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by

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Abstract

Let $S \subset \mathbb{R}^2$ be a bounded simply connected domain and let g be a Riemannian metric with positive Gauss curvature on S. We study the restriction of the Willmore functional to the class of isometric immersions of (S, g) into $\mathbb{R}^3$, and we derive Euler-Lagrange equations for its critical points.

1 Introduction

Let $S \subset \mathbb{R}^2$ be a bounded simply connected domain and let g be a Riemannian metric with positive Gauss curvature on S. In this paper we study the restriction of the Willmore functional (cf. e.g. [10, 9])

$$W(u) = \frac{1}{4}\int_S |H|^2 d\mu_g + \int_{\partial S} \kappa_g d\mu_{g_\partial} \tag{1}$$

to isometric immersions u of the fixed Riemannian manifold (S, g) into $\mathbb{R}^3$. Here H is the mean curvature of the immersion, κ_g is the geodesic curvature of ∂S and μ_g, μ_{g_∂} are the area measure on S and the induced boundary measure on ∂S, respectively.

More precisely, from now on $S \subset \mathbb{R}^2$ will denote a bounded simply connected domain with smooth boundary, and $g : \overline{S} \to \mathbb{R}^{2\times 2}$ will be a given smooth Riemannian metric on $\overline{S}$ with Gauss curvature $K_g > 0$ on $\overline{S}$. For $p \geq 2$ consider the class

$$W^{2,p}_g(S) = \Big\{u \in W^{2,p}(S, \mathbb{R}^3) : \partial_i u(x) \cdot \partial_j u(x) = g_{ij}(x) \text{ for } i, j = 1, 2 \text{ and almost every } x \in S\Big\}$$

of $W^{2,p}$ isometric immersions of (S, g) into $\mathbb{R}^3$. We will study the functional

$$\widetilde{\mathcal{W}}_g(u) = \begin{cases} W(u) & \text{if } u \in W^{2,2}_g(S) \\ +\infty & \text{otherwise.} \end{cases}$$

We will assume that the metric g is such that the set $W^{2,2}_g(S)$ is non-empty. The general approach to functionals of this type developed in [5] leads to 'abstract'

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Euler-Lagrange equations for an (a priori strict) subclass of all stationary points. In order to obtain explicit Euler-Lagrange equations satisfied by all (regular enough) stationary points, some information about the metric g is needed. Here we show that the condition of positive Gauss curvature $K_g > 0$ is enough.

The case when $K_g = 0$ was studied in [4]. Although the ideas in that paper cannot be applied to the present situation, one would actually have expected the convex case $K_g > 0$ to be much easier to handle than the highly degenerate flat case $K_g = 0$. Indeed, the analysis carried out in the present paper largely confirms this expectation. As expected, one is lead to consider certain elliptic partial differential equations, for which a well-established technical machinery is available. Hence, the analysis of the functionals $\widetilde{\mathcal{W}}_g$ for metrics g with positive Gauss curvature is technically much easier than the rather delicate analysis required for the case of zero Gauss curvature. What was yet missing in order to address the former case was the right conceptual framework given in [5].

Convex isometric immersions are heavily constrained, because each of their components satisfies an elliptic Monge-Ampère equation called the Darboux equation. It is therefore not clear how to find enough variations (i.e. bendings) in order to derive the Euler-Lagrange equation for a functional such as $\widetilde{\mathcal{W}}_g$. In fact, it is well-known that there are only trivial bendings (i.e. rigid motions) of convex surfaces without boundary, cf. [1, 8]. The same is true for certain convex surfaces with boundary, e.g. for 'convex caps' [8]. Such rigidity results show that the analysis of functionals like $\widetilde{\mathcal{W}}_g$ makes no sense within these classes of surfaces.

On the other hand, for convex surfaces with boundary, it is possible to find large classes of isometric immersions once it is know that there exists one. For instance, it is then possible to find an isometric immersion for prescribed values of the mean curvature on the boundary, cf. [3, 2]. The selection criterion for the 'right' isometric immersion which arises naturally in applications is not to impose any boundary conditions at all. Instead, the right isometric immersion is one that minimizes the Willmore functional within the class of all possible isometric immersions of (S, g) into $\mathbb{R}^3$, regardless of their boundary behaviour. Observe that the minimum must be strictly positive because the Gauss curvature differs from zero. In order to characterize the minimizer, one expects the Euler-Lagrange equation to determine its boundary conditions. Indeed, the Euler-Lagrange equations derived in this paper can be regarded as equations for the boundary values of the second fundamental form of the minimizing (or stationary) immersion.

2 Preliminaries: results for general metrics

We recall some concepts and results from [5] that will be needed in the present paper, and we will reformulate some of them. They apply regardless of the sign of the Gauss curvature of the given Riemannian metric g. First note that $\widetilde{\mathcal{W}}_g = \mathcal{W}_g +$ intrinsic quantities, where

$$\mathcal{W}_g(u) = \begin{cases} \int_S |\nabla^2 u|_g^2 d\mu_g & \text{if } u \in W_g^{2,2}(S) \\ +\infty & \text{otherwise,} \end{cases}$$

and where $|\nabla^2 u|^2 = \left\langle \nabla^2 u, \nabla^2 u \right\rangle_g$ and $\left\langle \nabla^2 u, \nabla^2 \tau \right\rangle_g = g^{ik} g^{jl} \partial_i \partial_j u \cdot \partial_k \partial_l \tau$. In particular, a surface u minimizes $\mathcal{W}_g$ if and only if it minimizes $\widetilde{\mathcal{W}}_g$, and the first variations (on isometric immersions) of these functionals agree. Existence of minimizers of $\mathcal{W}_g$ is easy to prove. Indeed, we have [5]:

Proposition 2.1 *The restriction of the functional $\mathcal{W}_g$ to the space*

$$\mathcal{A}_0 = \{u \in W^{2,2}_g(S) : \int_S u \; d\mu_g = 0\}$$

attains a global minimum on this space.

In order to derive the Euler-Lagrange equation satisfied by such minimizers, we will use infinitesimal bendings as admissible test functions:

Definition 2.2 *A vector field $\tau \in W^{2,2}(S, \mathbb{R}^3)$ is called an* infinitesimal bending *of an immersion $u \in W^{2,2}(S, \mathbb{R}^3)$ if it satisfies*

$$\partial_i \tau \cdot \partial_j u + \partial_j \tau \cdot \partial_i u = 0 \text{ almost everywere on } S \text{ for } i, j = 1, 2. \tag{2}$$

An infinitesimal bending is said to be trivial if it is the velocity field of a rigid motion.

A bending of a given immersion u is a one parameter-family of immersions which are isometric to u. Bendings constitute the admissible variations of a given critical point.

Definition 2.3 *A $W^{2,2}$-bending of an immersion $u \in W^{2,2}_g(S)$ is a strongly $W^{2,2}$-continuous one-parameter family $\{u_t\}_{t \in (-1,1)} \subset W^{2,2}_g(S)$ satisfying $u_0 = u$ and which is such that the weak $W^{2,2}$-limit*

$$\tau = \lim_{t \to 0} \frac{1}{t}(u_t - u_0) \tag{3}$$

exists. The vector field τ is called the infinitesimal bending field induced by the bending $\{u_t\}_{t \in (-1,1)}$.
Any vector field τ induced as in (3) by a $W^{2,2}$-bending of u is called a continuable *infinitesimal $W^{2,2}$-bending of u.*

Now we can formulate the concept of non-minimizing stationary points:

Definition 2.4 *An immersion $u \in W^{2,2}_g(S)$ is called a stationary point of $\mathcal{W}_g$ if*

$$\frac{d}{dt}\Big|_{t=0} \mathcal{W}_g(u_t) = 0 \text{ for all } W^{2,2}\text{-bendings } \{u_t\}_{t \in (-1,1)} \text{ of } u. \tag{4}$$

We recall the following definition from [5]:

Definition 2.5 *An immersion $u \in W^{2,2}_g(S)$ is said to be stationary for $\mathcal{W}_g$ under infinitesimal bendings if*

$$\int_S \left\langle \nabla^2 u, \nabla^2 \tau \right\rangle_g \; d\mu_g = 0 \text{ for all infinitesimal bendings } \tau \in W^{2,2}(S, \mathbb{R}^3) \text{ of } u.$$

The following simple observations were proven in [5]:

Proposition 2.6 *If $\{u_t\}_{t\in(-1,1)}$ is a $W^{2,2}$ bending of $u \in W^{2,2}_g(S)$ inducing the continuable infinitesimal bending $\tau \in W^{2,2}(S,\mathbb{R}^3)$, then*

$$\frac{d}{dt}\Big|_{t=0} \mathcal{W}_g(u_t) = \int_S \left\langle \nabla^2 u, \nabla^2 \tau \right\rangle_g \, d\mu_g. \tag{5}$$

In particular, we have the following:

(i) An immersion $u \in W^{2,2}_g(S)$ is a stationary point of $\mathcal{W}_g$ precisely if

$$\int_S \left\langle \nabla^2 u, \nabla^2 \tau \right\rangle_g \, d\mu_g = 0 \text{ for all continuable infinitesimal bendings } \tau \in W^{2,2}(S,\mathbb{R}^3) \text{ of } u. \tag{6}$$

(ii) Every $u \in W^{2,2}_g(S)$ that is a stationary for $\mathcal{W}_g$ under infinitesimal bendings is a stationary point of $\mathcal{W}_g$.

(iii) Conversely, if u is a stationary point of $\mathcal{W}_g$ and if the set of $W^{2,2}$ continuable infinitesimal bendings is weakly $W^{2,2}$-dense in the set of infinitesimal bendings, then u is stationary under infinitesimal bendings.

From now on we denote by $\langle\cdot,\cdot\rangle_g$ the scalar product associated with the metric g, and its extension to tensors is denoted by the same symbol. The Christoffel symbols are denoted by Γ^k_{ij} and D denotes the metric connection. With a slight abuse of notation, we will write df to denote the gradient vector field grad$f$ of a function f along u. By n we denote the normal to a given immersion u, by h we denote its second fundamental. If $K_g \neq 0$ then we set $a^{ik} = (h^{-1})_{ik}$, and we define the quadratic form B on u by setting

$$B(X,Y) = a^{ij} X_i Y_j = a_{ij} X^i X^j.$$

for all tangent vector fields $X = X^i \partial_i u$ and $Y = Y^i \partial_i u$. In coordinates, we have

$$(B_{ij}) = \frac{|g|}{K_g} \operatorname{cof}\left(g^{-1} h g^{-1}\right) = \frac{|g|}{K_g} \begin{pmatrix} h^{22} & -h^{12} \\ -h^{12} & h^{11} \end{pmatrix}. \tag{7}$$

In fact, the matrix with entries B_{ij} equals

$$g h^{-1} g = \frac{\operatorname{cof}(g^{-1} h g^{-1})}{\det(g^{-1} h g^{-1})},$$

and $g^{-1} h g^{-1}$ is the matrix with entries h^{ij}. Without the restriction $K_g \neq 0$ we can still define '$K_g B$'. More precisely, we denote by $\widehat{B}$ the quadratic form on u given in coordinates by

$$(\widehat{B}_{ij}) = |g| \operatorname{cof}\left(g^{-1} h g^{-1}\right) = |g| \begin{pmatrix} h^{22} & -h^{12} \\ -h^{12} & h^{11} \end{pmatrix}. \tag{8}$$

When $K_g \neq 0$ then $\widehat{B} = K_g B$.
For any quadratic form q the 1-form $div_g q$ is given in coordinates by

$$(D_i q)^{ij} = \partial_i q^{ij} + \Gamma^i_{ik} q^{kj} + \Gamma^j_{ik} q^{ik}$$

For any tangent vector field $X = X^i \partial_i u$ we define the contraction $q\lfloor X$ to be the 1-form given by

$$(q\lfloor X)(Y) = q(X, Y)$$

for all tangent vectorfields Y. In coordinates, $(q\lfloor X)_i = q_{ij}X^j$. For a given tangent vectorfield X we denote the Lie derivative of the metric g by $\mathcal{L}_X g$. In coordinates,

$$(\mathcal{L}_X g)_{ij} = \frac{1}{2}\left((D_i X)_j + (D_j X)_i\right).$$

We also introduce the natural almost complex structure J on u defined by $J(X) = n \wedge X$ for all tangent vectorfields X. In coordinates it is given by

$$J_{ij} = \begin{cases} -\sqrt{g} & \text{if } (i,j) = (1,2) \\ \sqrt{g} & \text{if } (i,j) = (2,1) \\ 0 & \text{otherwise.} \end{cases}$$

Explictly, consider the tangent vector $X = X^i \partial_i u$. Then $J(X) = (J(X))^i \partial_i u$, where

$$(J(X))^i = J^{ij} X_j.$$

In particular, if the coordinate functions X^i of X are smooth then so are the coordinate functions $(J(X))^i$ of $J(X)$.
Our first formulation of the Euler-Lagrange equation will be a consequence of the following result, which is a modification of a result in [5]:

Proposition 2.7 *Let $g \in C^\infty(\overline{S}, \mathbb{R}^{2\times 2})$ be a Riemannian metric on S and let $u \in W^{2,p}_g(S)$ for some $p > 2$. If u is stationary for $\mathcal{W}_g$ under infinitesimal bendings, then there exist sequences $\varphi_0^{(n)}$, $Y_1^{(n)}$, $Y_2^{(n)} \in C_0^\infty(S)$ such that the tangent vector fields*

$$Y^{(n)} = g^{ij} Y_j^{(n)} \partial_i u$$

satisfy

$$\varphi_0^{(n)} h + \mathcal{L}_{Y^{(n)}} g \rightharpoonup \widehat{B} \tag{9}$$

weakly in L^2 as $n \to \infty$.

Proof. By [5, Corollary 2.3] we know that there exist sequences $\varphi_0^{(n)}$, $\varphi_1^{(n)}$, $\varphi_2^{(n)} \in C_0^\infty(S)$ such that

$$\begin{aligned} &\varphi_0^{(n)} \operatorname{cof} h + \varphi_i^{(n)} \begin{pmatrix} -\Gamma_{i2}^1 & \frac{1}{2}(\Gamma_{i1}^1 - \Gamma_{i2}^2) \\ \frac{1}{2}(\Gamma_{i1}^1 - \Gamma_{i2}^2) & \Gamma_{i1}^2 \end{pmatrix} + \\ &\qquad + \begin{pmatrix} -\partial_2 \varphi_1^{(n)} & \frac{1}{2}(\partial_1 \varphi_1^{(n)} - \partial_2 \varphi_2^{(n)}) \\ \frac{1}{2}(\partial_1 \varphi_1^{(n)} - \partial_2 \varphi_2^{(n)}) & \partial_1 \varphi_2^{(n)} \end{pmatrix} \rightharpoonup \sqrt{g} \cdot g^{-1} h g^{-1} \end{aligned} \tag{10}$$

weakly in $L^2(S, \mathbb{R}^{2\times 2})$ as $n \to \infty$. We introduce the vectorfields $X^{(n)} = (X^{(n)})^i \partial_i u$ with coordinates $(X^{(n)})^i := \varphi_i^{(n)}$ for $i = 1, 2$. It is easy to check that the left-hand side of (10) then equals (we omit the index n)

$$\varphi_0 \operatorname{cof} h + \begin{pmatrix} -(D_2 X)^1 & \frac{1}{2}\left((D_1 X)^1 - (D_2 X)^2\right) \\ \frac{1}{2}\left((D_1 X)^1 - (D_2 X)^2\right) & (D_1 X)^2 \end{pmatrix}. \tag{11}$$

Now we define $Y = -J(X)$, so that $X = J(Y)$. Then, using the fact that

$$J^{12} = -\frac{1}{\sqrt{g}},$$

we see that

$$\begin{aligned}(D_i X)^1 &= -\frac{1}{\sqrt{g}}(D_i Y)_2 \\ (D_i X)^2 &= \frac{1}{\sqrt{g}}(D_i Y)_1.\end{aligned}$$

Hence (11) equals

$$\varphi_0 \operatorname{cof} h + \frac{1}{\sqrt{g}} \begin{pmatrix} (D_2 Y)_2 & -\frac{1}{2}\left((D_1 Y)_2 + (D_2 Y)_1\right) \\ -\frac{1}{2}\left((D_1 Y)_2 + (D_2 Y)_1\right) & (D_1 Y)_1 \end{pmatrix}.$$

Thus we conclude from (10) that

$$\varphi_0^{(n)} \operatorname{cof} h + \frac{1}{\sqrt{g}} \operatorname{cof}\left(\mathcal{L}_{Y^{(n)}} g\right) \rightharpoonup \sqrt{g} \cdot g^{-1} h g^{-1}$$

weakly in L^2. Applying cof on both sides, multiplying by the smooth scalar $\sqrt{g}$ and absorbing it into $\varphi_0^{(n)}$ and recalling (8) we arrive at (9). $\square$

Our second formulation of the Euler-Lagrange equation is based upon the following result from [5]:

Proposition 2.8 *Let $g \in C^\infty(\overline{S}, \mathbb{R}^{2\times 2})$ be a Riemannian metric on S with $K_g \neq 0$ on $\overline{S}$ and let $u \in W^{2,2}_g(S)$. If $u \in C^\infty(\overline{S}, \mathbb{R}^3)$ is stationary for $\mathcal{W}_g$ under infinitesimal bendings then there exist $\varphi_n \in C_0^\infty(S)$ such that*

$$\mathcal{M}^* \varphi_n \overset{*}{\rightharpoonup} F \text{ weakly-* in } W^{-2,2}(S).$$

Here, $\mathcal{M}^*$ denotes the formal adjoint (with respect to the flat scalar product in $\mathbb{R}^2$) of the differential operator $\mathcal{M}$ given by

$$\mathcal{M}\psi = \sqrt{g} div_g(h^{-1}\nabla\psi) + 2H\sqrt{g}\psi, \tag{12}$$

and $F \in W^{-2,2}(S)$ is the functional given by

$$\begin{aligned} F(\psi) = {} & 2\int_S div_g\left(B\lfloor J(dH)\right)\ \psi\ d\mu_g \\ & + 2\int_{\partial S} HB\left(d\psi, J(\nu)\right) - B\left(\nu, J(dH)\right)\ \psi\ d\mu_{g\partial} \end{aligned} \tag{13}$$

for all $\psi \in W^{2,2}(S)$.

Proof. By [5, Corollary 6.6 (ii)] the proposition is satisfied with

$$\begin{aligned} F(\psi) = {} & -2\int_S J^{ik}\psi D_j(a_k^j \partial_i H)\ d\mu_g \\ & + 2\int_{\partial S} J^{ik} a_k^l \left(\nu_l(\partial_i H)\psi - \nu_i H(\partial_l\psi)\right)\ d\mu_{g\partial}. \end{aligned}$$

As J is parallel and since (writing dH to denote the gradient vector field of H),

$$(J^{ki}\partial_i H)\partial_k u = J(dH),$$

the right-hand side equals

$$\begin{aligned}
2\int_S \psi D_j \left(a^j_k J^{ki}\partial_i H\right) d\mu_g - 2\int_{\partial S} a^j_k \nu_j (J^{ki}\partial_i H)\psi - H a^j_k J^{ki}\nu_i \partial_j \psi \; d\mu_{g_\partial} \\
= 2\int_S \psi D_i \left(a^{ik}(JdH)_k\right) \; d\mu_g - 2\int_{\partial S} a^{jk}\nu_j (JdH)_k \psi - H a^{jk}\partial_j \psi (J\nu)_k d\mu_{g_\partial}.
\end{aligned}$$

□

For future reference we also include the following result.

Proposition 2.9 *Under the hypotheses of Proposition 2.8 there exist quadratic forms $\sigma^{(n)}$ along u with coordinates satisfying*

$$\sigma^{(n)}(\partial_i u, \partial_j u) \in C^\infty_0(S)$$

and such that

$$\left\langle \sigma^{(n)}, h \right\rangle_g \overset{*}{\rightharpoonup} \Delta_g H + 2H(H^2 - K_g) + \lambda^{(1)} \tag{14}$$

$$div_g \sigma^{(n)} \overset{*}{\rightharpoonup} \lambda^{(2)} \tag{15}$$

weakly- in $W^{-2,2}$ as $n \to \infty$. Here, $\lambda^{(1)} \in W^{-2,2}(S)$ is defined by*

$$\lambda^{(1)}(\Phi) = \int_{\partial S} H \langle \nu, d\Phi \rangle_g - \Phi \langle \nu, dH \rangle_g \; d\mu_{g_\partial}$$

for all $\Phi \in W^{2,2}(S)$ and $\lambda^{(2)}$ is the $W^{-2,2}$-valued 1-form defined by

$$\lambda^{(2)}(V) = \int_{\partial S} H^2 \langle \nu, V \rangle_g \; d\mu_{g\partial}$$

for all tangent vectorfields $V = V^i \partial_i u$ with $V^i \in W^{2,2}(S)$.

Proof. By [5, Corollary 6.6 (i)] there exist $\varphi^{(n)}_1$, $\varphi^{(n)}_2$, $\varphi^{(n)}_3 \in C^\infty_0(S)$ such that

$$\begin{aligned}
\int_S \left\{ h_{11}\varphi^{(n)}_1 + h_{12}\varphi^{(n)}_2 + h_{22}\varphi^{(n)}_3 \right\} \Phi \; dx \\
\longrightarrow \int_S \Big(\Delta_g H + 2H(H^2 - K_g) \Big) \; \Phi \; d\mu_g + \lambda^{(1)}(\Phi)
\end{aligned} \tag{16}$$

for all $\Phi \in W^{2,2}(S)$ and

$$\begin{aligned}
\int_S \left\{ \Gamma^i_{11}\varphi^{(n)}_1 + \Gamma^i_{12}\varphi^{(n)}_2 + \Gamma^i_{22}\varphi^{(n)}_3 \right\} V_i + \int_S (\partial_1 \varphi^{(n)}_1 + \frac{1}{2}\partial_2 \varphi^{(n)}_2) V_1 \\
+ \int_S (\frac{1}{2}\partial_1 \varphi^{(n)}_2 + \partial_2 \varphi^{(n)}_3) V_2 \longrightarrow \lambda^{(2)}(V^i \partial_i u)
\end{aligned} \tag{17}$$

for all V_1, $V_2 \in W^{2,2}(S)$. In what follows we omit the index n. The formula

$$\int_S \langle \mathcal{L}_V g - \Phi h, \sigma \rangle_g \, d\mu_g = -\int_S (D_i \sigma)^{ij} V_j + \Phi \langle h, \sigma \rangle_g \; d\mu_g$$

motivates testing (17) with

$$\begin{aligned}\varphi_1 &= \sqrt{g}\sigma^{11}\\ \varphi_2 &= 2\sqrt{g}\sigma^{12}\\ \varphi_3 &= \sqrt{g}\sigma^{22},\end{aligned}$$

where $\sigma^{ij} \in C_0^\infty(S)$ are arbitrary coordinates of some quadratic form σ on u. We find that the left-hand side of (17) equals

$$\begin{aligned}&\int_S \left(\Gamma^i_{11}\sigma^{11} + 2\Gamma^i_{12}\sigma^{12} + \Gamma^i_{22}\sigma^{22}\right) V_i \, d\mu_g\\ &\quad + \int_S \left(\partial_1\sigma^{11} + \partial_2\sigma^{12}\right) V_1 + \left(\partial_1\sigma^{12} + \partial_2\sigma^{22}\right) V_2 d\mu_g\\ &\quad + \int_S (\partial_1\sqrt{g})\left(\sigma^{11}V_1 + \sigma^{12}V_2\right) + (\partial_2\sqrt{g})\left(\sigma^{12}V_1 + \sigma^{22}V_2\right)\, dx.\end{aligned}$$

The last term equals

$$\int_S \Gamma^k_{ki}\sigma^{ij}V_j \, d\mu_g.$$

We conclude that with the above choices of φ_i the left-hand side of (17) equals

$$\int_S (div_g\sigma)(V)\, d\mu_g.$$

And clearly the left-hand side of (16) equals

$$\int_S \langle h, \sigma\rangle_g \, d\mu_g.$$

□

3 Main results: convex metrics

In this section we derive the main results of this paper, which are concerned with metrics with positive Gauss curvature.

3.1 Existence of smooth minimizers and continuation of infinitesimal bendings

From now on we use the notation $C^k_g(S)$ to denote the space of C^k isometric immersions of (S, g) into $\mathbb{R}^3$, and we use similar notations to denote isometric immersions with other regularities. Stationarity and stationarity under infinitesimal bendings turn out to be equivalent notions for regular convex isometric immersions. Hence every (regular enough) stationary point of $\mathcal{W}_g$ satisfies the hypotheses of Proposition 2.8. More precisely, in this short section we will prove the following result:

Proposition 3.1 *Let $g \in C^\infty(\overline{S}, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ be a Riemannian metric on S with $K_g > 0$ on $\overline{S}$. Then the functional $\mathcal{W}_g$ attains a minimum on the set $\mathcal{A}_0 \cap C^2(S, \mathbb{R}^3)$. This set agrees with $\mathcal{A}_0 \cap C^\infty(S, \mathbb{R}^3)$.*
If $\alpha > 0$ and if $u \in C^{2,\alpha}_g(\overline{S})$ is a stationary point of $\mathcal{W}_g$ then it is stationary under infinitesimal bendings.

It is well-known that if $u \in C^{2,\alpha}(\overline{S}, \mathbb{R}^3)$ for some $\alpha > 0$ is an immersion whose induced metric has positive Gauss curvature on $\overline{S}$, then every regular infinitesimal bending of u is continuable. The following assertion about continuability of infinitesimal bendings $\tau \in C^{2,\alpha}(\overline{S}, \mathbb{R}^3)$ on the manifold (S, g) is a particular case of Theorem 3 in [6]. Another proof of this result can be found in [7].

Lemma 3.2 *Let $\alpha > 0$, let $u \in C^{2,\alpha}(\overline{S})$ be an immersion inducing a metric g whose Gauss curvature satisfies $K_g > 0$ on $\overline{S}$. Then every infinitesimal bending $\tau \in C^{2,\alpha}(\overline{S}, \mathbb{R}^3)$ of u is $C^{2,\alpha}$-continuable.*

We obtain the following consequence:

Lemma 3.3 *Let $\alpha > 0$, let $g \in C^\infty(\overline{S}, \mathbb{R}^{2\times 2})$ be a Riemannian metric with $K_g > 0$ on $\overline{S}$. If $u \in C^{2,\alpha}_g(\overline{S})$ is a stationary point of $\mathcal{W}_g$ then u is stationary under infinitesimal bendings.*

Proof. Lemma 3.2 implies that u satisfies (6) for all infinitesimal bendings $\tau \in C^{2,\alpha}(\overline{S}, \mathbb{R}^3)$. But by Theorem 1.2 in [7] such maps are (even strongly) $W^{2,2}$-dense in the set of $W^{2,2}$ infinitesimal bendings. Hence the claim follows from Proposition 2.6 (iii). □

Proof of Proposition 3.1. We must only show that $\mathcal{A}_0 \cap C^2(S, \mathbb{R}^3)$ is closed under weak $W^{2,2}$-convergence. But each surface in this space is either convex or concave. This property is stable under weak $W^{2,2}$-convergence, hence weak $W^{2,2}$-accumulation points of $\mathcal{A}_0 \cap C^2(S, \mathbb{R}^3)$ belong to $W^{2,2}$ and are either convex or concave. Thus by well-known regularity results about convex solutions of Monge-Ampère equations, we conclude that accumulation points even belong to $C^\infty(S, \mathbb{R}^3)$. The remaining claims follow from Lemma 3.3. □

3.2 Euler-Lagrange equations

When $K_g > 0$ on $\overline{S}$, then the operator (12) is elliptic. A similar assertion is true about the system behind (9). Combining the above results with some general facts about elliptic PDEs, in this section we will derive the following Euler-Lagrange equations.

Theorem 3.4 *Let $g \in C^\infty(\overline{S}, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ be a Riemannian metric on S with $K_g > 0$ on $\overline{S}$ and let $u \in C^\infty_g(\overline{S})$ be a stationary point of $\mathcal{W}_g$. Then the following are satisfied:*

(i) There exists a vectorfield $Y = Y^i \partial_i u$ with coordinates $Y^i \in H^1_0(S)$ solving the PDE system

$$\mathcal{L}_Y g - \frac{\langle \mathcal{L}_Y g, h \rangle_g}{|h|_g^2}\, h = K_g B - \frac{K_g \langle B, h \rangle_g}{|h|_g^2}\, h. \tag{18}$$

(ii) There exists a function $\varphi \in C^\infty(\overline{S})$ solving the over-determined uniformly elliptic boundary value problem

$$div_g(B \lfloor d\varphi) + 2H\varphi = 2div_g\left(B \lfloor J(dH)\right) \text{ in } S \tag{19}$$

$$\varphi = 2H \frac{B(\tau, \nu)}{B(\nu, \nu)} \text{ on } \partial S \tag{20}$$

and

$$
\begin{aligned}
&B(\nu, d\varphi) + div_{g_\partial}(\varphi\ B(\nu,\tau)\,\tau) \\
&\qquad = 2B(\nu, J(dH)) + div_{g_\partial}(2H\ B(\tau,\tau)\,\tau) \ \ \text{on } \partial S.
\end{aligned} \tag{21}
$$

Here ν denotes the outer unit co-normal to u along ∂S and $\tau = J(\nu)$.

Remarks.

1. The Euler-Lagrange equations from Theorem 3.4 are the counterpart (when $K_g > 0$) to the Euler-Lagrange equation obtained in [4] (when $K_g = 0$).

2. Equation (18) means that $\mathcal{L}_Y g - K_g B$ is a multiple of h at each point. So (18) is equivalent to the assertion that

   $$\langle \mathcal{L}_Y g - K_g B, q\rangle_g = 0$$

   for all quadratic forms $q \in L^2$ on u satisfying $\langle h, q\rangle_g = 0$ almost everywhere. Since there exist smooth quadratic forms $q^{(1)}$ and $q^{(2)}$ which at each point span the orthogonal complement of h (cf. the proof of Theorem 3.4), this is equivalent to the two scalar equations

   $$\left\langle \mathcal{L}_Y g, q^{(\alpha)}\right\rangle_g = \left\langle K_g B, q^{(\alpha)}\right\rangle_g \ \text{ for } \alpha = 1, 2.$$

   Indeed, for all q with $\langle q, h\rangle_g = 0$ there exist scalar functions $f^{(1)}$, $f^{(2)}$ such that $q(x) = f^{(\alpha)}(x) q^{(\alpha)}(x)$.

3. As the differential operator on the left-hand side of (19) does not admit a maximum principle, in general we do not know if the solution to the boundary value problem (19), (20) is unique. However, if the Dirichlet boundary value problem consisting only of (19) and (20) is uniquely solvable, then its solution φ satisfies the equation (21) on ∂S. Eliminating φ by expressing it in terms of the right-hand side of (19), this yields an equation for the second fundamental form h on ∂S.

4. The assertion of Theorem 3.4 (i) is not void because the system (18) with zero Dirichlet data is over-determined. This follows from general facts about elliptic first order systems. For instance, if we assume conjugate isometric coordinates, i.e., that the second fundamental form h of u satisfies $h_{11} = h_{22}$ and $h_{12} = 0$, then the orthocomplement with respect to (flat) matrix multiplication of the matrix (h_{ij}) in $\mathbb{R}^{2\times 2}_{\text{sym}}$ is spanned by the matrices

   $$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

   Flat scalar multiplication of (18) with these matrices leads to the two equations

   $$
   \begin{aligned}
   &(D_2 Y)_2 - (D_1 Y)_1 = K_g(B_{22} - B_{11}) \\
   &(D_1 Y)_2 + (D_2 Y)_1 = 2K_g B_{12}.
   \end{aligned}
   $$

By (7) we therefore see that Theorem 3.4 (i) asserts the existence of a variational solution Y of the elliptic boundary value problem

$$\partial_2 Y_2 - \partial_1 Y_1 + (\Gamma^i_{11} - \Gamma^i_{22}) Y_i = |g|(h^{22} - h^{11}) \text{ in } S \tag{22}$$

$$\partial_1 Y_2 + \partial_2 Y_2 - 2\Gamma^i_{12} Y_i = -2|g|h^{12} \text{ in } S \tag{23}$$

$$Y_1 = Y_2 = 0 \text{ on } \partial S. \tag{24}$$

One can rewrite this system in complex form with $w = \varphi_1 - i\varphi_2$ and $z = x_1 + ix_2$. Then it becomes the following (overdetermined) generalized Riemann-Hilbert problem:

$$\partial_{\overline{z}} w + Aw + B\overline{w} = F \text{ in } S \tag{25}$$

$$w = 0 \text{ on } \partial S, \tag{26}$$

where A, B, $F \in C^\infty(\overline{S}, \mathbb{C})$ are easily determined from the original system (22), (23). A well posed Riemann-Hilbert problem is obtained replacing the two equations (26) with the single real boundary equation

$$\Re\Big(e^{i\alpha(y)} w(y)\Big) = 0 \text{ for all } y \in \partial S \tag{27}$$

for some $\alpha \in C^\infty(\partial S)$. It is well-known that the operator taking $w \in W^{1,2}(S, \mathbb{C})$ into

$$\Big(\partial_{\overline{z}} w + Aw + B\overline{w}, \Re\Big(e^{i\alpha} w|_{\partial S}\Big)\Big) \in L^2(S) \times H^{1/2}(\partial S)$$

is Fredholm for any such α. (Its index depends on the winding number of $e^{i\alpha}$ along ∂S.) If the generalized Riemann-Hilbert problem (25), (27) admits a unique solution w, then the over-determined problem (25), (26) asserts that w satisfies the equation

$$\Im(e^{i\alpha(y)} w(y)) = 0 \text{ for all } y \in \partial S.$$

Returning to the original variables, this can in principle be written as an equation on ∂S involving the second fundamental form h of u as an unknown.

Proof of Theorem 3.4 (i). Since the hypotheses imply that $u \in W^{2,\infty}$, we can apply Proposition 2.7 with any $p \in (2, \infty)$ to conclude that there exist $\varphi_0^{(n)}$, $Y^{(n)}$ such that (9) is satisfied. At each point on the surface, the orthogonal complement of h is spanned by the quadratic forms q, $\widetilde{q}$ with coordinates

$$(q^{ij}) = \begin{pmatrix} -h_{22} & 0 \\ 0 & h_{11} \end{pmatrix} \text{ and } (\widetilde{q}^{ij}) = \begin{pmatrix} 2h_{12} & -h_{11} \\ -h_{11} & 0 \end{pmatrix}. \tag{28}$$

They are clearly linearly independent at each point and satisfy $\langle h, q\rangle_g = \langle h, \widetilde{q}\rangle_g = 0$ pointwise. Hence scalar multiplying (9) with q and recalling that Γ, $h \in L^\infty$, we find that

$$\langle \mathcal{L}_{Y^{(n)}} g, q\rangle_g \rightharpoonup K_g \langle B, q\rangle_g \tag{29}$$

weakly in $L^2(S)$, and the same for $\widetilde{q}$. In particular, the left-hand sides are uniformly bounded in L^2. In coordinates we have

$$\langle \mathcal{L}_{Y^{(n)}} g, q\rangle_g = -h_{22}(D_1 Y^{(n)})_1 + h_{11}(D_2 Y^{(n)})_2$$

and

$$\langle \mathcal{L}_{Y^{(n)}} g, \widetilde{q} \rangle_g = 2h_{12}(D_1 Y^{(n)})_1 - h_{11}\left((D_1 Y^{(n)})_2 + (D_2 Y^{(n)})_1 \right).$$

Introduce the differential operator $\mathcal{R} : H^1_0(S, \mathbb{R}^2) \to L^2(S, \mathbb{R}^2)$ by setting

$$\mathcal{R}\psi := \begin{pmatrix} -h_{22} & 0 \\ 2h_{12} & -h_{11} \end{pmatrix} \partial_1 \psi + \begin{pmatrix} 0 & h_{11} \\ -h_{11} & 0 \end{pmatrix} \partial_2 \psi + G\psi,$$

where

$$G := \begin{pmatrix} -h_{22} & 0 \\ 2h_{12} & -h_{11} \end{pmatrix} \begin{pmatrix} -\Gamma^1_{11} & -\Gamma^2_{11} \\ -\Gamma^1_{12} & -\Gamma^2_{12} \end{pmatrix} + \begin{pmatrix} 0 & h_{11} \\ -h_{11} & 0 \end{pmatrix} \begin{pmatrix} -\Gamma^1_{21} & -\Gamma^2_{21} \\ -\Gamma^1_{22} & -\Gamma^2_{22} \end{pmatrix}.$$

Define $\psi^{(n)} \in H^1(S, \mathbb{R}^2)$ by setting $\psi^{(n)} = \left((Y^{(n)})_1, (Y^{(n)})_2 \right)^T$. Then by the above observations $\mathcal{R}\psi^{(n)}$ are uniformly bounded in $L^2(S, \mathbb{R}^2)$. But $\mathcal{R}$ is easily seen to be uniformly elliptic, so by standard standard estimates for elliptic systems, there exists a constant C such that the following estimate is satisfied by all $\psi \in H^1_0(S, \mathbb{R}^2)$:

$$\|\nabla \psi\|_{L^2(S, \mathbb{R}^{2\times 2})} \leq C \left(\|\mathcal{R}\psi\|_{L^2(S,\mathbb{R}^2)} + \|\psi\|_{L^2(S,\mathbb{R}^2)} \right). \tag{30}$$

This readily implies that the range of the restriction $\mathcal{R}|_{H^1_0}$ is weakly closed in L^2. Hence (after passing to subsequences) there exists a $\psi \in H^1_0(S, \mathbb{R}^2)$ such that $\mathcal{R}\psi^{(n)} \rightharpoonup \mathcal{R}\psi$ weakly in $L^2(S, \mathbb{R}^2)$. Defining $Y = \psi_i g^{ij} \partial_j u$ and returning to our earlier notation, this means that

$$\langle \mathcal{L}_{Y^{(n)}}, q \rangle_g \rightharpoonup \langle \mathcal{L}_Y, q \rangle_g$$

weakly in L^2, and the same for $\widetilde{q}$. The claim (18) follows by combining this with (29).
Finally, note that the projection of (9) onto h is void, because for given $Y^{(n)}$ it is always possible to find $\varphi_0^{(n)}$ such that the projection of (9) onto h is satisfied, simply because $C^\infty_0(S)$ is dense in $L^2(S)$ and because $|h|$, $|h|^{-1} \in L^\infty(S)$.

□

Recall the definition (12) of the differential operator $\mathcal{M}$:

$$\mathcal{M}\psi = \sqrt{g} div_g (h^{-1} \nabla \psi) + 2H\sqrt{g}\psi.$$

Such an expression has a natural boundary operator $B_{\mathcal{M}}$ corresponding to it which arises in the Green formula. Here we have

$$B_{\mathcal{M}} = \sqrt{g} \nu_i a^{ij} \partial_j. \tag{31}$$

As usual, the symbol $\mathcal{M}^*$ will denote the formal adjoint of $\mathcal{M}$. Observe that $\mathcal{M}$ is formally self-adjoint, i.e. $\mathcal{M} = \mathcal{M}^*$. Nevertheless we will still sometimes use the symbol $\mathcal{M}^*$ instead of $\mathcal{M}$.

Lemma 3.5 *Let $u \in C^\infty_g(\overline{S})$, let $\mathcal{M}$ be given by (12) and $B_{\mathcal{M}}$ by (31). Let $M \in C^\infty(\overline{S})$ and Z_1, Z_2, $N \in C^\infty(\partial S)$. Let the functional $F \in W^{-2,2}(S)$ be given by*

$$F(\psi) = \int_S M\psi \, dx + \int_{\partial S} N\psi \, d\mathcal{H}^1 + \int_{\partial S} Z_i \partial_i \psi \, d\mathcal{H}^1 \text{ for all } \psi \in W^{2,2}(S).$$

If $\varphi \in L^2(S)$ satisfies

$$\mathcal{M}^*\varphi = F \text{ in } W^{-2,2}(S), \tag{32}$$

then $\varphi \in C^\infty(\overline{S})$ and $\mathcal{M}^*\varphi = M$ in S and

$$\int_{\partial S} \Big\{ (N + B_{\mathcal{M}}\varphi)\psi + (Z_i\partial_i\psi - \varphi B_{\mathcal{M}}\psi) \Big\}\, d\mathcal{H}^1 = 0 \text{ for all } \psi \in W^{2,2}(S). \tag{33}$$

Proof. Observe that (32) is an equality in $W^{-2,2}(S)$ and not just in $H^{-2}(S)$. Hence e.g. by extending the problem to a domain containing $\overline{S}$, and using interior elliptic regularity on that larger domain it is not difficult to see that (32) implies that $\varphi \in W^{2,2}(S)$ and that it solves (33). As all data are smooth, standard elliptic regularity on S implies that $\varphi \in C^\infty(\overline{S})$. $\square$

Proof of Theorem 3.4 (ii). Clearly $\mathcal{M}$ is uniformly elliptic. Let $\varphi_n \in C_0^\infty$ be the sequence obtained by Proposition 2.8, and let $F \in W^{-2,2}(S)$ be given by (13). The bounded operator

$$\begin{aligned} T : W^{2,2}(S) &\to L^2(S) \times H^{3/2}(\partial S) \\ \varphi &\mapsto (\mathcal{M}\varphi, \varphi|_{\partial S}) \end{aligned}$$

is Fredholm. Hence its dual

$$T : L^2(S) \times H^{3/2}(\partial S) \to W^{-2,2}(S)$$

is Fredholm, too. It is defined by

$$\begin{aligned} T'(f,b)(\psi) &= \langle (f,b), T\psi \rangle_{L^2(S)\times H^{3/2}(\partial S)} \\ &= \int_S f\mathcal{M}\psi + \langle b, \psi \rangle_{H^{3/2}(\partial S)} \text{ for all } \psi \in W^{2,2}(S). \end{aligned}$$

That is,

$$T'(f,b) = \mathcal{M}^* f + \langle b, \cdot \rangle_{H^{3/2}(\partial S)}\,.$$

Since $\varphi_n \in C_0^\infty(S)$ we have

$$T'(\varphi_n, 0) = \mathcal{M}^*\varphi_n \overset{*}{\rightharpoonup} F \text{ weakly-* in } W^{-2,2}(S).$$

Set $N = \ker T'$. Since T' is Fredholm, its restriction

$$T' : N^\perp \to \operatorname{im} T'$$

is invertible. Hence, denoting by

$$P_N : L^2(S) \times H^{3/2}(\partial S) \to N$$

the orthogonal projection onto N, we have

$$\|(\varphi_n, 0) - P_N(\varphi_n, 0)\|_{L^2(S)\times H^{3/2}(\partial S)} \le C\|T'(\varphi_n, 0)\|_{W^{-2,2}(S)}. \tag{34}$$

As N is finite dimensional, there is $K \in \mathbb{N}$ and there is an $(L^2(S) \times H^{3/2}(\partial S))$-orthonormal basis $\{(f^{(k)}, b^{(k)})\}_{k=1}^K$ of N. So

$$P_N(\varphi_n, 0) = \sum_{k=1}^K a_n^{(k)} (f^{(k)}, b^{(k)}),$$

where

$$a_n^{(k)} = \left\langle (f^{(k)}, b^{(k)}), (\varphi_n, 0) \right\rangle_{L^2(S)\times H^{3/2}(\partial S)} = \left\langle f^{(k)}, \varphi_n \right\rangle_{L^2(S)}.$$

By (34) the sequence

$$\tilde{\varphi}_n := \varphi_n - \sum_{k=1}^{K} a_n^{(k)} f^{(k)}$$

is bounded in $L^2(S)$. And the sequence

$$\tilde{b}_n := \sum_{k=1}^{K} a_n^{(k)} b^{(k)}$$

is bounded in $H^{3/2}(\partial S)$. After passing to subsequences, there is $(\tilde{\varphi}, \tilde{b}) \in L^2(S) \times H^{3/2}(\partial S)$ such that

$$(\tilde{\varphi}_n, \tilde{b}_n) \rightharpoonup (\tilde{\varphi}, \tilde{b}) \text{ weakly in } L^2(S) \times H^{3/2}(\partial S).$$

But since $(\tilde{\varphi}_n, \tilde{b}_n)$ and $(\varphi_n, 0)$ differ by a term in N, we have

$$T'(\tilde{\varphi}_n, \tilde{b}_n) = T'(\varphi_n, 0) \overset{*}{\rightharpoonup} F \text{ weakly-* in } W^{-2,2}(S).$$

Hence

$$T'(\tilde{\varphi}, \tilde{b}) = F \text{ in } W^{-2,2}(S). \tag{35}$$

But $\tilde{b}_n$ is contained in the (finite dimensional) span of $\{b^{(1)}, ..., b^{(K)}\}$. As finite dimensional subspaces are closed, this implies that also $\tilde{b}$ is contained in this span. Since by definition $T'(f^{(k)}, b^{(k)}) = 0$, this implies that there are constants $a^{(1)}, ..., a^{(K)}$ such that

$$T'(0, \tilde{b}) = \sum_{k=1}^{K} a^{(k)} T'(0, b^{(k)}) = T'\left(-\sum_{k=1}^{K} a^{(k)} f^{(k)}, 0\right).$$

Thus (35) can be written in the form

$$F = T'(\tilde{\varphi}, 0) + T'(0, \tilde{b}) = T'\left(\tilde{\varphi} - \sum_{k=1}^{K} a^{(k)} f^{(k)}, 0\right).$$

Setting

$$\varphi := \tilde{\varphi} - \sum_{k=1}^{K} a^{(k)} f^{(k)} \in L^2(S)$$

we therefore see that

$$\mathcal{M}^* \varphi = F \text{ in } W^{-2,2}(S).$$

By definition of F we therefore see that the hypotheses of Lemma 3.5 are satisfied with

$$\begin{aligned} M &= 2\sqrt{g} div_g \left(B \lfloor J(dH)\right) \\ N &= -2\sqrt{g} B\left(\nu, J(dH)\right) \\ Z_i &= 2\sqrt{g} H a^{ik} \tau_k, \end{aligned}$$

where we have introduced the vector field $\tau = J(\nu)$. It is readily seen that $|\tau|_g^2 = 1$ and $\langle \nu, \tau \rangle_g = 0$. Hence τ is a unit tangent vectorfield to the curve $u|_{\partial S}$.

We conclude that (33) is satisfied for Z, N, M as above. Inserting these definitions we find:

$$0 = \int_{\partial S} B\left(\nu, d\varphi - 2J(dH)\right) \psi \ d\mu_{g\partial} + \int_{\partial S} B\left(2H\tau - \varphi\nu, d\psi\right) \ d\mu_{g\partial}. \tag{36}$$

Introducing the vector field

$$X = B\left(2H\tau - \varphi\nu, \tau\right)\tau,$$

the second term on the right-hand side equals

$$\begin{aligned} &\int_{\partial S} B\left(2H\tau - \varphi\nu, \nu\right) \partial_\nu \psi \ d\mu_{g\partial} + \int_{\partial S} \langle X, d\psi \rangle_g \ d\mu_{g\partial} \\ &= \int_{\partial S} B\left(2H\tau - \varphi\nu, \nu\right) \partial_\nu \psi \ d\mu_{g\partial} - \int_{\partial S} div_{g_\partial}(X) \ \psi \ d\mu_{g\partial}. \end{aligned}$$

Here $div_{g_\partial}(X)$ is the intrinsic divergence of the tangent vector field X to $u|_{\partial S}$. Since ψ and $\partial_\nu \psi$ are independent and since $\psi \in W^{2,2}$ is arbitrary, we therefore conclude from (36) that on ∂S we have

$$\begin{aligned} 0 &= B\left(\nu, d\varphi - 2J(dH)\right) - div_{g_\partial}(X) \\ 0 &= B\left(2H\tau - \varphi\nu, \nu\right). \end{aligned}$$

Finally, note that by definition $\frac{1}{\sqrt{g}}\mathcal{M}\varphi = div_g(B\lfloor d\varphi) + 2H\varphi$. □

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