{D} + 1\right) T_{-1} T_\alpha \bar{f} \\
&= \left(\bar{D} - \alpha\right) T_\alpha \bar{f} = \bar{f},
\end{aligned}$$

the so defined $\bar{u}(x, y)$ solves (28). In order to prove estimates (29), observe that for $(k, l) \in \mathbb{N}_0^2 - \{(0,0), (0,1)\}$

$$
\begin{aligned}
\sum_{m=0}^{3} \left\| \partial_x^k \partial_y^l \bar{D}^m T \bar{f} \right\| &\leq \sum_{m_1=0}^{1} \sum_{m_2=0}^{1} \sum_{m_3=0}^{1} \left\| \partial_x^k \partial_y^l \bar{D}^{m_1} T_\beta \bar{D}^{m_2} T_{-1} \bar{D}^{m_3} T_\alpha \bar{f} \right\| \\
&\lesssim \sum_{m_2=0}^{1} \sum_{m_3=0}^{1} \left\| \partial_x^k \partial_y^l \bar{D}^{m_2} T_{-1} \bar{D}^{m_3} T_\alpha \bar{f} \right\| \\
&\lesssim \sum_{m_3=0}^{1} \left\| \partial_x^k \partial_y^l \bar{D}^{m_3} T_\alpha \bar{f} \right\| \lesssim \left\| \bar{f} \right\|,
\end{aligned}
$$

by estimates (31) and (36) and the commutation relations of Lemmas 1 and 2. $\qquad\square$

4 Nonlinear theory

We apply the unfolding $u(x) = \bar{u}\left(x, bx^\beta\right)$ to equation (18) and obtain the boundary value problem

$$
\begin{aligned}
p\left(\bar{D}\right) \bar{u} = Ax &- \left((1 + \bar{u})^{n-1} - 1\right) q(\bar{D})\bar{u} \\
&+ A\left((1 + \bar{u})^{n-1} - 1 - (n-1)\bar{u}\right) \quad \text{for} \quad x, y > 0, \quad (37\mathrm{a}) \\
(\bar{u}, \partial_y \bar{u})(0, 0) &= (0, -1). \quad (37\mathrm{b})
\end{aligned}
$$

Here we fixed the derivative $\partial_y \bar{u}(0, 0)$ as in (27b).

Proposition 2. *There exist $\varepsilon \in (0, 1)$ and $\bar{u}(x, y)$ analytic in $[0, \varepsilon^2] \times [0, \varepsilon]$ such that $\bar{u}(x, y)$ solves problem (37) in $[0, \varepsilon^2] \times [0, \varepsilon]$. Furthermore*

$$
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{3} \frac{\varepsilon^{2k+l}}{k! l!} \left\| \partial_x^k \partial_y^l \bar{D}^m \bar{u} \right\| \lesssim \varepsilon, \quad (38)
$$

where we write

$$
\|\bar{v}\| := \sup_{(x,y) \in [0, \varepsilon^2] \times [0, \varepsilon]} |\bar{v}(x, y)|, \quad \varepsilon > 0.
$$

Proof. We write $\bar{u}(x, y) = -y + \bar{u}_0(x, y)$ and arrive at the equivalent formulation (with homogeneous boundary conditions)

$$
\begin{aligned}
p\left(\bar{D}\right) \bar{u}_0 &= \bar{f}_{\bar{u}} \quad \text{for} \quad x, y > 0, \\
(\bar{u}_0, \partial_y \bar{u}_0)(0, 0) &= (0, 0),
\end{aligned}
$$

where

$$\bar{f}_{\bar{u}} := Ax - \left((1+\bar{u})^{n-1} - 1\right) q(\bar{D})\bar{u} + A\left((1+\bar{u})^{n-1} - 1 - (n-1)\bar{u}\right).$$

Our goal is to apply the contraction mapping theorem. We fix $K \geq 1$, $L \geq 2$, and set

$$\left|\bar{f}\right|_0 := \sum_{k=0}^{K} \sum_{l=0}^{L} \frac{\varepsilon^{2k+l}}{k!l!} \left\|\partial_x^k \partial_y^l \bar{f}\right\| \tag{39a}$$

for a certain $\varepsilon \in (0,1)$, which will be specified later. The norm $\left|\bar{f}\right|_0$ mimics the Taylor series of $\bar{f}(x,y)$ in the rectangle $[0,\varepsilon] \times [0,\varepsilon^2]$ to order K in x and L in y. We also introduce

$$\left|\bar{v}\right|_1 := \sum_{m=0}^{3} \left|\bar{D}^m \bar{v}\right|_0 = \sum_{k=0}^{K} \sum_{l=0}^{L} \sum_{m=0}^{3} \frac{\varepsilon^{2k+l}}{k!l!} \left\|\partial_x^k \partial_y^l \bar{D}^m \bar{v}\right\| \tag{39b}$$

and the complete metric space S as the closure with respect to $\left|\cdot\right|_1$ of

$$\{\bar{u} \text{ smooth in } [0,\varepsilon^2] \times [0,\varepsilon] : \ \left|\bar{u}_0\right|_1 \leq \varepsilon, \ (\bar{u}_0, \partial_y \bar{u}_0)(0,0) = (0,0)\}.$$

Note that

$$\bar{u} \in S \implies \left(\bar{f}_{\bar{u}}, \partial_y \bar{f}_{\bar{u}}\right)(0,0) = (0,0). \tag{40}$$

The argument for this implication merely uses the product rule, the commutation relation $\partial_y q(\bar{D}) = q(\bar{D} + \beta)\partial_y$, and the fact that

$$Q(\bar{D})\bar{v}\big|_{(x,y)=(0,0)} = Q(0)\bar{v}(0,0) = 0$$

for a polynomial $Q(\zeta)$ and a function $\bar{v}(x,y)$ being smooth enough at $(x,y) = (0,0)$:

$$
\begin{aligned}
\bar{f}_{\bar{u}}(0,0) &= A \cdot 0 - \left((1+\bar{u}(0,0))^{n-1} - 1\right) q(0)\bar{u}(0,0) \\
&\quad + A\left((1+\bar{u}(0,0))^{n-1} - 1 - (n-1)\bar{u}(0,0)\right) = 0, \\
\partial_y \bar{f}_{\bar{u}}(0,0) &= -(n-1)(1+\bar{u}(0,0))^{n-2} \partial_y \bar{u}(0,0) q(0)\bar{u}(0,0) \\
&\quad - \left((1+\bar{u}(0,0))^{n-1} - 1\right) q(\beta)\partial_y \bar{u}(0,0) \\
&\quad + A\left((n-1)(1+\bar{u}(0,0))^{n-2}\partial_y \bar{u}(0,0) - (n-1)\partial_y \bar{u}(0,0)\right) = 0.
\end{aligned}
$$

Hence — up to approximation by smooth functions — the linear solution operator T given by Proposition 1 is applicable, which yields the fixed point equation

$$\bar{u} = -y + T\bar{f}_{\bar{u}} =: \mathcal{T}(\bar{u}).$$

In order to prove the self-mapping and contraction properties, we also need the following two estimates:

Lemma 3. *Let $\bar{f}(x,y)$, $\bar{g}(x,y)$ be smooth. Then the following inequalities hold:*

(a) if $(\bar{f}, \partial_y \bar{f})(0,0) = (0,0)$, then $\|\bar{f}\| + \varepsilon \|\partial_y \bar{f}\| \lesssim \varepsilon^2 \left(\|\partial_x \bar{f}\| + \|\partial_y^2 \bar{f}\|\right)$;

(b) sub-multiplicativity:

$$|\bar{f}\bar{g}|_0 \leq |\bar{f}|_0 \, |\bar{g}|_0 \,. \tag{41}$$

Proof. By scaling $x = \varepsilon^2 \hat{x}$ and $y = \varepsilon \hat{y}$, it is enough to treat the case of $\varepsilon = 1$. The first estimate is an immediate consequence of the representations

$$\bar{f}(x,y) = \int_0^x \partial_x \bar{f}\,(x',y)\,\mathrm{d}x' + \int_0^y \int_0^{y'} \partial_y^2 \bar{f}\,(0,y'')\,\mathrm{d}y''\mathrm{d}y',$$

which yields $\|\bar{f}\| \leq \|\partial_x \bar{f}\| + \|\partial_y^2 \bar{f}\|$, and

$$
\begin{aligned}
\partial_y \bar{f}(x,y) \;=\;& \bar{f}(x,1) - \bar{f}(x,0) + \int_0^y y' \partial_y^2 \bar{f}\,(x,y')\,\mathrm{d}y' \\
& - \int_y^1 (1-y')\,\partial_y^2 \bar{f}\,(x,y')\,\mathrm{d}y',
\end{aligned}
$$

showing $\|\partial_y \bar{f}\| \lesssim \|\bar{f}\| + \|\partial_y^2 \bar{f}\|$.

For Part *(b)* of the lemma we use Leibniz' rule and the sub-multiplicativity of $\|\cdot\|$:

$$
\begin{aligned}
|\bar{f}\bar{g}|_0 \;\leq\;& \sum_{k=0}^{K}\sum_{l=0}^{L} \frac{\varepsilon^{2k+l}}{k!l!} \sum_{k'=0}^{k}\sum_{l'=0}^{l} \binom{k}{k'}\binom{l}{l'} \left\|\left(\partial_x^{k'}\partial_y^{l'}\bar{f}\right)\left(\partial_x^{k-k'}\partial_y^{l-l'}\bar{g}\right)\right\| \\
\leq\;& \sum_{k=0}^{K}\sum_{l=0}^{L}\sum_{k'=0}^{k}\sum_{l'=0}^{l} \frac{\varepsilon^{2k'+l'}}{k'!l'!} \left\|\partial_x^{k'}\partial_y^{l'}\bar{f}\right\| \frac{\varepsilon^{2(k-k')+(l-l')}}{(k-k')!(l-l')!} \left\|\partial_x^{k-k'}\partial_y^{l-l'}\bar{g}\right\|.
\end{aligned}
$$

Part *(b)* now follows at once from estimating the 1-norm of a discrete convolution:

$$
\begin{aligned}
\sum_{k=0}^{K}\sum_{l=0}^{L}\sum_{k'=0}^{k}\sum_{l'=0}^{l} |a_{k',l'}b_{k-k',l-l'}| \;\leq\;& \sum_{k=0}^{K}\sum_{l=0}^{L}\sum_{k'=0}^{K}\sum_{l'=0}^{L} |a_{k',l'}b_{k,l}| \\
=\;& \left(\sum_{k=0}^{K}\sum_{l=0}^{L} |a_{k,l}|\right)\left(\sum_{k=0}^{K}\sum_{l=0}^{L} |b_{k,l}|\right).
\end{aligned}
$$

$\square$

The following lemma is an easy consequence of the sub-multiplicativity of $|\cdot|_0$.

Lemma 4. *Provided $\bar{f}(x, y)$, $\bar{g}(x, y)$ are smooth with $\left|\bar{f}\right|_0, |\bar{g}|_0 \leq \frac{1}{2}$, we have for any exponent $m \in \mathbb{R}$:*

$$\left|(1 + \bar{f})^m - 1\right|_0 \;\lesssim_m\; \left|\bar{f}\right|_0, \tag{42a}$$

$$\left|(1 + \bar{f})^m - (1 + \bar{g})^m\right|_0 \;\lesssim_m\; \left|\bar{f} - \bar{g}\right|_0, \tag{42b}$$

$$\left|(1 + \bar{f})^m - m\bar{f} - (1 + \bar{g})^m + m\bar{g}\right|_0 \;\lesssim_m\; \max\left\{\left|\bar{f}\right|_0, |\bar{g}|_0\right\} \cdot \left|\bar{f} - \bar{g}\right|_0 \tag{42c}$$

In fact we will apply inequalities (42) for $m = n - 1$.

Proof of Lemma 4. **Step 1**. We first prove that for an analytic function

$$G(z) = \sum_{k=0}^{\infty} a_k z^k$$

with a radius of convergence larger than $\frac{1}{2}$, we have

$$\left|G\left(\bar{f}\right)\right|_0 \;\leq\; G_1\left(\left|\bar{f}\right|_0\right), \tag{43a}$$

$$\left|G\left(\bar{f}\right) - G\left(\bar{g}\right)\right|_0 \;\leq\; G_2\left(\max\{\left|\bar{f}\right|_0, |\bar{g}|_0\}\right)\left|\bar{f} - \bar{g}\right|_0, \tag{43b}$$

where

$$G_1(z) := \sum_{k=0}^{\infty} |a_k| \, z^k \quad \text{and} \quad G_2(z) := \sum_{k=0}^{\infty} (k + 1) \, |a_{k+1}| \, z^k$$

have a radius of convergence larger than $\frac{1}{2}$, too. Inequality (43a) follows immediately by applying the sub-multiplicativity (41):

$$\left|G\left(\bar{f}\right)\right|_0 \leq \sum_{k=0}^{\infty} |a_k| \, \left|\bar{f}\right|_0^k = G_1\left(\left|\bar{f}\right|_0\right).$$

Inequality (43b) follows from

$$G\left(\bar{f}\right) - G\left(\bar{g}\right) = \sum_{k=0}^{\infty} a_k \left(\bar{f}^k - \bar{g}^k\right) = \left(\bar{f} - \bar{g}\right) \sum_{k=0}^{\infty} a_{k+1} \sum_{j=0}^{k} \bar{f}^j \bar{g}^{k-j},$$

hence, by (41),

$$\begin{aligned} \left|G\left(\bar{f}\right) - G\left(\bar{g}\right)\right|_0 \;&\leq\; \left|\bar{f} - \bar{g}\right|_0 \sum_{k=0}^{\infty} |a_{k+1}| \sum_{j=0}^{k} \left|\bar{f}\right|_0^j |\bar{g}|_0^{k-j} \\ &\leq\; \left|\bar{f} - \bar{g}\right|_0 \sum_{k=0}^{\infty} (k + 1) \, |a_{k+1}| \left(\max\left\{\left|\bar{f}\right|_0, |\bar{g}|_0\right\}\right)^k \\ &=\; G_2\left(\max\left\{\left|\bar{f}\right|_0, |\bar{g}|_0\right\}\right)\left|\bar{f} - \bar{g}\right|_0. \end{aligned}$$

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Step 2. We use the Taylor series

$$(1+z)^m = \sum_{k=0}^{\infty} a_k z^k \quad \text{where} \quad a_k := \frac{m(m-1)\cdots(m-k)}{k!}. \qquad (44)$$

We note that this series converges absolutely for $|z| < 1$. From (44) we then obtain the representations

$$(1+\bar{f})^m = \sum_{k=0}^{\infty} a_k \bar{f}^k,$$

$$(1+\bar{f})^m - 1 = \bar{f} \sum_{k=0}^{\infty} a_{k+1} \bar{f}^k,$$

$$(1+\bar{f})^m - 1 - m\bar{f} = \bar{f}^2 \sum_{k=0}^{\infty} a_{k+2} \bar{f}^k.$$

Hence we can apply estimates (43) of Step 1, immediately showing inequalities (42). $\qquad \square$

Proof of Proposition 2 (continuation). By Part *(a)* of Lemma 3 we have

$$\left|\bar{f}\right|_0 \sim \sum_{\substack{k=0,\cdots,K;\, l=0,\cdots,L \\ (k,l)\neq(0,0),(0,1)}} \frac{\varepsilon^{2k+l}}{k!l!} \left\|\partial_x^k \partial_y^l \bar{f}\right\| \quad \text{if} \quad (\bar{f}, \partial_y \bar{f})(0,0) = (0,0). \qquad (45)$$

Then, by the maximal regularity estimates (29) of Proposition 1, the definition of the norms $|\cdot|_0$ and $|\cdot|_1$ (cf. (39)), boundary conditions (40) for $\bar{f}_{\bar{u}}$, and the linearity of T, we obtain

$$\left|y + \mathcal{T}(\bar{u})\right|_1 \lesssim \left|\bar{f}_{\bar{u}}\right|_0 \quad \text{and} \quad \left|\mathcal{T}(\bar{u}) - \mathcal{T}(\bar{v})\right|_1 \lesssim \left|\bar{f}_{\bar{u}} - \bar{f}_{\bar{v}}\right|_0 \qquad (46)$$

for $\bar{u}, \bar{v} \in S$. We claim that

$$\left|y + \mathcal{T}(\bar{u})\right|_1 \lesssim \varepsilon^2 \qquad (47)$$

and

$$\left|\mathcal{T}(\bar{u}) - \mathcal{T}(\bar{v})\right|_1 \lesssim \varepsilon \left|\bar{u} - \bar{v}\right|_1 \qquad (48)$$

for $\varepsilon > 0$ sufficiently small. Furthermore noting that $\left(\mathcal{T}(\bar{u}), \partial_y \mathcal{T}(\bar{u})\right)(0,0) = (0,-1)$ for $\bar{u} \in S$ by construction of T (cf. Proposition 1), we can infer that

$\mathcal{T}$ is a contraction for $\varepsilon \ll 1$. It therefore remains to estimate the different terms of $\bar{f}_{\bar{u}}$ in (46) separately, i.e.

$$
\begin{aligned}
|y + \mathcal{T}(\bar{u})|_1 &\leq |A| \, |x|_0 + \left|\left((1+\bar{u})^{n-1} - 1\right) q(\bar{D})\bar{u}\right|_0 \\
&\quad + |A| \left|(1+\bar{u})^{n-1} - 1 - (n-1)\bar{u}\right|_0, \quad\quad (49a) \\
|\mathcal{T}(\bar{u}) - \mathcal{T}(\bar{v})|_1 &\leq \left|\left((1+\bar{u})^{n-1} - 1\right) q(\bar{D})\bar{u} - \left((1+\bar{v})^{n-1} - 1\right) q(\bar{D})\bar{v}\right|_0 \\
&\quad + |A| \left|\left((1+\bar{u})^{n-1} - (n-1)\bar{u}\right)\right. \\
&\quad\quad \left. - \left((1+\bar{v})^{n-1} - (n-1)\bar{v}\right)\right|_0. \quad\quad (49b)
\end{aligned}
$$

Trivially

$$
|A| \, |x|_0 \lesssim \varepsilon^2. \quad\quad (50)
$$

Applying the triangle inequality, the sub-multiplicativity (41) (cf. Lemma 3), and estimates (42a) and (42b) of Lemma 4, we obtain

$$
\begin{aligned}
&\left|\left((1+\bar{u})^{n-1} - 1\right) q(\bar{D})\bar{u} - \left((1+\bar{v})^{n-1} - 1\right) q(\bar{D})\bar{v}\right|_0 \\
&\leq \left|(1+\bar{u})^{n-1} - (1+\bar{v})^{n-1}\right|_0 \left|q(\bar{D})\bar{u}\right|_0 + \left|(1+\bar{v})^{n-1} - 1\right|_0 \left|q(\bar{D})(\bar{u} - \bar{v})\right|_0 \\
&\lesssim \max\left\{|\bar{u}|_1, |\bar{v}|_1\right\} \cdot |\bar{u} - \bar{v}|_1 \lesssim \varepsilon \, |\bar{u} - \bar{v}|_1 \quad\quad (51)
\end{aligned}
$$

for $\varepsilon \ll 1$, where in the last inequality we used that $|\bar{u}|_1 \leq |y|_1 + |\bar{u}_0|_1 \lesssim \varepsilon$ (since $\bar{u} \in S$ and $|y|_1 \lesssim \varepsilon$) and in the same way $|\bar{v}|_1 \lesssim \varepsilon$.

Finally, using (42c) of Lemma 4, we learn that

$$
\begin{aligned}
&\left|\left((1+\bar{u})^{n-1} - (n-1)\bar{u}\right) - \left((1+\bar{v})^{n-1} - (n-1)\bar{v}\right)\right|_0 \\
&\lesssim \max\left\{|\bar{u}|_1, |\bar{v}|_1\right\} \cdot |\bar{u} - \bar{v}|_1 \lesssim \varepsilon \, |\bar{u} - \bar{v}|_1 \quad\quad (52)
\end{aligned}
$$

for $\varepsilon \ll 1$. Collecting (50), (51), and (52) for $\bar{v} = 0$, we learn from (49a) that (47) is true. Inserting (51) and (52) into (49b), we recognise that also (48) holds.

We further notice that:

- since all the estimates are in terms of constants independent of K and L, ε does not depend on K and L, too;

- the spaces S are nested as K and L increase, hence they all share the same unique fixed point.

Then indeed the unique fixed point $\bar{u} \in C^\infty([0, \varepsilon^2] \times [0, \varepsilon])$ obeys

$$
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{3} \frac{\varepsilon^{2k+l}}{k!l!} \left\|\partial_x^k \partial_y^l \bar{D}^m \bar{u}_0\right\| \leq \varepsilon, \quad\quad (53)
$$

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This also implies that the Taylor series

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\partial_x^k \partial_y^l \bar{u}_0(0,0)}{k! l!} x^k y^l$$

converges absolutely for $(x,y) \in [0, \varepsilon^2] \times [0, \varepsilon]$. As the error terms

$$\frac{1}{k! l!} \left| \partial_\xi^k \partial_\eta^l \bar{u}_0(\xi, \eta) \right| x^k y^l \quad \text{with} \quad \xi \in (0, x), \quad \eta \in (0, y)$$

converge to zero for $k, l \to \infty$ and $(x,y) \in [0, \varepsilon^2] \times [0, \varepsilon]$ due to (53), the Taylor series also represents $\bar{u}_0(x, y)$.

Finally, we recall that $\bar{u} = -y + \bar{u}_0$ and since $|y|_1 \lesssim \varepsilon$, (38) follows from (53) and the triangle inequality. $\qquad \square$

5 Shooting argument

We now prove Theorem 1 by a shooting argument. By Proposition 2 we have constructed a solution of (7a) of the form

$$H_b(x) = A^{-\frac{\nu}{3}} x^\nu \left(1 + u_b(x) \right), \tag{54}$$

where

$$\nu = \frac{3}{n}, \quad A = \nu(\nu - 1)(2 - \nu), \quad \text{and} \quad u_b(x) = \bar{u}\left(x, b x^\beta \right).$$

Here $\bar{u}(x, y)$ is analytic in $[0, \varepsilon_0^2] \times [0, \varepsilon_0]$, and hence $H_b(x)$ and $u_b(x)$ are defined for

$$0 \leq x \leq \hat{x}_b := \min \left\{ \varepsilon_0^2, \left(\frac{\varepsilon_0}{b} \right)^{\frac{1}{\beta}} \right\}.$$

By standard ode theory we can extend this solution of (7a) to a smooth solution on a maximal interval $[0, x_b)$, with

$$H_b > 0 \text{ in } (0, x_b) \quad \text{and either} \quad x_b = +\infty \quad \text{or} \quad H_b(x_b-) = 0. \tag{55}$$

Note that $u_b(x)$ and $H_b(x)$ are functions of two independent variables, b and x. However, in order to facilitate readability and avoid any possible confusion with respect to the previous sections, we use the total derivative notation to denote (partial) differentiation with respect to each of them; e.g.,

for sufficiently smooth functions $v_b(x)$ and $\bar{v}(x, y)$ with $v_b(x) = \bar{v}\left(x, bx^\beta\right)$, we write

$$\frac{\mathrm{d}}{\mathrm{d}b} v_b(x) := \lim_{b' \to b} \frac{v_b(x) - v_{b'}(x)}{b - b'} = x^\beta \partial_y \bar{v}\left(x, bx^\beta\right), \tag{56a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} v_b(x) := \lim_{x' \to x} \frac{v_b(x) - v_b(x')}{x - x'} = \partial_x \bar{v}(x, bx^\beta) + b\beta x^{\beta-1} \partial_y \bar{v}\left(x, bx^\beta\right). \tag{56b}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}x} H_b(x) \overset{(56b)}{=} A^{-\frac{\nu}{3}} x^{\nu-1} \left(\nu(1 + \bar{u}) + \bar{D}\bar{u}\right)(x, bx^\beta), \tag{57}$$

i.e. boundary condition (7b) at $x = 0$ is satisfied since $\nu > 1$ and the bracket in (57) is bounded (even analytic) at $x = 0$. Therefore, our aim is to show that there exists a parameter value $b > 0$ such that boundary condition (7c) is fulfilled.

It is convenient to subtract the travelling-wave solution (cf. (11)) $H_{\mathrm{TW}}(x) = A^{-\frac{\nu}{3}} x^\nu$:

$$(H_b - H_{\mathrm{TW}})(x) = A^{-\frac{\nu}{3}} x^\nu u_b(x) \quad \text{for} \quad 0 \le x \le x_b.$$

We use the commutation relation

$$\frac{\mathrm{d}}{\mathrm{d}x} x^\mu = x^\mu \frac{\mathrm{d}}{\mathrm{d}x} + \mu x^{\mu-1} = x^{\mu-1}(D + \mu), \quad \mu \in \mathbb{R}$$

k-times so that we obtain

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} x^\nu u_b(x) = x^{\nu-k}(\bar{D} + \nu) \cdots (\bar{D} + \nu + 1 - k)\bar{u}\left(x, bx^\beta\right). \tag{58a}$$

We also have

$$\frac{\mathrm{d}}{\mathrm{d}b} u_b(x) \overset{(56a)}{=} \frac{1}{b} y\partial_y \bar{u}\left(x, bx^\beta\right). \tag{58b}$$

By construction (cf. (37b)) we know that $(\bar{u}, \partial_y \bar{u})(0, 0) = (0, -1)$. Further applying ∂_x to equation (37a), using $\partial_x \bar{D} = (\bar{D} + 1)\partial_x$, and evaluating at $(x, y) = (0, 0)$, we learn that $\partial_x \bar{u}(0, 0) = \frac{A}{p(1)} > 0$, due to (8) and the fact that all roots of $p(\zeta)$ are smaller than 1 (cf. (20) and (21)). The analyticity of $\bar{u}(x, y)$ implies that

$$\bar{D}^k \bar{u}(x, y) = \frac{A}{p(1)}(1 + O(\varepsilon))x - \beta^k (1 + O(\varepsilon)) y, \tag{59a}$$

$$y\partial_y \bar{D}^k \bar{u}(x, y) = -\beta^k(1 + O(\varepsilon))y \tag{59b}$$

for $\varepsilon \le \varepsilon_0$ and $(x, y) \in [0, \varepsilon^2] \times [0, \varepsilon]$. From (58a) and (59a), we obtain

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}(H_b - H_{\mathrm{TW}})(x) = \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(A^{-\frac{\nu}{3}} x^\nu u_b(x)\right)$$

$$\stackrel{(58a)}{=} A^{-\frac{\nu}{3}} x^{\nu-k} \left(\prod_{l=0}^{k-1}(\bar{D} + \nu - l)\right) \bar{u}(x, bx^\beta)$$

$$\stackrel{(59a)}{=} A^{-\frac{\nu}{3}} x^{\nu-k} \left(\prod_{l=0}^{k-1}(D + \nu - l)\right) \left(\frac{A}{p(1)}(1 + O(\varepsilon))x - b(1 + O(\varepsilon))x^\beta\right)$$

$$\stackrel{(58a)}{=} A^{-\frac{\nu}{3}} \frac{\mathrm{d}^k}{\mathrm{d}x^k} x^\nu \left(\frac{A}{p(1)}(1 + O(\varepsilon))x - b(1 + O(\varepsilon))x^\beta\right)$$

$$= A^{-\frac{\nu}{3}} \left(\frac{A}{p(1)}(1 + O(\varepsilon))\frac{\mathrm{d}^k}{\mathrm{d}x^k} x^{\nu+1} - b(1 + O(\varepsilon))\frac{\mathrm{d}^k}{\mathrm{d}x^k} x^{\nu+\beta}\right) \qquad (60a)$$

for $\varepsilon \le \varepsilon_0$ and

$$0 \le x \le x_b^*(\varepsilon) := \min\left\{\varepsilon^2, \left(\frac{\varepsilon}{b}\right)^{\frac{1}{\beta}}\right\}. \qquad (60b)$$

Analogously, from (58b) and (59b), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}b}\frac{\mathrm{d}^k}{\mathrm{d}x^k} H_b(x) = -A^{-\frac{\nu}{3}}\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k} x^{\nu+\beta}\right)(1 + O(\varepsilon)) \qquad (60c)$$

for $\varepsilon \le \varepsilon_0$ and $0 \le x \le x_b^*(\varepsilon)$. The leading order expansions (60) are essential in order to show:

Lemma 5. *The function $H_b(x)$ defined by equation (54) obeys:*

(a) *overshooting for $b = 0$, i.e. $x_0 = \infty$ and $\frac{\mathrm{d}^k H_0}{\mathrm{d}x^k}(x) > \frac{\mathrm{d}^k H_{\mathrm{TW}}}{\mathrm{d}x^k}(x)$ for $k = 0, 1, 2, 3$ and $x \in (0, \infty)$;*

(b) *monotonicity in b, i.e. $\frac{\mathrm{d}}{\mathrm{d}b}\frac{\mathrm{d}^k H_b}{\mathrm{d}x^k}(x) < 0$ for $k = 0, 1, 2$ and $x \in (0, 1] \cap (0, x_b)$;*

(c) *undershooting for $b \nearrow \infty$, i.e. $x_b \searrow 0$ for $b \nearrow \infty$.*

Proof. Recall that $\nu > 1$ and thus $\frac{\mathrm{d}^k}{\mathrm{d}x^k} x^{\nu+1} > 0$ on $(0, \infty)$ for $k = 0, 1, 2$. Hence we obtain from equation (60a) for $\varepsilon > 0$ sufficiently small

$$\frac{\mathrm{d}^k H_0}{\mathrm{d}x^k} > \frac{\mathrm{d}^k H_{\mathrm{TW}}}{\mathrm{d}x^k} \quad \text{on } (0, x_0^*(\varepsilon)] \quad \text{for } k = 0, 1, 2. \qquad (61)$$

We now use equations (7a) and (12) and obtain

$$\frac{\mathrm{d}^3 H_0}{\mathrm{d}x^3} - \frac{\mathrm{d}^3 H_{\mathrm{TW}}}{\mathrm{d}x^3} = \frac{H_0^{n-1} - H_{\mathrm{TW}}^{n-1}}{H_0^{n-1} H_{\mathrm{TW}}^{n-1}} + \frac{x}{H_0^{n-1}} \overset{(55)}{>} \frac{H_0^{n-1} - H_{\mathrm{TW}}^{n-1}}{H_0^{n-1} H_{\mathrm{TW}}^{n-1}} \quad \text{on} \ \ (0, x_0).$$

$$(62)$$

From the differential inequality (62) and the ordering (61) of the initial data, it follows by an ode argument that $\frac{\mathrm{d}^k H_0}{\mathrm{d}x^k} > \frac{\mathrm{d}^k H_{\mathrm{TW}}}{\mathrm{d}x^k}$ on $(0, x_0)$ for $k = 0, 1, 2, 3$ and thus in particular $x_0 = \infty$, yielding *(a)*.

Note that $\nu > 1$, $\beta > 0$ (cf. (8) and (21b)) so that $\frac{\mathrm{d}^k}{\mathrm{d}x^k} x^{\nu+\beta} > 0$ on $(0, \infty)$ for $k = 0, 1, 2$. Hence, by equation (60c) and for ε sufficiently small, we have

$$\frac{\mathrm{d}}{\mathrm{d}b} \frac{\mathrm{d}^k H_b}{\mathrm{d}x^k} < 0 \quad \text{on} \ \ (0, x_b^*(\varepsilon)] \quad \text{for} \ \ k = 0, 1, 2. \tag{63}$$

Differentiation of (7a) w.r.t. b further yields

$$\frac{\mathrm{d}}{\mathrm{d}b} \frac{\mathrm{d}^3 H_b}{\mathrm{d}x^3} = (n-1) \frac{1 - x}{H_b^n} \frac{\mathrm{d}H_b}{\mathrm{d}b} \quad \text{on} \ \ (0, x_b). \tag{64}$$

By an ode argument, as above, (63) and (64) yield Part *(b)*.

We finally turn to Part *(c)*. Since $\beta < 1$ (cf. (21b)), it follows from (60b) that for $b \gtrsim 1$ the second contribution proportional to $x^{\nu+\beta}$ in (60a) dominates the contribution proportional to $x^{\nu+1}$. For $k = 0, 1, 2$ and as $\nu > 1$, the contribution is negative so that we have

$$\frac{\mathrm{d}^k H_b}{\mathrm{d}x^k}(x) - \frac{\mathrm{d}^k H_{\mathrm{TW}}}{\mathrm{d}x^k}(x) \lesssim -b x^{\nu+\beta-k} \quad \text{for} \ \ x \in (0, x_b^*(\varepsilon)].$$

For $b \geq \varepsilon^{1-2\beta}$ we have $x_b^*(\varepsilon) = \left(\frac{\varepsilon}{b}\right)^{\frac{1}{\beta}}$ so that we obtain in particular

$$\left. \begin{array}{rcl} H_b - H_{\mathrm{TW}} & \leq & 0 \\ \frac{\mathrm{d}H_b}{\mathrm{d}x} - \frac{\mathrm{d}H_{\mathrm{TW}}}{\mathrm{d}x} & \leq & 0 \\ \frac{\mathrm{d}^2 H_b}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 H_{\mathrm{TW}}}{\mathrm{d}x^2} & \lesssim & -b^{\frac{2-\nu}{\beta}} \varepsilon^{\frac{\nu+\beta-2}{\beta}} \end{array} \right\} \quad \text{at} \ \ x = x_b^*(\varepsilon). \tag{65}$$

For the third derivatives we observe that

$$
\begin{aligned}
\frac{\mathrm{d}^3 H_b}{\mathrm{d}x^3} - \frac{\mathrm{d}^3 H_{\mathrm{TW}}}{\mathrm{d}x^3} &= \frac{-1 + x}{H_b^{n-1}} + \frac{1}{H_{\mathrm{TW}}^{n-1}} \\
&\leq \frac{-1 + x}{H_0^{n-1}} + \frac{1}{H_{\mathrm{TW}}^{n-1}} \quad \text{for } x \leq 1 \ (\text{by } (b)) \\
&= \frac{(1 - x)(H_0^{n-1} - H_{\mathrm{TW}}^{n-1})}{(H_0 H_{\mathrm{TW}})^{n-1}} + \frac{x}{H_{\mathrm{TW}}^{n-1}} \\
&\lesssim \frac{(1 - x)(H_0 - H_{\mathrm{TW}})}{H_{\mathrm{TW}}^n} + \frac{x}{H_{\mathrm{TW}}^{n-1}} \quad (\text{by } (a)).
\end{aligned}
$$

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By (60a) and the definition of H_{TW},

$$\frac{(1-x)(H_0 - H_{\mathrm{TW}})}{H_{\mathrm{TW}}^n} \sim \frac{x^{\nu+1}}{x^{\nu n}} = x^{\nu-2} \quad \text{as } x \searrow 0,$$

$$\frac{x}{H_{\mathrm{TW}}^{n-1}} \sim \frac{x}{x^{\nu(n-1)}} = x^{\nu-2} \quad \text{as } x \searrow 0,$$

which implies that

$$\frac{\mathrm{d}^3 H_b}{\mathrm{d}x^3} - \frac{\mathrm{d}^3 H_{\mathrm{TW}}}{\mathrm{d}x^3} \lesssim x^{\nu-2} \quad \text{for } x \in (x_b^*, \min\{1, x_b\}), \ b \geq \varepsilon^{1-2\beta}.$$

This leads to the following third order remainder term in a Taylor expansion of $H_b - H_{\mathrm{TW}}$ around $x = x_b^*$:

$$\frac{1}{2} \int_{x_b^*(\varepsilon)}^{x} \left(\frac{\mathrm{d}^3 H_b}{\mathrm{d}x^3} - \frac{\mathrm{d}^3 H_{\mathrm{TW}}}{\mathrm{d}x^3} \right)(x')\, (x' - x_b^*)^2 \,\mathrm{d}x' \lesssim \int_0^x (x')^\nu \,\mathrm{d}x' \sim x^{\nu+1}. \quad (66)$$

It follows from (65), (66), and the Taylor expansion of $(H_b - H_{\mathrm{TW}})(x)$ around $x = x_b^*(\varepsilon)$ that there exist constants $c_1, c_2 > 0$ s.t.

$$H_b(x) \leq H_{\mathrm{TW}}(x) - c_1 b^{\frac{2-\nu}{\beta}} \varepsilon^{\frac{\nu+\beta-2}{\beta}} (x - x_b^*)^2 + c_2 x^{\nu+1}.$$

for all $x \in (x_b^*(\varepsilon), \min\{1, x_b\})$. Recalling that $\nu < 2$, the factor $b^{\frac{2-\nu}{\beta}}$ diverges for $b \nearrow \infty$ so that we obtain the claim of Part *(c)*. $\qquad\square$

Proof of Theorem 1. It follows immediately from the definition of u_b and H_b (near $x = 0$) and standard ode theory (in the bulk) that

$$\frac{\mathrm{d}^k H_b}{\mathrm{d}x^k} \ (0 \leq k \leq 3) \text{ depend continuously on } b \text{ on compact subsets of } (0, x_b).$$

$$(67)$$

Part *(c)* of Lemma 5 implies that $x_b \leq 1$ for $b \gg 1$. Having $H_b(0) = H_b(x_b) = 0$, it follows from the mean value theorem that $\frac{\mathrm{d}H_b}{\mathrm{d}x}(x) = 0$ for some $x \in (0, x_b)$. Hence

$$b^* = \inf \mathcal{B}, \quad \mathcal{B} := \left\{ b \geq 0 : \frac{\mathrm{d}H_b}{\mathrm{d}x}(x) = 0 \text{ for some } x \in (0, x_b) \cap (0, 1] \right\}$$

are well defined. By equation (60a), we know $\frac{\mathrm{d}H_b}{\mathrm{d}x}(x) > 0$ for $0 < x \ll 1$. Hence for $b \in \mathcal{B}$

$$x_b^{**} \in (0, 1] : \frac{\mathrm{d}H_b}{\mathrm{d}x}(x_b^{**}) = 0 \text{ and } \frac{\mathrm{d}H_b}{\mathrm{d}x} > 0 \text{ on } (0, x_b^{**})$$

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is well defined. By Part *(a)* of Lemma 5, $\frac{\mathrm{d}H_0}{\mathrm{d}x} > \frac{\mathrm{d}H_{\mathrm{TW}}}{\mathrm{d}x}$, hence $b^* > 0$ by (67) and Part *(b)* of Lemma 5. Again, by (67), the infimum is attained, i.e. $b^* \in \mathcal{B}$.

We claim that $x_{b^*}^{**} = 1$, i.e. that H_{b^*} is the desired solution. Assume by contradiction that $x_{b^*}^{**} < 1$. By (7a), $\frac{\mathrm{d}^3 H_{b^*}}{\mathrm{d}x^3} < 0$ in $(0, x_{b^*}^{**})$, i.e. $\frac{\mathrm{d}H_{b^*}}{\mathrm{d}x}$ is strictly concave. Hence, since $\frac{\mathrm{d}H_{b^*}}{\mathrm{d}x}(0) = \frac{\mathrm{d}H_{b^*}}{\mathrm{d}x}(x_{b^*}^{**}) = 0$, we obtain that $\frac{\mathrm{d}^2 H_{b^*}}{\mathrm{d}x^2}(x_{b^*}^{**}) < 0$ and that $\frac{\mathrm{d}H_{b^*}}{\mathrm{d}x} < 0$ in a right-neighbourhood of $x_{b^*}^{**}$. Therefore

$$\frac{\mathrm{d}H_{b^*}}{\mathrm{d}x}(\xi_1) > 0 \quad \text{and} \quad \frac{\mathrm{d}H_{b^*}}{\mathrm{d}x}(\xi_2) < 0 \quad \text{for some} \ \ 0 < \xi_1 < \xi_2 < 1.$$

Appealing once more to (67), we conclude that a neighbourhood I of b^* exists such that $\frac{\mathrm{d}H_b}{\mathrm{d}x}$ satisfies the same property for all $b \in I$. Hence, for all $b \in I$ there would be $\xi_b \in (\xi_1, \xi_2)$ such that $\frac{\mathrm{d}H_b}{\mathrm{d}x}(\xi_b) = 0$: this contradicts the definition of b^* and completes the proof of Theorem 1. $\qquad \square$

6 Conclusions

We investigated source-type (self-similar) solutions H to the thin-film equation with mobility exponent $n \in \left(\frac{3}{2}, 3\right)$ in one space dimension (cf. (7)). We proved that H has the following regularity (cf. Theorem 1): $H(x) = H_{\mathrm{TW}}(x)(1 + v(x, x^\beta))$, where $H_{\mathrm{TW}}(x) \sim x^{\frac{3}{n}}$ is the travelling-wave profile for the thin-film equation, β covers the range $(0, 1)$ as a function of n, and $v(x, y)$ is an *analytic* function in a neighbourhood of $(x, y) = (0, 0)$ with $v(0, 0) = 0$. Furthermore $\partial_y v(0, 0) < 0$, which shows that the x^β-contribution is present: Therefore H is *not* smooth, even when the travelling wave solution is factored off and even when $n = 2$ (corresponding to the Navier slip condition). The exponent β is dictated by the linearisation of the thin-film equation around H_{TW}.

Besides its relevance as an independent result, we consider our analysis as a natural first step towards a regularity theory for the full thin-film equation with zero contact angle (cf. (1)). More precisely, we expect that the ratio between a solution of (1) and $|z - z_\pm(t)|^{\frac{3}{n}}$ is *not* smooth as a function of $(z - z_\pm(t))$ at the contact lines, but *is* smooth as a function of $(z - z_\pm(t))$ and $|z - z_\pm(t)|^\beta$. We are also convinced that techniques similar to the ones used here can be employed to investigate the regularity of solutions to related problems, such as the travelling wave for films thicker than the slippage length (cf. (14)) or the Stokes problem for a moving cusp.

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