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Degenerate-elliptic operators in mathematical finance and higher-order regularity for solutions to variational equations

by

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DEGENERATE-ELLIPTIC OPERATORS IN MATHEMATICAL FINANCE AND HIGHER-ORDER REGULARITY FOR SOLUTIONS TO VARIATIONAL EQUATIONS

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ABSTRACT. We establish higher-order weighted Sobolev and Hölder regularity for solutions to variational equations defined by the elliptic Heston operator, a linear second-order degenerate-elliptic operator arising in mathematical finance [19]. Furthermore, given C^∞-smooth data, we prove C^∞-regularity of solutions up to the portion of the boundary where the operator is degenerate. In mathematical finance, solutions to obstacle problems for the elliptic Heston operator correspond to value functions for perpetual American-style options on the underlying asset.

CONTENTS

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1. Introduction

Suppose $\mathscr{O} \subsetneqq \mathbb{H}$ is a domain (possibly unbounded) in the open upper half-space $\mathbb{H} := \mathbb{R}^{d-1} \times \mathbb{R}_+$ (where $d \geq 2$ and $\mathbb{R}_+ := (0, \infty)$), and $\partial_1 \mathscr{O} := \partial \mathscr{O} \cap \mathbb{H}$ is the portion of the boundary $\partial \mathscr{O}$ of $\mathscr{O}$ which lies in $\mathbb{H}$, and $\partial_0 \mathscr{O}$ is the interior of $\partial \mathbb{H} \cap \partial \mathscr{O}$, where $\partial \mathbb{H} = \mathbb{R}^{d-1} \times \{0\}$ is the boundary of $\bar{\mathbb{H}} := \mathbb{R}^{d-1} \times \bar{\mathbb{R}}_+$ and $\bar{\mathbb{R}}_+ := [0, \infty)$. We allow $\partial_0 \mathscr{O}$ to be non-empty and consider a second-order, linear elliptic differential operator, A, on $\mathscr{O}$ which is degenerate along $\partial_0 \mathscr{O}$. In this article, when $d = 2$ and the operator A is given by (1.3), we prove higher-order regularity up to the boundary portion, $\partial_0 \mathscr{O}$ — as measured by certain weighted Sobolev spaces, $\mathscr{H}^{k+2}(\mathscr{O}, \mathfrak{w})$ (Definition 4.3), and weighted Hölder spaces, $C_s^{k,2+\alpha}(\bar{\mathscr{O}})$ (Definition 2.15) — for suitably defined *weak* solutions, $u \in H^1(\mathscr{O}, \mathfrak{w})$ (see (2.2) for its definition), to the elliptic boundary value problem,

$$Au = f \quad \text{(a.e.) on } \mathscr{O}, \tag{1.1}$$

$$u = g \quad \text{on } \partial_1 \mathscr{O}, \tag{1.2}$$

where $f : \mathscr{O} \to \mathbb{R}$ is a source function and the function $g : \partial_1 \mathscr{O} \to \mathbb{R}$ prescribes a Dirichlet boundary condition. We denote $\underline{\mathscr{O}} := \mathscr{O} \cup \partial_0 \mathscr{O}$ throughout our article. Furthermore, when $f \in C^\infty(\underline{\mathscr{O}})$, we will also show that $u \in C^\infty(\underline{\mathscr{O}})$ (see Corollary 1.7). Since A becomes degenerate along $\partial_0 \mathscr{O}$, such regularity results do not follow from the standard theory for strictly elliptic differential operators [18, 22].

Because $\kappa\theta > 0$ (see Assumption 1.1 below), *no boundary condition* is prescribed for the equation (1.1) along $\partial_0 \mathscr{O}$. Indeed, we recall from [3] that the problem (1.1), (1.2) is well-posed, given $f \in L^2(\mathscr{O}, \mathfrak{w})$ and $g \in H^1(\mathscr{O}, \mathfrak{w})$ obeying mild pointwise growth conditions, when we seek weak solutions in $H^1(\mathscr{O}, \mathfrak{w})$ or strong solutions in $H^2(\mathscr{O}, \mathfrak{w})$. The *elliptic Heston operator* is defined by

$$Av := -\frac{y}{2}\left(v_{xx} + 2\varrho\sigma v_{xy} + \sigma^2 v_{yy}\right) - \left(c_0 - q - \frac{y}{2}\right) v_x - \kappa(\theta - y) v_y + c_0 v, \quad v \in C^\infty(\mathbb{H}), \tag{1.3}$$

and $-A$ is the generator of the two-dimensional Heston stochastic volatility process with killing [19], a degenerate diffusion process well known in mathematical finance and a paradigm for a broad class of degenerate Markov processes, driven by d-dimensional Brownian motion, and corresponding generators which are degenerate-elliptic integro-differential operators. The coefficients of A are required to satisfy the

Assumption 1.1 (Ellipticity condition for the coefficients of the Heston operator)**.** The coefficients defining A in (1.3) are constants obeying

$$\sigma \neq 0, \quad -1 < \varrho < 1, \tag{1.4}$$

and $\kappa > 0$, $\theta > 0$, $c_0 \geq 0$, and[1] $q \in \mathbb{R}$.

In [12], we proved that a weak solution, $u \in H^1(\mathscr{O}, \mathfrak{w})$, to (1.1), (1.2) is Hölder continuous up to $\partial\mathscr{O}$ in the sense that $u \in C^{\alpha}_{s,\mathrm{loc}}(\bar{\mathscr{O}})$ (Definition 2.9), while in [3], we proved that $u \in H^2(\mathscr{O}, \mathfrak{w})$, for suitable f and g in both cases. In §1.1, we state the main results of our article and set them in context in §1.2, where we provide a survey of previous related research by other authors. We point out some of the mathematical difficulties and issues of broader interest in §1.3. The results of this article may be generalized to a broader class of degenerate-elliptic operators and expected extensions of our results to such a class are discussed in §1.4. We provide a guide in §1.5 to the remainder of this article. We refer the reader to §1.6 for our notational conventions.

1.1. **Summary of main results.** We summarize our main results concerning interior higher-order Sobolev regularity in §1.1.1, while our results on interior higher-order Hölder regularity are given in §1.1.2. Here, our use of the term "interior" is in the sense intended by [4], for example, $U \subset \mathscr{O}$ is an *interior subdomain* of a domain $\mathscr{O} \subseteqq \mathbb{H}$ if $\bar{U} \subset \underline{\mathscr{O}}$ and by "interior regularity" of a function u on $\mathscr{O}$, we mean regularity of u up to $\partial_0\mathscr{O}$ — see Figure 1.1.

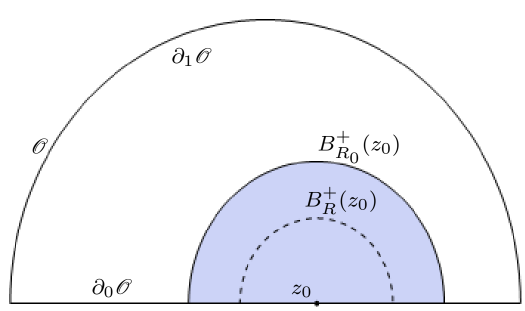

FIGURE 1.1. Boundaries and regions in the statement of Theorem 1.2.

[1] Although q has a financial interpretation as a dividend yield, which is non-negative, our analysis allows $q \in \mathbb{R}$.

1.1.1. *Higher-order interior Sobolev regularity.* We explain in §2.1 how solutions, $u \in H^1(\mathscr{O}, \mathfrak{w})$, to a variational equation (2.11) defined by the operator, A, may be interpreted as weak solutions to (1.1), where the Sobolev space, $H^1(\mathscr{O}, \mathfrak{w})$, is defined in (2.2). See Definitions 4.3 and 4.4 for the descriptions of the weighted Sobolev spaces, $\mathscr{H}^{k+2}(\mathscr{O}, \mathfrak{w})$ and $W^{k,p}(\mathscr{O}, \mathfrak{w})$, respectively.

Theorem 1.2 (Interior $\mathscr{H}^{k+2}$ regularity on half-balls)**.** *Let $R_0 > R$ be positive constants and let $k \geq 0$ be an integer. Then there is a positive constant, $C = C(A, k, R, R_0)$, such that the following holds. Let $\mathscr{O} \subseteqq \mathbb{H}$ be a domain and let $z_0 \in \partial_0\mathscr{O}$ be such that*

$$\mathbb{H} \cap B_{R_0}(z_0) \subset \mathscr{O}.$$

Suppose that $f \in L^2(\mathscr{O}, \mathfrak{w})$ and that $u \in H^1(\mathscr{O}, \mathfrak{w})$ is a solution to the variational equation (2.11)*. If $f \in W^{k,2}(B^+_{R_0}(z_0), \mathfrak{w})$, then*

$$u \in \mathscr{H}^{k+2}(B^+_R(z_0), \mathfrak{w}),$$

and u solves (1.1) *on $B^+_R(z_0)$ and*

$$\|u\|_{\mathscr{H}^{k+2}(B^+_R(z_0),\mathfrak{w})} \leq C \left(\|f\|_{W^{k,2}(B^+_{R_0}(z_0),\mathfrak{w})} + \|u\|_{L^2(B^+_{R_0}(z_0),\mathfrak{w})} \right). \tag{1.5}$$

We also have the following analogues of [18, Theorem 8.10].

Theorem 1.3 (Interior $\mathscr{H}^{k+2}$ regularity on domains)**.** *Let $k \geq 0$ be an integer and let $\mathscr{O} \subseteqq \mathbb{H}$ be a domain. Suppose that $f \in L^2(\mathscr{O}, \mathfrak{w})$ and $u \in H^1(\mathscr{O}, \mathfrak{w})$ is a solution to the variational equation* (2.11)*. If $f \in W^{k,2}_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w})$, then*

$$u \in \mathscr{H}^{k+2}_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w}),$$

and u solves (1.1)*. Moreover, for positive constants $d_1 < \Lambda$ and each pair of subdomains, $\mathscr{O}' \subset \mathscr{O}'' \subset \mathscr{O}$ with $\mathscr{O}' \Subset \underline{\mathscr{O}}''$ and $\operatorname{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O}'') \geq d_1$ and $\operatorname{height}(\mathscr{O}'') \leq \Lambda$, there is a positive constant, $C = C(A, d_1, k, \Lambda)$, such that u obeys*

$$\|u\|_{\mathscr{H}^{k+2}(\mathscr{O}',\mathfrak{w})} \leq C \left(\|f\|_{W^{k,2}(\mathscr{O}'',\mathfrak{w})} + \|u\|_{L^2(\mathscr{O}'',\mathfrak{w})} \right). \tag{1.6}$$

Remark 1.4 (Regularity up to the "non-degenerate boundary")*.* Of course, regarding the conclusion of Theorem 1.3, standard elliptic regularity results for linear, second-order, strictly elliptic operators [18, Theorem 8.13] also imply, when $k \geq 0$, that $u \in W^{k+2,2}_{\mathrm{loc}}(\mathscr{O} \cup \partial_1\mathscr{O})$ if $f \in W^{k,2}_{\mathrm{loc}}(\mathscr{O} \cup \partial_1\mathscr{O})$, and $g \in W^{k+2,2}_{\mathrm{loc}}(\mathscr{O} \cup \partial_1\mathscr{O}) \cap H^1(\mathscr{O}, \mathfrak{w})$, and $\partial_1\mathscr{O}$ is C^{k+2}, and $u - g \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w})$. However, our focus in this article is on regularity of u up to the "degenerate boundary", $\partial_0\mathscr{O}$, so we shall omit further mention of this or other similarly straightforward generalizations.

Finally, we have an analogue of [18, Theorem 8.9].

Theorem 1.5 (Existence and uniqueness of solutions with interior $\mathscr{H}^{k+2}$ regularity)**.** *Let $k \geq 0$ be an integer and let $\mathscr{O} \subseteqq \mathbb{H}$. Suppose that*[2] *$f \in L^\infty(\mathscr{O}) \cap W^{k,2}_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w})$, and $(1+y)g \in W^{2,\infty}(\mathscr{O})$, and the constant, c_0, in* (1.3) *obeys $c_0 > 0$. Then there exists a unique solution $u \in H^1(\mathscr{O}, \mathfrak{w}) \cap \mathscr{H}^{k+2}_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w})$ to the variational equation* (2.11) *with boundary condition $u - g \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w})$. Moreover, u solves* (1.1) *and, for positive constants $d_1 < \Lambda$ and each pair of subdomains, $\mathscr{O}' \subset \mathscr{O}'' \subset \mathscr{O}$ with $\mathscr{O}' \Subset \underline{\mathscr{O}}''$ and $\operatorname{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O}'') \geq d_1$ and $\operatorname{height}(\mathscr{O}'') \leq \Lambda$, there is a positive constant, $C = C(A, d_1, k, \Lambda)$, such that the estimate* (1.6) *holds.*

[2] The hypotheses on f and g are relaxed in [3, 11] to allow for unbounded f and g with suitable growth properties.

1.1.2. *Higher-order interior Hölder regularity.* See Definitions 2.9 and 2.14 for descriptions of the Daskalopoulos-Hamilton family of $C^{k,\alpha}_s$ Hölder norms. We have the following analogue of Theorem 1.2.

Theorem 1.6 (Interior $C^{k,\alpha}_s$ regularity for a solution to the variational equation)**.** *Let $k \geq 0$ be an integer, let $p > \max\{4, 2+k+\beta\}$, and let R_0 be a positive constant. Then there are positive constants $R_1 = R_1(k, R_0) < R_0$, and $C = C(A, k, p, R_0)$, and $\alpha = \alpha(A, k, p, R_0) \in (0,1)$ such that the following holds. Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain. If $f \in L^2(\mathscr{O}, \mathfrak{w})$, and $u \in H^1(\mathscr{O}, \mathfrak{w})$ is a solution to the variational equation* (2.11)*, and $z_0 \in \partial_0\mathscr{O}$ is such that*

$$\mathbb{H} \cap B_{R_0}(z_0) \subset \mathscr{O},$$

and $f \in W^{2k,p}(B^+_{R_0}(z_0), y^{\beta-1})$, then

$$u \in C^{k,\alpha}_s(\bar{B}^+_{R_1}(z_0)),$$

and u solves (1.1) *on $B^+_{R_1}(z_0)$. Moreover, u obeys*

$$\|u\|_{C^{k,\alpha}_s(\bar{B}^+_{R_1}(z_0))} \leq C\left(\|f\|_{W^{2k,p}(B^+_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(B^+_{R_0}(z_0), y^{\beta-1})}\right). \tag{1.7}$$

Given Theorem 1.6, one easily obtains — but via purely Sobolev space and Moser iteration methods — the following degenerate-elliptic analogue of the C^∞-regularity result for the degenerate-parabolic model for the linearization of the porous medium equation [4, Theorem I.1.1].

Corollary 1.7 (Interior C^∞-regularity)**.** *Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain. If $f \in L^2(\mathscr{O}, \mathfrak{w})$ and $u \in H^1(\mathscr{O}, \mathfrak{w})$ is a solution to the variational equation* (2.11)*, and $f \in C^\infty(\underline{\mathscr{O}})$, then $u \in C^\infty(\underline{\mathscr{O}})$ and u solves* (1.1)*.*

We also have an analogue of Theorem 1.3 and of [18, Theorem 6.17].

Theorem 1.8 (Interior $C^{k,\alpha}_s$ regularity on domains)**.** *Let $k \geq 0$ be an integer and let $p > \max\{4, 3+k+\beta\}$. Then there is a positive constant $\alpha = \alpha(A, k, p) \in (0,1)$ such that the following holds. Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain. If $f \in L^2(\mathscr{O}, \mathfrak{w})$ and $u \in H^1(\mathscr{O}, \mathfrak{w})$ is a solution to the variational equation* (2.11)*, and $f \in W^{2k+2,p}_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w})$, then*

$$u \in C^{k,\alpha}_s(\underline{\mathscr{O}}).$$

Moreover, u solves (1.1) *and, for positive constants $d_1 < \Lambda$ and each pair of subdomains, $\mathscr{O}' \subset \mathscr{O}'' \subset \mathscr{O}$ with $\mathscr{O}' \Subset \underline{\mathscr{O}}''$ and* $\operatorname{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O}'') \geq d_1$ *and* [3] *$\mathscr{O}'' \subset (-\Lambda, \Lambda) \times (0, \Lambda)$, there is a positive constant, $C = C(A, d_1, k, p, \Lambda)$, such that*

$$\|u\|_{C^{k,\alpha}_s(\bar{\mathscr{O}}')} \leq C\left(\|f\|_{W^{2k+2,p}(\mathscr{O}'', \mathfrak{w})} + \|u\|_{L^2(\mathscr{O}'', \mathfrak{w})}\right). \tag{1.8}$$

Corollary 1.9 (Interior a priori $C^{k,\alpha}_s$ estimate on domains of finite height)**.** *If in addition to the hypotheses of Theorem 1.8, the hypothesis on f is strengthened to $f \in W^{2k+2,p}(\mathscr{O}, y^{\beta-1})$, then for positive constants $d_1 < \Lambda$ and each pair of subdomains, $\mathscr{O}' \subset \mathscr{O}'' \subset \mathscr{O}$ with $\mathscr{O}' \Subset \underline{\mathscr{O}}''$ and* $\operatorname{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O}'') \geq d_1$ *and* $\operatorname{height}(\mathscr{O}'') \leq \Lambda$*, there is a positive constant, $C = C(A, d_1, k, p, \Lambda)$, such that*

$$\|u\|_{C^{k,\alpha}_s(\bar{\mathscr{O}}')} \leq C\left(\|f\|_{W^{2k+2,p}(\mathscr{O}'', y^{\beta-1})} + \|u\|_{L^2(\mathscr{O}'', y^{\beta-1})}\right). \tag{1.9}$$

[3]While the equation (1.1) is translation-invariant in the x-direction, the estimate (1.8) is not when $\gamma \neq 0$ in the definition (2.5) of the weight, $\mathfrak{w}$.

Remark 1.10 (Regularity up to the "non-degenerate boundary"). Regarding the conclusion of Theorem 1.8, standard elliptic regularity results for linear, second-order, strictly elliptic operators [18, Theorems 6.19 & 9.19] also imply, when $k \geq 0$, that $u \in W^{k+2,p}_{\mathrm{loc}}(\mathscr{O})$ $(C^{k+2,\alpha}(\mathscr{O}))$ if $f \in W^{k,p}_{\mathrm{loc}}(\mathscr{O})$ (respectively, $C^{k,\alpha}(\mathscr{O})$), while if $f \in W^{k,p}_{\mathrm{loc}}(\mathscr{O} \cup \partial_1\mathscr{O})$, and $g \in W^{k+2,p}_{\mathrm{loc}}(\mathscr{O} \cup \partial_1\mathscr{O}) \cap H^1(\mathscr{O}, \mathfrak{w})$, and $\partial_1\mathscr{O}$ is $C^{k+1,1}$ (respectively, $C^{k+2,\alpha}$), and $u - g \in H^1(\mathscr{O}, \mathfrak{w})$, then $u \in W^{k+2,p}_{\mathrm{loc}}(\mathscr{O} \cup \partial_1\mathscr{O})$ (respectively, $C^{k+2,\alpha}(\mathscr{O} \cup \partial_1\mathscr{O})$). As before, because our focus in this article is on regularity of u up to the "degenerate boundary", $\partial_0\mathscr{O}$, we shall omit further mention of such straightforward generalizations.

Lastly, we give an analogue of the existence and interior Schauder a priori estimate results [4, Theorems I.1.1, I.1.2, & I.12.2] for the initial value problem for a degenerate-parabolic model for the linearization of the porous medium equation on a half-space, and of [18, Theorems 6.13 & 6.19], in the case of boundary value problems for strictly elliptic operators. We recall the description of the weighted Hölder space, $C^{k,2+\alpha}_s(\underline{\mathscr{O}})$, due to Daskalopoulos and Hamilton [4], in Definition 2.15. While the interior a priori estimate (1.10) is stated in Theorem 1.11 for bounded subdomains, $\mathscr{O}'' \subset \mathbb{H}$, for the sake of clarity, the estimate should easily extend to unbounded domains using the family of Hölder spaces and Hölder norms defined in [13].

Theorem 1.11 (Existence and uniqueness of solutions with interior $C^{k,2+\alpha}_s$ regularity)**.** *Let $k \geq 0$ be an integer and let K be a finite right-circular cone. Then there is a positive constant $\alpha = \alpha(A, k, K) \in (0, 1)$ such that the following holds. Let $\mathscr{O} \subseteqq \mathbb{H}$ be a domain obeying a uniform exterior cone condition along $\partial_1\mathscr{O}$ with cone K. If $f \in C^{2k+6,\alpha}_s(\underline{\mathscr{O}}) \cap C(\bar{\mathscr{O}})$ and $g \in C(\bar{\mathscr{O}})$ with $(1 + y)g \in C^2(\bar{\mathscr{O}})$, and the constant, c_0, in* (1.3) *obeys $c_0 > 0$, then there is a unique solution,*

$$u \in C^{k,2+\alpha}_s(\underline{\mathscr{O}}) \cap C^{\alpha}(\mathscr{O} \cup \partial_1\mathscr{O}) \cap L^{\infty}(\mathscr{O}),$$

to the boundary value problem, (1.1), (1.2). *Moreover, for positive constants $d_1 < \Lambda$ and each pair of subdomains, $\mathscr{O}' \subset \mathscr{O}'' \subset \mathscr{O}$ with $\mathscr{O}' \Subset \underline{\mathscr{O}}''$ and $\operatorname{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O}'') \geq d_1$ and $\operatorname{diam}(\mathscr{O}'') \leq \Lambda$, there is a positive constant, $C = C(A, d_1, k, p, \Lambda)$, such that*

$$\|u\|_{C^{k,2+\alpha}_s(\bar{\mathscr{O}}')} \leq C \left(\|f\|_{C^{2k+6,\alpha}_s(\bar{\mathscr{O}}'')} + \|u\|_{C(\bar{\mathscr{O}}'')} \right). \tag{1.10}$$

Remark 1.12 (Schauder a priori estimates and approach to existence of solutions). As we explain in [14], the proof of existence of solutions, $u \in C^{k,2+\alpha}_s(\underline{\mathscr{O}}) \cap C(\bar{\mathscr{O}})$, to the boundary value problem, (1.1), (1.2), given $f \in C^{k,\alpha}_s(\underline{\mathscr{O}})$ and $g \in C(\bar{\mathscr{O}})$, is considerably more difficult when $\partial_1\mathscr{O}$ is non-empty because, unlike in [4], one must consider the impact of the "corner" points, $\overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$, of the subdomain, $\mathscr{O} \subset \mathbb{H}$, where the "non-degenerate boundary", $\partial_1\mathscr{O}$, intersects the "degenerate boundary", $\partial\mathbb{H}$.

Remark 1.13 (Refinements of Theorem 1.11). Our existence result and Schauder a priori estimate in Theorem 1.11 may appear far from optimal because of the strong hypothesis that $f \in C^{2k+6,\alpha}_s(\underline{\mathscr{O}})$, the fact that the Hölder exponent, $\alpha \in (0, 1)$, is not arbitrary, and the presence of the cone condition hypothesis. However, the regularity hypothesis for f in Theorem 1.11 may be relaxed to $f \in C^{k,\alpha}_s(\underline{\mathscr{O}})$, with $\alpha \in (0, 1)$ arbitrary, and the cone conditions on $\mathscr{O}$ removed, using an interior Schauder a priori estimate which we develop by quite different methods in [14].

Remark 1.14 (Campanato spaces). In the context of non-degenerate elliptic equations, Campanato spaces [32] provide a natural bridge between Sobolev spaces and Hölder spaces and allow one to prove Schauder a priori estimates and Hölder regularity using Sobolev space methods. It would be interesting to explore whether the conclusions of Theorems 1.6 and 1.8, and thus

Theorem 1.11 in particular, could be sharpened with the aid of a suitable version of Campanato spaces adapted to the weights appearing in our definitions of weighted Sobolev and Hölder spaces.

Given an additional geometric hypothesis on $\mathscr{O}$ near points in $\overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$, the property $u \in C^\alpha(\mathscr{O}\cup\partial_1\mathscr{O})\cap L^\infty(\mathscr{O})$ simplifies to $u \in C^\alpha_{s,\mathrm{loc}}(\bar{\mathscr{O}})\cap C(\bar{\mathscr{O}})$.

Corollary 1.15 (Existence and uniqueness of globally continuous $C^{k,2+\alpha}_s(\underline{\mathscr{O}})$ solutions)**.** *Suppose, in addition to the hypotheses of Theorem 1.11, that the domain, $\mathscr{O}$, satisfies a uniform exterior and interior cone condition on $\overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$ with cone K in the sense of Definition 2.3. Then the solution, u, obeys*

$$u \in C^{k,2+\alpha}_s(\underline{\mathscr{O}})\cap C^\alpha_{s,\mathrm{loc}}(\bar{\mathscr{O}})\cap C(\bar{\mathscr{O}}),$$

and, if $\mathscr{O}$ is bounded, then $u \in C^{k,2+\alpha}_s(\underline{\mathscr{O}})\cap C^\alpha_s(\bar{\mathscr{O}})$.

In a different direction, given additional hypotheses on f, we easily obtain

Corollary 1.16 (Interior a priori $C^{k,2+\alpha}_s$ estimate on domains of finite height)**.** *If in addition to the hypotheses of Theorem 1.11, the hypothesis on f is strengthened to $f \in C^{2k+6,\alpha}_s(\bar{\mathscr{O}})$ then, for positive constants $d_1 < \Lambda$ and each pair of subdomains, $\mathscr{O}' \subset \mathscr{O}'' \subset \mathscr{O}$ with $\mathscr{O}' \Subset \underline{\mathscr{O}}''$ and* $\mathrm{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O}'') \geq d_1$ *and* $\mathrm{height}(\mathscr{O}'') \leq \Lambda$*, there is a positive constant, $C = C(A, d_1, k, p, \Lambda)$, such that*

$$\|u\|_{C^{k,2+\alpha}_s(\bar{\mathscr{O}}')} \leq C\left(\|f\|_{C^{2k+6,\alpha}_s(\bar{\mathscr{O}}'')} + \|u\|_{C(\bar{\mathscr{O}}'')}\right). \tag{1.11}$$

1.2. **Survey of previous related research.** We provide a brief survey of some related research by other authors on regularity theory for solutions to degenerate elliptic and parabolic partial differential equations most closely related to the results described in our article. For a discussion of previous research related to supremum bounds and Hölder continuity near the boundary for weak solutions to the Heston equation, we refer the reader to our article [12].

Naturally, the principal feature which distinguishes the equation (1.1), when the operator A is given by (1.3), from the linear, second-order, strictly elliptic operators in [18], is the fact that A becomes degenerate when $y = 0$ and, because $\kappa\theta > 0$ in (1.3), boundary conditions may be omitted along $y = 0$.

The literature on degenerate elliptic and parabolic equations is vast, with the well-known articles of Fabes, Kenig, and Serapioni [7, 8], Fichera [15, 16], Kohn and Nirenberg [21], Murthy and Stampacchia [27, 28] and the monographs of Levendorskiĭ [24] and Oleĭnik and Radkevič [29, 30, 31], being merely the tip of the iceberg. As far as the authors can tell, however, there has been far less research on higher-order regularity of solutions up to the portion of the domain boundary where the operator becomes degenerate. In this context, the work of Daskalopoulos and her collaborators [4, 5] and of Koch stands out in recent years because of their introduction of the cycloidal metric on the upper-half space, weighted Hölder norms, and weighted Sobolev norms which provide the key ingredients required to unlock the existence, uniqueness, and higher-order regularity theory for solutions to the porous medium equation and the degenerate-parabolic model equation on the upper half-space for the linearization of the porous medium equation.

Koch [20] develops a regularity theory for certain linear, degenerate elliptic and parabolic partial differential operators in divergence form (with a degeneracy similar to that of (1.3)) which serve as models for the linearization of the porous medium equation. However, while Koch uses Sobolev weights which are comparable to ours, his methods — which use Moser iteration and pointwise estimates for fundamental solutions — are very different from those we employ in [12], which use Moser iteration and the abstract John-Nirenberg inequality. Since our approach avoids the use of potential theory and its pointwise estimates, we circumvent any need to consider pointwise estimates for the fundamental solution for the Heston operator (1.3) which, although tantalizingly explicit in its various forms [9, 10, 14, 19], appears quite intractable for the analysis required to emulate the role of the fundamental solution for the Laplace operator in the development of Schauder or L^p theory in [18]. Moreover, the structure of the lower-order terms in the linear operators is simpler in [20], whereas the new u_x term present in (1.3) causes considerable difficulty. Finally, Koch does not consider the case where $\partial\mathscr{O} = \partial_0\mathscr{O} \cup \overline{\partial_1\mathscr{O}}$, where A is degenerate along $\partial_0\mathscr{O}$ but non-degenerate along $\partial_1\mathscr{O}$.

1.3. **Some mathematical highlights of this article.** Our approach in §3 and §4 of our article to the higher-order Sobolev regularity theory for weak solutions to equation (1.1), that is, solutions to the variational equation (2.11), may appear to proceed by adapting a traditional strategy (such as that of [18, §8.3 & §8.4]), but the degeneracy of the operator, A, in (1.3) makes this strategy far more complicated than one might expect from [18].

As we explained in [3], it is surprisingly difficult to improve the $L^2(\mathscr{O},\mathfrak{w})$-estimate for $y^{1/2}Du$ implicit in the a priori $H^1(\mathscr{O},\mathfrak{w})$-estimate for a solution to (2.11) to an $L^2(\mathscr{O},\mathfrak{w})$-estimate for the gradient, Du. For this purpose, we use a trick due to Koch [20, Lemma 4.6.1] which works nicely when the domain, $\mathscr{O}$, is the upper half-plane, $\mathbb{H}$. However, while elementary methods then yield an interior $L^2(\mathscr{O}',\mathfrak{w})$-estimate for Du on a subdomain $\mathscr{O}' \subset \mathscr{O} \subsetneqq \mathbb{H}$ with $\bar{\mathscr{O}}' \subset \underline{\mathscr{O}}$ and $\operatorname{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O}) > 0$, it is unclear how to justify a global $L^2(\mathscr{O},\mathfrak{w})$-estimate for Du on a subdomain $\mathscr{O} \subsetneqq \mathbb{H}$ without a priori knowledge of the existence of smooth solutions, $u \in C^\infty(\bar{\mathscr{O}})$ to the boundary value problem (1.1), (1.2). Therefore, we instead confine our attention in this article to "interior" regularity of solutions to the variational equation (2.11), that is, regularity of such solutions up to the "degenerate boundary", $\partial_0\mathscr{O}$, and defer a discussion of global a priori estimates and regularity of solutions up to $\partial\mathscr{O}$ to later work. While we adapt the finite-difference methods of [18, §8.3 & §8.4] in §3.2 to prove $H^2_{\mathrm{loc}}(\underline{\mathscr{O}},\mathfrak{w})$-regularity of solutions to the variational equation (2.11), finite-difference methods do not extend to give $H^2(\mathscr{O},\mathfrak{w})$-regularity of solutions, u, on neighborhoods of the "corner points", $\overline{\partial_0\mathscr{O}} \cap \overline{\partial_0\mathscr{O}}$, where the degenerate and non-degenerate boundary portions intersect. (As noted earlier, standard methods from [18] give $H^2_{\mathrm{loc}}(\mathscr{O}\cup\partial_1\mathscr{O},\mathfrak{w})$-regularity of solutions to the variational equation (2.11), that is, regularity of solutions up to the non-degenerate boundary, $\partial_1\mathscr{O}$.)

While the essential idea in §4 underlying the development of higher-order Sobolev regularity, $\mathscr{H}^{k+2}_{\mathrm{loc}}(\underline{\mathscr{O}},\mathfrak{w})$ with $k \geq 1$, of a solution $u \in H^1(\mathscr{O},\mathfrak{w})$ to the variational equation (2.11) is to take derivatives of the equation (1.1) and estimate $k+2$ derivatives of u in terms of k derivatives of f, such an approach is complicated by the presence of the degeneracy factor, y, multiplying the second-order derivatives, $yu_{xx}, yu_{xy}, yu_{yy}$, in (1.3). For example, differentiating (1.1) once with respect to y yields unweighted second-order derivative terms, u_{xx}, u_{xy}, u_{yy}, *without* the degeneracy factor, y, and these are even harder to estimate, precisely because the operator, A, in (1.3) is degenerate elliptic. It is this feature which partly accounts for the complexity of our Definition 4.3 of the higher-order weighted Sobolev spaces, $\mathscr{H}^{k+2}(\mathscr{O},\mathfrak{w})$.

Naturally, the same difficulty arises in §5 when we consider higher-order Hölder regularity, $C^{k,\alpha}_s(\underline{\mathscr{O}})$ with $k \geq 1$ and $C^{k,2+\alpha}_s(\underline{\mathscr{O}})$, of a solution $u \in H^1(\mathscr{O},\mathfrak{w})$. However, at this stage, the difficulties have largely been overcome in §4. While it may be unorthodox to prove higher-order Hölder regularity, Schauder a priori estimates, and Schauder existence results parallel to those of [18, §6.1 & §6.3] using a variational approach, it is not without precedent as illustrated by previous applications of Campanato spaces in the context of linear, second order, strictly elliptic operators [32].

As we explain in more detail in [14], it is a challenging problem to prove existence of solutions, u — whether in $C^{k,2+\alpha}_s(\underline{\mathscr{O}})\cap C(\bar{\mathscr{O}})$ or $C^{k,2+\alpha}_s(\bar{\mathscr{O}})$ — to the elliptic boundary value problem (1.1), (1.2) entirely within a Schauder framework parallel to that of [4] (where boundary conditions such as (1.2) along the "fixed boundary" do not arise since $\partial_1\mathscr{O}$ is empty in their application) and this motivates the variational approach which we employ here. One reason for the difficulty is due to complications which emerge when one attempts to apply the continuity method to prove existence of solutions $u \in C^{k,2+\alpha}_s(\bar{\mathscr{O}})$ to (1.1), (1.2), say with $g = 0$ on $\partial_1\mathscr{O}$, by analogy with the method of proof of [18, Theorem 6.8]. While the reflection principle (across the axis $x = 0$) does not hold for the operator, A, in (1.3), it does hold for the simpler model operator,

$$A_0 v := -\frac{y}{2}\left(v_{xx} + \sigma^2 v_{yy}\right) - \kappa(\theta - y)v_y + c_0 v, \quad v \in C^\infty(\mathbb{H}),$$

since the v_{xy} and v_x terms are absent and so, provided f obeys $f(-x,y) = -f(x,y)$ for $(x,y) \in \mathbb{H}$ (and thus $f(0,\cdot) = 0$), one can solve

$$A_0 u_0 = f \quad \text{on } \mathscr{O}, \quad u = 0 \quad \text{on } \partial_1\mathscr{O},$$

for a solution, u_0, when the domain, $\mathscr{O}$, is the quadrant $\mathbb{R}_+ \times \mathbb{R}_+$.

However, if $u \in C^{2+\alpha}_s(\bar{\mathscr{O}})$ solves (1.1), (1.2) then, letting $y \to 0$ in (1.1), we find that

$$-(c_0 - q)u_x(0,0) = f(0,0),$$

since $u_y(0,0) = 0$ (because $u(0,\cdot) = 0$) and as $u \in C^{2+\alpha}_s(\bar{\mathscr{O}})$ implies $\lim_{(x,y)\to(0,0)} yD^2u = 0$ [4, 13]. Hence, when $c_0 - q = 0$, we see that we can only solve (1.1), (1.2) when f obeys the compatibility condition $f(0,0) = 0$, whereas this compatibility condition for f is not present when $c_0 - q \neq 0$.

Furthermore, we can only use the method of continuity to produce a solution $u \in C^{k,2+\alpha}_s(\bar{\mathscr{O}})$ when we already have a global a priori Schauder estimate analogous to that of [18, Theorem 6.6] and developing such an estimate is a challenging problem, albeit one we address elsewhere. Finally, the continuity method in the proof of [18, Theorem 6.8] is justified because the first-order derivative terms in linear, second-order, strictly elliptic operators with variable coefficients can be treated as lower-order terms due to interpolation inequalities [18, Lemma 6.35]. In the case of the Heston operator (and operators with similar structure), the first-order derivative terms cannot be treated as lower-order, as we can observe from the interpolation inequality [13, Lemma 3.2 & Equation (3.8)].

1.4. **Extensions.** The Heston stochastic volatility process and its associated generator serve as paradigms for degenerate Markov processes and their degenerate-elliptic generators which appear widely in mathematical finance, so we briefly comment on two directions for extending our work in this article.

1.4.1. *Degenerate elliptic and parabolic operators in higher dimensions.* Generalizations of the Heston process to higher-dimensional, degenerate diffusion processes may be accommodated by extending the framework developed in this article and we shall describe extensions in a sequel. First, the two-dimensional Heston process has natural d-dimensional analogues [17] defined, for example, by coupling non-degenerate $(d-1)$-diffusion processes with degenerate one-dimensional processes [2, 26, 34]. Elliptic differential operators arising in this way have time-independent, affine coefficients but, as one can see from standard theory [18, 22, 23, 25] and previous work of Daskalopoulos and her collaborators [4, 5] on the porous medium equation, we would not expect significant new difficulties to arise when extending the methods and results of this article to the case of elliptic and parabolic operators in higher dimensions and variable coefficients, depending on both spatial variables or time and possessing suitable regularity and growth properties.

Specifically, as we explain further in Remark 2.8, we expect that all of the main results of this article should extend to the case of a degenerate-elliptic operator on a subdomain $\mathscr{O}$ of a half-space $\mathbb{H} = \mathbb{R}^{d-1} \times \mathbb{R}_+$,

$$Av := -x_d a^{ij} v_{x_i x_j} - b^i v_{x_i} + cv, \quad v \in C^\infty(\mathscr{O}), \tag{1.12}$$

under the assumptions that the matrix (a^{ij}) is strictly elliptic, $b^d \geq \nu > 0$, for some constant $\nu > 0$, and $c \geq 0$ and the coefficients (a^{ij}), (b^i), and c have suitable growth and regularity properties. See [13] for an analysis with applications to probability theory based on a parabolic version of this type of elliptic operator as well as [11] for weak maximum principles for a general class of degenerate-elliptic operators.

1.4.2. *Regularity near the fixed boundary and global a priori estimates.* The important question of higher-order regularity for solutions u to the elliptic boundary value problem (1.1), (1.2) — for example, whether solutions belong to $C_s^{k,2+\alpha}(\bar{\mathscr{O}})$ when f belongs to $C_s^{k,\alpha}(\bar{\mathscr{O}})$ and g belongs to $C_s^{k,2+\alpha}(\bar{\mathscr{O}})$ and $\mathscr{O}$ is bounded with $\partial_1\mathscr{O}$ of class $C^{k+2,\alpha}$ and $C^{k,2+\alpha}$-transverse to $\partial\mathbb{H}$ — is one which we defer to a subsequent article.

1.5. **Outline of the article.** For the convenience of the reader, we provide a brief outline of the article. In §2, we review local supremum estimates and local $C_s^\alpha(\underline{\mathscr{O}})$-regularity results for solutions, $u \in H^1(\mathscr{O}, \mathfrak{w})$, to the variational equation (2.11), and which we proved in [12]. In §3, we establish the $H^2(\underline{\mathscr{O}}, \mathfrak{w})$-regularity for solutions, $u \in H^1(\mathscr{O}, \mathfrak{w})$, concluding with Theorem 3.16. In §4, we establish the $\mathscr{H}^{k+2}(\underline{\mathscr{O}}, \mathfrak{w})$-regularity for solutions, $u \in H^1(\mathscr{O}, \mathfrak{w})$, for all $k \geq 0$, with proofs of Theorems 1.2 and 1.3, together with Theorem 1.5 and Corollary 1.7. Section 5 contains our proofs of $C_s^{k,\alpha}(\underline{\mathscr{O}})$-regularity of solutions, $u \in H^1(\mathscr{O}, \mathfrak{w})$, in the form of Theorems 1.6 and 1.8, together with proofs of $C_s^{k,2+\alpha}(\underline{\mathscr{O}})$-regularity and a Schauder a priori estimate, as part of Theorem 1.11, and Corollary 1.15. Appendix A collects some useful facts from our earlier articles, together with proofs of a more technical nature.

1.6. **Notation and conventions.** In the definition and naming of function spaces, including spaces of continuous functions, Hölder spaces, or Sobolev spaces, we follow Adams [1] and alert the reader to occasional differences in definitions between [1] and standard references such as Gilbarg and Trudinger [18] or Krylov [22, 23].

We let $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$ denote the set of non-negative integers. For $r > 0$ and $P_0 = (x_0, y_0) \in \mathbb{R}^2$, we let $B_r(P_0) := \{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 < r^2\}$ denote the open ball with center P_0 and radius r and, given a domain $\mathscr{O} \subset \mathbb{R}^2$, we denote $B_r^+(P_0) := \mathscr{O} \cap B_r(P_0)$, when the domain $\mathscr{O}$ is understood from the context.

If $V \subset U \subset \mathbb{R}^d$ are open subsets, we write $V \Subset U$ when U is bounded with closure $\bar{U} \subset V$. By $\operatorname{supp}\zeta$, for any $\zeta \in C(\mathbb{R}^2)$, we mean the *closure* in $\mathbb{R}^2$ of the set of points where $\zeta \neq 0$.

We use $C = C(*, \ldots, *)$ to denote a constant which depends at most on the quantities appearing on the parentheses. In a given context, a constant denoted by C may have different values depending on the same set of arguments and may increase from one inequality to the next. We let $C(A)$, $C(A, *)$, and so on, denote constants which may depend on one or more of the constant coefficients of the operator A (that is, $c_0, q, \kappa, \theta, \varrho, \sigma$).

2. Review of supremum estimates and Hölder regularity results

We describe the main results in [12] concerning boundedness and C_s^α Hölder regularity of "weak solutions" to (1.1) (and (1.2)). In §2.1, we review our definition of the variational equation (2.11) corresponding to (1.2), together with the required Sobolev spaces. In §2.2, we recall our local supremum estimates for solutions, $u \in H^1(\mathscr{O},\mathfrak{w})$, to the variational equation (2.11) which we proved in [12], while in §2.3, we review our Hölder continuity results for solutions, $u \in H^1(\mathscr{O},\mathfrak{w})$, which we also proved in [12]. Finally, in §2.4, we give the definitions of higher-order weighted Hölder spaces due to Daskalopoulos and Hamilton [4].

2.1. **Preliminaries.** We review our definitions of weighted Sobolev spaces from [3, Definition 2.20]. For $1 \leq p < \infty$, let

$$L^p(\mathscr{O},\mathfrak{w}) := \{u \in L^1_{\mathrm{loc}}(\mathscr{O}) : \|u\|_{L^p(\mathscr{O},\mathfrak{w})} < \infty\}, \tag{2.1}$$

$$H^1(\mathscr{O},\mathfrak{w}) := \{u \in W^{1,2}_{\mathrm{loc}}(\mathscr{O}) : \|u\|_{H^1(\mathscr{O},\mathfrak{w})} < \infty\}, \tag{2.2}$$

where

$$\|u\|^p_{L^p(\mathscr{O},\mathfrak{w})} := \int_{\mathscr{O}} |u|^p \mathfrak{w}\, dx\, dy, \tag{2.3}$$

$$\|u\|^2_{H^1(\mathscr{O},\mathfrak{w})} := \int_{\mathscr{O}} \left(y|Du|^2 + (1+y)u^2\right) \mathfrak{w}\, dx\, dy, \tag{2.4}$$

with weight function $\mathfrak{w} : \mathbb{H} \to (0,\infty)$ given by[4]

$$\mathfrak{w}(x,y) := y^{\beta-1} e^{-\gamma\sqrt{1+x^2}-\mu y}, \quad (x,y) \in \mathbb{H}, \tag{2.5}$$

where

$$\beta := \frac{2\kappa\theta}{\sigma^2} \quad \text{and} \quad \mu := \frac{2\kappa}{\sigma^2}, \tag{2.6}$$

and $0 < \gamma < \gamma_0(A)$, where γ_0 depends only on the constant coefficients of A in (1.3). We denote $H^0(\mathscr{O},\mathfrak{w}) = L^2(\mathscr{O},\mathfrak{w})$.

Remark 2.1 (Role of γ). In [3] and [11], we require the constant, γ, in (2.5) to be positive for the purpose of proving existence and uniqueness, respectively, for solutions to (2.11) when $\mathscr{O}$ is unbounded. However, while we shall continue to assume $\gamma > 0$ in this article for consistency, this constant plays no role in regularity arguments or when $\mathscr{O}$ is bounded and, for the latter purposes, one could set $\gamma = 0$.

We recall that [3, Definition 2.22]

$$\begin{aligned}
\mathfrak{a}(u,v) := \frac{1}{2}\int_{\mathscr{O}} &\left(u_x v_x + \varrho\sigma u_y v_x + \varrho\sigma u_x v_y + \sigma^2 u_y v_y\right) y\, \mathfrak{w}\, dx\, dy \\
&- \frac{\gamma}{2}\int_{\mathscr{O}} (u_x + \varrho\sigma u_y)\, v\, \frac{x}{\sqrt{1+x^2}} y\, \mathfrak{w}\, dx\, dy \\
&- \int_{\mathscr{O}} (a_1 y + b_1) u_x v\, \mathfrak{w}\, dx\, dy + \int_{\mathscr{O}} c_0 uv\, \mathfrak{w}\, dx\, dy, \quad \forall u,v \in H^1(\mathscr{O},\mathfrak{w}),
\end{aligned} \tag{2.7}$$

is the *bilinear form associated with the Heston operator*, A, in (1.3), where

$$a_1 := \frac{\kappa\varrho}{\sigma} - \frac{1}{2} \quad \text{and} \quad b_1 := c_0 - q - \frac{\kappa\theta\varrho}{\sigma}. \tag{2.8}$$

We shall also avail of the

Assumption 2.2 (Condition on the coefficients of the Heston operator)**.** The coefficients defining A in (1.3) have the property that $b_1 = 0$ in (2.8).

[4]In [3], we used the equivalent factor, $|x|$, but we use $\sqrt{1+x^2}$ here since the resulting weight is in $C^\infty(\mathbb{H})$.

Assumption 2.2 involves no loss of generality because, using a simple affine changes of variables on $\mathbb{R}^2$ which map $(\mathbb{H}, \partial\mathbb{H})$ onto $(\mathbb{H}, \partial\mathbb{H})$ (see [3, Lemma 2.2]), we can arrange that $b_1 = 0$. The conditions (1.4) ensure that $y^{-1}A$ is uniformly and strictly elliptic on $\mathbb{H}$. Indeed,

$$\frac{y}{2}(\xi_1^2 + 2\varrho\sigma\xi_1\xi_2 + \sigma^2\xi_2^2) \geq \nu_0 y(\xi_1^2 + \xi_2^2), \quad \forall(\xi_1, \xi_2) \in \mathbb{R}^2, \tag{2.9}$$

where

$$\nu_0 := \frac{1}{2}(1 - |\varrho|)\min\{1, \sigma^2\}, \tag{2.10}$$

and $\nu_0 > 0$ by Assumption 1.1.

Given a subset $T \subset \partial\mathscr{O}$, we let $H^1_0(\mathscr{O} \cup T, \mathfrak{w})$ be the closure[5] in $H^1(\mathscr{O}, \mathfrak{w})$ of $C^\infty_0(\mathscr{O} \cup T)$. Given a source function $f \in L^2(\mathscr{O}, \mathfrak{w})$ and recalling that $\underline{\mathscr{O}} = \mathscr{O} \cup \partial_0\mathscr{O}$, we call a function $u \in H^1(\mathscr{O}, \mathfrak{w})$ a *solution to the variational equation* for the Heston operator if

$$\mathfrak{a}(u, v) = (f, v)_{L^2(\mathscr{O}, \mathfrak{w})}, \quad \forall v \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w}). \tag{2.11}$$

Given a subset $T \subset \partial\mathscr{O}$ and $g \in H^1(\mathscr{O}, \mathfrak{w})$, we say that $u \in H^1(\mathscr{O}, \mathfrak{w})$ obeys $u = g$ *on* $T \subset \partial\mathscr{O}$ *in the sense of* H^1 if

$$u - g \in H^1_0(\mathscr{O} \cup T^c, \mathfrak{w}), \tag{2.12}$$

where $T^c := \partial\mathscr{O} \setminus T$. In our application, we shall only consider $T \subsetneqq \partial_1\mathscr{O}$. If $u \in H^2(\mathscr{O}, \mathfrak{w})$ (see (3.9) for its definition) and $g \in H^1(\mathscr{O}, \mathfrak{w})$, we recall from [3, Lemma 2.29] that u is a solution to (1.1) (a.e. on $\mathscr{O}$) and (1.2) (in the sense of H^1) if and only if $u - g \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w})$ and u is a solution to the variational equation (2.11).

2.2. **Local supremum bounds near the degenerate boundary.** We say that a domain, $U \subset \mathbb{H}$, obeys an *exterior cone condition relative to* $\mathbb{H}$ *at a point* $z_0 \in \partial U$ if there exists a finite, right circular cone $K = K_{z_0} \subset \bar{\mathbb{H}}$ with vertex z_0 such that $\bar{U} \cap K_{z_0} = \{z_0\}$ (compare [18, p. 203]). A domain, U, obeys a *uniform exterior cone condition relative to* $\mathbb{H}$ *on* $T \subset \partial U$ if U satisfies an exterior cone condition relative to $\mathbb{H}$ at every point $z_0 \in T$ and the cones K_{z_0} are all congruent to some fixed finite cone, K (compare [18, p. 205]).

Definition 2.3 (Interior and exterior cone conditions)**.** Let K be a finite, right circular cone. We say that $\mathscr{O}$ obeys *interior and exterior cone conditions at* $z_0 \in \overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$ *with cone* K if the domains $\mathscr{O}$ and $\mathbb{H} \setminus \bar{\mathscr{O}}$ obey exterior cone conditions relative to $\mathbb{H}$ at z_0 with cones congruent to K. We say that $\mathscr{O}$ obeys *uniform interior and exterior cone conditions on* $\overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$ *with cone* K if the domains $\mathscr{O}$ and $\mathbb{H} \setminus \bar{\mathscr{O}}$ obey exterior cone conditions relative to $\mathbb{H}$ at each point $z_0 \in \overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$ with cones congruent to K.

In the statement of the supremum estimates, we use the following

Definition 2.4 (Volume of sets)**.** If $S \subset \bar{\mathbb{H}}$ is a Borel measurable subset, we let $|S|_\beta$ denote the volume of S with respect to the measure $y^\beta\, dx\, dy$, and $|S|_{\mathfrak{w}}$ denote the volume of S with respect to the measure $\mathfrak{w}\, dx\, dy$.

We recall from [12] the following analogues of [20, Proposition 4.5.1] and [18, Theorem 8.15]; we have reformulated the results here in terms of Euclidean balls in order to make them more readily applicable in our present article (see Appendix A.4 for details).

[5]Note that $H^1_0(\mathscr{O} \cup \bar{T}, \mathfrak{w}) = H^1_0(\mathscr{O} \cup T, \mathfrak{w}) = H^1_0(\mathscr{O} \cup \mathring{T}, \mathfrak{w})$, since $C^\infty_0(\mathscr{O} \cup \bar{T}) = C^\infty_0(\mathscr{O} \cup T) = C^\infty_0(\mathscr{O} \cup \mathring{T})$, where $\mathring{T}$ and $\bar{T}$ denote the interior and closure, respectively, of T in $\partial\mathscr{O}$.

Theorem 2.5 (Supremum estimates near points in $\partial_0\mathscr{O}$)**.** [12, Theorem 1.7 & Remark 1.10] *Let $p > 2+\beta$ and let R_0 be a positive constant. Then there are positive constants, $C = C(A, p, R_0)$ and $R_1 = R_1(R_0) < R_0$, such that the following holds. Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain. If $u \in H^1(\mathscr{O}, \mathfrak{w})$ satisfies the variational equation* (2.11) *with source function $f \in L^2(\mathscr{O}, \mathfrak{w})$, and $z_0 \in \partial_0\mathscr{O}$ is such that*

$$\mathbb{H} \cap B_{R_0}(z_0) \subset \mathscr{O},$$

and f obeys

$$f \in L^p(B^+_{R_0}(z_0), y^{\beta-1}), \tag{2.13}$$

then $u \in L^\infty(B^+_{R_1}(z_0))$, and

$$\|u\|_{L^\infty(B^+_{R_1}(z_0))} \leq C\left(\|f\|_{L^p(B^+_{R_0}(z_0),y^{\beta-1})} + \|u\|_{L^2(B^+_{R_0}(z_0),y^{\beta-1})}\right). \tag{2.14}$$

Theorem 2.6 (Supremum estimates near points in $\overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$)**.** [12, Theorem 1.7] *Let K be a finite right circular cone, let $p > 2+\beta$, and let $R_0 > 0$ be a positive constant. Then there are positive constants, $C = C(A, K, p, R_0)$ and $R_1 = R_1(K, R_0)$, such that the following holds. Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain. If $u \in H^1(\mathscr{O}, \mathfrak{w})$ satisfies the variational equation* (2.11) *with source function $f \in L^2(\mathscr{O}, \mathfrak{w})$ and $z_0 \in \overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$ is such that $\mathscr{O}$ obeys an interior cone condition at z_0 with cone K, and*

$$u = 0 \text{ on } \partial_1\mathscr{O} \cap B_{R_0}(z_0) \quad \text{(in the sense of } H^1\text{)},$$

and f obeys (2.13)*, then $u \in L^\infty(B^+_{R_1}(z_0))$ and u satisfies* (2.14).

Remark 2.7 (Use of the weight $y^{\beta-1}$ versus $\mathfrak{w}$ in Theorems 2.5 and 2.6). Notice that on the right-hand-side of estimate (2.14) we have $\|f\|_{L^p(B_{R_0}(z_0),y^{\beta-1})}$ instead of $\|f\|_{L^p(B_{R_0}(z_0),\mathfrak{w})}$. This allows us to conclude that the constant C appearing in (2.14) is independent of the point $z_0 \in \partial_0\mathscr{O}$. By (2.5), the weight $\mathfrak{w}$ contains the factor $e^{-\gamma\sqrt{1+x^2}}$, which means that the constant C will depend on the x-coordinate of the point $z_0 \in \partial_0\mathscr{O}$, if we replace $\|f\|_{L^p(B_{R_0}(z_0),y^{\beta-1})}$ by $\|f\|_{L^p(B_{R_0}(z_0),\mathfrak{w})}$ on the right-hand-side of (2.14).

Remark 2.8 (Supremum estimates and Hölder regularity in higher dimensions). Theorems 2.5, 2.6, 2.10, 2.11, 2.12, and 2.13 are stated for the case $d = 2$. However, as we noted in [12], when $d > 2$ and the operator A in (1.3) is replaced by one of the form (1.12), then the conclusions of these theorems (and hence their consequences in this article) remain valid, with virtually no change in the proofs, expect for slight changes in the hypotheses. For example, when $d > 2$, the hypothesis $p > 2 + \beta$ in Theorems 2.5 and 2.6 is replaced by $p > d + \beta$ and the constants C and R_1 now additionally depend on d. Similarly, in Theorems 2.10, 2.11, 2.12, and 2.13, the hypothesis $p > \max\{4, 2+\beta\}$ is replaced by $p > \max\{2d, d+\beta\}$ and the constants C, R_1, and α now additionally depend on d.

2.3. **Hölder continuity up to the degenerate boundary for solutions to the variational equation.** We recall the definition of the *Koch distance function*, $s(\cdot,\cdot)$, on $\mathbb{H}$ introduced by Koch in [20, p. 11],

$$s(z, z_0) := \frac{|z - z_0|}{\sqrt{y + y_0 + |z - z_0|}}, \quad \forall z = (x, y), z_0 = (x_0, y_0) \in \bar{\mathbb{H}}, \tag{2.15}$$

where $|z - z_0|^2 = (x - x_0)^2 + (y - y_0)^2$. The Koch distance function is equivalent to the *cycloidal distance function* introduced by Daskalopoulos and Hamilton in [4, p. 901] for the study of the porous medium equation. For $r>0$ and $z_0 \in \bar{\mathscr{O}}$, we define the corresponding *cycloidal balls* and their intersections with subdomains of $\mathbb{H}$ by

$$\begin{aligned} \mathscr{B}_r(z_0) &:= \{z \in \mathbb{H} : s(z,z_0) < r\}, \\ \mathscr{B}_r^+(z_0) &:= \mathscr{O} \cap \mathscr{B}_r(z_0), \end{aligned} \tag{2.16}$$

while $\bar{\mathscr{B}}_r(z_0) = \{z \in \bar{\mathbb{H}} : s(z,z_0) \leq r\}$ and $\bar{\mathscr{B}}_r^+(z_0) = \bar{\mathscr{O}} \cap \bar{\mathscr{B}}_r(z_0)$ denote the closures these subsets in $\bar{\mathscr{O}}$ and $\bar{\mathbb{H}}$, respectively.

Observe that

$$s(z,z_0) \leq |z-z_0|^{1/2}, \quad \forall z, z_0 \in \bar{\mathbb{H}}, \tag{2.17}$$

and thus

$$\mathbb{H} \cap B_{r^2}(z_0) \subset \mathscr{B}_r(z_0), \quad \forall z_0 \in \bar{\mathbb{H}},\ r>0. \tag{2.18}$$

The reverse inequality and inclusion take their simplest form when $y_0 = 0$, in which case the inequalities $y \leq |z-z_0|$ and

$$|z-z_0| = s(z,z_0)\sqrt{y+|z-z_0|} \leq s(z,z_0)\sqrt{2|z-z_0|},$$

give

$$|z-z_0| \leq 2s(z,z_0)^2, \quad \forall z \in \bar{\mathbb{H}},\ z_0 \in \partial\mathbb{H}, \tag{2.19}$$

and

$$\mathscr{B}_r(z_0) \subset \mathbb{H} \cap B_{2r^2}(z_0), \quad \forall z_0 \in \partial\mathbb{H},\ r>0. \tag{2.20}$$

(Analogues of (2.19) and (2.20) when $y_0 > 0$ are given in Appendix A.3.)

Following [1, §1.26], for a domain $U \subset \mathbb{H}$, we let $C(U)$ denote the vector space of continuous functions on U and let $C(\bar{U})$ denote the Banach space of functions in $C(U)$ which are bounded and uniformly continuous on U, and thus have unique bounded, continuous extensions to $\bar{U}$, with norm

$$\|u\|_{C(\bar{U})} := \sup_U |u|.$$

Noting that U may be *unbounded*, we let $C_{\mathrm{loc}}(\bar{U})$ denote the linear subspace of functions $u \in C(U)$ such that $u \in C(\bar{V})$ for every precompact open subset $V \Subset \bar{U}$. Daskalopoulos and Hamilton provide the

Definition 2.9 (C_s^α norm and Banach space)**.** [4, p. 901] Given $\alpha \in (0,1)$ and a domain $U \subset \mathbb{H}$, we say that $u \in C_s^\alpha(\bar{U})$ if $u \in C(\bar{U})$ and

$$\|u\|_{C_s^\alpha(\bar{U})} < \infty,$$

where

$$\|u\|_{C_s^\alpha(\bar{U})} := [u]_{C_s^\alpha(\bar{U})} + \|u\|_{C(\bar{U})}, \tag{2.21}$$

and

$$[u]_{C_s^\alpha(\bar{U})} := \sup_{\substack{z_1,z_2 \in U \\ z_1 \neq z_2}} \frac{|u(z_1)-u(z_2)|}{s(z_1,z_2)^\alpha}. \tag{2.22}$$

We say that $u \in C_s^\alpha(\underline{U})$ if $u \in C_s^\alpha(\bar{V})$ for all precompact open subsets $V \Subset \underline{U}$, recalling that $\underline{U} := U \cup \partial_0 U$. We let $C_{s,\mathrm{loc}}^\alpha(\bar{U})$ denote the linear subspace of functions $u \in C_s^\alpha(U)$ such that $u \in C_s^\alpha(\bar{V})$ for every precompact open subset $V \Subset \bar{U}$.

It is known that $C_s^\alpha(\bar{U})$ is a Banach space [4, §I.1] with respect to the norm (2.21). We recall the following analogues of [18, Theorem 8.27 & 8.29] and [20, Theorem 4.5.5 & 4.5.6].

Theorem 2.10 (Hölder continuity near points in $\partial_0\mathscr{O}$ for solutions to the variational equation)**.** [12, Theorem 1.11] *Let $p > \max\{4, 2+\beta\}$ and let R_0 be a positive constant. Then there are positive constants, $R_1 = R_1(R_0) < R_0$, and $C = C(A,p,R_0)$, and $\alpha = \alpha(A,p,R_0) \in (0,1)$ such that the following holds. Let $\mathscr{O} \subseteqq \mathbb{H}$ be a domain. If $u \in H^1(\mathscr{O},\mathfrak{w})$ satisfies the variational equation* (2.11) *with source function $f \in L^2(\mathscr{O},\mathfrak{w})$ and $z_0 \in \partial_0\mathscr{O}$ is such that*

$$\mathbb{H} \cap B_{R_0}(z_0) \subset \mathscr{O},$$

and f obeys (2.13)*, then $u \in C^\alpha_s(\bar{B}^+_{R_1}(z_0))$, and*

$$[u]_{C^\alpha_s(\bar{B}^+_{R_1}(z_0))} \leq C\left(\|f\|_{L^p(B^+_{R_0}(z_0),y^{\beta-1})} + \|u\|_{L^\infty(B^+_{R_0}(z_0))}\right). \tag{2.23}$$

Theorem 2.11 (Hölder continuity near points in $\overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$ for solutions to the variational equation)**.** [12, Theorem 1.11] *Let K be a finite, right circular cone, let $p > \max\{4, 2+\beta\}$, and let R_0 be a positive constant. Then there are positive constants, $R_1 = R_1(K,R_0) < R_0$, and $C = C(A,K,R_0,p)$, and $\alpha = \alpha(A,K,p,R_0) \in (0,1)$, such that the following holds. Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain. If $u \in H^1(\mathscr{O},\mathfrak{w})$ satisfies the variational equation* (2.11) *with source function $f \in L^2(\mathscr{O},\mathfrak{w})$ and $z_0 \in \overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$ is such that f obeys* (2.13)*, and*

$$u = 0 \text{ on } \partial_1\mathscr{O} \cap B_{R_0}(z_0) \quad \text{(in the sense of } H^1\text{)},$$

and $\mathscr{O}$ obeys an interior and exterior cone condition with cone K at z_0 and a uniform exterior cone condition with cone K along $\overline{\partial_1\mathscr{O}} \cap B_{R_0}(z_0)$, then $u \in C^\alpha_s(\bar{B}^+_{R_1}(z_0))$ and satisfies (2.23)*.*

Theorems 2.10 and 2.11 are not stated in the form we need for our application, since the estimate (2.23) has an L^∞ rather than an L^2 norm of u on the right-hand side and a Hölder semi-norm rather than a norm of u on the left-hand side. However, by combining Theorems 2.5 and 2.10, we obtain

Theorem 2.12 (Hölder continuity near points in $\partial_0\mathscr{O}$ for solutions to the variational equation)**.** *Let $p > \max\{4, 2+\beta\}$ and let R_0 be a positive constant. Then there are positive constants, $R_1 = R_1(R_0) < R_0$, and $C = C(A,p,R_0)$, and $\alpha = \alpha(A,p,R_0) \in (0,1)$ such that the following holds. Let $\mathscr{O} \subseteqq \mathbb{H}$ be a domain. If $u \in H^1(\mathscr{O},\mathfrak{w})$ satisfies the variational equation* (2.11) *with source function $f \in L^2(\mathscr{O},\mathfrak{w})$ and $z_0 \in \partial_0\mathscr{O}$ is such that*

$$\mathbb{H} \cap B_{R_0}(z_0) \subset \mathscr{O},$$

and f obeys (2.13)*, then $u \in C^\alpha_s(\bar{B}^+_{R_1}(z_0))$ and*

$$\|u\|_{C^\alpha_s(\bar{B}^+_{R_1}(z_0))} \leq C\left(\|f\|_{L^p(B^+_{R_0}(z_0),y^{\beta-1})} + \|u\|_{L^2(B^+_{R_0}(z_0))}\right). \tag{2.24}$$

Similarly, by combining Theorems 2.6 and 2.11 we obtain

Theorem 2.13 (Hölder continuity near points in $\overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$ for solutions to the variational equation)**.** *Let K be a finite, right circular cone, let $p > \max\{4, 2+\beta\}$, and let R_0 be a positive constant. Then there are positive constants, $R_1 = R_1(K,R_0) < R_0$, and $C = C(A,K,R_0,p)$, and $\alpha = \alpha(A,K,p,R_0) \in (0,1)$, such that the following holds. Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain. If $u \in H^1(\mathscr{O},\mathfrak{w})$ satisfies the variational equation* (2.11) *with source function $f \in L^2(\mathscr{O},\mathfrak{w})$ and $z_0 \in \overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$ is such that f obeys* (2.13)*, and*

$$u = 0 \text{ on } \partial_1\mathscr{O} \cap B_{R_0}(z_0) \quad \text{(in the sense of } H^1\text{)},$$

and $\mathscr{O}$ obeys an interior and exterior cone condition with cone K at z_0 and a uniform exterior cone condition with cone K along $\overline{\partial_1\mathscr{O}} \cap B_{R_0}(z_0)$, then $u \in C^\alpha_s(\bar{B}^+_{R_1}(z_0))$ and satisfies (2.24)*.*

The translation of [12, Theorem 1.11] into the forms stated in Theorems 2.10, 2.11, 2.12, and 2.13 is given in Appendix A.4.

2.4. **Higher-order Daskalopoulos-Hamilton Hölder spaces.** We shall need the following higher-order weighted Hölder $C^{k,\alpha}_s$ and $C^{k,2+\alpha}_s$ norms and Banach spaces pioneered by Daskalopoulos and Hamilton [4]. We record their definition here for later reference.

Definition 2.14 ($C^{k,\alpha}_s$ norms and Banach spaces)**.** [4, p. 902] Given an integer $k \geq 0$, $\alpha \in (0,1)$, and a domain $U \subset \mathbb{H}$, we say that $u \in C^{k,\alpha}_s(\bar U)$ if $u \in C^k(\bar U)$ and

$$\|u\|_{C^{k,\alpha}_s(\bar U)} < \infty,$$

where

$$\|u\|_{C^{k,\alpha}_s(\bar U)} := \sum_{j=0}^{k} \|D^j u\|_{C^{\alpha}_s(\bar U)}. \tag{2.25}$$

When $k = 0$, we denote $C^{0,\alpha}_s(\bar U) = C^{\alpha}_s(\bar U)$.

Definition 2.15 ($C^{k,2+\alpha}_s$ norms and Banach spaces)**.** [4, pp. 901–902] Given an integer $k \geq 0$, $\alpha \in (0,1)$, and a domain $U \subset \mathbb{H}$, we say that $u \in C^{k,2+\alpha}_s(\bar U)$ if $u \in C^{k+1,\alpha}_s(\bar U)$, the derivatives, $D_x^{k+2-m}D_y^m$, $0 \leq m \leq k+2$, of order $k+2$ are continuous on U, and the functions, $yD_x^{k+2-m}D_y^m$, $0 \leq m \leq k+2$, extend continuously up to the boundary, ∂U, and those extensions belong to $C^{\alpha}_s(\bar U)$. We define

$$\|u\|_{C^{k,2+\alpha}_s(\bar U)} := \|u\|_{C^{k+1,\alpha}_s(\bar U)} + \|yD^2 u\|_{C^{k,\alpha}_s(\bar U)}.$$

We say that[6] $u \in C^{k,2+\alpha}_s(\underline{U})$ if $u \in C^{k,2+\alpha}_s(\bar V)$ for all precompact open subsets $V \Subset \underline{U}$. When $k = 0$, we denote $C^{0,2+\alpha}_s(\bar U) = C^{2+\alpha}_s(\bar U)$.

For any non-negative integer k, we let $C^k_0(\underline{U})$ denote the linear subspace of functions $u \in C^k(U)$ such that $u \in C^k(\bar V)$ for every precompact open subset $V \Subset \underline{U}$ and define $C^\infty_0(\underline{U}) := \cap_{k\geq 0} C^k_0(\underline{U})$. Note that we also have $C^\infty_0(\underline{U}) = \cap_{k\geq 0} C^{k,\alpha}_s(\underline{U}) = \cap_{k\geq 0} C^{k,2+\alpha}_s(\underline{U})$.

3. H^2 regularity for solutions to the variational equation

In this section, we develop $H^2_{\text{loc}}(\underline{\mathscr{O}},\mathfrak{w})$-regularity results for a solution, $u \in H^1(\mathscr{O},\mathfrak{w})$, to the variational equation (2.11), whose existence was established in [3]. In §3.1, we give a self-contained proof (that is, independent of results in [3]) of an important and powerful interior a priori estimate (Proposition 3.8) for solutions, $u \in H^1(\mathscr{O},\mathfrak{w})$, to the variational equation (2.11) by exploiting an idea of Koch [20]. In §3.2, we prove $H^2_{\text{loc}}(\underline{\mathscr{O}},\mathfrak{w})$-regularity (Theorem 3.16) for a solution, $u \in H^1(\mathscr{O},\mathfrak{w})$, using finite-difference methods independent of results in [3]).

3.1. **Interior Koch estimate and interior $W^{1,2}$ regularity.** We first recall the following elementary a priori estimate for a solution to the variational equation (2.11).

Lemma 3.1 (A priori estimate for solutions to the variational equation)**.** [3, Lemma 3.20] *Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain. Then there is a positive constant, $C = C(A)$, such that the following holds. If $f \in L^2(\mathscr{O},\mathfrak{w})$ and $u \in H^1(\mathscr{O},\mathfrak{w})$ is a solution to the variational equation* (2.11)*, then*[7]

$$\|u\|_{H^1(\mathscr{O},\mathfrak{w})} \leq C\left(\|f\|_{L^2(\mathscr{O},\mathfrak{w})} + \|(1+y)u\|_{L^2(\mathscr{O},\mathfrak{w})}\right). \tag{3.1}$$

---

[6] In [4, p. 901], when defining the spaces $C^{k,\alpha}_s(\mathscr{A})$ and $C^{k,2+\alpha}_s(\mathscr{A})$, it is assumed that $\mathscr{A}$ is a compact subset of the *closed* half-plane, $\{y \geq 0\}$.

[7] The result trivially holds if $(1+y)u \notin L^2(\mathscr{O},\mathfrak{w})$.

Recall from [3, Definition 3.1] that $A_\lambda := A + \lambda(1+y)$, for any constant $\lambda \geq 0$.

Theorem 3.2 (Existence of smooth solutions on the half-plane)**.** [14] *Let $f \in C_0^\infty(\underline{\mathbb{H}})$ and $\lambda \geq 0$. Then there is a function $u \in C^\infty(\bar{\mathbb{H}})$ such that*

$$A_\lambda u = f \quad \text{on } \mathbb{H}.$$

Proposition 3.3 (Koch estimate on the half-plane)**.** *There is a positive constant, $C = C(A)$, such that the following holds. If $f \in L^2(\mathbb{H}, \mathfrak{w})$ and $u \in H^1(\mathbb{H}, \mathfrak{w})$ is a solution to the variational equation* (2.11) *with $\mathscr{O} = \mathbb{H}$, then* [8]

$$\|Du\|_{L^2(\mathbb{H},\mathfrak{w})} \leq C \left(\|f\|_{L^2(\mathbb{H},\mathfrak{w})} + \|(1+y)u\|_{L^2(\mathbb{H},\mathfrak{w})} \right).$$

Remark 3.4 (Proof of the Koch estimate on the half-plane). Proposition 3.3 is proved as [3, Proposition 5.8] when $\mathscr{O} = \mathbb{H}$ with the aid of Theorem 3.2 (this is [3, Theorem 5.2] when $\mathscr{O} = \mathbb{H}$). However, the hypothesis in [3, Proposition 5.8] that [3, Theorem 5.2] holds for $\mathscr{O} \subsetneqq \mathbb{H}$ and $u \in H_0^1(\underline{\mathscr{O}}, \mathfrak{w})$ appears difficult to verify.

In order to prove an interior version of the Koch estimate on subdomains of the half-plane, we shall need the following commutator identity.

Lemma 3.5 (Heston bilinear map commutator identity)**.** [3, Corollary 2.46] *Let $u, v \in H^1(\mathscr{O}, \mathfrak{w})$ and let $\zeta \in C^\infty(\bar{\mathscr{O}})$ be such that $\operatorname{supp}\zeta \subset \underline{\mathscr{O}}$. Then*[9]

$$\mathfrak{a}(\zeta u, v) = \mathfrak{a}(u, \zeta v) + ([A, \zeta]u, v)_{L^2(\mathscr{O},\mathfrak{w})}. \tag{3.2}$$

Proof. For $u \in C^\infty(\bar{\mathscr{O}})$ and $v \in C^\infty(\underline{\mathscr{O}})$, then

$$\begin{aligned}
\mathfrak{a}(\zeta u, v) &= (A(\zeta u), v)_{L^2(\mathscr{O},\mathfrak{w})} \quad \text{(by Lemma A.3)} \\
&= (\zeta A u, v)_{L^2(\mathscr{O},\mathfrak{w})} + ([A, \zeta]u, v)_{L^2(\mathscr{O},\mathfrak{w})} \\
&= (Au, \zeta v)_{L^2(\mathscr{O},\mathfrak{w})} + ([A, \zeta]u, v)_{L^2(\mathscr{O},\mathfrak{w})} \\
&= \mathfrak{a}(u, \zeta v) + ([A, \zeta]u, v)_{L^2(\mathscr{O},\mathfrak{w})}.
\end{aligned}$$

By approximation, the result continues to hold for $u \in H^1(\mathscr{O}, \mathfrak{w})$ and $v \in H^1(\mathscr{O}, \mathfrak{w})$. □

Hence, if $u \in H^1(\mathscr{O}, \mathfrak{w})$ is a solution to the variational equation (2.11) and $\zeta \in C^\infty(\bar{\mathscr{O}})$ is such that $\operatorname{supp}\zeta \subset \underline{\mathscr{O}}$, then $\zeta u \in H^1(\mathbb{H}, \mathfrak{w})$ obeys, for all $v \in H_0^1(\underline{\mathscr{O}}, \mathfrak{w})$,

$$\begin{aligned}
\mathfrak{a}(\zeta u, v) &= \mathfrak{a}(u, \zeta v) + ([A, \zeta]u, v)_{L^2(\mathscr{O},\mathfrak{w})} \\
&= (f, \zeta v)_{L^2(\mathscr{O},\mathfrak{w})} + ([A, \zeta]u, v)_{L^2(\mathscr{O},\mathfrak{w})} \\
&= (\zeta f + [A, \zeta]u, v)_{L^2(\mathbb{H},\mathfrak{w})}.
\end{aligned}$$

Because $\operatorname{supp}\zeta \subset \underline{\mathscr{O}}$, the preceding variational equation remains unchanged when the space of test functions, $H_0^1(\underline{\mathscr{O}}, \mathfrak{w})$, is replaced by $H^1(\mathbb{H}, \mathfrak{w})$ and so $\zeta u \in H^1(\mathbb{H}, \mathfrak{w})$ obeys the variational equation,

$$\mathfrak{a}(\zeta u, v) = (f_{\zeta,u}, v)_{L^2(\mathbb{H},\mathfrak{w})}, \quad \forall v \in H^1(\mathbb{H}, \mathfrak{w}), \tag{3.3}$$

where, if $\operatorname{height}(\operatorname{supp}\zeta) < \infty$,

$$f_{\zeta,u} := \zeta f + [A, \zeta]u \in L^2(\mathbb{H}, \mathfrak{w}). \tag{3.4}$$

[8] The result trivially holds if $(1+y)u \notin L^2(\mathscr{O}, \mathfrak{w})$.

[9] This a simpler version of [3, Corollary 2.46].

We recall from [3, Equation (2.33)]) that

$$\begin{aligned}[A,\zeta]v = -y\left((\zeta_x+\varrho\sigma\zeta_y)v_x+(\varrho\sigma\zeta_x+\sigma^2\zeta_y)v_y\right)\\ -\frac{y}{2}\left(\zeta_{xx}+2\varrho\sigma\zeta_{xy}+\sigma^2\zeta_{yy}\right)v\\ -(r-q-y/2)\zeta_x v-\kappa(\theta-y)\zeta_y v.\end{aligned} \tag{3.5}$$

Noting that the derivatives v_x and v_y in (3.5) are multiplied by the factor y, we immediately obtain the

Lemma 3.6 (L^p commutator estimate)**.** *Let $\mathscr{O}\subseteqq\mathbb{H}$ be a domain and let M be a positive constant. Then there is a positive constant, $C=C(A,M)$, such that the following holds. If $\zeta\in C^\infty(\bar{\mathscr{O}})$ is such that $\|\zeta\|_{C^2(\bar{\mathscr{O}})}\le M$ and $v\in W^{1,p}_{\mathrm{loc}}(\mathscr{O})$, for $1\le p\le\infty$, then*[10]

$$\|[A,\zeta]v\|_{L^p(\mathscr{O},\mathfrak{w})}\le C\left(\|yDv\|_{L^p(\mathscr{O},\mathfrak{w})}+\|(1+y)v\|_{L^p(\mathscr{O},\mathfrak{w})}\right).$$

Moreover, if $\operatorname{height}(\operatorname{supp}\zeta)\le\Lambda<\infty$ *and $p=2$, then there is a positive constant, $C=C(A,M,\Lambda)$, such that*

$$\|[A,\zeta]v\|_{L^2(\mathscr{O},\mathfrak{w})}\le C\|v\|_{H^1(\mathscr{O},\mathfrak{w})}.$$

Recall from [3, Proposition 5.1] that, for a constant $C=C(A)$,

Lemma 3.7 (Weighted a priori first-order derivative estimate for a solution to the variational equation)**.** [3, Proposition 5.1 (1)] *There is a positive constant, $C=C(A)$, such that the following holds. Let $\mathscr{O}\subseteqq\mathbb{H}$ be a domain, $f\in L^2(\mathscr{O},\mathfrak{w})$, and $u\in H^1(\mathscr{O},\mathfrak{w})$ be a solution to the variational equation* (2.11)*. Then*[11]

$$\|yDu\|_{L^2(\mathscr{O},\mathfrak{w})}\le C\left(\|y^{1/2}f\|_{L^2(\mathscr{O},\mathfrak{w})}+\|(1+y)u\|_{L^2(\mathscr{O},\mathfrak{w})}\right), \tag{3.6}$$

We have the following interior version of Proposition 3.3 for a solution $u\in H^1(\mathscr{O},\mathfrak{w})$ to the variational equation (2.11), given $f\in L^2(\mathscr{O},\mathfrak{w})$.

Proposition 3.8 (Interior Koch estimate)**.** *Let $\mathscr{O}\subseteqq\mathbb{H}$ be a domain and let $d_1>0$. Then there is a constant $C=C(A,d_1)$ such that the following holds. Let $f\in L^2(\mathscr{O},\mathfrak{w})$ and suppose that $u\in H^1(\mathscr{O},\mathfrak{w})$ satisfies the variational equation* (2.11)*. If $\mathscr{O}'\subset\mathscr{O}$ is a subdomain such that $\bar{\mathscr{O}}'\subset\underline{\mathscr{O}}$ and $\operatorname{dist}(\partial_1\mathscr{O}',\partial_1\mathscr{O})\ge d_1$, then*[11]

$$\|Du\|_{L^2(\mathscr{O}',\mathfrak{w})}\le C\left(\|(1+y)^{1/2}f\|_{L^2(\mathscr{O},\mathfrak{w})}+\|(1+y)u\|_{L^2(\mathscr{O},\mathfrak{w})}\right). \tag{3.7}$$

Proof. Choose a cutoff function $\zeta\in C^\infty(\bar{\mathscr{O}})$ such that $0\le\zeta\le1$ on $\mathscr{O}$ and $\zeta=1$ on $\mathscr{O}'$ and $\operatorname{supp}\zeta\subset\underline{\mathscr{O}}''$, for a subdomain $\mathscr{O}''\subset\mathscr{O}$ such that $\bar{\mathscr{O}}''\subset\underline{\mathscr{O}}$ and $\operatorname{dist}(\partial_1\mathscr{O}'',\partial_1\mathscr{O})\ge d_1/2$. The conclusion now follows from Proposition 3.3, Equations (3.3) and (3.4), and Lemmas 3.6 and 3.7. □

The far more elementary a priori estimate in Lemma 3.1 may also be localized by an argument very similar to that used to prove Proposition 3.8.

Lemma 3.9 (Interior H^1 a priori estimate for a solution to the variational equation)**.** *Let $\mathscr{O}\subseteqq\mathbb{H}$ be a domain and let $d_1>0$. Then there is a constant $C=C(A,d_1)$ such that the following holds.*

[10] The result trivially holds if $yDv\notin L^p(\mathscr{O},\mathfrak{w})$ or $(1+y)v\notin L^p(\mathscr{O},\mathfrak{w})$.

[11] The result trivially holds if $(1+y)^{1/2}f\notin L^2(\mathscr{O},\mathfrak{w})$ or $(1+y)u\notin L^2(\mathscr{O},\mathfrak{w})$.

Let $f \in L^2(\mathscr{O},\mathfrak{w})$ and suppose that $u \in H^1(\mathscr{O},\mathfrak{w})$ satisfies the variational equation (2.11)*. If $\mathscr{O}' \subset \mathscr{O}$ is a subdomain such that $\bar{\mathscr{O}}' \subset \underline{\mathscr{O}}$ and $\operatorname{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O}) \geq d_1$, then*[11]

$$\|u\|_{H^1(\mathscr{O}',\mathfrak{w})} \leq C\left(\|(1+y)^{1/2}f\|_{L^2(\mathscr{O},\mathfrak{w})} + \|(1+y)u\|_{L^2(\mathscr{O},\mathfrak{w})}\right). \tag{3.8}$$

Proof. Choose a cutoff function $\zeta \in C^\infty(\bar{\mathscr{O}})$ such that $0 \leq \zeta \leq 1$ on $\mathscr{O}$ and $\zeta = 1$ on $\mathscr{O}'$ and $\operatorname{supp}\zeta \subset \underline{\mathscr{O}}''$, for a subdomain $\mathscr{O}'' \subset \mathscr{O}$ such that $\bar{\mathscr{O}}'' \subset \underline{\mathscr{O}}$ and $\operatorname{dist}(\partial_1\mathscr{O}'', \partial_1\mathscr{O}) \geq d_1/2$. The conclusion now follows from Lemma 3.1, Equations (3.3) and (3.4), and Lemmas 3.6 and 3.7. ☐

3.2. **Interior H^2 regularity.** Recall from [3, Definition 2.20] that

$$H^2(\mathscr{O},\mathfrak{w}) := \{u \in W^{2,2}_{\mathrm{loc}}(\mathscr{O}) : \|u\|_{H^2(\mathscr{O},\mathfrak{w})} < \infty\}, \tag{3.9}$$

where

$$\|u\|^2_{H^2(\mathscr{O},\mathfrak{w})} := \int_{\mathscr{O}} \left(y^2|D^2u|^2 + (1+y)^2|Du|^2 + (1+y)u^2\right) \mathfrak{w}\,dx\,dy. \tag{3.10}$$

We say that $u \in H^2_{\mathrm{loc}}(\underline{\mathscr{O}},\mathfrak{w})$ if $u \in H^2(U,\mathfrak{w})$ for every $U \Subset \underline{\mathscr{O}}$.

We denote the finite difference with respect to x of a function v on $\mathscr{O}$ by

$$\delta^h_x v(x,y) := \frac{1}{h}\left(v(x+h,y) - v(x,y)\right), \tag{3.11}$$

for $h \in \mathbb{R}\setminus\{0\}$ and all $(x,y) \in \mathscr{O}$ with $(x+h,y) \in \mathscr{O}$. We have the following analogue and extension of [6, Theorem 5.8.3] or [18, Lemmas 7.23 & 7.24].

Lemma 3.10 (Convergence and bounds on finite differences)**.** *Let $\mathscr{O} \subseteqq \mathbb{H}$ be a domain and let $\mathscr{O}' \subset \mathscr{O}$ be a subdomain such that $\bar{\mathscr{O}}' \subset \underline{\mathscr{O}}$.*

(1) *There is a constant $C = C(\operatorname{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O}))$ such that the following holds. If $u \in L^2(\mathscr{O},\mathfrak{w})$ with $u_x \in L^2(\mathscr{O},\mathfrak{w})$, then*
$$\|\delta^h_x u\|_{L^2(\mathscr{O}',\mathfrak{w})} \leq C\|u_x\|_{L^2(\mathscr{O},\mathfrak{w})},$$
for all $h \in \mathbb{R}$ such that $0 < 2|h| < \operatorname{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O})$.

(2) *If $u \in L^2(\mathscr{O},\mathfrak{w})$ and there is a constant $K > 0$ such that*
$$\|\delta^h_x u\|_{L^2(\mathscr{O}',\mathfrak{w})} \leq K,$$
for all $h \in \mathbb{R}$ such that $0 < 2|h| < \operatorname{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O})$, then $u_x \in L^2(\mathscr{O}',\mathfrak{w})$ exists and obeys $\|u_x\|_{L^2(\mathscr{O}',\mathfrak{w})} \leq K$.

Proof. The proof of Item (1) adapts line-by-line from the proofs of [6, Theorem 5.8.3 (i)] or [18, Lemma 7.23]. To prove Item (2), it is enough to notice that $L^2(\mathscr{O},\mathfrak{w})$ is a separable Hilbert space (therefore, reflexive also) and so [18, Problem 5.4] applies. The proof of [18, Lemma 7.24] now adapts line-by-line. ☐

We shall adapt the proof of [18, Theorem 8.8] in order to establish

Theorem 3.11 (Interior regularity of second-order derivatives parallel to the degenerate boundary)**.** *Let $\mathscr{O} \subseteqq \mathbb{H}$ be a domain and let $d_1 > 0$. Then there is a constant $C = C(A, d_1)$ such that the following holds. Let $f \in L^2(\mathscr{O},\mathfrak{w})$ and suppose that $u \in H^1(\mathscr{O},\mathfrak{w})$ satisfies the variational equation* (2.11)*. If $\mathscr{O}' \subset \mathscr{O}$ is a subdomain such that $\bar{\mathscr{O}}' \subset \underline{\mathscr{O}}$ and $\operatorname{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O}) \geq d_1$, then*[12]

$$yu_{xx},\ yu_{xy} \in L^2(\mathscr{O}',\mathfrak{w}),$$

[12] The result trivially holds if $(1+y)u_x \notin L^2(\mathscr{O},\mathfrak{w})$.

and

$$\|yDu_x\|_{L^2(\mathscr{O}',\mathfrak{w})} \leq C\left(\|f\|_{L^2(\mathscr{O},\mathfrak{w})} + \|y^{1/2}Du\|_{L^2(\mathscr{O},\mathfrak{w})} + \|(1+y)u_x\|_{L^2(\mathscr{O},\mathfrak{w})} + \|u\|_{L^2(\mathscr{O},\mathfrak{w})}\right).$$

Remark 3.12 (Comparison with regularity results and their proofs in [3]). While stronger results than Theorem 3.11 are proved as Corollary 5.15 and Theorem 5.17 in [3], where (in the case of [3, Corollary 5.15]) the subdomain $\mathscr{O}'$ is replaced by $\mathscr{O}$ under suitable hypotheses on $\partial_1\mathscr{O}$ and Du_x is replaced by D^2u, the proof of [3, Corollary 5.15] relies on a hypothesis (see [3, Theorem 5.2]) in [3] that there exist solutions $u \in C^\infty(\bar{\mathscr{O}})$ to $Au = f$ on $\mathscr{O}$ and $u = 0$ on $\partial_1\mathscr{O}$ when $f \in C_0^\infty(\bar{\mathscr{O}})$ and $\partial_1\mathscr{O}$ is C^∞-transverse to $\partial\mathbb{H}$. In contrast, our proof of Theorem 3.11 does not rely on [3, Theorem 5.2], whose proof appears difficult, and instead uses more elementary methods (finite differences, in particular). See also Remark 3.4.

Using the $L^2(\mathscr{O},\mathfrak{w})$-analogue[13] of the finite-difference integration-by-parts formula [6, Equation (6.3.16)], we find that, for $f, v \in L^2(\mathscr{O},\mathfrak{w})$,

$$-\left(f, \delta_x^{-h}v\right)_{L^2(\mathscr{O},\mathfrak{w})} = \left((\mathfrak{w}^h/\mathfrak{w})\delta_x^h f, v\right)_{L^2(\mathscr{O},\mathfrak{w})} + \left((\delta_x^h\mathfrak{w}/\mathfrak{w})f, v\right)_{L^2(\mathscr{O},\mathfrak{w})}, \tag{3.12}$$

where the finite-difference product rule [6, Equation (6.3.17)] gives

$$\delta_x^h(\mathfrak{w}f) = \mathfrak{w}^h\delta_x^h f + f\delta_x^h\mathfrak{w} \quad \text{a.e. on } \mathscr{O},$$

with $\mathfrak{w}^h(x,y) := \mathfrak{w}(x+h,y)$.

Proof of Theorem 3.11. We may assume without loss of generality that $(1+y)u_x \in L^2(\mathscr{O},\mathfrak{w})$. From the integral identities (2.7) and (2.11) (using our Assumption 2.2 that $b_1 = 0$), we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathscr{O}}\left(u_xv_x + \varrho\sigma u_yv_x + \varrho\sigma u_xv_y + \sigma^2u_yv_y\right)y\,\mathfrak{w}\,dx\,dy \\
&= \frac{\gamma}{2}\int_{\mathscr{O}}\left(u_x + \varrho\sigma u_y\right)v\,\frac{x}{\sqrt{1+x^2}}y\,\mathfrak{w}\,dx\,dy \\
&\quad + \int_{\mathscr{O}} a_1u_xvy\,\mathfrak{w}\,dx\,dy - \int_{\mathscr{O}} c_0uv\,\mathfrak{w}\,dx\,dy + \int_{\mathscr{O}} fv\,\mathfrak{w}\,dx\,dy, \quad \forall v \in C_0^\infty(\underline{\mathscr{O}}).
\end{aligned}$$

We may replace v by the difference quotient, $\delta_x^{-h}v$, in the preceding identity, provided $|h| < \frac{1}{2}\operatorname{dist}(\operatorname{supp} v, \partial_1\mathscr{O})$, and use the $L^2(\mathscr{O},\mathfrak{w})$ finite-difference integration-by-parts formula (3.12) to find that

$$\begin{aligned}
&\int_{\mathscr{O}}(\mathfrak{w}^h/\mathfrak{w})\left((\delta_x^hu_x)v_x + \varrho\sigma(\delta_x^hu_y)v_x + \varrho\sigma(\delta_x^hu_x)v_y + \sigma^2(\delta_x^hu_y)v_y\right)y\,\mathfrak{w}\,dx\,dy \\
&+ \int_{\mathscr{O}}(\delta_x^h\mathfrak{w}/\mathfrak{w})\left(u_xv_x + \varrho\sigma u_yv_x + \varrho\sigma u_xv_y + \sigma^2u_yv_y\right)y\,\mathfrak{w}\,dx\,dy \\
&= -\int_{\mathscr{O}}\left(u_x(\delta_x^{-h}v_x) + \varrho\sigma u_y(\delta_x^{-h}v_x) + \varrho\sigma u_x(\delta_x^{-h}v_y) + \sigma^2u_y(\delta_x^{-h}v_y)\right)y\,\mathfrak{w}\,dx\,dy
\end{aligned}$$

[13] The proof adapts line-by-line.

and give

$$
\begin{aligned}
&-\frac{1}{2}\int_{\mathscr{O}}(\mathfrak{w}^h/\mathfrak{w})\left((\delta_x^h u_x)v_x+\varrho\sigma(\delta_x^h u_y)v_x+\varrho\sigma(\delta_x^h u_x)v_y+\sigma^2(\delta_x^h u_y)v_y\right)y\,\mathfrak{w}\,dx\,dy\\
&=\frac{1}{2}\int_{\mathscr{O}}(\delta_x^h\mathfrak{w}/\mathfrak{w})\left(u_xv_x+\varrho\sigma u_yv_x+\varrho\sigma u_xv_y+\sigma^2u_yv_y\right)y\,\mathfrak{w}\,dx\,dy\\
&\quad+\frac{\gamma}{2}\int_{\mathscr{O}}(u_x+\varrho\sigma u_y)\,(\delta_x^{-h}v)\,\frac{x}{\sqrt{1+x^2}}y\,\mathfrak{w}\,dx\,dy\\
&\quad+\int_{\mathscr{O}}a_1u_x(\delta_x^{-h}v)y\,\mathfrak{w}\,dx\,dy-\int_{\mathscr{O}}c_0u(\delta_x^{-h}v)\,\mathfrak{w}\,dx\,dy+\int_{\mathscr{O}}f(\delta_x^{-h}v)\,\mathfrak{w}\,dx\,dy.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathscr{O}}(\mathfrak{w}^h/\mathfrak{w})\left((\delta_x^h u_x)v_x+\varrho\sigma(\delta_x^h u_y)v_x+\varrho\sigma(\delta_x^h u_x)v_y+\sigma^2(\delta_x^h u_y)v_y\right)y\,\mathfrak{w}\,dx\,dy\\
&\le C\|y^{1/2}Du\|_{L^2(\mathscr{O},\mathfrak{w})}\left(\|y^{1/2}Dv\|_{L^2(\mathscr{O},\mathfrak{w})}+\|y^{1/2}\delta_x^{-h}v\|_{L^2(\mathscr{O},\mathfrak{w})}\right)\\
&\quad+C\left(\|u\|_{L^2(\mathscr{O},\mathfrak{w})}+\|f\|_{L^2(\mathscr{O},\mathfrak{w})}\right)\|\delta_x^{-h}v\|_{L^2(\mathscr{O},\mathfrak{w})}\\
&\le C\|y^{1/2}Du\|_{L^2(\mathscr{O},\mathfrak{w})}\|y^{1/2}Dv\|_{L^2(\mathscr{O},\mathfrak{w})}+C\left(\|u\|_{L^2(\mathscr{O},\mathfrak{w})}+\|f\|_{L^2(\mathscr{O},\mathfrak{w})}\right)\|v_x\|_{L^2(\mathscr{O},\mathfrak{w})},
\end{aligned}
$$

where the final inequality follows from Lemma 3.10 (1). Now choose $\zeta\in C^\infty(\underline{\mathscr{O}})$ with $0\le\zeta\le 1$ on $\mathscr{O}$ and $\zeta=1$ on $\mathscr{O}'$ and $\operatorname{supp}\zeta\subset\underline{\mathscr{O}}$, and set $v=y\zeta^2\delta_x^h u$ with $|h|<\frac{1}{2}\operatorname{dist}(\operatorname{supp}\zeta,\partial_1\mathscr{O})$. Therefore, applying (2.9), we obtain

$$
\begin{aligned}
&\nu_0\int_{\mathscr{O}}(\mathfrak{w}^h/\mathfrak{w})|\zeta D\delta_x^h u|^2y^2\,\mathfrak{w}\,dx\,dy\\
&\le\frac{1}{2}\int_{\mathscr{O}}(\mathfrak{w}^h/\mathfrak{w})\zeta^2\left((\delta_x^h u_x)^2+2\varrho\sigma(\delta_x^h u_y)(\delta_x^h u_x)+\sigma^2(\delta_x^h u_y)^2\right)y^2\,\mathfrak{w}\,dx\,dy,
\end{aligned}
$$

and using

$$
\begin{aligned}
y\zeta^2\delta_x^h u_x&=(y\zeta^2\delta_x^h u)_x-2y\zeta\zeta_x\delta_x^h u=v_x-2y\zeta\zeta_x\delta_x^h u,\\
y\zeta^2\delta_x^h u_y&=(y\zeta^2\delta_x^h u)_y-\zeta^2\delta_x^h u-2y\zeta\zeta_y\delta_x^h u=v_y-(\zeta^2+2y\zeta\zeta_y)\delta_x^h u,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\nu_0\int_{\mathscr{O}}(\mathfrak{w}^h/\mathfrak{w})|\zeta D\delta_x^h u|^2y^2\,\mathfrak{w}\,dx\,dy\\
&\le\frac{1}{2}\int_{\mathscr{O}}(\mathfrak{w}^h/\mathfrak{w})\Big((\delta_x^h u_x)\left(v_x-2y\zeta\zeta_x\delta_x^h u\right)+\varrho\sigma(\delta_x^h u_y)\left(v_x-2y\zeta\zeta_x\delta_x^h u\right)\\
&\qquad+\varrho\sigma(\delta_x^h u_x)\left(v_y-(\zeta^2+2y\zeta\zeta_y)\delta_x^h u\right)+\sigma^2(\delta_x^h u_y)\left(v_y-(\zeta^2+2y\zeta\zeta_y)\delta_x^h u\right)\Big)y\,\mathfrak{w}\,dx\,dy\\
&=\frac{1}{2}\int_{\mathscr{O}}(\mathfrak{w}^h/\mathfrak{w})\left((\delta_x^h u_x)v_x+\varrho\sigma(\delta_x^h u_y)v_x+\varrho\sigma(\delta_x^h u_x)v_y+\sigma^2(\delta_x^h u_y)v_y\right)y\,\mathfrak{w}\,dx\,dy\\
&\quad-\frac{1}{2}\int_{\mathscr{O}}(\mathfrak{w}^h/\mathfrak{w})\Big((\delta_x^h u_x)2y\zeta\zeta_x\delta_x^h u+\varrho\sigma(\delta_x^h u_y)2y\zeta\zeta_x\delta_x^h u\\
&\qquad+\varrho\sigma(\delta_x^h u_x)(\zeta^2+2y\zeta\zeta_y)\delta_x^h u+\sigma^2(\delta_x^h u_y)(\zeta^2+2y\zeta\zeta_y)\delta_x^h u\Big)y\,\mathfrak{w}\,dx\,dy.
\end{aligned}
$$

Hence, there is a positive constant $C = C(A, \operatorname{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O}))$ such that

$$\begin{aligned}
&\|\zeta y D\delta_x^h u\|^2_{L^2(\mathscr{O},\mathfrak{w})}\\
&\quad\leq C\|y^{1/2}Du\|_{L^2(\mathscr{O},\mathfrak{w})}\|y^{1/2}Dv\|_{L^2(\mathscr{O},\mathfrak{w})} + C\left(\|u\|_{L^2(\mathscr{O},\mathfrak{w})} + \|f\|_{L^2(\mathscr{O},\mathfrak{w})}\right)\|v_x\|_{L^2(\mathscr{O},\mathfrak{w})}\\
&\qquad + C\|\zeta y D\delta_x^h u\|_{L^2(\mathscr{O},\mathfrak{w})}\|(1+y)\delta_x^h u\|_{L^2(\operatorname{supp}\zeta,\mathfrak{w})}\\
&\quad\leq C\|y^{1/2}Du\|_{L^2(\mathscr{O},\mathfrak{w})}\left(\|\zeta y D\delta_x^h u\|_{L^2(\mathscr{O},\mathfrak{w})} + \|(1+y)\delta_x^h u\|_{L^2(\operatorname{supp}\zeta,\mathfrak{w})}\right)\\
&\qquad + C\left(\|u\|_{L^2(\mathscr{O},\mathfrak{w})} + \|f\|_{L^2(\mathscr{O},\mathfrak{w})}\right)\left(\|\zeta y\delta_x^h u_x\|_{L^2(\mathscr{O},\mathfrak{w})} + \|y\delta_x^h u\|_{L^2(\operatorname{supp}\zeta,\mathfrak{w})}\right)\\
&\qquad + C\|\zeta y D\delta_x^h u\|_{L^2(\mathscr{O},\mathfrak{w})}\|(1+y)\delta_x^h u\|_{L^2(\operatorname{supp}\zeta,\mathfrak{w})},
\end{aligned}$$

where, to obtain the final inequality, we used

$$v_x = y\zeta^2\delta_x^h u_x + 2y\zeta\zeta_x\delta_x^h u, \quad v_y = y\zeta^2\delta_x^h u_y + (\zeta^2 + 2y\zeta\zeta_y)\delta_x^h u.$$

Applying Young's inequality (that is, $2ab \leq \varepsilon a^2 + \varepsilon^{-1}b^2$, for any $\varepsilon > 0$ and $a, b \in \mathbb{R}$) to the terms on the right containing the factor $\|\zeta y D\delta_x^h u\|_{L^2(\mathscr{O},\mathfrak{w})}$ and rearranging, we obtain

$$\begin{aligned}
&\|\zeta y D\delta_x^h u\|^2_{L^2(\mathscr{O},\mathfrak{w})}\\
&\quad\leq C\|y^{1/2}Du\|^2_{L^2(\mathscr{O},\mathfrak{w})} + C\|y^{1/2}Du\|_{L^2(\mathscr{O},\mathfrak{w})}\|(1+y)\delta_x^h u\|_{L^2(\operatorname{supp}\zeta,\mathfrak{w})}\\
&\qquad + C\left(\|u\|_{L^2(\mathscr{O},\mathfrak{w})} + \|f\|_{L^2(\mathscr{O},\mathfrak{w})}\right)^2 + C\left(\|u\|_{L^2(\mathscr{O},\mathfrak{w})} + \|f\|_{L^2(\mathscr{O},\mathfrak{w})}\right)\|y\delta_x^h u\|_{L^2(\operatorname{supp}\zeta,\mathfrak{w})}\\
&\qquad + C\|(1+y)\delta_x^h u\|^2_{L^2(\operatorname{supp}\zeta,\mathfrak{w})},
\end{aligned}$$

for all $h \in \mathbb{R}$ such that $|h| < \frac{1}{2}\operatorname{dist}(\operatorname{supp}\zeta, \partial_1\mathscr{O})$. Again applying Lemma 3.10 (1), we see that

$$\begin{aligned}
&\|yDu_x\|^2_{L^2(\mathscr{O}',\mathfrak{w})}\\
&\quad\leq C\|y^{1/2}Du\|_{L^2(\mathscr{O},\mathfrak{w})}\left(\|y^{1/2}Du\|_{L^2(\mathscr{O},\mathfrak{w})} + \|(1+y)u_x\|_{L^2(\mathscr{O},\mathfrak{w})}\right)\\
&\quad + C\left(\|u\|_{L^2(\mathscr{O},\mathfrak{w})} + \|f\|_{L^2(\mathscr{O},\mathfrak{w})}\right)\left(\|u\|_{L^2(\mathscr{O},\mathfrak{w})} + \|f\|_{L^2(\mathscr{O},\mathfrak{w})} + \|yu_x\|_{L^2(\mathscr{O},\mathfrak{w})}\right)\\
&\qquad + C\|(1+y)u_x\|^2_{L^2(\mathscr{O},\mathfrak{w})},
\end{aligned}$$

and taking square roots completes the proof. □

Proceeding by analogy with the proof of [18, Theorem 8.12] to estimate yu_{yy}, we obtain

Lemma 3.13 (Interior regularity of second-order derivatives orthogonal to the degenerate boundary)**.** *There is a constant $C = C(A)$ such that the following holds. Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain and let $\mathscr{O}' \subsetneqq \mathscr{O}$ be a subdomain. Let $f \in L^2(\mathscr{O}, \mathfrak{w})$ and suppose that $u \in H^1(\mathscr{O}, \mathfrak{w})$ satisfies the variational equation* (2.11)*. If $yu_{xx},\ yu_{xy} \in L^2(\mathscr{O}', \mathfrak{w})$, then*

$$yu_{yy} \in L^2(\mathscr{O}', \mathfrak{w}),$$

and

$$\begin{aligned}
\|yu_{yy}\|_{L^2(\mathscr{O}',\mathfrak{w})} \leq C\big(&\|yu_{xx}\|_{L^2(\mathscr{O}',\mathfrak{w})} + \|yu_{xy}\|_{L^2(\mathscr{O}',\mathfrak{w})} + \|(1+y)Du\|_{L^2(\mathscr{O}',\mathfrak{w})}\\
&+ \|u\|_{L^2(\mathscr{O}',\mathfrak{w})} + \|f\|_{L^2(\mathscr{O}',\mathfrak{w})}\big).
\end{aligned}\tag{3.13}$$

Proof. From [18, Theorem 8.8], we know that $u \in W^{2,2}_{\mathrm{loc}}(\mathscr{O})$ and $Au = f$ a.e. on $\mathscr{O}$, and thus by (1.3), we have

$$\frac{\sigma^2}{2}yu_{yy} = -\frac{y}{2}\left(u_{xx} + 2\varrho\sigma u_{xy}\right) - \left(c_0 - q - \frac{y}{2}\right)u_x - \kappa(\theta - y)u_y + c_0u - f.$$

Hence, there is a constant, $C = C(A)$, such that (3.13) holds. $\square$

Therefore, we find that

Theorem 3.14 (Interior regularity of second-order derivatives)**.** *Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain and let $d_1 > 0$. Then there is a constant $C = C(A, d_1)$ such that the following holds. Let $f \in L^2(\mathscr{O}, \mathfrak{w})$ and suppose that $u \in H^1(\mathscr{O}, \mathfrak{w})$ satisfies the variational equation* (2.11)*. If $(1+y)^{1/2} f$ and $(1+y)u$ belong to $L^2(\mathscr{O}, \mathfrak{w})$ and $\mathscr{O}' \subset \mathscr{O}$ is a subdomain such that $\bar{\mathscr{O}}' \subset \underline{\mathscr{O}}$ and $\operatorname{dist}(\partial_1 \mathscr{O}', \partial_1 \mathscr{O}) \geq d_1$, then*

$$yu_{xx},\ yu_{xy},\ yu_{yy} \in L^2(\mathscr{O}', \mathfrak{w}),$$

and

$$\|yD^2 u\|_{L^2(\mathscr{O}', \mathfrak{w})} \leq C \left(\|(1+y)^{1/2} f\|_{L^2(\mathscr{O}, \mathfrak{w})} + \|(1+y)u\|_{L^2(\mathscr{O}, \mathfrak{w})} \right).$$

Proof. The conclusion follows by combining the estimates in Proposition 3.8, Theorem 3.11, Lemma 3.13 and the a priori $H^1(\mathscr{O}, \mathfrak{w})$-estimate for a solution u given by Lemma 3.1 and the $L^2(\mathscr{O}, \mathfrak{w})$-estimate for yDu in Lemma 3.7. $\square$

Consequently, we have the

Theorem 3.15 (Interior H^2 regularity and a priori estimate)**.** *Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain and let $d_1 > 0$. Then there is a constant $C = C(A, d_1)$ such that the following holds. If $f \in L^2(\mathscr{O}, \mathfrak{w})$ and $u \in H^1(\mathscr{O}, \mathfrak{w})$ satisfies the variational equation* (2.11)*, then $u \in H^2_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w})$. Moreover, if $(1+y)^{1/2} f$ and $(1+y)u$ belong to $L^2(\mathscr{O}, \mathfrak{w})$ and $\mathscr{O}' \subset \mathscr{O}$ is a subdomain such that $\bar{\mathscr{O}}' \subset \underline{\mathscr{O}}$ and $\operatorname{dist}(\partial_1 \mathscr{O}', \partial_1 \mathscr{O}) \geq d_1$, then $u \in H^2(\mathscr{O}', \mathfrak{w})$ and*

$$\|u\|_{H^2(\mathscr{O}', \mathfrak{w})} \leq C \left(\|(1+y)^{1/2} f\|_{L^2(\mathscr{O}, \mathfrak{w})} + \|(1+y)u\|_{L^2(\mathscr{O}, \mathfrak{w})} \right).$$

Proof. The conclusion follows by combining Proposition 3.8 with Theorem 3.14, the a priori $H^1(\mathscr{O}, \mathfrak{w})$-estimate for a solution u given by Lemma 3.1, and the $L^2(\mathscr{O}, \mathfrak{w})$-estimate for yDu in Lemma 3.7. $\square$

In the sequel, we shall most often apply Theorem 3.15 in the following special form.

Theorem 3.16 (Interior H^2 regularity for a solution to the variational equation)**.** *Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain and let $R < R_0$ be positive constants. Then there is a positive constant, $C = C(A, R, R_0)$, such that the following holds. If $f \in L^2(\mathscr{O}, \mathfrak{w})$ and $u \in H^1(\mathscr{O}, \mathfrak{w})$ is a solution to the variational equation* (2.11)*, and $z_0 \in \partial_0 \mathscr{O}$ is such that*

$$\mathbb{H} \cap B_{R_0}(z_0) \subset \mathscr{O},$$

then

$$u \in H^2(B^+_R(z_0), \mathfrak{w}),$$

and

$$\|u\|_{H^2(B^+_R(z_0), \mathfrak{w})} \leq C \left(\|f\|_{L^2(B^+_{R_0}(z_0), \mathfrak{w})} + \|u\|_{L^2(B^+_{R_0}(z_0), \mathfrak{w})} \right). \tag{3.14}$$

4. Higher-order Sobolev regularity for solutions to the variational equation

In this section, we develop higher-order "interior" regularity results for a solution, $u \in H^1(\mathscr{O}, \mathfrak{w})$, to the variational equation (2.11). After providing motivation for their construction in §4.1, we describe the families of higher-order weighted Sobolev spaces which we shall need for this article, namely $\mathscr{H}^k(\mathscr{O}, \mathfrak{w})$ (Definition 4.3) and $W^{k,p}(\mathscr{O}, \mathfrak{w})$ (Definition 4.4).

We begin our development of higher-order Sobolev regularity theory in §4.2, where we establish $H^2_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w})$-regularity (Proposition 4.11) of the derivatives, $D_x^k u$, of a solution, $u \in H^1(\mathscr{O}, \mathfrak{w})$, to the variational equation (2.11), while in §4.3, we establish $H^2_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w})$-regularity (Proposition 4.13) of the derivative, u_y. The preceding regularity results are combined in §4.4 to give $\mathscr{H}^3_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w})$-regularity (Theorem 4.14) of a solution, $u \in H^1(\mathscr{O}, \mathfrak{w})$. We conclude in §4.5 with a proof of two of the main results of our article, Theorems 1.3 and Theorem 1.5, and $\mathscr{H}^{k+2}_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w})$-regularity of a solution, $u \in H^1(\mathscr{O}, \mathfrak{w})$, for any integer $k \geq 0$.

4.1. Motivation and definition of higher-order weighted Sobolev norms.

We now extend our previous definition of $H^\ell(\mathscr{O}, \mathfrak{w})$ when $\ell = 0, 1, 2$ (see [3, Definitions 2.15 & 2.20]) to allow $\ell \geq 2$. For $k \geq 0$, it is natural to define $H^{k+2}(\mathscr{O}, \mathfrak{w})$ as a Sobolev space contained in the domain of $D_x^{k-m} D_y^m A$, so the operators

$$D_x^{k-m} D_y^m A : H^{k+2}(\mathscr{O}, \mathfrak{w}) \to L^2(\mathscr{O}, \mathfrak{w}), \quad m \in \mathbb{N}, \quad 0 \leq m \leq k,$$

are bounded, and we use this principle as a guide to our definition.

From the expression (1.3) for A, we have, for $v \in C^\infty(\mathscr{O})$,

$$[D_x, A]v := D_x A v - A D_x v = 0 \quad \text{on } \mathscr{O},$$

and so

$$[D_x^m, A]v \equiv A D_x^m v - D_x^m A v = 0 \quad \text{on } \mathscr{O}, \quad \forall m \in \mathbb{N},$$

whereas

$$[D_y, A]v \equiv D_y A v - A D_y v = -\frac{1}{2}\left(v_{xx} + 2\varrho\sigma v_{xy} + \sigma^2 v_{yy}\right) + \frac{1}{2} v_x + \kappa v_y \quad \text{on } \mathscr{O}, \tag{4.1}$$

is a (non-trivial) second-order, elliptic operator with constant coefficients (and therefore commutes with both D_x and D_y). Hence,

$$\begin{aligned} D_y^2 A v &= D_y A D_y v + D_y [D_y, A] v \\ &= A D_y^2 v + [D_y, A] D_y v + [D_y, A] D_y v \\ &= A D_y^2 v + 2[D_y, A] D_y v, \end{aligned}$$

while

$$\begin{aligned} D_y^3 A v &= D_y^2 A D_y v + D_y^2 [D_y, A] v \\ &= A D_y^3 v + 2[D_y, A] D_y^2 v + [D_y, A] D_y^2 v \\ &= A D_y^3 v + 3[D_y, A] D_y^2 v, \end{aligned}$$

and, by induction,

$$[D_y^m, A]v \equiv D_y^m A v - A D_y^m v = m[D_y, A] D_y^{m-1} v \quad \text{on } \mathscr{O}, \quad \forall m \in \mathbb{N}.$$

By combining the two cases, we obtain

$$[D_x^{k-m} D_y^m, A]v \equiv D_x^{k-m} D_y^m A v - A D_x^{k-m} D_y^m v = m[D_y, A] D_x^{k-m} D_y^{m-1} v \quad \text{on } \mathscr{O}, \tag{4.2}$$

for all $k, m \in \mathbb{N}$ with $0 \leq m \leq k$.

Given $v \in H^k(\mathscr{O}, \mathfrak{w})$ and $k \geq 2$ and a suitable definition of $H^k(\mathscr{O}, \mathfrak{w})$, we should expect that

$$D_x^m D_y^n Av \in L^2(\mathscr{O}, \mathfrak{w}), \quad 0 \leq m+n \leq k-2,$$

and so,

$$AD_x^m D_y^n v,\ [D_y, A] D_x^m D_y^{n-1} v \in L^2(\mathscr{O}, \mathfrak{w}), \quad 0 \leq m+n \leq k-2.$$

The second condition is fulfilled when

$$D_x^m D_y^n v \in L^2(\mathscr{O}, \mathfrak{w}), \quad 0 \leq m+n \leq k-1,$$

whereas the expression (1.3) for A implies that the first condition is fulfilled when

$$\begin{aligned} yD_x^m D_y^n v &\in L^2(\mathscr{O}, \mathfrak{w}), \quad 1 \leq m+n \leq k, \\ D_x^m D_y^n v &\in L^2(\mathscr{O}, \mathfrak{w}), \quad 0 \leq m+n \leq k-1. \end{aligned}$$

Therefore, when $k \geq 2$, and keeping in mind that we want $H^k(\mathscr{O}, \mathfrak{w}) \subset H^1(\mathscr{O}, \mathfrak{w})$, for all $k \geq 2$, we make the

Definition 4.1 (Higher-order weighted Sobolev spaces)**.** Let $\mathscr{O} \subseteqq \mathbb{H}$ be a domain. For any integer $k \geq 1$, set

$$H^{k+2}(\mathscr{O}, \mathfrak{w}) := \left\{ v \in W^{k+2,2}_{\text{loc}}(\mathscr{O}) : \|v\|_{H^{k+2}(\mathscr{O}, \mathfrak{w})} < \infty \right\},$$

where

$$\begin{aligned} \|v\|^2_{H^{k+2}(\mathscr{O}, \mathfrak{w})} := &\int_{\mathscr{O}} y^2 |D^{k+2} v|^2 \,\mathfrak{w}\, dx\, dy + \sum_{j=1}^{k+1} \int_{\mathscr{O}} (1+y)^2 |D^j v|^2 \,\mathfrak{w}\, dx\, dy \\ &+ \int_{\mathscr{O}} (1+y) v^2 \,\mathfrak{w}\, dx\, dy, \end{aligned} \tag{4.3}$$

and $D^j v$ denotes the vector $(D_x^{j-m} D_y^m v : 0 \leq m \leq j)$, for $0 \leq j \leq k$.

As we shall later see, Definition 4.1 is not well-adapted to a development of a higher-order regularity theory for solutions to (1.1) or (2.11), and it is best regarded as a stepping stone to the one we ultimately adopt for our regularity theory, namely Definition 4.3. By way of motivation, we observe that the expression (4.1) for the commutator $[D_y, A]$ involves both derivatives with respect to x and y. The alternative "commutator" provided by (4.4) will prove more useful than (4.1) in our approach to the higher-order regularity since it only involves derivatives with respect to x.

Lemma 4.2 (Alternative commutator of A and D_y)**.** *For any integer $m \geq 1$,*

$$D_y^m Av - A_m D_y^m v = mBD_y^{m-1} v \quad \text{on } \mathscr{O}, \quad \forall v \in C^\infty(\mathbb{H}), \tag{4.4}$$

where

$$Bv := -\frac{1}{2} v_{xx} + \frac{1}{2} v_x, \tag{4.5}$$

and A_m is obtained from the expression for A in (1.3) by replacing θ by $\theta_m := \theta + m\sigma^2/(2\kappa)$ (and β by $\beta_m := \beta + m$), and q by $q_m := q - m\varrho\sigma$, and c_0 by $c_{0,m} := c_0 + m\kappa$.

Proof. We compute that

$$\begin{aligned}
D_yAv &= -\frac{y}{2}\left(v_{xxy}+2\varrho\sigma v_{xyy}+\sigma^2 v_{yyy}\right)-\left(c_0-q-\frac{y}{2}\right)v_{xy}-\kappa(\theta-y)v_{yy}\\
&\quad -\frac{1}{2}\left(v_{xx}+2\varrho\sigma v_{xy}+\sigma^2 v_{yy}\right)+\frac{1}{2}v_x+\kappa v_y+c_0v_y\\
&= -\frac{y}{2}\left(v_{xxy}+2\varrho\sigma v_{xyy}+\sigma^2 v_{yyy}\right)-\left(c_0-q+\varrho\sigma-\frac{y}{2}\right)v_{xy}-\kappa\left(\theta+\frac{\sigma^2}{2\kappa}-y\right)v_{yy}\\
&\quad -\frac{1}{2}v_{xx}+\frac{1}{2}v_x+(c_0+\kappa)v_y\\
&= A_1D_yv+Bv,
\end{aligned}$$

where A_1 is defined by replacing θ by $\theta_1=\theta+\sigma^2/(2\kappa)$ (and β by $\beta_1=\beta+1$), and q by $q_1=q-\varrho\sigma$, and the coefficient, c_0, of v by $c_{0,1}=c_0+\kappa$, and $Bv=-\frac{1}{2}v_{xx}+\frac{1}{2}v_x$. Note that B is a linear, second-order differential operator which commutes with D_y. Computing D_y^2Av, we see that

$$\begin{aligned}
D_y^2Av &= D_y(D_yAv)=D_y\left(A_1D_yv+Bv\right)\\
&= A_2D_y^2v+BD_yv+D_yBv\\
&= A_2D_y^2v+2BD_yv,
\end{aligned}$$

where A_2 is defined by replacing θ_1 by $\theta_2=\theta_1+\sigma^2/(2\kappa)=\theta+2\sigma^2/(2\kappa)$ (and β_1 by $\beta_2=\beta_1+1=\beta+2$), and q_1 by $q_2=q_1-\varrho\sigma=q-2\varrho\sigma$, and the coefficient, $c_{0,1}$, of v by $c_{0,2}=c_{0,1}+\kappa=c_0+2\kappa$. It is now clear that the stated formula for D_y^mAv follows by induction. □

Recall that the weight function (2.5) for our weighted Sobolev spaces is given by

$$\mathfrak{w}(x,y)=y^{\beta-1}e^{-\gamma\sqrt{1+x^2}-\mu y},\quad (x,y)\in\mathbb{H},$$

where $\beta=2\kappa\theta/\sigma^2$ and $\mu=2\kappa/\sigma^2$, and thus is defined by the coefficients of A (and γ). We denote the weight defined by the corresponding coefficients of the operator A_m by

$$\mathfrak{w}_m(x,y):=y^m\mathfrak{w}(x,y)=y^{\beta+m-1}e^{-\gamma\sqrt{1+x^2}-\mu y},\quad (x,y)\in\mathbb{H},\quad m\geq 1, \tag{4.6}$$

noting that $\beta_m=\beta+m$ and $\kappa_m=\kappa$. Similarly, the bilinear map, $\mathfrak{a}(u,v)$, defined in (2.7) by the coefficients of A (and γ) has an analogue, which we denote by $\mathfrak{a}_m(u,v)$, defined by the coefficients of A_m (and γ), for $u\in C^\infty(\bar{\mathscr{O}})$ and $v\in C_0^\infty(\underline{\mathscr{O}})$, with the property that

$$\mathfrak{a}_m(u,v)=(A_mu,v)_{L^2(\mathscr{O},\mathfrak{w}_m)},\quad \forall u\in C^\infty(\bar{\mathscr{O}}),\ v\in C_0^\infty(\underline{\mathscr{O}}). \tag{4.7}$$

When $k,m\in\mathbb{N}$, one may define $H^k(\mathscr{O},\mathfrak{w}_m)$ by simply replacing the weight $\mathfrak{w}$ by $\mathfrak{w}_m$ in the definitions of $H^k(\mathscr{O},\mathfrak{w})$. We shall need to introduce the following alternative definition of higher-order Sobolev spaces which lie between $H^{k+2}(\mathscr{O},\mathfrak{w}_k)$ and $H^{k+2}(\mathscr{O},\mathfrak{w})$ when $k\geq 1$.

Definition 4.3 (Alternative higher-order weighted Sobolev spaces)**.** Let $\mathscr{O}\subseteqq\mathbb{H}$ be a domain. For $\ell=0,1,2$, define $\mathscr{H}^\ell(\mathscr{O},\mathfrak{w}):=H^\ell(\mathscr{O},\mathfrak{w})$ and, for any integer $k\geq 1$, set

$$\mathscr{H}^{k+2}(\mathscr{O},\mathfrak{w}):=\left\{v\in W^{k+2,2}_{\mathrm{loc}}(\mathscr{O}):\|v\|_{\mathscr{H}^{k+2}(\mathscr{O},\mathfrak{w})}<\infty\right\},$$

where

$$\begin{aligned}\|v\|^2_{\mathscr{H}^{k+2}(\mathscr{O},\mathfrak{w})} := &\int_{\mathscr{O}} y^2 \left(|D_x^{k+2}v|^2 + |D_x^{k+1}D_yv|^2 + |D_x^kD_y^2v|^2\right) \mathfrak{w}\,dx\,dy\\ &+ \sum_{m=1}^k \int_{\mathscr{O}} y^2|D_x^{k-m}D_y^{m+2}v|^2\,\mathfrak{w}_m\,dx\,dy\\ &+ \sum_{j=0}^k \int_{\mathscr{O}} (1+y)^2 \left(|D_x^{j+1}v|^2 + |D_x^jD_yv|^2\right) \mathfrak{w}\,dx\,dy\\ &+ \sum_{j=1}^k\sum_{m=1}^j \int_{\mathscr{O}} (1+y)^2|D_x^{j-m}D_y^{m+1}v|^2\,\mathfrak{w}_m\,dx\,dy\\ &+ \int_{\mathscr{O}} (1+y)v^2\,\mathfrak{w}\,dx\,dy.\end{aligned} \tag{4.8}$$

We denote $\mathscr{H}^2(\mathscr{O},\mathfrak{w}) = H^2(\mathscr{O},\mathfrak{w})$ when $k=0$.

For example, if $k=1$,

$$\begin{aligned}\|v\|^2_{\mathscr{H}^3(\mathscr{O},\mathfrak{w})} := &\int_{\mathscr{O}} y^2 \left(v_{xxx}^2 + v_{xxy}^2 + v_{xyy}^2 + yv_{yyy}^2\right) \mathfrak{w}\,dx\,dy\\ &+ \int_{\mathscr{O}} (1+y)^2 \left(v_{xx}^2 + v_{xy}^2 + yv_{yy}^2\right) \mathfrak{w}\,dx\,dy\\ &+ \int_{\mathscr{O}} (1+y)^2 \left(v_x^2 + v_y^2\right) \mathfrak{w}\,dx\,dy + \int_{\mathscr{O}} (1+y)v^2\,\mathfrak{w}\,dx\,dy.\end{aligned} \tag{4.9}$$

Observe that if $\mathscr{O} \subset \mathbb{H}$ is a subdomain of finite height, then

$$L^2(\mathscr{O},\mathfrak{w}) \subset L^2(\mathscr{O},\mathfrak{w}_m), \tag{4.10}$$

for all $m \in \mathbb{N}$. When $k \geq 1$,

$$\|v\|_{H^{k+2}(\mathscr{O},\mathfrak{w}_k)} \leq C\|v\|_{\mathscr{H}^{k+2}(\mathscr{O},\mathfrak{w})} \leq C\|v\|_{H^{k+2}(\mathscr{O},\mathfrak{w})},$$

and so

$$H^{k+2}(\mathscr{O},\mathfrak{w}) \subset \mathscr{H}^{k+2}(\mathscr{O},\mathfrak{w}) \subset H^{k+2}(\mathscr{O},\mathfrak{w}_k),$$

when $\mathscr{O}$ has finite height. Definition 4.3 gives the following inductive inequality,

$$\begin{aligned}\|v\|^2_{\mathscr{H}^{k+3}(\mathscr{O},\mathfrak{w})} \leq &\int_{\mathscr{O}} y^2 \left(|D_x^{k+3}v|^2 + |D_x^{k+2}D_yv|^2 + |D_x^{k+1}D_y^2v|^2\right) \mathfrak{w}\,dx\,dy\\ &+ \sum_{m=1}^{k+1} \int_{\mathscr{O}} y^2|D_x^{k+1-m}D_y^{m+2}v|^2\,\mathfrak{w}_m\,dx\,dy\\ &+ \int_{\mathscr{O}} (1+y)^2 \left(|D_x^{k+2}v|^2 + |D_x^{k+1}D_yv|^2\right) \mathfrak{w}\,dx\,dy\\ &+ \sum_{m=1}^{k+1} \int_{\mathscr{O}} (1+y)^2|D_x^{k+1-m}D_y^{m+1}v|^2\,\mathfrak{w}_m\,dx\,dy\\ &+ \|v\|^2_{\mathscr{H}^{k+2}(\mathscr{O},\mathfrak{w})},\end{aligned} \tag{4.11}$$

for $k \geq 0$. Equation (4.22) gives the inductive inequality for $\mathscr{H}^3(\mathscr{O},\mathfrak{w})$.

We recall the definition of a weighted Sobolev space and norm, where the weight is the same for all derivatives of the function (denoted $W^{k,p}_w(\mathscr{O})$ in [33, Definition 2.1.1], though we shall not require that w be an A_p weight in this article).

Definition 4.4 (Higher-order weighted Sobolev spaces with a single weight)**.** Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain and let $w \in L^1_{\mathrm{loc}}(\mathscr{O})$ be a *weight function*, so that $w > 0$ a.e. on $\mathscr{O}$. For any $1 \leq p < \infty$ and integer $k \geq 0$, set

$$W^{k,p}(\mathscr{O}, w) := \left\{ v \in W^{k,p}_{\mathrm{loc}}(\mathscr{O}) : \|v\|_{W^{k,p}(\mathscr{O},w)} < \infty \right\},$$

where

$$\|v\|_{W^{k,p}(\mathscr{O},w)} := \left(\sum_{j=0}^k \int_{\mathscr{O}} |D^j v|^p \, w \, dx \, dy \right)^{1/p}. \tag{4.12}$$

We denote $W^{k,p}(\mathscr{O}, w) = L^p(\mathscr{O}, w)$ when $k = 0$.

Finally, we shall need the following "interior" versions of the weighted Sobolev spaces defined in this subsection.

Definition 4.5 (Interior weighted Sobolev norms)**.** Let $T \subset \partial\mathscr{O}$ be relatively open in $\mathbb{R}^2$ and let $k \geq 0$ be an integer. We say that $v \in H^k_{\mathrm{loc}}(\mathscr{O} \cup T, \mathfrak{w})$ (respectively, $\mathscr{H}^k_{\mathrm{loc}}(\mathscr{O} \cup T, \mathfrak{w})$ or $W^{k,p}_{\mathrm{loc}}(\mathscr{O} \cup T, \mathfrak{w})$) if for every subdomain $U \subset \mathscr{O}$ such that $U \Subset \mathscr{O} \cup T$, we have $v \in H^k(U, \mathfrak{w})$ (respectively, $\mathscr{H}^k(U, \mathfrak{w})$ or $W^{k,p}(U, \mathfrak{w})$).

4.2. **Interior H^2 regularity for first-order derivatives parallel to the degenerate boundary.** We proceed in a manner similar to that in [18, p. 186].

Lemma 4.6 (Variational equation for the derivative of a solution with respect to x)**.** *Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain with* finite height,[14] *let $f \in L^2(\mathscr{O}, \mathfrak{w})$, and suppose that $u \in H^1(\mathscr{O}, \mathfrak{w})$ satisfies the variational equation* (2.11)*. If*[15]

$$f \in W^{1,2}(\mathscr{O}, \mathfrak{w}), \quad u \in H^2(\mathscr{O}, \mathfrak{w}), \quad \text{and} \quad u_x \in H^1(\mathscr{O}, \mathfrak{w}),$$

then $u_x \in H^1(\mathscr{O}, \mathfrak{w})$ obeys

$$\mathfrak{a}(u_x, v) = (f_x, v)_{L^2(\mathscr{O},\mathfrak{w})}, \tag{4.13}$$

for all $v \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w})$.

Proof. Suppose first that $u \in C^\infty(\bar{\mathscr{O}})$ and $v \in C^\infty_0(\underline{\mathscr{O}})$. Then $v_x \in C^\infty_0(\underline{\mathscr{O}})$ and

$$\begin{aligned}
\mathfrak{a}(u, -v_x) &= (Au, -v_x)_{L^2(\mathscr{O},\mathfrak{w})} \quad \text{(by Lemma A.3)} \\
&= (Au_x, v)_{L^2(\mathscr{O},\mathfrak{w})} + (Au, v(\log \mathfrak{w})_x)_{L^2(\mathscr{O},\mathfrak{w})} \\
&= \mathfrak{a}(u_x, v) + (Au, v(\log \mathfrak{w})_x)_{L^2(\mathscr{O},\mathfrak{w})},
\end{aligned}$$

where from (2.5) we see that

$$(\log \mathfrak{w})_x = -\gamma \frac{x}{\sqrt{1 + x^2}} \quad \text{on } \mathbb{H}.$$

[14] As one can see from the proof, the hypothesis that $\mathscr{O}$ has finite height is only used in a very mild way and the condition could be removed using more precise bounds, but we shall not need such an extension.

[15] While the right-hand side of the identity (4.13) is well-defined when $f_x \in L^2(\mathscr{O}, \mathfrak{w})$, we appeal to an approximation argument requiring $f \in W^{1,2}(\mathscr{O}, \mathfrak{w})$.

Now suppose, more generally, that $u \in H^2(\mathscr{O},\mathfrak{w})$ with $u_x \in H^1(\mathscr{O},\mathfrak{w})$, as in our hypotheses. For $v \in C_0^\infty(\underline{\mathscr{O}})$, we may choose a subdomain $\mathscr{O}' \Subset \underline{\mathscr{O}}$ such that $\operatorname{supp} v \subset \underline{\mathscr{O}}'$ and $\partial_1\mathscr{O}'$ is C^1-orthogonal to $\partial\mathbb{H}$ in the sense of Definition A.1. According to Theorem A.2, there is a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty(\bar{\mathscr{O}}')$ such that $u_n \to u$ in $H^2(\mathscr{O}',\mathfrak{w})$ as $n \to \infty$ and hence, for each $v \in C_0^\infty(\underline{\mathscr{O}})$ with $\operatorname{supp} v \subset \underline{\mathscr{O}}'$,

$$(Au_n, v)_{L^2(\mathscr{O},\mathfrak{w})} \to (Au, v)_{L^2(\mathscr{O},\mathfrak{w})} \quad \text{and} \quad \mathfrak{a}(u_{n,x}, v) \to \mathfrak{a}(u_x, v), \quad n \to \infty,$$

since, in the second case, by (2.7) we see that

$$\begin{aligned}|\mathfrak{a}(u_{n,x} - u_x, v)| &\leq C\left(\|yD(u_{n,x} - u_x)\|_{L^2(\mathscr{O},\mathfrak{w})} + \|u_{n,x} - u_x\|_{L^2(\mathscr{O},\mathfrak{w})}\right)\|v\|_{W^{1,2}(\mathscr{O},\mathfrak{w})}\\ &\leq C\|u_n - u\|_{H^2(\mathscr{O},\mathfrak{w})}\|v\|_{W^{1,2}(\mathscr{O},\mathfrak{w})},\end{aligned}$$

where $C = C(\operatorname{height}(\mathscr{O}))$. Therefore, by approximation and also the fact that $v \in C_0^\infty(\underline{\mathscr{O}})$ is arbitrary, the preceding variational equation continues to hold for $u \in H^2(\mathscr{O},\mathfrak{w})$ with $u_x \in H^1(\mathscr{O},\mathfrak{w})$, that is,

$$\mathfrak{a}(u, -v_x) = \mathfrak{a}(u_x, v) + (Au, v(\log\mathfrak{w})_x)_{L^2(\mathscr{O},\mathfrak{w})}, \quad \forall v \in C_0^\infty(\underline{\mathscr{O}}).$$

Since $u \in H^2(\mathscr{O},\mathfrak{w})$, then (2.11) implies that $Au = f$ a.e. on $\mathscr{O}$ by Lemma A.3 and thus the preceding identity gives

$$\mathfrak{a}(u, -v_x) = \mathfrak{a}(u_x, v) + (f, v(\log\mathfrak{w})_x)_{L^2(\mathscr{O},\mathfrak{w})}, \quad \forall v \in C_0^\infty(\underline{\mathscr{O}}).$$

Moreover, by substituting $-v_x$ for v in (2.11), using the fact that $f \in W^{1,2}(\mathscr{O},\mathfrak{w})$ by hypothesis, so $f_x \in L^2(\mathscr{O},\mathfrak{w})$, and appealing to Theorem A.2 to choose a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C^\infty(\bar{\mathscr{O}})$ such that $f_n \to f$ in $W^{1,2}(\mathscr{O},\mathfrak{w})$ and so $f_{n,x} \to f_x$ in $L^2(\mathscr{O},\mathfrak{w})$ as $n \to \infty$, we obtain

$$\begin{aligned}\mathfrak{a}(u, -v_x) &= (f, -v_x)_{L^2(\mathscr{O},\mathfrak{w})}\\ &= (f_x, v)_{L^2(\mathscr{O},\mathfrak{w})} + (f, v(\log\mathfrak{w})_x)_{L^2(\mathscr{O},\mathfrak{w})}, \quad \forall v \in C_0^\infty(\underline{\mathscr{O}}),\end{aligned}$$

where the integration-by-parts identity is justified by approximation, just as in the proof of Lemma A.3. Combining these identities yields (4.13) for all $v \in C_0^\infty(\underline{\mathscr{O}})$ and hence for all $v \in H_0^1(\underline{\mathscr{O}},\mathfrak{w})$. □

Remark 4.7 (Need for the regularity condition on u in Lemma 4.6). If we only knew that $u \in H^2(\mathscr{O},\mathfrak{w})$, then the definition (3.9) of $H^2(\mathscr{O},\mathfrak{w})$ would imply that $y|D^2u|$, $(1+y)|Du| \in L^2(\mathscr{O},\mathfrak{w})$ and so $y^{1/2}|Du_x|$, $(1+y)^{1/2}u_x \in L^2(\mathscr{O},y\mathfrak{w}) = L^2(\mathscr{O},\mathfrak{w}_1)$ and thus $u_x \in H^1(\mathscr{O},\mathfrak{w}_1)$, *but not necessarily* $H^1(\mathscr{O},\mathfrak{w})$, by the definition (2.2) of $H^1(\mathscr{O},\mathfrak{w})$.

Lemma 4.8 (Variational equation for higher-order derivatives of a solution with respect to x)**.** *Let $\mathscr{O} \subseteq \mathbb{H}$ be a domain with* finite height*, let $k \geq 1$ be an integer, let $f \in L^2(\mathscr{O},\mathfrak{w})$, and suppose that $u \in H^1(\mathscr{O},\mathfrak{w})$ satisfies the variational equation* (2.11)*. If*

$$D_x^k f \in L^2(\mathscr{O},\mathfrak{w}), \quad u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w}), \quad \textit{and} \quad D_x^k u \in H^1(\mathscr{O},\mathfrak{w}),$$

then $D_x^k u \in H^1(\mathscr{O},\mathfrak{w})$ obeys

$$\mathfrak{a}(D_x^k u, v) = (D_x^k f, v)_{L^2(\mathscr{O},\mathfrak{w})}, \quad v \in H_0^1(\underline{\mathscr{O}},\mathfrak{w}). \tag{4.14}$$

Proof. By hypothesis,

$$f_x \in L^2(\mathscr{O},\mathfrak{w}), \quad u \in H^2(\mathscr{O},\mathfrak{w}), \quad \text{and} \quad u_x \in H^1(\mathscr{O},\mathfrak{w}),$$

and so Lemma 4.6 implies that u_x obeys

$$\mathfrak{a}(u_x, v) = (f_x, v)_{L^2(\mathscr{O},\mathfrak{w})}, \quad \forall v \in H_0^1(\underline{\mathscr{O}},\mathfrak{w}).$$

By induction we may assume that the conclusion holds when k is replaced by $k-1$. Note that $u \in \mathscr{H}^{k+1}(\mathscr{O}, \mathfrak{w})$ by hypothesis and so, by Definition 4.3, we see that u_x obeys

$$yD_x^k u_x,\ yD_x^{k-1}D_y u_x,\ yD_x^{k-2}D_y^2 u_x \in L^2(\mathscr{O}, \mathfrak{w}),$$
$$yD_x^{k-m}D_y^m u_x \in L^2(\mathscr{O}, \mathfrak{w}_{m-2}) \quad 3 \le m \le k.$$

Therefore, $u \in \mathscr{H}^{k+1}(\mathscr{O}, \mathfrak{w}) \implies u_x \in \mathscr{H}^k(\mathscr{O}, \mathfrak{w})$ when $k \ge 2$. Since

$$D_x^{k-1} f_x = D_x^k f \in L^2(\mathscr{O}, \mathfrak{w}), \quad u_x \in \mathscr{H}^k(\mathscr{O}, \mathfrak{w}), \quad \text{and} \quad D_x^{k-1} u_x = D_x^k u \in H^1(\mathscr{O}, \mathfrak{w}),$$

we can apply Lemma 4.8 to the preceding variational equation, with $k-1$ and $u_x \in H^1(\mathscr{O}, \mathfrak{w})$ and $f_x \in L^2(\mathscr{O}, \mathfrak{w})$ replacing k and $u \in H^1(\mathscr{O}, \mathfrak{w})$ and $f \in L^2(\mathscr{O}, \mathfrak{w})$, respectively, to give

$$\mathfrak{a}(D_x^k u, v) = \mathfrak{a}(D_x^{k-1} u_x, v) = (D_x^{k-1} f_x, v)_{L^2(\mathscr{O}, \mathfrak{w})} = (D_x^k f, v)_{L^2(\mathscr{O}, \mathfrak{w})}, \quad \forall v \in H_0^1(\underline{\mathscr{O}}, \mathfrak{w}).$$

This completes the proof. $\square$

In order to establish a refinement of Lemma 4.6 which yields $u_x \in H^1(\mathscr{O}, \mathfrak{w})$ as a conclusion, assuming only $u \in H^2(\mathscr{O}, \mathfrak{w})$, we shall need to substitute $v \in C_0^\infty(\underline{\mathscr{O}}')$, where $\mathscr{O}' \Subset \underline{\mathscr{O}}$, by a finite difference, $-\delta_x^{-h} v$, rather than $-v_x$, by analogy with the proof of [18, Theorem 8.8].

Proposition 4.9 (Variational equation for the derivative of a solution with respect to x)**.** *Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain and let d_1, Λ be positive constants. Then there is a positive constant, $C = C(A, d_1, \Lambda)$, such that the following holds. Let $f \in L^2(\mathscr{O}, \mathfrak{w})$ and suppose that $u \in H^1(\mathscr{O}, \mathfrak{w})$ satisfies the variational equation* (2.11)*. If*

$$f_x \in L^2(\mathscr{O}, \mathfrak{w}),$$

then $u_x \in H^1_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w})$ and, for any subdomain $\mathscr{O}' \subset \mathscr{O}$ with $\bar{\mathscr{O}}' \subset \underline{\mathscr{O}}$ and $\operatorname{dist}(\partial_1 \mathscr{O}', \partial_1 \mathscr{O}) \ge d_1$ *and* $\operatorname{height}(\mathscr{O}') \le \Lambda$*, one has*

$$u_x \in H^1(\mathscr{O}', \mathfrak{w}),$$

and

$$\mathfrak{a}(u_x, v) = (f_x, v)_{L^2(\mathscr{O}', \mathfrak{w})}, \quad \forall v \in H_0^1(\underline{\mathscr{O}}', \mathfrak{w}), \tag{4.15}$$

and

$$\|u_x\|_{H^1(\mathscr{O}', \mathfrak{w})} \le C \left(\|f_x\|_{L^2(\mathscr{O}, \mathfrak{w})} + \|f\|_{L^2(\mathscr{O}, \mathfrak{w})} + \|u\|_{L^2(\mathscr{O}, \mathfrak{w})} \right).$$

Proof. We partially follow the idea of the proof of [18, Theorem 8.8], but the argument is simpler here because of the relatively strong hypothesis that $f_x \in L^2(\mathscr{O}, \mathfrak{w})$ as well as $f \in L^2(\mathscr{O}, \mathfrak{w})$).

Choose a subdomain $\mathscr{O}'' \subset \mathscr{O}$ such that $\bar{\mathscr{O}}' \subset \underline{\mathscr{O}}''$ and $\bar{\mathscr{O}}'' \subset \underline{\mathscr{O}}$ and $\partial_1 \mathscr{O}''$ is C^1-orthogonal to $\partial \mathbb{H}$, while $\operatorname{dist}(\partial_1 \mathscr{O}', \partial_1 \mathscr{O}'') \ge d_1/4$ and $\operatorname{dist}(\partial_1 \mathscr{O}'', \partial_1 \mathscr{O}) \ge d_1/2$ and $\operatorname{height}(\mathscr{O}'') \le 2\Lambda$. Observe that if $(x, y) \in \mathscr{O}''$, then $(x \pm h, y) \in \mathscr{O}$ provided $\operatorname{dist}((x, y), \partial_1 \mathscr{O}) < |h|$, so in choosing h, we shall always assume that $0 < |h| < \frac{1}{2} \operatorname{dist}(\mathscr{O}'', \partial_1 \mathscr{O})$. For any $v \in C_0^\infty(\underline{\mathscr{O}}'')$, observe that $\delta_x^{-h} v \in C_0^\infty(\underline{\mathscr{O}})$, so we may substitute $-\delta_x^{-h} v$ for v as a test function in (2.11).

For any $w \in C^\infty(\bar{\mathscr{O}})$, noting that $\delta_x^h A w = A \delta_x^h w$ on $\mathscr{O}''$, we obtain

$$\begin{aligned}
-\mathfrak{a}(w, \delta_x^{-h} v) &= -(Aw, \delta_x^{-h} v)_{L^2(\mathscr{O}, \mathfrak{w})} \quad \text{(by Lemma A.3)} \\
&= (\delta_x^h A w, (\mathfrak{w}^h/\mathfrak{w}) v)_{L^2(\mathscr{O}, \mathfrak{w})} + (Aw, (\delta_x^h \mathfrak{w}/\mathfrak{w}) v)_{L^2(\mathscr{O}, \mathfrak{w})} \quad \text{(by (3.12))} \\
&= (A \delta_x^h w, (\mathfrak{w}^h/\mathfrak{w}) v)_{L^2(\mathscr{O}, \mathfrak{w})} + (Aw, (\delta_x^h \mathfrak{w}/\mathfrak{w}) v)_{L^2(\mathscr{O}, \mathfrak{w})} \\
&= \mathfrak{a}(\delta_x^h w, (\mathfrak{w}^h/\mathfrak{w}) v) + \mathfrak{a}(w, (\delta_x^h \mathfrak{w}/\mathfrak{w}) v), \quad \forall v \in C_0^\infty(\underline{\mathscr{O}}''),
\end{aligned}$$

where, in the last equality, we use the fact that $(\mathfrak{w}^h/\mathfrak{w})v \in C_0^\infty(\underline{\mathscr{O}}'')$ when $v \in C_0^\infty(\underline{\mathscr{O}}'')$, recalling by (2.5) that

$$\mathfrak{w}(x,y) = y^{\beta-1}e^{-\gamma\sqrt{1+x^2}-\mu y}, \quad (x,y) \in \mathbb{H}.$$

Recall that $\operatorname{supp} v \subset \underline{\mathscr{O}}''$. Since $C^\infty(\bar{\mathscr{O}}'')$ is dense in $H^1(\mathscr{O}'',\mathfrak{w})$ by Theorem A.2, we may choose $\{w_n\}_{n\in\mathbb{N}} \subset C^\infty(\bar{\mathscr{O}}'')$ with $w_n \to u$ strongly in $H^1(\mathscr{O}'',\mathfrak{w})$ to see that

$$-\mathfrak{a}(u,\delta_x^{-h}v) = \mathfrak{a}(\delta_x^h u,(\mathfrak{w}^h/\mathfrak{w})v) + \mathfrak{a}(u,(\delta_x^h\mathfrak{w}/\mathfrak{w})v), \quad \forall v \in C_0^\infty(\underline{\mathscr{O}}'').$$

Therefore, for all $v \in C_0^\infty(\underline{\mathscr{O}}'')$, the preceding identity yields

$$\begin{aligned}\mathfrak{a}(\delta_x^h u,(\mathfrak{w}^h/\mathfrak{w})v) &= -\mathfrak{a}(u,\delta_x^{-h}v) - \mathfrak{a}(u,(\delta_x^h\mathfrak{w}/\mathfrak{w})v)\\ &= -(f,\delta_x^{-h}v)_{L^2(\mathscr{O},\mathfrak{w})} - (f,(\delta_x^h\mathfrak{w}/\mathfrak{w})v)_{L^2(\mathscr{O},\mathfrak{w})} \quad \text{(by (2.11))}\\ &= (\delta_x^h f,(\mathfrak{w}^h/\mathfrak{w})v)_{L^2(\mathscr{O},\mathfrak{w})} \quad \text{(by (3.12))},\end{aligned}$$

and consequently,

$$\mathfrak{a}(\delta_x^h u, v) = (\delta_x^h f, v)_{L^2(\mathscr{O}'',\mathfrak{w})}, \quad \forall v \in C_0^\infty(\underline{\mathscr{O}}'').$$

Since $C_0^\infty(\underline{\mathscr{O}}'')$ is dense in $H_0^1(\underline{\mathscr{O}}'',\mathfrak{w})$ by definition, we obtain

$$\mathfrak{a}(\delta_x^h u, v) = (\delta_x^h f, v)_{L^2(\mathscr{O}'',\mathfrak{w})}, \quad \forall v \in H_0^1(\underline{\mathscr{O}}'',\mathfrak{w}). \tag{4.16}$$

The interior a priori estimate (3.8) for solutions to the preceding equation yields

$$\|\delta_x^h u\|_{H^1(\mathscr{O}',\mathfrak{w})} \le C\left(\|\delta_x^h f\|_{L^2(\mathscr{O}'',\mathfrak{w})} + \|\delta_x^h u\|_{L^2(\mathscr{O}'',\mathfrak{w})}\right), \quad 0<2|h|<\operatorname{dist}(\mathscr{O}'',\partial_1\mathscr{O}),$$

where $C = C(A,\Lambda)$. Choose a subdomain $\mathscr{O}''' \subset \mathscr{O}$ such that $\bar{\mathscr{O}}'' \subset \underline{\mathscr{O}}'''$ and $\bar{\mathscr{O}}''' \subset \underline{\mathscr{O}}$, while $\operatorname{dist}(\partial_1\mathscr{O}''',\partial_1\mathscr{O}) \ge d_1/4$ and $\operatorname{dist}(\partial_1\mathscr{O}'',\partial_1\mathscr{O}''') \ge d_1/8$ and $\operatorname{height}(\mathscr{O}'') \le 4\Lambda$. By Lemma 3.10 (1) and the facts that $f_x \in L^2(\mathscr{O},\mathfrak{w})$ by hypothesis and $u_x \in L^2(\mathscr{O}''',\mathfrak{w})$ by Proposition 3.8, we see that

$$\|\delta_x^h f\|_{L^2(\mathscr{O}'',\mathfrak{w})} + \|\delta_x^h u\|_{L^2(\mathscr{O}'',\mathfrak{w})} \le \|f_x\|_{L^2(\mathscr{O}''',\mathfrak{w})} + \|u_x\|_{L^2(\mathscr{O}''',\mathfrak{w})}, \quad 0<2|h|<\operatorname{dist}(\mathscr{O}'',\partial_1\mathscr{O}'''),$$

and so

$$\|\delta_x^h u\|_{H^1(\mathscr{O}',\mathfrak{w})} \le C\left(\|f_x\|_{L^2(\mathscr{O}''',\mathfrak{w})} + \|u_x\|_{L^2(\mathscr{O}''',\mathfrak{w})}\right), \quad 0<2|h|<\operatorname{dist}(\mathscr{O}'',\partial_1\mathscr{O}''').$$

Therefore, since $(\delta_x^h u)_x = \delta_x^h u_x$ and $(\delta_x^h u)_y = \delta_x^h u_y$, we have

$$\begin{aligned}\|y^{1/2}\delta_x^h u_x\|_{L^2(\mathscr{O}',\mathfrak{w})} &\le C_1,\\ \|y^{1/2}\delta_x^h u_y\|_{L^2(\mathscr{O}',\mathfrak{w})} &\le C_1,\\ \|\delta_x^h u\|_{L^2(\mathscr{O}',\mathfrak{w})} &\le C_1, \quad 0<2|h|<\operatorname{dist}(\mathscr{O}',\partial_1\mathscr{O}''),\end{aligned}$$

where $C_1 := C(\|f_x\|_{L^2(\mathscr{O}''',\mathfrak{w})} + \|u_x\|_{L^2(\mathscr{O}''',\mathfrak{w})})$. Lemma 3.10 (2) (and its proof) gives $u_x \in H^1(\mathscr{O}',\mathfrak{w})$ and weak convergence, after passing to a diagonal subsequence,

$$y^{1/2}\delta_x^h u_x \rightharpoonup y^{1/2}u_{xx}, \quad y^{1/2}\delta_x^h u_y \rightharpoonup y^{1/2}u_{xy}, \quad \delta_x^h u \rightharpoonup u_x \quad \text{weakly in } L^2(\mathscr{O}',\mathfrak{w}) \text{ as } h\to 0,$$

and thus,

$$\delta_x^h u \rightharpoonup u_x \quad \text{weakly in } H^1(\mathscr{O}',\mathfrak{w}) \text{ as } h \to 0.$$

Therefore,

$$\|u_x\|_{H^1(\mathscr{O}',\mathfrak{w})} \le \liminf_{h\to 0}\|\delta_x^h u\|_{H^1(\mathscr{O}',\mathfrak{w})}.$$

and, by combining the preceding inequalities,

$$\|u_x\|_{H^1(\mathscr{O}',\mathfrak{w})} \le C\left(\|f_x\|_{L^2(\mathscr{O}''',\mathfrak{w})} + \|u_x\|_{L^2(\mathscr{O}''',\mathfrak{w})}\right).$$

Finally, Proposition 3.8 yields

$$\|u_x\|_{L^2(\mathscr{O}''',\mathfrak{w})} \leq C \left(\|f\|_{L^2(\mathscr{O},\mathfrak{w})} + \|u\|_{L^2(\mathscr{O},\mathfrak{w})} \right),$$

for $C = C(A, d_1, \Lambda)$ and the conclusion follows from the preceding two estimates. $\square$

Clearly, by repeatedly applying Proposition 4.9, induction on $k \geq 1$ yields the following refinement of Lemma 4.8.

Proposition 4.10 (Variational equation for higher-order derivatives of a solution with respect to x)**.** *Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain, let d_1, Λ be positive constants, and let $k \geq 1$ be an integer. Then there is a positive constant, $C = C(A, d_1, k, \Lambda)$, such that the following holds. Let $f \in L^2(\mathscr{O}, \mathfrak{w})$ and suppose that $u \in H^1(\mathscr{O}, \mathfrak{w})$ satisfies the variational equation* (2.11)*. If*

$$D_x^j f \in L^2(\mathscr{O}, \mathfrak{w}), \quad 1 \leq j \leq k,$$

then $D_x^k u \in H^1_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w})$ and, for any subdomain $\mathscr{O}' \subset \mathscr{O}$ with $\bar{\mathscr{O}}' \subset \underline{\mathscr{O}}$ and $\operatorname{dist}(\partial_1 \mathscr{O}', \partial_1 \mathscr{O}) \geq d_1$ *and* $\operatorname{height}(\mathscr{O}') \leq \Lambda$*, one has*

$$D_x^k u \in H^1(\mathscr{O}', \mathfrak{w}),$$

and

$$\mathfrak{a}(D_x^k u, v) = (D_x^k f, v)_{L^2(\mathscr{O}',\mathfrak{w})}, \quad \forall v \in H^1_0(\underline{\mathscr{O}}', \mathfrak{w}),$$

and

$$\|D_x^k u\|_{H^1(\mathscr{O}',\mathfrak{w})} \leq C \left(\sum_{j=0}^{k} \|D_x^j f\|_{L^2(\mathscr{O},\mathfrak{w})} + \|u\|_{L^2(\mathscr{O},\mathfrak{w})} \right).$$

Proof. Proposition 4.9 yields the conclusion when $k = 1$ and so we can take $k \geq 2$ and assume, by induction, that the result holds for $k-1$ in place of k. Choose a subdomain $\mathscr{O}'' \subset \mathscr{O}$ with $\bar{\mathscr{O}}' \subset \underline{\mathscr{O}}''$ and $\bar{\mathscr{O}}'' \subset \underline{\mathscr{O}}$, while $\operatorname{dist}(\partial_1 \mathscr{O}', \partial_1 \mathscr{O}'') \geq d_1/4$ and $\operatorname{dist}(\partial_1 \mathscr{O}'', \partial_1 \mathscr{O}) \geq d_1/2$ and $\operatorname{height}(\mathscr{O}'') \leq 2\Lambda$. By the induction hypothesis, $D_x^{k-1} u \in H^1_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w}) \cap H^1(\mathscr{O}'', \mathfrak{w})$ and $D_x^{k-1} u$ obeys

$$\mathfrak{a}(D_x^{k-1} u, v) = (D_x^{k-1} f, v)_{L^2(\mathscr{O}'',\mathfrak{w})}, \quad \forall v \in H^1_0(\underline{\mathscr{O}}'', \mathfrak{w}).$$

Hence, by applying Proposition 4.9 to the preceding variational equation in place of (2.11), we see that $D_x^k u \in H^1(\mathscr{O}', \mathfrak{w})$ and, because the choice of subdomain $\mathscr{O}'' \subset \mathscr{O}$ with $\bar{\mathscr{O}}'' \subset \underline{\mathscr{O}}$ was arbitrary, that also $D_x^k u \in H^1_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w})$. Moreover, Proposition 4.9 yields

$$\|D_x^k u\|_{H^1(\mathscr{O}',\mathfrak{w})} \leq C \left(\|D_x^k f\|_{L^2(\mathscr{O}'',\mathfrak{w})} + \|D_x^{k-1} f\|_{L^2(\mathscr{O}'',\mathfrak{w})} + \|D_x^{k-1} u\|_{L^2(\mathscr{O}'',\mathfrak{w})} \right),$$

for $C = C(A, d_1, \Lambda)$, while the induction hypothesis gives

$$\|D_x^{k-1} u\|_{H^1(\mathscr{O}'',\mathfrak{w})} \leq C \left(\sum_{j=0}^{k-1} \|D_x^j f\|_{L^2(\mathscr{O},\mathfrak{w})} + \|u\|_{L^2(\mathscr{O},\mathfrak{w})} \right),$$

for $C = C(A, d_1, k, \Lambda)$. We obtain the conclusion by combining the preceding estimates. $\square$

As the regularity questions of interest to us only concern regularity of a solution to the variational equation (2.11), it will be convenient to consider, for $z_0 \in \partial\mathbb{H}$ and $0 < R < R_1 < R_0$, half-balls $U \subset U' \subset V$, where

$$U := B^+_R(z_0), \quad U' := B^+_{R_1}(z_0), \quad \text{and} \quad V = B^+_{R_0}(z_0), \tag{4.17}$$

and we recall that $B^+_R(z_0) := B_R(z_0) \cap \mathscr{O}$, for any $R > 0$ and $z_0 \in \mathbb{R}^2$. Note that $U \Subset \underline{U}'$ and $U' \Subset \underline{V}$.

Proposition 4.11 (Interior H^2 regularity for higher-order derivatives of a solution with respect to x)**.** *Let $R < R_0$ be positive constants and let $k \geq 1$ be an integer. Then there is a positive constant, $C = C(A, k, R, R_0)$, such that the following holds. Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain, let $f \in L^2(\mathscr{O}, \mathfrak{w})$, and suppose that $u \in H^1(\mathscr{O}, \mathfrak{w})$ is a solution to the variational equation* (2.11)*. If $U \subset V$ are as in* (4.17) *with $V \subset \mathscr{O}$, and*

$$D_x^j f \in L^2(V, \mathfrak{w}), \quad 1 \leq j \leq k,$$

then

$$D_x^k u \in H^2(U, \mathfrak{w}),$$

and

$$\|D_x^k u\|_{H^2(U,\mathfrak{w})} \leq C \left(\sum_{j=0}^{k} \|D_x^j f\|_{L^2(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})} \right). \tag{4.18}$$

Proof. Choose an auxiliary half-ball, U' as in (4.17), with $U' = B^+_{R_1}(z_0)$ and $U \subset U' \subset V$, and fix $R_1 = (R + R_0)/2$. Since $D_x^j f \in L^2(V, \mathfrak{w})$, $1 \leq j \leq k$ by hypothesis, we can apply Proposition 4.10 to give $D_x^k u \in H^1(U', \mathfrak{w})$ and

$$\mathfrak{a}(D_x^k u, v) = (D_x^k f, v)_{L^2(U',\mathfrak{w})}, \quad \forall v \in H^1_0(\underline{U}', \mathfrak{w}).$$

We can now apply Theorem 3.16 to the preceding variational equation to give $D_x^k u \in H^2(U, \mathfrak{w})$ and

$$\|D_x^k u\|_{H^2(U,\mathfrak{w})} \leq C \left(\|D_x^k f\|_{L^2(U',\mathfrak{w})} + \|D_x^k u\|_{L^2(U',\mathfrak{w})} \right),$$

where $C = C(A, R, R_1)$. But

$$\|D_x^k u\|_{L^2(U',\mathfrak{w})} \leq \|D_x^k u\|_{H^1(U',\mathfrak{w})},$$

and by Proposition 4.10, we obtain

$$\|D_x^k u\|_{H^1(U',\mathfrak{w})} \leq C \left(\sum_{j=0}^{k} \|D_x^j f\|_{L^2(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})} \right),$$

where $C = C(A, k, R_1, R_0)$. Combining the preceding estimates completes the proof. □

4.3. **Interior H^2 regularity for first-order derivatives orthogonal to the degenerate boundary.** We have the following analogue of Lemma 4.6. Observe that if $u \in H^2(\mathscr{O}, \mathfrak{w})$, then the definition (3.9) of $H^2(\mathscr{O}, \mathfrak{w})$ implies that $y|D^2 u|$, $(1+y)|Du| \in L^2(\mathscr{O}, \mathfrak{w})$ and so $y^{1/2}|Du_y|$, $(1+y)^{1/2} u_y \in L^2(\mathscr{O}, y\mathfrak{w}) = L^2(\mathscr{O}, \mathfrak{w}_1)$ and thus $u_y \in H^1(\mathscr{O}, \mathfrak{w}_1)$ by the definition (2.2) of $H^1(\mathscr{O}, \mathfrak{w})$.

Lemma 4.12 (Variational equation for the derivative of a solution with respect to y)**.** *Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain, let $f \in L^2(\mathscr{O}, \mathfrak{w})$, and suppose that $u \in H^1(\mathscr{O}, \mathfrak{w})$ satisfies the variational equation* (2.11)*. If*[16]

$$f \in H^1(\mathscr{O}, \mathfrak{w}), \quad u \in H^2(\mathscr{O}, \mathfrak{w}), \quad \text{and} \quad u_{xx} \in L^2(\mathscr{O}, \mathfrak{w}_1),$$

then u_y obeys

$$\mathfrak{a}_1(u_y, v) = (f_y, v)_{L^2(\mathscr{O},\mathfrak{w}_1)} - (Bu, v)_{L^2(\mathscr{O},\mathfrak{w}_1)}, \tag{4.19}$$

for all $v \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w}_1)$.

[16]While the right-hand side of the identity (4.19) is well-defined when $f_y \in L^2(\mathscr{O}, \mathfrak{w}_1)$, we appeal to an approximation argument requiring at least $f \in H^1(\mathscr{O}, \mathfrak{w})$ to justify integration by parts involving f.

Proof. Again, suppose first that $u \in C^\infty(\bar{\mathscr{O}})$ and $v \in C_0^\infty(\underline{\mathscr{O}})$. Then $(yv)_y \in C_0^\infty(\underline{\mathscr{O}})$ too and

$$\begin{aligned}
\mathfrak{a}(u,(yv)_y) &= (Au,(yv)_y)_{L^2(\mathscr{O},\mathfrak{w})} \quad \text{(by Lemma A.3)}\\
&= -((Au)_y, yv)_{L^2(\mathscr{O},\mathfrak{w})} - (Au, yv(\log\mathfrak{w})_y)_{L^2(\mathscr{O},\mathfrak{w})}\\
&= -(A_1u_y, yv)_{L^2(\mathscr{O},\mathfrak{w})} - (Bu, yv)_{L^2(\mathscr{O},\mathfrak{w})} - (Au, yv(\log\mathfrak{w})_y)_{L^2(\mathscr{O},\mathfrak{w})} \quad \text{(by (4.4))}\\
&= -(A_1u_y, v)_{L^2(\mathscr{O},\mathfrak{w}_1)} - (Bu, v)_{L^2(\mathscr{O},\mathfrak{w}_1)} - (Au, yv(\log\mathfrak{w})_y)_{L^2(\mathscr{O},\mathfrak{w})} \quad \text{(by (4.6))}\\
&= -\mathfrak{a}_1(u_y, v) - (Bu, v)_{L^2(\mathscr{O},\mathfrak{w}_1)} - (Au, yv(\log\mathfrak{w})_y)_{L^2(\mathscr{O},\mathfrak{w})} \quad \text{(by (4.7))},
\end{aligned}$$

where from (2.5) we see that

$$(\log\mathfrak{w})_y = (\beta-1)y^{-1} - \mu \quad \text{on } \mathbb{H}.$$

As in the proof of Lemma 4.6, for $v \in C_0^\infty(\underline{\mathscr{O}})$, we may choose a subdomain $\mathscr{O}' \Subset \underline{\mathscr{O}}$ such that $\operatorname{supp} v \subset \underline{\mathscr{O}}'$ and $\partial_1\mathscr{O}'$ is C^1-orthogonal to $\partial\mathbb{H}$. If we now assume only that $u \in H^2(\mathscr{O},\mathfrak{w})$ and $u_{xx} \in L^2(\mathscr{O},\mathfrak{w}_1)$, as in our hypotheses, there is a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty(\bar{\mathscr{O}}')$ such that $u_n \to u$ in $H^2(\mathscr{O}',\mathfrak{w})$ as $n\to\infty$ by Theorem A.2. But then $u_{n,xx} \rightharpoonup u_{xx}$ weakly in $L^2(\mathscr{O}',\mathfrak{w}_1)$ as $n\to\infty$ since, for $v \in C_0^\infty(\underline{\mathscr{O}})$ with $\operatorname{supp} \subset \underline{\mathscr{O}}'$ and all $n \in \mathbb{N}$,

$$\begin{aligned}
\left|(u_{n,xx} - u_{xx}, v)_{L^2(\mathscr{O},\mathfrak{w}_1)}\right| &= \left|(yu_{n,xx} - yu_{xx}, v)_{L^2(\mathscr{O},\mathfrak{w})}\right|\\
&\le \|y(u_{n,xx} - u_{xx})\|_{L^2(\mathscr{O},\mathfrak{w})}\|v\|_{L^2(\mathscr{O},\mathfrak{w})}\\
&\le \|u_n - u\|_{H^2(\mathscr{O},\mathfrak{w})}\|v\|_{L^2(\mathscr{O},\mathfrak{w})}.
\end{aligned}$$

Therefore, by approximation, the variational identity continues to hold for $u \in H^2(\mathscr{O},\mathfrak{w})$, which ensures $u_x \in L^2(\mathscr{O},\mathfrak{w}) \subset L^2(\mathscr{O},\mathfrak{w}_1)$, and $u_{xx} \in L^2(\mathscr{O},\mathfrak{w}_1)$ (thus $Bu \in L^2(\mathscr{O},\mathfrak{w}_1)$), that is,

$$\begin{aligned}
\mathfrak{a}(u,(yv)_y) = -\mathfrak{a}_1(u_y, v) &- (Bu, v)_{L^2(\mathscr{O},\mathfrak{w}_1)}\\
&- (Au, yv(\log\mathfrak{w})_y)_{L^2(\mathscr{O},\mathfrak{w})}, \quad \forall v \in C_0^\infty(\underline{\mathscr{O}}).
\end{aligned}$$

Also, since $u \in H^2(\mathscr{O},\mathfrak{w})$, then (2.11) implies that $Au = f$ a.e. on $\mathscr{O}$ by Lemma A.3. Hence, (2.11) and the fact that $f \in H^1(\mathscr{O},\mathfrak{w})$ and thus $f_y \in L^2(\mathscr{O},\mathfrak{w}_1)$ by hypothesis, yields

$$\begin{aligned}
\mathfrak{a}(u,(yv)_y) &= (f,(yv)_y)_{L^2(\mathscr{O},\mathfrak{w})}\\
&= -(f_y, yv)_{L^2(\mathscr{O},\mathfrak{w})} - (f, yv(\log\mathfrak{w})_y)_{L^2(\mathscr{O},\mathfrak{w})}\\
&= -(f_y, v)_{L^2(\mathscr{O},\mathfrak{w}_1)} - (f, yv(\log\mathfrak{w})_y)_{L^2(\mathscr{O},\mathfrak{w})}, \quad \forall v \in C_0^\infty(\underline{\mathscr{O}}),
\end{aligned}$$

while the preceding variational identity gives

$$\begin{aligned}
\mathfrak{a}(u,(yv)_y) = -\mathfrak{a}_1(u_y, v) &- (Bu, v)_{L^2(\mathscr{O},\mathfrak{w}_1)}\\
&- (f, yv(\log\mathfrak{w})_y)_{L^2(\mathscr{O},\mathfrak{w})}, \quad \forall v \in C_0^\infty(\underline{\mathscr{O}}).
\end{aligned}$$

Combining these variational identities yields (4.19), for all $v \in C_0^\infty(\underline{\mathscr{O}})$, and hence the variational identity holds for all $v \in H_0^1(\underline{\mathscr{O}},\mathfrak{w})$. □

Proposition 4.13 (Interior H^2 regularity for a derivative of a solution with respect to y)**.** *Let $R < R_0$ be positive constants. Then there is a positive constant, $C = C(A,R,R_0)$, such that the following holds. Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain and let $U \subset V$ be as in (4.17), with $V \subset \mathscr{O}$. Suppose that $f \in L^2(\mathscr{O},\mathfrak{w})$ and $u \in H^1(\mathscr{O},\mathfrak{w})$ is a solution to the variational equation (2.11). If*

$$f \in W^{1,2}(V,\mathfrak{w}), \quad u \in H^2(V,\mathfrak{w}), \quad \text{and} \quad u_x \in H^1(V,\mathfrak{w}),$$

then $u_y \in H^2(U,\mathfrak{w}_1)$ and

$$\|u_y\|_{H^2(U,\mathfrak{w}_1)} \le C\left(\|f\|_{W^{1,2}(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})}\right). \tag{4.20}$$

Proof. The argument is similar to the proof of Proposition 4.11, except that the appeal to Proposition 4.9 is replaced by an appeal to Lemma 4.12 and we need to keep track of the different Sobolev weights which now arise. Notice that $u \in H^1(\mathscr{O}, \mathfrak{w})$ by hypothesis, and so $y^{1/2}u_x \in L^2(V, \mathfrak{w})$ or equivalently $u_x \in L^2(V, \mathfrak{w}_1)$. Moreover, $f \in H^1(V, \mathfrak{w})$, since $f \in W^{1,2}(V, \mathfrak{w})$ by hypothesis. Also, $u_x \in H^1(V, \mathfrak{w})$ by hypothesis, and so $y^{1/2}u_{xx} \in L^2(V, \mathfrak{w})$ or, equivalently, $u_{xx} \in L^2(V, \mathfrak{w}_1)$. Finally, the hypothesis $u \in H^2(\mathscr{O}, \mathfrak{w})$ implies $u_y \in H^1(V, \mathfrak{w}_1)$. Therefore, Lemma 4.12, with V in place of $\mathscr{O}$, gives

$$\mathfrak{a}_1(u_y, v) = (f_y, v)_{L^2(V,\mathfrak{w}_1)} - (Bu, v)_{L^2(V,\mathfrak{w}_1)}, \quad \forall v \in H^1_0(\underline{V}, \mathfrak{w}_1).$$

Choose an auxiliary half-ball, U' as in (4.17), with $U' = B^+_{R_1}(z_0)$ and $U \subset U' \subset V$, and fix $R_1 = (R + R_0)/2$. We can apply Theorem 3.16 to the preceding equation in place of (2.11) to deduce that $u_y \in H^2(U, \mathfrak{w}_1)$ and

$$\begin{aligned} \|u_y\|_{H^2(U,\mathfrak{w}_1)} &\leq C \left(\|f_y - Bu\|_{L^2(U',\mathfrak{w}_1)} + \|u_y\|_{L^2(U',\mathfrak{w}_1)}\right), \\ &\leq C \left(\|f_y\|_{L^2(U',\mathfrak{w}_1)} + \|u_x\|_{L^2(U',\mathfrak{w}_1)} + \|u_{xx}\|_{L^2(U',\mathfrak{w}_1)} + \|u_y\|_{L^2(U',\mathfrak{w}_1)}\right), \end{aligned}$$

where $C = C(A, R, R_1)$. But

$$\|Du\|_{L^2(U',\mathfrak{w}_1)} \leq \|u\|_{H^2(U',\mathfrak{w}_1)} \leq C\|u\|_{H^2(U',\mathfrak{w})},$$

where the first inequality follows from (3.9) and the second from (4.10), with $C = C(R_1)$. By Theorem 3.16, since u obeys (2.11), we obtain

$$\|u\|_{H^2(U',\mathfrak{w})} \leq C \left(\|f\|_{L^2(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})}\right),$$

where $C = C(A, R_1, R_0)$. Finally,

$$\|u_{xx}\|_{L^2(U',\mathfrak{w}_1)} \leq \|u_x\|_{H^2(U',\mathfrak{w}_1)} \leq C\|u_x\|_{H^2(U',\mathfrak{w})},$$

and applying Proposition 4.11, we obtain

$$\|u_x\|_{H^2(U',\mathfrak{w})} \leq C \left(\|f_x\|_{L^2(V,\mathfrak{w})} + \|f\|_{L^2(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})}\right).$$

Combining the preceding estimates gives

$$\begin{aligned} \|u_y\|_{H^2(U,\mathfrak{w}_1)} &\leq C \left(\|f_y\|_{L^2(U',\mathfrak{w}_1)} + \|f_x\|_{L^2(V,\mathfrak{w})} + \|f\|_{L^2(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})}\right) \\ &\leq C \left(\|f_y\|_{L^2(V,\mathfrak{w})} + \|f_x\|_{L^2(V,\mathfrak{w})} + \|f\|_{L^2(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})}\right), \end{aligned}$$

and this completes the proof. □

4.4. **Interior $\mathscr{H}^3$ regularity.** By combining Propositions 4.11 and 4.13, we obtain

Theorem 4.14 (Interior $\mathscr{H}^3$ regularity)**.** *Let $R < R_0$ be positive constants. Then there is a positive constant, $C = C(A, R, R_0)$, such that the following holds. Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain and let $U \subset V$ be as in* (4.17)*, with $V \subset \mathscr{O}$. Suppose that $f \in L^2(\mathscr{O}, \mathfrak{w})$ and that $u \in H^1(\mathscr{O}, \mathfrak{w})$ is a solution to the variational equation* (2.11)*. If*

$$f \in W^{1,2}(V, \mathfrak{w}),$$

then $u \in \mathscr{H}^3(U, \mathfrak{w})$ and

$$\|u\|_{\mathscr{H}^3(U,\mathfrak{w})} \leq C \left(\|f\|_{W^{1,2}(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})}\right). \tag{4.21}$$

Proof. Since $f \in L^2(\mathscr{O}, \mathfrak{w})$, Theorem 3.16 implies that $u \in H^2(V, \mathfrak{w})$. Choose an auxiliary half-ball, U' as in (4.17), with $U' = B^+_{R_1}(z_0)$ and $U \subset U' \subset V$, and fix $R_1 = (R + R_0)/2$. By hypothesis, we have $f \in W^{1,2}(V, \mathfrak{w})$ and so Proposition 4.11 yields $u_x \in H^2(U', \mathfrak{w})$ and

$$\begin{aligned}\|yu_{xxx}\|_{L^2(U',\mathfrak{w})} &+ \|yu_{xxy}\|_{L^2(U',\mathfrak{w})} + \|(1+y)u_{xx}\|_{L^2(U',\mathfrak{w})} + \|(1+y)u_{xy}\|_{L^2(U',\mathfrak{w})}\\ &\leq \|u_x\|_{H^2(U',\mathfrak{w})}\\ &\leq C\left(\|f_x\|_{L^2(V,\mathfrak{w})} + \|f\|_{L^2(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})}\right).\end{aligned}$$

Because $f \in W^{1,2}(U', \mathfrak{w})$ by hypothesis, and $u \in H^2(U', \mathfrak{w})$, and $u_x \in H^1(U', \mathfrak{w})$ (since $u_x \in H^2(U', \mathfrak{w})$), then Proposition 4.13 gives $u_y \in H^2(U, \mathfrak{w}_1)$ and

$$\begin{aligned}\|yu_{yyy}\|_{L^2(U,\mathfrak{w}_1)} &+ \|(1+y)u_{yy}\|_{L^2(U,\mathfrak{w}_1)}\\ &\leq \|u_y\|_{H^2(U,\mathfrak{w}_1)}\\ &\leq C\left(\|Df\|_{L^2(V,\mathfrak{w})} + \|f\|_{L^2(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})}\right).\end{aligned}$$

Because $u \in H^2(V, \mathfrak{w})$ by hypothesis, we obtain $u \in \mathscr{H}^3(U, \mathfrak{w})$ from Definition 4.3 since

$$\begin{aligned}\|u\|^2_{\mathscr{H}^3(U,\mathfrak{w})} = \|yu_{xxx}\|^2_{L^2(U,\mathfrak{w})} &+ \|yu_{xxy}\|^2_{L^2(U,\mathfrak{w})} + \|yu_{yyy}\|^2_{L^2(U,\mathfrak{w}_1)}\\ &+ \|(1+y)u_{xx}\|^2_{L^2(U,\mathfrak{w})} + \|(1+y)u_{xy}\|^2_{L^2(U,\mathfrak{w})} + \|(1+y)u_{yy}\|^2_{L^2(U,\mathfrak{w}_1)}\\ &+ \|(1+y)u_{x}\|^2_{L^2(U,\mathfrak{w})} + \|(1+y)u_{y}\|^2_{L^2(U,\mathfrak{w})} + \|(1+y)^{1/2}u\|^2_{L^2(U,\mathfrak{w})},\end{aligned}$$

and hence

$$\begin{aligned}\|u\|^2_{\mathscr{H}^3(U,\mathfrak{w})} \leq \|yu_{xxx}\|^2_{L^2(U,\mathfrak{w})} &+ \|yu_{xxy}\|^2_{L^2(U,\mathfrak{w})} + \|yu_{yyy}\|^2_{L^2(U,\mathfrak{w}_1)}\\ &+ \|(1+y)u_{xx}\|^2_{L^2(U,\mathfrak{w})} + \|(1+y)u_{xy}\|^2_{L^2(U,\mathfrak{w})} + \|(1+y)u_{yy}\|^2_{L^2(U,\mathfrak{w}_1)}\\ &+ \|u\|^2_{H^2(U,\mathfrak{w})}.\end{aligned} \tag{4.22}$$

Since u obeys (2.11), Theorem 3.16 yields

$$\|u\|_{H^2(U,\mathfrak{w})} \leq C\left(\|f\|_{L^2(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})}\right),$$

and combining the preceding estimates gives (4.21). □

4.5. **Interior $\mathscr{H}^{k+2}$ regularity.** We can iterate the preceding arguments, used to establish $u \in \mathscr{H}^3(U, \mathfrak{w})$ given $u \in H^2(V, \mathfrak{w})$ and additional hypotheses on f, to give higher-order Sobolev regularity, where $U \subset V$ are as in (4.17) and $V \subset \mathscr{O}$. We begin with the following combined generalization of Lemmas 4.8 and 4.12.

Proposition 4.15 (Variational equation for higher-order derivatives of a solution with respect to x and y)**.** *Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain with* finite height[17], *let $k \geq 1$ and $0 \leq m \leq k$ be integers, let $f \in L^2(\mathscr{O}, \mathfrak{w})$, and suppose that $u \in H^1(\mathscr{O}, \mathfrak{w})$ satisfies the variational equation* (2.11)*. If*[18]

$$f \in W^{k,2}(\mathscr{O}, \mathfrak{w}), \quad u \in \mathscr{H}^{k+1}(\mathscr{O}, \mathfrak{w}), \quad \text{and} \quad D_x^k u \in H^1(\mathscr{O}, \mathfrak{w}) \quad (m = 0, 1),$$

then $D_x^{k-m} D_y^m u \in H^1(\mathscr{O}, \mathfrak{w})$ obeys

$$\mathfrak{a}_m(D_x^{k-m} D_y^m u, v) = (D_x^{k-m} D_y^m f, v)_{L^2(\mathscr{O},\mathfrak{w}_m)} - m(BD_x^{k-m} D_y^{m-1} u, v)_{L^2(\mathscr{O},\mathfrak{w}_m)}, \tag{4.23}$$

for all $v \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w}_m)$.

[17] Proposition 4.15 should, of course, hold without a hypothesis that $\mathscr{O}$ has finite height, but its already technical proof is simpler with this hypothesis included and we shall only apply the result to domains of finite height.

[18] The need for the auxiliary condition, $D_x^k u \in H^1(\mathscr{O}, \mathfrak{w})$, when $m = 0, 1$ is explained in Appendix A.5.

Remark 4.16 (Need for the auxiliary regularity condition when $m = 0, 1$). The role of the auxiliary regularity condition, $D_x^k u \in H^1(\mathscr{O}, \mathfrak{w})$, when $m = 0$ or 1 is explained in Appendix A.5.

Proof of Proposition 4.15. Lemma 4.8 implies that (4.23) holds when $m = 0$ and *any* $k \geq 1$, while Lemma 4.12 gives the conclusion when $k = m = 1$. So we may assume without loss of generality that $k \geq 2$ and $m \geq 1$ in our proof of Proposition 4.15. Therefore, to establish (4.23), it suffices to consider the inductive step $(k, m-1) \implies (k, m)$ (one extra derivative with respect to y), assuming (4.23) holds with m replaced by $m - 1$. The argument for this inductive step follows the pattern of proof of Lemma 4.12.

As usual, suppose first that $u \in C^\infty(\bar{\mathscr{O}})$ and $v \in C_0^\infty(\underline{\mathscr{O}})$. Then $(yv)_y \in C_0^\infty(\underline{\mathscr{O}})$ too and

$$
\begin{aligned}
&\mathfrak{a}_{m-1}(D_x^{k-m} D_y^{m-1} u, (yv)_y) \\
&= (A_{m-1} D_x^{k-m} D_y^{m-1} u, (yv)_y)_{L^2(\mathscr{O}, \mathfrak{w}_{m-1})} \quad \text{(by Lemma A.3)} \\
&= -((A_{m-1} D_x^{k-m} D_y^{m-1} u)_y, yv)_{L^2(\mathscr{O}, \mathfrak{w}_{m-1})} - (A_{m-1} D_x^{k-m} D_y^{m-1} u, yv(\log \mathfrak{w})_y)_{L^2(\mathscr{O}, \mathfrak{w}_{m-1})} \\
&= -(A_m D_x^{k-m} D_y^m u, yv)_{L^2(\mathscr{O}, \mathfrak{w}_{m-1})} - (B D_x^{k-m} D_y^{m-1} u, yv)_{L^2(\mathscr{O}, \mathfrak{w}_{m-1})} \\
&\quad - (A_{m-1} D_x^{k-m} D_y^{m-1} u, yv(\log \mathfrak{w})_y)_{L^2(\mathscr{O}, \mathfrak{w}_{m-1})} \quad \text{(by (4.4))},
\end{aligned}
$$

that is, by Lemma A.3 and (4.6),

$$
\begin{aligned}
\mathfrak{a}_{m-1}(D_x^{k-m} D_y^{m-1} u, (yv)_y) = -\mathfrak{a}_m(D_x^{k-m} D_y^m u, v) &- (B D_x^{k-m} D_y^{m-1} u, v)_{L^2(\mathscr{O}, \mathfrak{w}_m)} \\
&- (A_{m-1} D_x^{k-m} D_y^{m-1} u, yv(\log \mathfrak{w})_y)_{L^2(\mathscr{O}, \mathfrak{w}_{m-1})}, \\
&\forall v \in C_0^\infty(\underline{\mathscr{O}}).
\end{aligned} \tag{4.24}
$$

We next establish the

Claim 4.17. *The identity* (4.24) *continues to hold when the requirement* $u \in C^\infty(\bar{\mathscr{O}})$ *is relaxed to* $u \in \mathscr{H}^{k+1}(\mathscr{O}, \mathfrak{w})$ *together with, when* $m = 1$, $D_x^{k+1} u \in L^2(\mathscr{O}, \mathfrak{w}_1)$.

Proof. The terms in the right and left-hand sides of the identity (4.24) are well-defined when

$$
\begin{gathered}
D_x^{k-m} D_y^{m-1} u \in H^1(\mathscr{O}, \mathfrak{w}_{m-1}), \quad D_x^{k-m} D_y^m u \in H^1(\mathscr{O}, \mathfrak{w}_m), \\
D_x^{k+1-m} D_y^{m-1} u,\ D_x^{k+2-m} D_y^{m-1} u \in L^2(\mathscr{O}, \mathfrak{w}_m), \\
D_x^{k-m} D_y^{m-1} u \in H^2(\mathscr{O}, \mathfrak{w}_{m-1}).
\end{gathered}
$$

We consider each of the five preceding terms in turn. First, according to Definition 4.3, we have that $u \in \mathscr{H}^{k+1}(\mathscr{O}, \mathfrak{w})$ implies

$$
\begin{aligned}
D_x^{k-m} D_y^{m-1} u &\in \begin{cases} L^2(\mathscr{O}, \mathfrak{w}_{m-2}), & m \geq 3, \\ L^2(\mathscr{O}, \mathfrak{w}), & m = 1, 2, \end{cases} \\
D_x^{k+1-m} D_y^{m-1} u &\in \begin{cases} L^2(\mathscr{O}, \mathfrak{w}_{m-2}), & m \geq 3, \\ L^2(\mathscr{O}, \mathfrak{w}), & m = 1, 2, \end{cases} \\
D_x^{k-m} D_y^m u &\in \begin{cases} L^2(\mathscr{O}, \mathfrak{w}_{m-1}), & m \geq 2, \\ L^2(\mathscr{O}, \mathfrak{w}), & m = 1. \end{cases}
\end{aligned}
$$

Since $L^2(\mathscr{O}, \mathfrak{w}_{m-2}) \subset L^2(\mathscr{O}, \mathfrak{w}_{m-1})$ (any $m \geq 2$) and $L^2(\mathscr{O}, \mathfrak{w}) \subset L^2(\mathscr{O}, \mathfrak{w}_{m-1})$ (any $m \geq 1$), then

$$
D_x^{k-m} D_y^{m-1} u, \quad y^{1/2} D_x^{k+1-m} D_y^{m-1} u, \quad y^{1/2} D_x^{k-m} D_y^m u \in L^2(\mathscr{O}, \mathfrak{w}_{m-1}), \quad 1 \leq m \leq k,
$$

and the definition (2.2) of $H^1(\mathscr{O},\mathfrak{w}_{m-1})$ gives

$$u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w}) \implies D_x^{k-m}D_y^{m-1}u \in H^1(\mathscr{O},\mathfrak{w}_{m-1}), \quad 1 \leq m \leq k.$$

Second, according to Definition 4.3, we see $u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w})$ implies

$$\begin{aligned} D_x^{k-m}D_y^m u &\in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-1}), & m \geq 2, \\ L^2(\mathscr{O},\mathfrak{w}), & m = 1, \end{cases}, \\ yD_x^{k+1-m}D_y^m u &\in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-2}), & m \geq 3, \\ L^2(\mathscr{O},\mathfrak{w}), & m = 1,2, \end{cases} \\ yD_x^{k-m}D_y^{m+1} u &\in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-1}), & m \geq 2, \\ L^2(\mathscr{O},\mathfrak{w}), & m = 1, \end{cases} \end{aligned}$$

that is,

$$\begin{aligned} D_x^{k-m}D_y^m u &\in L^2(\mathscr{O},\mathfrak{w}_{m-1}), \quad m \geq 1, \\ D_x^{k+1-m}D_y^m u &\in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m}), & m \geq 3, \\ L^2(\mathscr{O},\mathfrak{w}_2), & m = 1,2, \end{cases} \\ D_x^{k-m}D_y^{m+1} u &\in L^2(\mathscr{O},\mathfrak{w}_{m+1}), \quad m \geq 1. \end{aligned}$$

Therefore, using $L^2(\mathscr{O},\mathfrak{w}_{m-1}) \subset L^2(\mathscr{O},\mathfrak{w}_m)$ (any $m \geq 1$), we obtain

$$D_x^{k-m}D_y^m u, \quad y^{1/2}D_x^{k+1-m}D_y^m u, \quad y^{1/2}D_x^{k-m}D_y^{m+1}u \in L^2(\mathscr{O},\mathfrak{w}_m), \quad 1 \leq m \leq k,$$

and the definition (2.2) of $H^1(\mathscr{O},\mathfrak{w}_m)$ gives

$$u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w}) \implies D_x^{k-m}D_y^m u \in H^1(\mathscr{O},\mathfrak{w}_m), \quad 1 \leq m \leq k.$$

Third, we have seen that $u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w})$ implies

$$D_x^{k+1-m}D_y^{m-1}u \in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-2}), & m \geq 3, \\ L^2(\mathscr{O},\mathfrak{w}), & m = 1,2, \end{cases}$$

and so, using $L^2(\mathscr{O},\mathfrak{w}_{m-2}) \subset L^2(\mathscr{O},\mathfrak{w}_m)$ (any $m \geq 2$), we obtain

$$D_x^{k+1-m}D_y^{m-1}u \in L^2(\mathscr{O},\mathfrak{w}_m), \quad 1 \leq m \leq k.$$

For the fifth term, $D_x^{k-m}D_y^{m-1}u$ (we shall consider the fourth term last), observe that $u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w})$ implies

$$yD_x^{k+2-m}D_y^{m-1}u \in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-3}), & m \geq 4,\\ L^2(\mathscr{O},\mathfrak{w}), & m = 1,2,3,\end{cases}$$

$$yD_x^{k+1-m}D_y^{m}u \in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-2}), & m \geq 3,\\ L^2(\mathscr{O},\mathfrak{w}), & m = 1,2,\end{cases}$$

$$yD_x^{k-m}D_y^{m+1}u \in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-1}), & m \geq 2,\\ L^2(\mathscr{O},\mathfrak{w}), & m = 1,\end{cases}$$

$$D_x^{k+1-m}D_y^{m-1}u \in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-2}), & m \geq 3,\\ L^2(\mathscr{O},\mathfrak{w}), & m = 1,2,\end{cases}$$

$$D_x^{k-m}D_y^{m}u \in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-1}), & m \geq 2,\\ L^2(\mathscr{O},\mathfrak{w}), & m = 1,\end{cases}$$

$$D_x^{k-m}D_y^{m-1}u \in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-2}), & m \geq 3,\\ L^2(\mathscr{O},\mathfrak{w}), & m = 1,2.\end{cases}$$

Hence, from the definition (3.9) of $H^2(\mathscr{O},\mathfrak{w}_{m-1})$, we see that

$$D_x^{k-m}D_y^{m-1}u \in H^2(\mathscr{O},\mathfrak{w}_{m-1}), \quad 1 \leq m \leq k.$$

Finally, considering the fourth term[19], observe that for each $v \in C_0^\infty(\underline{\mathscr{O}})$, we may choose a subdomain $\mathscr{O}' \Subset \underline{\mathscr{O}}$ such that $\operatorname{supp} v \subset \underline{\mathscr{O}}'$ and $\partial_1\mathscr{O}'$ is C^1-orthogonal to $\partial\mathbb{H}$ in the sense of Definition A.1. According to Theorem A.2, there is a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty(\bar{\mathscr{O}}')$ such that $u_n \to u$ in $\mathscr{H}^{k+1}(\mathscr{O}',\mathfrak{w})$ as $n \to \infty$ and hence, for each $v \in C_0^\infty(\underline{\mathscr{O}})$ with $\operatorname{supp} v \subset \underline{\mathscr{O}}'$, we have

$$(D_x^{k+2-m}D_y^{m-1}u_n, v)_{L^2(\mathscr{O},\mathfrak{w}_m)} \to (D_x^{k+2-m}D_y^{m-1}u, v)_{L^2(\mathscr{O},\mathfrak{w}_m)} \quad \text{as } n \to \infty,$$

since, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\left|(D_x^{k+2-m}D_y^{m-1}(u_n-u), v)_{L^2(\mathscr{O},\mathfrak{w}_m)}\right| \\
&= \left|(y^m D_x^{k+2-m}D_y^{m-1}(u_n-u), v)_{L^2(\mathscr{O},\mathfrak{w})}\right| \\
&\leq \|y^m D_x^{k+2-m}D_y^{m-1}(u_n-u)\|_{L^2(\mathscr{O},\mathfrak{w})}\|v\|_{L^2(\mathscr{O},\mathfrak{w})} \\
&\leq C\|u_n-u\|_{\mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w})}\|v\|_{L^2(\mathscr{O},\mathfrak{w})},
\end{aligned}$$

where $C = C(\operatorname{height}(\mathscr{O}))$, noting that $u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w})$ implies, by Definition 4.3,

$$yD_x^{k+2-m}D_y^{m-1}u \in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-3}), & m \geq 4,\\ L^2(\mathscr{O},\mathfrak{w}), & m = 1,2,3,\end{cases}$$

[19] As explained in Appendix A.5, it is only in the case $m=1$ that $D_x^{k+2-m}D_y^{m-1}u \in L^2(\mathscr{O},\mathfrak{w}_m)$ is not implied by $u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w})$ and this case is explicitly covered by the additional hypothesis, $D_x^k u \in H^1(\mathscr{O},\mathfrak{w})$, which ensures, by definition (2.2) of $H^1(\mathscr{O},\mathfrak{w})$, that $y^{1/2}D_x^{k+1}u \in L^2(\mathscr{O},\mathfrak{w})$ or, equivalently, $D_x^{k+1}u \in L^2(\mathscr{O},\mathfrak{w}_1)$.

and, for $C = C(\text{height}(\mathscr{O}))$,

$$\begin{aligned}\|y^m D_x^{k+2-m} D_y^{m-1}(u_n - u)\|_{L^2(\mathscr{O},\mathfrak{w})} &= \|y^{3+(m-3)/2} D_x^{k+2-m} D_y^{m-1}(u_n - u)\|_{L^2(\mathscr{O},\mathfrak{w}_{m-3})}\\ &\leq C\|y D_x^{k+2-m} D_y^{m-1}(u_n - u)\|_{L^2(\mathscr{O},\mathfrak{w}_{m-3})}\\ &\leq C\|u_n - u\|_{\mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w})}, \quad m \geq 4,\\ \|y^m D_x^{k+2-m} D_y^{m-1}(u_n - u)\|_{L^2(\mathscr{O},\mathfrak{w})} &= \|y^{1+(m-1)} D_x^{k+2-m} D_y^{m-1}(u_n - u)\|_{L^2(\mathscr{O},\mathfrak{w})}\\ &\leq \|y D_x^{k+2-m} D_y^{m-1}(u_n - u)\|_{L^2(\mathscr{O},\mathfrak{w})}\\ &\leq C\|u_n - u\|_{\mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w})}, \quad m = 1, 2, 3.\end{aligned}$$

Therefore, by approximation, the identity (4.24) continues to hold for $u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w})$ and, when $m = 1$, that $D_x^{k+1} u \in L^2(\mathscr{O},\mathfrak{w}_1)$. This completes the proof of Claim 4.17. $\square$

By induction on m, the identity (4.23) holds for $(k-1, m-1)$ in place of (k, m), and so for all $v \in C_0^\infty(\underline{\mathscr{O}})$ and thus $(yv)_y \in C_0^\infty(\underline{\mathscr{O}})$, we have

$$\begin{aligned}&\mathfrak{a}_{m-1}(D_x^{k-m} D_y^{m-1} u, (yv)_y)\\ &= (D_x^{k-m} D_y^{m-1} f, (yv)_y)_{L^2(\mathscr{O},\mathfrak{w}_{m-1})} - (m-1)(B D_x^{k-m} D_y^{m-2} u, (yv)_y)_{L^2(\mathscr{O},\mathfrak{w}_{m-1})}.\end{aligned}$$

Therefore, integrating by parts with respect to y on the right-hand side of the preceding identity and applying (4.6) yields

$$\begin{aligned}&\mathfrak{a}_{m-1}(D_x^{k-m} D_y^{m-1} u, (yv)_y)\\ &= -(D_x^{k-m} D_y^{m} f, v)_{L^2(\mathscr{O},\mathfrak{w}_m)} + (m-1)(B D_x^{k-m} D_y^{m-1} u, v)_{L^2(\mathscr{O},\mathfrak{w}_m)}\\ &\quad - (D_x^{k-m} D_y^{m-1} f, yv(\log\mathfrak{w})_y)_{L^2(\mathscr{O},\mathfrak{w}_{m-1})} + (m-1)(B D_x^{k-m} D_y^{m-2} u, yv(\log\mathfrak{w})_y)_{L^2(\mathscr{O},\mathfrak{w}_{m-1})},\\ &\qquad \forall v \in C_0^\infty(\underline{\mathscr{O}}).\end{aligned}$$

But $D_x^{k-m} D_y^{m-1} f \in L^2(\mathscr{O},\mathfrak{w}_{m-1})$, since $f \in W^{k,2}(\mathscr{O},\mathfrak{w})$ hypothesis, and $Au = f$ a.e. on $\mathscr{O}$ yields

$$\begin{aligned}D_x^{k-m} D_y^{m-1} f &= D_x^{k-m} D_y^{m-1} Au \quad \text{a.e. on } \mathscr{O}\\ &= A_{m-1} D_x^{k-m} D_y^{m-1} u + (m-1) B D_x^{k-m} D_y^{m-2} u \quad \text{a.e. on } \mathscr{O} \quad (\text{by } (4.4)),\end{aligned}$$

noting that $u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w})$ by hypothesis and so (by an analysis very similar to that in the proof of Claim 4.17),

$$A_{m-1} D_x^{k-m} D_y^{m-1} u,\ B D_x^{k-m} D_y^{m-2} u \in L^2(\mathscr{O},\mathfrak{w}_{m-1}).$$

Substituting this identity for $D_x^{k-m} D_y^{m-1} f$ into the preceding variational equation yields

$$\begin{aligned}&\mathfrak{a}_{m-1}(D_x^{k-m} D_y^{m-1} u, (yv)_y)\\ &= -(D_x^{k-m} D_y^{m} f, v)_{L^2(\mathscr{O},\mathfrak{w}_m)} + (m-1)(B D_x^{k-m} D_y^{m-1} u, v)_{L^2(\mathscr{O},\mathfrak{w}_m)}\\ &\quad - (A_{m-1} D_x^{k-m} D_y^{m-1} u, yv(\log\mathfrak{w})_y)_{L^2(\mathscr{O},\mathfrak{w}_{m-1})}, \quad \forall v \in C_0^\infty(\underline{\mathscr{O}}).\end{aligned} \tag{4.25}$$

Combining the variational equations (4.24) and (4.25) yields

$$\mathfrak{a}_m(D_x^{k-m} D_y^m u, v) = (D_x^{k-m} D_y^m f, v)_{L^2(\mathscr{O},\mathfrak{w}_m)} - m(B D_x^{k-m} D_y^{m-1} u, v)_{L^2(\mathscr{O},\mathfrak{w}_m)}, \quad v \in C_0^\infty(\underline{\mathscr{O}}),$$

and hence (4.23) holds for all $v \in H_0^1(\underline{\mathscr{O}},\mathfrak{w})$. This completes the proof of Proposition 4.15. $\square$

We now show that $D_x^{k-m} D_y^m u \in H^2(U,\mathfrak{w}_m)$, where $U \Subset \underline{\mathscr{O}}$ is as in (4.17), for any $k \geq 1$ and $0 \leq m \leq k$, and provide estimates for these derivatives analogous to those in Propositions 4.11 and Propositions 4.13.

Proposition 4.18 (Interior H^2 regularity for higher-order derivatives of a solution with respect to x and y)**.** *Let $R < R_0$ be positive constants and let $k \geq 1$ and $0 \leq m \leq k$ be integers. Then there is a positive constant, $C = C(A, k, m, R, R_0)$, such that the following holds. Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain and let $U \subset V$ be as in* (4.17)*, with $V \subset \mathscr{O}$. Suppose that $f \in L^2(\mathscr{O}, \mathfrak{w})$ and $u \in H^1(\mathscr{O}, \mathfrak{w})$ is a solution to the variational equation* (2.11)*. If*

$$f \in W^{k,2}(V, \mathfrak{w}) \quad \textit{and} \quad u \in H^2(V, \mathfrak{w}),$$

then

$$D_x^{k-m} D_y^m u \in H^2(U, \mathfrak{w}_m),$$

and

$$\|D_x^{k-m} D_y^m u\|_{H^2(U,\mathfrak{w}_m)} \leq C \left(\|f\|_{W^{k,2}(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})} \right). \tag{4.26}$$

Proof. Proposition 4.11 yields the conclusion for any $k \geq 1$, when $m = 0$, while Proposition 4.13 gives the conclusion when $k = m = 1$. Therefore, we may assume that $k \geq 2$ and $m \geq 1$.

By Proposition 4.15, we see that $D_x^{k-m} D_y^m u \in H^1(\mathscr{O}, \mathfrak{w})$ obeys (4.23), that is

$$\mathfrak{a}_m(D_x^{k-m} D_y^m u, v) = (D_x^{k-m} D_y^m f, v)_{L^2(V,\mathfrak{w}_m)} - m(B D_x^{k-m} D_y^{m-1} u, v)_{L^2(V,\mathfrak{w}_m)},$$

for all $v \in H^1_0(\underline{V}, \mathfrak{w}_m)$, *provided* (in addition to $f \in W^{k,2}(V, \mathfrak{w})$)

$$\begin{aligned} u &\in \mathscr{H}^{k+1}(V, \mathfrak{w}), \\ D_x^k u &\in H^1(\mathscr{O}, \mathfrak{w}) \quad \text{when } m = 0, 1. \end{aligned}$$

The condition $D_x^k u \in H^1(\mathscr{O}, \mathfrak{w})$ follows from Proposition 4.11, since $f \in W^{k,2}(V, \mathfrak{w})$ by hypothesis.

Thus, it remains to verify that $u \in \mathscr{H}^{k+1}(V, \mathfrak{w})$ and justify this application of Proposition 4.15, noting that, by induction on k, we may assume Proposition 4.18 holds for k replaced by $k-1$ and so we may assume $u \in \mathscr{H}^k(V, \mathfrak{w})$.

Claim 4.19. *$u \in \mathscr{H}^{k+1}(V, \mathfrak{w})$, for $k \geq 2$.*

Proof. According to (4.11), we have

$$\begin{aligned} \|v\|^2_{\mathscr{H}^{k+1}(U,\mathfrak{w})} \leq{}& \|y D_x^{k+1} u\|^2_{L^2(U,\mathfrak{w})} + \|y D_x^k D_y u\|^2_{L^2(U,\mathfrak{w})} + \|y D_x^{k-1} D_y^2 u\|^2_{L^2(U,\mathfrak{w})} \\ &+ \sum_{m=1}^{k-1} \|y D_x^{k-1-m} D_y^{m+2} u\|^2_{L^2(U,\mathfrak{w}_m)} \\ &+ \|(1+y) D_x^k u\|^2_{L^2(U,\mathfrak{w})} + \|(1+y) D_x^{k-1} D_y u\|^2_{L^2(U,\mathfrak{w})} \\ &+ \sum_{m=1}^{k-1} \|(1+y) D_x^{k-1-m} D_y^{m+1} u\|^2_{L^2(U,\mathfrak{w}_m)} + \|v\|^2_{\mathscr{H}^k(U,\mathfrak{w})}, \end{aligned}$$

and so we may conclude that $u \in \mathscr{H}^{k+1}(U, \mathfrak{w})$ if the terms on the right hand side are finite.

By induction on k, Proposition 4.18 gives $D_x^{k-1-m} D_y^m u \in H^2(U, \mathfrak{w}_m)$, for $0 \leq m \leq k-1$, and

$$\|D_x^{k-1-m} D_y^m u\|_{H^2(U,\mathfrak{w}_m)} \leq C \left(\|f\|_{W^{k-1,2}(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})} \right),$$

where $C = C(A, k, m, R, R_0)$. The preceding estimate yields

$$\begin{aligned}
&\|yD_x^{k+1}u\|^2_{L^2(U,\mathfrak{w})} + \|yD_x^kD_yu\|^2_{L^2(U,\mathfrak{w})} + \|yD_x^{k-1}D_y^2u\|^2_{L^2(U,\mathfrak{w})}\\
&+ \sum_{m=1}^{k-1} \|yD_x^{k-1-m}D_y^{m+2}u\|^2_{L^2(U,\mathfrak{w}_m)}\\
&+ \|(1+y)D_x^ku\|^2_{L^2(U,\mathfrak{w})} + \|(1+y)D_x^{k-1}D_yu\|^2_{L^2(U,\mathfrak{w})}\\
&+ \sum_{m=1}^{k-1} \|(1+y)D_x^{k-1-m}D_y^{m+1}u\|^2_{L^2(U,\mathfrak{w}_m)}\\
&\quad \le C\left(\|f\|_{W^{k-1,2}(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})}\right)^2.
\end{aligned}$$

Combining the preceding estimates yields $u \in \mathscr{H}^{k+1}(U,\mathfrak{w})$, for $k \ge 2$, and completes the proof of Claim 4.19. □

We now proceed to verify the estimate (4.26). Because $D_x^{k-m}D_y^mf \in L^2(V,\mathfrak{w}_m)$ by hypothesis, we can apply Theorem 3.16 to (4.23) and conclude that $D_x^{k-m}D_y^mu \in H^2(U,\mathfrak{w}_m)$ and

$$\begin{aligned}
\|D_x^{k-m}D_y^mu\|_{H^2(U,\mathfrak{w}_m)} \le C\Big(&\|D_x^{k-m}D_y^mf\|_{L^2(U',\mathfrak{w}_m)} + \|D_x^{k+1-m}D_y^{m-1}u\|_{L^2(U',\mathfrak{w}_m)}\\
&+ \|D_x^{k+2-m}D_y^{m-1}u\|_{L^2(U',\mathfrak{w}_m)} + \|D_x^{k-m}D_y^mu\|_{L^2(U',\mathfrak{w}_m)}\Big),
\end{aligned}$$

where U' is as in (4.17), with $U' = B^+_{R_1}(z_0)$ and $U \subset U' \subset V$ and $R_1 = (R+R_0)/2$, and $C = C(A, R, R_1)$.

We now estimate the terms on the right-hand side of the preceding inequality. Observe that

$$\|D_x^{k+1-m}D_y^{m-1}u\|_{L^2(U',\mathfrak{w}_m)} \le \|D_x^{k-m}D_y^{m-1}u\|_{H^2(U',\mathfrak{w}_m)} \le C\|D_x^{k-m}D_y^{m-1}u\|_{H^2(U',\mathfrak{w}_{m-1})},$$

where the first inequality follows from (3.9) and the second from (4.10), with $C = C(R_1)$. By induction on k and m, we may assume that Proposition 4.18 holds for $k-1$ in place of k and $m-1$ in place of m and so $D_x^{k-m}D_y^{m-1}u = D_x^{k-1-(m-1)}D_y^{m-1}u \in H^2(U',\mathfrak{w}_{m-1})$ with

$$\|D_x^{k-m}D_y^{m-1}u\|_{H^2(U',\mathfrak{w}_{m-1})} \le C\left(\|f\|_{W^{k-1,2}(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})}\right),$$

where $C = C(A, k, R_1, R_0)$. Similarly, observe that

$$\|D_x^{k-m}D_y^mu\|_{L^2(U',\mathfrak{w}_m)} \le \|D_x^{k-m}D_y^{m-1}u\|_{H^2(U',\mathfrak{w}_m)} \le C\|D_x^{k-m}D_y^{m-1}u\|_{H^2(U',\mathfrak{w}_{m-1})},$$

where the last term is estimated above. Finally, we notice that

$$\|D_x^{k+2-m}D_y^{m-1}u\|_{L^2(U',\mathfrak{w}_m)} \le \|D_x^{k+1-m}D_y^{m-1}u\|_{H^2(U',\mathfrak{w}_m)} \le C\|D_x^{k+1-m}D_y^{m-1}u\|_{H^2(U',\mathfrak{w}_{m-1})},$$

where $C = C(R_1)$. For a given $k \ge 2$, we may assume by induction on m that Proposition 4.18 holds for $m-1$ in place of m and so

$$D_x^{k-(m-1)}D_y^{m-1}u = D_x^{k+1-m}D_y^{m-1}u \in H^2(U',\mathfrak{w}_{m-1}),$$

with

$$\|D_x^{k+1-m}D_y^{m-1}u\|_{H^2(U',\mathfrak{w}_{m-1})} \le C\left(\|f\|_{W^{k,2}(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})}\right),$$

where $C = C(A, k, m, R_1, R_0)$. Combining the preceding estimates gives (4.26). □

We can now combine our results for higher-order derivatives with respect to x and y to prove the extension, Theorem 1.2, of Theorem 4.14 from the case $k = 1$ to $k \ge 1$.

Proof of Theorem 1.2. When $k=0$, the conclusion is given by Theorem 3.16 while if $k=1$, the conclusion follows from Theorem 4.14, so we may assume that $k \geq 2$. According to (4.11), we have

$$\begin{aligned}
\|v\|^2_{\mathscr{H}^{k+2}(U,\mathfrak{w})} \leq \|yD_x^{k+2}u\|^2_{L^2(U,\mathfrak{w})} &+ \|yD_x^{k+1}D_y u\|^2_{L^2(U,\mathfrak{w})} + \|yD_x^k D_y^2 u\|^2_{L^2(U,\mathfrak{w})} \\
&+ \sum_{m=1}^{k} \|yD_x^{k-m}D_y^{m+2}u\|^2_{L^2(U,\mathfrak{w}_m)} \\
&+ \|(1+y)D_x^{k+1}u\|^2_{L^2(U,\mathfrak{w})} + \|(1+y)D_x^k D_y u\|^2_{L^2(U,\mathfrak{w})} \\
&+ \sum_{m=1}^{k} \|(1+y)D_x^{k-m}D_y^{m+1}u\|^2_{L^2(U,\mathfrak{w}_m)} \\
&\quad + \|v\|^2_{\mathscr{H}^{k+1}(U,\mathfrak{w})},
\end{aligned}$$

and so we may conclude that $u \in \mathscr{H}^{k+2}(U,\mathfrak{w})$ if the terms on the right hand side are finite.

By induction on k, we may assume that Theorem 1.2 holds for $k-1$ in place of k and so $u \in \mathscr{H}^{k+1}(U,\mathfrak{w})$ and

$$\|u\|_{\mathscr{H}^{k+1}(U,\mathfrak{w})} \leq C\left(\|f\|_{W^{k-1,2}(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})}\right).$$

Moreover, Proposition 4.18 gives $D_x^{k-m}D_y^m u \in H^2(U,\mathfrak{w}_m)$, for $0 \leq m \leq k$, and

$$\|D_x^{k-m}D_y^m u\|_{H^2(U,\mathfrak{w}_m)} \leq C\left(\|f\|_{W^{k,2}(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})}\right),$$

where $C = C(A,k,m,R,R_0)$. The preceding estimate yields

$$\begin{aligned}
&\|yD_x^{k+2}u\|^2_{L^2(U,\mathfrak{w})} + \|yD_x^{k+1}D_y u\|^2_{L^2(U,\mathfrak{w})} + \|yD_x^k D_y^2 u\|^2_{L^2(U,\mathfrak{w})} \\
&+ \sum_{m=1}^{k} \|yD_x^{k-m}D_y^{m+2}u\|^2_{L^2(U,\mathfrak{w}_m)} \\
&+ \|(1+y)D_x^{k+1}u\|^2_{L^2(U,\mathfrak{w})} + \|(1+y)D_x^k D_y u\|^2_{L^2(U,\mathfrak{w})} \\
&+ \sum_{m=1}^{k} \|(1+y)D_x^{k-m}D_y^{m+1}u\|^2_{L^2(U,\mathfrak{w}_m)} \\
&\quad \leq C\left(\|f\|_{W^{k,2}(V,\mathfrak{w})} + \|u\|_{L^2(V,\mathfrak{w})}\right)^2.
\end{aligned}$$

Combining the preceding estimates yields $u \in \mathscr{H}^{k+2}(U,\mathfrak{w})$ and (1.5) for $k \geq 2$. $\square$

Next, we have

Proof of Theorem 1.3. For any $z_1 \in \mathscr{O}$, there is a constant $R_0 > 0$ and a ball $B_{R_0}(z_1)$ such that $B_{R_0}(z_1) \Subset \mathscr{O}$ and [18, Theorem 8.10] implies that $u \in W^{k+2,2}(B_{R_0/2}(z_1))$ and

$$\|u\|_{W^{k+2,2}(B_{R_0/2}(z_1))} \leq C\left(\|f\|_{W^{k,2}(B_{R_0}(z_1))} + \|u\|_{L^2(B_{R_0}(z_1))}\right), \tag{4.27}$$

for some positive constant, $C = C(A,k,R_0)$. If $z_0 \in \partial_0\mathscr{O}$, there is a constant $R_0 > 0$ such that $\mathbb{H} \cap B_{R_0}(z_0) \subset \mathscr{O}$ (since $\partial_0\mathscr{O}$ is defined to be the *interior* of $\partial\mathbb{H} \cap \partial\mathscr{O}$) and Theorem 1.2 implies that $u \in \mathscr{H}^{k+2,2}(B^+_{R_0/2}(z_0),\mathfrak{w})$ and that inequality (1.5) holds. Hence, by combining these observations, $u \in \mathscr{H}^{k+2,2}_{\text{loc}}(\underline{\mathscr{O}},\mathfrak{w})$.

Recalling that $\operatorname{dist}(\partial\mathscr{O}', \partial\mathscr{O}'') \geq d_1 > 0$ by hypothesis, we choose $R_0 = d_1/2$. There are sequences of points, $\{z_{0,i}\} \subset \partial_0\mathscr{O}'$ with $\mathbb{H} \cap B_{R_0}(z_{0,j}) \subset \mathscr{O}''$ and $\{z_{1,j}\} \subset \mathscr{O}'$ with $B_{R_0}(z_{1,j}) \Subset \mathscr{O}''$, such that

$$\bar{\mathscr{O}}' \subset \bigcup_{i,j} B^+_{R_0/2}(z_{0,i}) \cup B_{R_0/2}(z_{1,j}) \subset \mathscr{O}''.$$

The preceding covering of $\mathscr{O}'$ can be chosen to be uniformly locally finite (for example, by locating the ball centers on a rectangular grid with square cells of width d_1) with constant[20] $N = 4$, in the sense that each open ball in the covering intersects at most 4 other balls in the covering.

The definition (2.5) of the weight, $\mathfrak{w}$, and the definitions (2.1), (4.8), and (4.12) of the $L^2(\mathscr{O}, \mathfrak{w})$, $\mathscr{H}^{k+2}(\mathscr{O}, \mathfrak{w})$, and $W^{k,2}(\mathscr{O}, \mathfrak{w})$ norms, respectively, and the estimate (4.27) combine to give[21]

$$\begin{aligned}
\|u\|_{\mathscr{H}^{k+2}(B_{R_0/2}(z_{1,j}),\mathfrak{w})} &\leq Ce^{-\frac{\gamma}{2}|x_{1,j}|}\|u\|_{W^{k+2,2}(B_{R_0/2}(z_{1,j}))} \\
&\leq Ce^{-\frac{\gamma}{2}|x_{1,j}|}\left(\|f\|_{W^{k,2}(B_{R_0}(z_{1,j}))} + \|u\|_{L^2(B_{R_0}(z_{1,j}))}\right) \quad \text{(by (4.27))} \\
&\leq Ce^{-\frac{\gamma}{2}|x_{1,j}|}e^{\frac{\gamma}{2}|x_{1,j}|}\left(\|f\|_{W^{k,2}(B_{R_0}(z_{1,j}),\mathfrak{w})} + \|u\|_{L^2(B_{R_0}(z_{1,j}),\mathfrak{w})}\right),
\end{aligned}$$

where $z_{1,j} = (x_{1,j}, y_{1,j})$ and $C = C(A, d_1, k, \Lambda)$, recalling that $\operatorname{height}(\mathscr{O}'') \leq \Lambda$ by hypothesis, that is

$$\|u\|_{\mathscr{H}^{k+2}(B_{R_0/2}(z_{1,j}),\mathfrak{w})} \leq C\left(\|f\|_{W^{k,2}(B_{R_0}(z_{1,j}),\mathfrak{w})} + \|u\|_{L^2(B_{R_0}(z_{1,j}),\mathfrak{w})}\right). \tag{4.28}$$

Therefore, we obtain the inequality (1.6) from (1.5) and (4.28) and the uniform local finiteness of the open covering of $\mathscr{O}'$. $\square$

Similarly, we obtain

Proof of Theorem 1.5. Uniqueness of a solution $u \in H^1(\mathscr{O}, \mathfrak{w})$ to the variational inequality (2.11) with boundary condition, $u - g \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w})$ defined by $g \in H^1(\mathscr{O}, \mathfrak{w})$, follows from [11, Theorem 8.15], noting that $f \in L^\infty(\mathscr{O}, \mathfrak{w})$ by hypothesis and that $L^\infty(\mathscr{O}) \subset L^2(\mathscr{O}, \mathfrak{w})$ since $\operatorname{vol}(\mathscr{O}, \mathfrak{w}) < \infty$, while $(1+y)g \in W^{2,\infty}(\mathscr{O})$ implies $g \in H^1(\mathscr{O}, \mathfrak{w})$. When $g \equiv 0$ on $\mathscr{O}$, existence of a solution $u \in H^1(\mathscr{O}, \mathfrak{w})$ to the variational inequality (2.11) follows from [3, Theorem 3.16], again noting that $f \in L^\infty(\mathscr{O})$ by hypothesis. For a non-zero g with $(1+y)g \in W^{2,\infty}(\mathscr{O})$, we have $g \in H^2(\mathscr{O}, \mathfrak{w})$ and

$$\mathfrak{a}(g, v) = (Ag, v)_{L^2(\mathscr{O},\mathfrak{w})}, \quad \forall v \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w}),$$

by Lemma A.3. By replacing $u \in H^1(\mathscr{O}, \mathfrak{w})$ with $\tilde{u} := u - g \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w})$ and noting that $\tilde{f} := f - Ag \in L^\infty(\mathscr{O})$, existence of a solution $\tilde{u} \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w})$ to the variational inequality,

$$\mathfrak{a}(\tilde{u}, v) = (\tilde{f}, v)_{L^2(\mathscr{O},\mathfrak{w})}, \quad \forall v \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w}),$$

again follows from [3, Theorem 3.16]. Therefore, we obtain existence of a solution $u \in H^1(\mathscr{O}, \mathfrak{w})$ to the variational inequality (2.11) with boundary condition, $u - g \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w})$. The facts that $u \in \mathscr{H}^{k+2}_{\mathrm{loc}}(\underline{\mathscr{O}})$ and u obeys (1.6) follow from Theorem 1.3. $\square$

[20]In higher dimensions, the constant $N(d)$ depends on the dimension, d, where $\mathscr{O} \subset \mathbb{H}$ and $\mathbb{H} = \mathbb{R}^{d-1} \times \mathbb{R}_+$.

[21]See the proof of Proposition 5.1 for a similar argument with additional details.

5. Higher-order Hölder regularity for solutions to the variational equation

In this section, we extend the $C^\alpha_s(\underline{\mathscr{O}})$-regularity results from [12] for solutions, $u \in H^1(\mathscr{O}, \mathfrak{w})$, to the variational equation (2.11) to $C^{k,\alpha}_s(\underline{\mathscr{O}})$-regularity, for any integer $k \geq 1$. We begin in §5.1 by proving $C^\alpha_s(\underline{\mathscr{O}})$-regularity (Theorem 5.4) for the gradient of a solution, $u \in H^1(\mathscr{O}, \mathfrak{w})$, to the variational equation (2.11). In §5.2, we establish $C^\alpha_s(\underline{\mathscr{O}})$-regularity (Proposition 5.6) for higher-order derivatives of a solution, $u \in H^1(\mathscr{O}, \mathfrak{w})$, concluding with a proof of one of our main results, Theorem 1.6, giving $C^{k,\alpha}_s(\underline{\mathscr{O}})$-regularity of a solution, $u \in H^1(\mathscr{O}, \mathfrak{w})$. We conclude in §5.3 with proofs of our remaining principal results, namely, Corollary 1.7, Theorem 1.8, Corollary 1.9, Theorem 1.11, and Corollaries 1.15 and 1.16.

5.1. **Hölder regularity for first-order derivatives of solutions to the variational equation.**

We begin with

Proposition 5.1 (Interior C^α_s Hölder continuity of u_x for a solution u to the variational equation)**.** *Let $p > \max\{4, 2+\beta\}$ and let R_0 be a positive constant. Then there are positive constants, $R_1 = R_1(R_0) < R_0$, and $C = C(A, p, R_0)$, and $\alpha = \alpha(A, p, R_0) \in (0,1)$ such that the following holds. Let $\mathscr{O} \subsetneqq \mathbb{H}$ be a domain. If $f \in L^2(\mathscr{O}, \mathfrak{w})$, and $u \in H^1(\mathscr{O}, \mathfrak{w})$ satisfies the variational equation* (2.11)*, and $z_0 \in \partial_0\mathscr{O}$ is such that*

$$B_{R_0}(z_0) \cap \mathbb{H} \subset \mathscr{O},$$

and

$$f_x \in L^p(B^+_{R_0}(z_0), y^{\beta-1}),$$

then

$$u_x \in C^\alpha_s(\bar{B}^+_{R_1}(z_0)),$$

and

$$\begin{aligned}\|u_x\|_{C^\alpha_s(\bar{B}^+_{R_1}(z_0))} \leq \Big(&\|f_x\|_{L^p(B^+_{R_0}(z_0), y^{\beta-1})} + \|f\|_{L^2(B^+_{R_0}(z_0), y^{\beta-1})} \\ &+ \|u\|_{L^2(B^+_{R_0}(z_0), y^{\beta-1})}\Big).\end{aligned} \tag{5.1}$$

Proof. By hypothesis, $f_x \in L^2(B^+_{R_0}(z_0), \mathfrak{w})$ since $p > 2$ and so Proposition 4.9 implies that $u_x \in H^1(B^+_{R_2}(z_0), \mathfrak{w})$, for any $R_2 = R_2(R_0)$, to be determined, in the range $0 < R_2 < R_0$ and u_x obeys

$$\mathfrak{a}(u_x, v) = (f_x, v)_{L^2(B^+_{R_2}(z_0), \mathfrak{w})}, \quad \forall v \in H^1_0(\underline{B}^+_{R_2}(z_0), \mathfrak{w}),$$

and

$$\|u_x\|_{H^1(B^+_{R_2}(z_0), \mathfrak{w})} \leq C\left(\|f_x\|_{L^2(B^+_{R_0}(z_0), \mathfrak{w})} + \|f\|_{L^2(B^+_{R_0}(z_0), \mathfrak{w})} + \|u\|_{L^2(B^+_{R_0}(z_0), \mathfrak{w})} \right).$$

The conclusion $u_x \in C^\alpha_s(\bar{B}^+_{R_1}(z_0))$ and estimate, with $C = C(A, p, R_0)$, and $\alpha = \alpha(A, p, R_0) \in (0,1)$, and $R_1 = R_1(R_2) < R_2$,

$$\|u_x\|_{C^\alpha_s(\bar{B}^+_{R_1}(z_0))} \leq C\left(\|f_x\|_{L^p(B^+_{R_2}(z_0), y^{\beta-1})} + \|u_x\|_{L^2(B^+_{R_2}(z_0), y^{\beta-1})} \right),$$

follow by applying Theorem 2.12 to the variational equation for u_x on $B^+_{R_2}(z_0)$. By definition (2.5) of $\mathfrak{w}$, we have

$$
\begin{aligned}
\|u_x\|_{L^2(B^+_{R_2}(z_0),y^{\beta-1})} &= \left(\int_{B^+_{R_0}(z_0)} u_x^2\ y^{\beta-1}\,dx\,dy\right)^{1/2}\\
&\leq e^{\frac{1}{2}\gamma\sqrt{1+\max\{(x_0+R_2)^2,(x_0-R_2)^2\}}+\frac{1}{2}\mu R_2}\left(\int_{B^+_{R_0}(z_0)} u_x^2\ y^{\beta-1}e^{-\gamma\sqrt{1+x^2}-\mu y}\,dx\,dy\right)^{1/2}\\
&\leq e^{\frac{1}{2}\gamma\sqrt{1+(|x_0|+R_2)^2}+\frac{1}{2}\mu R_2}\|u_x\|_{H^1(B^+_{R_2}(z_0),\mathfrak{w})}\\
&\leq Ce^{\frac{1}{2}\gamma|x_0|}\|u_x\|_{H^1(B^+_{R_2}(z_0),\mathfrak{w})},
\end{aligned}
$$

for $C = C(A, R_2)$. Finally, we note that

$$
\begin{aligned}
\|f_x\|_{L^2(B^+_{R_0}(z_0),\mathfrak{w})} &= \left(\int_{B^+_{R_0}(z_0)} f_x^2\ y^{\beta-1}e^{-\gamma\sqrt{1+x^2}-\mu y}\,dx\,dy\right)^{1/2}\\
&\leq e^{-\frac{1}{2}\gamma\sqrt{1+\min\{(x_0+R_2)^2,(x_0-R_2)^2\}}}\left(\int_{B^+_{R_0}(z_0)} f_x^2\ y^{\beta-1}\,dx\,dy\right)^{1/2}\\
&\leq Ce^{-\frac{1}{2}\gamma\sqrt{1+(|x_0|-R_2)^2}}\|f_x\|_{L^p(B^+_{R_0}(z_0),y^{\beta-1})}\\
&\leq Ce^{-\frac{1}{2}\gamma|x_0|}\|f_x\|_{L^p(B^+_{R_0}(z_0),y^{\beta-1})},
\end{aligned}
$$

where $C = C(p, R_0, R_2, \beta)$, and similarly for the terms $\|f\|_{L^2(B^+_{R_0}(z_0),\mathfrak{w})}$ and $\|u\|_{L^2(B^+_{R_0}(z_0),\mathfrak{w})}$. We may choose $R_2 = R_0/2$ and combining the preceding inequalities yields (5.1), with $C = C(A, p, R_0)$. $\square$

Next, we have

Proposition 5.2 (Interior C^α_s Hölder continuity of u_y for a solution u to the variational equation)**.** *Let $p > \max\{4, 3+\beta\}$ and let R_0 be a positive constant. Then there are positive constants, $R_1 = R_1(R_0) < R_0$, and $C = C(A, p, R_0)$, and $\alpha = \alpha(A, p, R_0) \in (0,1)$ such that the following holds. Let $\mathscr{O} \subseteqq \mathbb{H}$ be a domain. If $f \in L^2(\mathscr{O},\mathfrak{w})$, and $u \in H^1(\mathscr{O},\mathfrak{w})$ satisfies the variational equation (2.11), and $z_0 \in \partial_0\mathscr{O}$ is such that*

$$B_{R_0}(z_0) \cap \mathbb{H} \subset \mathscr{O},$$

and f obeys

$$f_x,\ f_y,\ f_{xx} \in L^p(B^+_{R_0}(z_0), y^{\beta-1}), \tag{5.2}$$

then

$$u_y \in C^\alpha_s(\bar{B}^+_{R_1}(z_0)),$$

and

$$
\begin{aligned}
\|u_y\|_{C^\alpha_s(\bar{B}^+_{R_1}(z_0))} \leq C\Big(&\|f_{xx}\|_{L^p(B^+_{R_0}(z_0),y^{\beta-1})} + \|f\|_{W^{1,p}(B^+_{R_0}(z_0),y^{\beta-1})}\\
&+ \|u\|_{L^2(B^+_{R_0}(z_0),y^{\beta-1})}\Big).
\end{aligned}
\tag{5.3}
$$

Proof. Since $p > 2$ and so $f \in W^{1,2}(B^+_{R_0}(z_0), y^{\beta-1}) = W^{1,2}(B^+_{R_0}(z_0), \mathfrak{w})$ by hypothesis, Theorem 4.14 ensures that $u \in \mathscr{H}^3(B^+_{R_0/2}(z_0), \mathfrak{w})$. By Definition 4.3, we therefore have $u \in H^2(B^+_{R_0/2}(z_0), \mathfrak{w})$ and $u_{xx} \in L^2(B^+_{R_0/2}(z_0), \mathfrak{w}_1)$ and so Lemma 4.12 implies that

$$u_y \in H^1(B^+_{R_0/2}(z_0), \mathfrak{w}_1),$$

and u_y obeys the variational equation,

$$\mathfrak{a}_1(u_y, v) = (f_y - Bu, v)_{L^2(B^+_{R_0/2}(z_0),\mathfrak{w}_1)}, \quad \forall v \in H^1_0(\underline{B}^+_{R_0/2}(z_0), \mathfrak{w}_1).$$

We note that the preceding variational equation continues to hold on $B^+_{R_2}(z_0)$, for any R_2 in the range $0 < R_2 \leq R_0/2$ and still to be determined. To apply Theorem 2.12 to the preceding variational equation and conclude that $u_y \in C^\alpha_s(\bar{B}^+_{R_1}(z_0))$ for some $R_1 = R_1(R_2) < R_2$ and, for a positive constant $C = C(A, p, R_2)$,

$$\begin{aligned}\|u_y\|_{C^\alpha_s(\bar{B}^+_{R_1}(z_0))} \leq C \Big(&\|f_y\|_{L^p(B^+_{R_2}(z_0),y^\beta)} + \|u_x\|_{L^p(B^+_{R_2}(z_0),y^\beta)} + \|u_{xx}\|_{L^p(B^+_{R_2}(z_0),y^\beta)} \\ &+ \|u_y\|_{L^2(B^+_{R_2}(z_0),y^\beta)} \Big),\end{aligned} \tag{5.4}$$

we need $f_y - Bu$ to obey the integrability condition (2.13) obeyed by f, with $\beta + 1$ and R_2 in place of β and R_0, respectively. In other words, u must obey

$$u_x,\ u_{xx} \in L^p(B^+_{R_2}(z_0), y^\beta), \tag{5.5}$$

while our hypothesis (5.2) on f ensures $f_y \in L^p(B^+_{R_2}(z_0), y^{\beta-1}) \subset L^p(B^+_{R_2}(z_0), y^\beta)$. For the condition (5.5) on u, it is enough to show that $u_x,\ u_{xx} \in L^\infty(B^+_{R_2}(z_0))$.

Since $f_x, f_{xx} \in L^2(B^+_{R_0}(z_0), y^{\beta-1})$ by hypothesis, Proposition 4.10 (with $k = 1, 2$) implies that $u_x,\ u_{xx} \in H^1_0(B^+_{R_0/2}(z_0), \mathfrak{w})$ and that they obey

$$\begin{aligned}\mathfrak{a}(u_x, v) &= (f_x, v)_{L^2(B^+_{R_0/2}(z_0),\mathfrak{w})}, \\ \mathfrak{a}(u_{xx}, v) &= (f_{xx}, v)_{L^2(B^+_{R_0/2}(z_0),\mathfrak{w})}, \quad \forall v \in H^1_0(\underline{B}^+_{R_0/2}(z_0), \mathfrak{w}).\end{aligned}$$

Also, because $f_x, f_{xx} \in L^p(B^+_{R_0}(z_0), y^{\beta-1})$ by hypothesis, we can apply Theorem 2.5 to the preceding variational equations to give an $R_3 = R_3(R_0) < R_0/2$ such that $u_x, u_{xx} \in L^\infty(B^+_{R_3}(z_0))$ and

$$\|u_x\|_{L^\infty(B^+_{R_3}(z_0))} \leq C \left(\|f_x\|_{L^p(B^+_{R_0/2}(z_0),y^{\beta-1})} + \|u_x\|_{L^2(B^+_{R_0/2}(z_0),y^{\beta-1})} \right), \tag{5.6}$$

$$\|u_{xx}\|_{L^\infty(B^+_{R_3}(z_0))} \leq C \left(\|f_{xx}\|_{L^p(B^+_{R_0/2}(z_0),y^{\beta-1})} + \|u_{xx}\|_{L^2(B^+_{R_0/2}(z_0),y^{\beta-1})} \right). \tag{5.7}$$

We now choose $R_2 = R_3$ and observe that condition (5.5) holds and the estimate (5.4) is justified with

$$\begin{aligned}\|u_x\|_{L^p(B^+_{R_3}(z_0),y^{\beta-1})} &\leq C\|u_x\|_{L^\infty(B^+_{R_3}(z_0))}, \\ \|u_{xx}\|_{L^p(B^+_{R_3}(z_0),y^{\beta-1})} &\leq C\|u_{xx}\|_{L^\infty(B^+_{R_3}(z_0))},\end{aligned} \tag{5.8}$$

where $C = C(\beta, p, R_3) = (\int_{B^+_{R_3}(z_0)} y^{\beta-1}\, dx\, dy)^{1/p}$. The definition (4.8) of $\mathscr{H}^3(\mathscr{O}, \mathfrak{w})$ and the definition (2.5) of $\mathfrak{w}$ imply that

$$
\begin{aligned}
&\|u_y\|_{L^2(B^+_{R_0/2}(z_0),y^{\beta-1})} + \|u_x\|_{L^2(B^+_{R_0/2}(z_0),y^{\beta-1})} + \|u_{xx}\|_{L^2(B^+_{R_0/2}(z_0),y^{\beta-1})} \\
&\leq Ce^{\frac{\gamma}{2}|x_0|}\left(\|u_y\|_{L^2(B^+_{R_0/2}(z_0),\mathfrak{w})} + \|u_x\|_{L^2(B^+_{R_0/2}(z_0),\mathfrak{w})} + \|u_{xx}\|_{L^2(B^+_{R_0/2}(z_0),\mathfrak{w})}\right) \\
&\leq Ce^{\frac{\gamma}{2}|x_0|}\|u\|_{\mathscr{H}^3(B^+_{R_0/2}(z_0),\mathfrak{w})},
\end{aligned}
$$

where $C = C(A, R_0)$ and the factors, $e^{\frac{\gamma}{2}|x_0|}$, arise just as in the proof of Proposition 5.1. Thus, by Theorem 4.14,

$$
\begin{aligned}
&\|u_y\|_{L^2(B^+_{R_0/2}(z_0),y^{\beta-1})} + \|u_x\|_{L^2(B^+_{R_0/2}(z_0),y^{\beta-1})} + \|u_{xx}\|_{L^2(B^+_{R_0/2}(z_0),y^{\beta-1})} \\
&\leq Ce^{\frac{\gamma}{2}|x_0|}\left(\|f\|_{W^{1,2}(B^+_{R_0}(z_0),\mathfrak{w})} + \|u\|_{L^2(B^+_{R_0}(z_0),\mathfrak{w})}\right), \\
&\leq Ce^{\frac{\gamma}{2}|x_0|}e^{-\frac{\gamma}{2}|x_0|}\left(\|f\|_{W^{1,2}(B^+_{R_0}(z_0),y^{\beta-1})} + \|u\|_{L^2(B^+_{R_0}(z_0),y^{\beta-1})}\right), \\
&\leq C\left(\|f\|_{W^{1,p}(B^+_{R_0}(z_0),y^{\beta-1})} + \|u\|_{L^2(B^+_{R_0}(z_0),y^{\beta-1})}\right),
\end{aligned}
$$

where $C = C(A, p, R_0))$ and the factors, $e^{-\frac{\gamma}{2}|x_0|}$, arise just as in the proof of Proposition 5.1. The estimate (5.3) is obtained by combining the preceding inequality with (5.4), (5.6), and (5.7), and (5.8). □

We may combine Propositions 5.1 and 5.2 to give

Proposition 5.3 (Interior C^α_s Hölder continuity of Du for a solution u to the variational equation)**.** *Let $p > \max\{4, 3+\beta\}$ and let R_0 be a positive constant. Then there are positive constants, $R_1 = R_1(R_0) < R_0$ and $C = C(A, p, R_0)$ and $\alpha = \alpha(A, p, R_0) \in (0, 1)$ such that the following holds. Let $\mathscr{O} \subseteqq \mathbb{H}$ be a domain. If $f \in L^2(\mathscr{O}, \mathfrak{w})$ and $u \in H^1(\mathscr{O}, \mathfrak{w})$ satisfies the variational equation* (2.11)*, and $z_0 \in \partial_0\mathscr{O}$ is such that*

$$B_{R_0}(z_0) \cap \mathbb{H} \subset \mathscr{O},$$

and f obeys (5.2)*, then*

$$u_x,\ u_y \in C^\alpha_s(\bar{B}^+_{R_1}(z_0)),$$

and

$$
\begin{aligned}
\|Du\|_{C^\alpha_s(\bar{B}^+_{R_1}(z_0))} \leq C\Big(&\|f_{xx}\|_{L^p(B^+_{R_0}(z_0),y^{\beta-1})} + \|f\|_{W^{1,p}(B^+_{R_0}(z_0),y^{\beta-1})} \\
&+ \|u\|_{L^2(B^+_{R_0}(z_0),y^{\beta-1})}\Big).
\end{aligned}
\tag{5.9}
$$

Proof. The conclusion $u_x,\ u_y \in C^\alpha_s(\bar{B}^+_{R_1}(z_0))$ and estimate (5.9) follow from Propositions 5.1 and 5.2. □

Finally, we may combine Theorems 2.12 and 4.14 and Proposition 5.3 to give

Theorem 5.4 (Interior $C^{1,\alpha}_s$ Hölder continuity for a solution u to the variational equation)**.** *Let $p > \max\{4, 3+\beta\}$ and let R_0 be a positive constant. Then there are positive constants, $R_1 = R_1(R_0) < R_0$ and $C = C(A, p, R_0)$ and $\alpha = \alpha(A, p, R_0) \in (0, 1)$ such that the following* *holds. Let $\mathscr{O} \subseteq \mathbb{H}$ be a domain. If $f \in L^2(\mathscr{O}, \mathfrak{w})$, and $u \in H^1(\mathscr{O}, \mathfrak{w})$ satisfies the variational equation* (2.11)*, and $z_0 \in \partial_0\mathscr{O}$ is such that*

$$B_{R_0}(z_0) \cap \mathbb{H} \subset \mathscr{O},$$

and f obeys (2.13) *and* (5.2)*, then*

$$u \in C_s^{1,\alpha}(\bar{B}^+_{R_1}(z_0)),$$

and

$$\|u\|_{C_s^{1,\alpha}(\bar{B}^+_{R_1}(z_0))} \leq C \left(\|f_{xx}\|_{L^p(B^+_{R_0}(z_0), y^{\beta-1})} + \|f\|_{W^{1,p}(B^+_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(B^+_{R_0}(z_0), y^{\beta-1})} \right). \tag{5.10}$$

Proof. Since $f, f_x, f_y \in L^p(B^+_{R_0}(z_0), y^{\beta-1}) \subset L^2(B^+_{R_0}(z_0), \mathfrak{w})$, Theorem 4.14 implies that $u \in \mathscr{H}^3(B^+_{R_2}(z_0), \mathfrak{w})$ for any $0 < R_2 < R_0$. By applying Theorem 2.12 and Proposition 5.3, with R_2 in place of R_0, we obtain $u \in C_s^{1,\alpha}(\bar{B}^+_{R_1}(z_0))$ for some $R_1 < R_2$ and, say, $R_2 = R_0/2$. The inequality (5.10) is obtained by combining (2.24) and (5.9). $\square$

5.2. **Hölder regularity for higher-order derivatives of solutions to the variational equation.** We first give an extension of Proposition 5.1 from the case $k = 1$ to arbitrary $k \geq 1$.

Proposition 5.5 (Interior C_s^α Hölder continuity of higher-order derivatives with respect to x for a solution to the variational equation)**.** *Let $p > \max\{4, 2+\beta\}$, let R_0 be a positive constant, and let $k \geq 1$ be an integer. Then there are positive constants, $R_1 = R_1(R_0) < R_0$, and $C = C(A, k, p, R_0)$, and $\alpha = \alpha(A, p, R_0) \in (0,1)$ such that the following holds. Let $\mathscr{O} \subseteq \mathbb{H}$ be a domain. If $f \in L^2(\mathscr{O}, \mathfrak{w})$, and $u \in H^1(\mathscr{O}, \mathfrak{w})$ satisfies the variational equation* (2.11)*, and $z_0 \in \partial_0\mathscr{O}$ is such that*

$$B_{R_0}(z_0) \cap \mathbb{H} \subset \mathscr{O},$$

and

$$D_x^j f \in L^p(B^+_{R_0}(z_0), y^{\beta-1}), \quad 1 \leq j \leq k,$$

then

$$D_x^k u \in C_s^\alpha(\bar{B}^+_{R_1}(z_0)),$$

and, if $z_0 = (x_0, 0)$,

$$\|D_x^k u\|_{C_s^\alpha(\bar{B}^+_{R_1}(z_0))} \leq C \left(\sum_{j=0}^k \|D_x^j f\|_{L^p(B^+_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(B^+_{R_0}(z_0), y^{\beta-1})} \right). \tag{5.11}$$

Proof. The argument is similar to the proof of Proposition 5.1. By hypothesis, $D_x^j f \in L^2(B^+_{R_0}(z_0), \mathfrak{w})$, $1 \leq j \leq k$, since $p > 2$ and so Proposition 4.10 implies that $D_x^k u \in H^1(B^+_{R_2}(z_0), \mathfrak{w})$, for any $R_2 = R_2(R_0)$, to be determined, in the range $0 < R_2 < R_0$, and that $D_x^k u$ obeys

$$\mathfrak{a}(D_x^k u, v) = (D_x^k f, v)_{L^2(B^+_{R_2}(z_0), \mathfrak{w})}, \quad \forall v \in H^1_0(\underline{B}^+_{R_2}(z_0), \mathfrak{w}),$$

and, for $C = C(A, k, R_0)$,

$$\|D_x^k u\|_{H^1(B^+_{R_2}(z_0), \mathfrak{w})} \leq C \left(\sum_{j=0}^k \|D_x^j f\|_{L^2(B^+_{R_0}(z_0), \mathfrak{w})} + \|u\|_{L^2(B^+_{R_0}(z_0), \mathfrak{w})} \right).$$

The conclusion $D_x^k u \in C_s^\alpha(\bar{B}_{R_1}^+(z_0))$ and estimate, with $C = C(A,p,R_0)$, and $\alpha = \alpha(A,p,R_0) \in (0,1)$, and $R_1 = R_1(R_2) < R_2$ (recall that $R_2 = R_2(R_0)$),

$$\|D_x^k u\|_{C_s^\alpha(\bar{B}_{R_1}^+(z_0))} \leq C\left(\|D_x^k f\|_{L^p(B_{R_2}^+(z_0),y^{\beta-1})} + \|D_x^k u\|_{L^2(B_{R_2}^+(z_0),y^{\beta-1})}\right),$$

follow by applying Theorem 2.12 to the variational equation for $D_x^k u$ on $B_{R_2}^+(z_0)$. As in the proof of Proposition 5.1, we have

$$\begin{aligned}\|D_x^k u\|_{L^2(B_{R_2}^+(z_0),y^{\beta-1})} &\leq Ce^{\frac{\gamma}{2}|x_0|}\|D_x^k u\|_{L^2(B_{R_2}^+(z_0),\mathfrak{w})}\\ &\leq Ce^{\frac{\gamma}{2}|x_0|}\|D_x^k u\|_{H^1(B_{R_2}^+(z_0),\mathfrak{w})},\end{aligned}$$

where $C = C(A,R_2)$. We may choose $R_2 = R_0/2$ and observe that, by combining the preceding inequalities, we obtain (5.11), with $C = C(A,k,p,R_0)$, noting that the factor, $e^{\frac{\gamma}{2}|x_0|}$, cancels just as in the proof of Proposition 5.1. □

The extension of Propositions 5.5 and 5.2 to the case of derivatives of the form $D_x^{k-m}D_y^m u$ when $1 \leq m \leq k$ is best illustrated by an example when $k = 2$ and $m = 0,1,2$. The case $k = 2, m = 0$ is given by Proposition 5.5, so

$$\|u_{xx}\|_{C_s^\alpha(\bar{B}_{R_1}^+(z_0))} \leq C\left(\sum_{j=0}^{2}\|D_x^j f\|_{L^p(B_{R_0}^+(z_0),y^{\beta-1})} + \|u\|_{L^2(B_{R_0}^+(z_0),y^{\beta-1})}\right).$$

For $k = 2, m = 1$, the pattern of proof of Proposition 5.2 shows that a C_s^α estimate of u_{xy} requires an L^p-bound on f_{xy}, u_{xx}, and u_{xxx}, thus an additional L^∞-bound on u_{xxx}, and hence an additional L^p bound on f_{xxx}, to give

$$\begin{aligned}\|u_{xy}\|_{C_s^\alpha(\bar{B}_{R_1}^+(z_0))} \leq C\Bigg(&\sum_{i=0}^{3}\|D_x^i f\|_{L^p(B_{R_0}^+(z_0),y^{\beta-1})} + \|f_{xy}\|_{L^p(B_{R_0}^+(z_0),y^{\beta-1})}\\ &+ \|u\|_{L^2(B_{R_0}^+(z_0),y^{\beta-1})}\Bigg),\end{aligned}$$

and thus the following will suffice,

$$\|u_{xy}\|_{C_s^\alpha(\bar{B}_{R_1}^+(z_0))} \leq C\left(\|f\|_{W^{3,p}(B_{R_0}^+(z_0),y^{\beta-1})} + \|u\|_{L^2(B_{R_0}^+(z_0),y^{\beta-1})}\right).$$

For $k = 2, m = 2$, the pattern of proof of Proposition 5.2 shows that a C_s^α estimate of u_{yy} requires an L^p-bound on f_{yy}, u_{xy}, and u_{xxy}, thus L^∞-bounds on u_{xy} and u_{xxy}, hence additional L^p bounds on f_{xxy}, f_{xxxx}, to give

$$\begin{aligned}\|u_{yy}\|_{C_s^\alpha(\bar{B}_{R_1}^+(z_0))} \leq C\Bigg(&\sum_{j=0}^{4}\|D_x^j f\|_{L^p(B_{R_0}^+(z_0),y^{\beta-1})} + \sum_{i=1}^{2}\|D_x^i f_y\|_{L^p(B_{R_0}^+(z_0),y^{\beta-1})}\\ &+ \|f_{yy}\|_{L^p(B_{R_0}^+(z_0),y^{\beta-1})} + \|u\|_{L^2(B_{R_0}^+(z_0),y^{\beta-1})}\Bigg),\end{aligned}$$

and thus the following will suffice,

$$\|u_{yy}\|_{C_s^\alpha(\bar{B}_{R_1}^+(z_0))} \leq C\left(\|f\|_{W^{4,p}(B_{R_0}^+(z_0),y^{\beta-1})} + \|u\|_{L^2(B_{R_0}^+(z_0),y^{\beta-1})}\right).$$

The preceding examples motivate the statement of the following combined extension of Propositions 5.2 and 5.5.

Proposition 5.6 (Interior C^α_s Hölder continuity of higher-order derivatives of a solution to the variational equation)**.** *Let R_0 be a positive constant, let m, k be integers with $k \geq 1$ and $0 \leq m \leq k$, and let $p > \max\{4, 2+m+\beta\}$. Then there are positive constants, $R_1 = R_1(m, R_0) < R_0$, and $C = C(A, k, m, p, R_0)$, and $\alpha = \alpha(A, m, p, R_0) \in (0,1)$ such that the following holds. Let $\mathscr{O} \subseteqq \mathbb{H}$ be a domain. If $f \in L^2(\mathscr{O}, \mathfrak{w})$, and $u \in H^1(\mathscr{O}, \mathfrak{w})$ satisfies the variational equation* (2.11)*, and $z_0 \in \partial_0\mathscr{O}$ is such that*

$$B_{R_0}(z_0) \cap \mathbb{H} \subset \mathscr{O},$$

and

$$f \in W^{k+m,p}(B^+_{R_0}(z_0), y^{\beta-1}),$$

then

$$D^{k-m}_x D^m_y u \in C^\alpha_s(\bar{B}^+_{R_1}(z_0)),$$

and

$$\|D^{k-m}_x D^m_y u\|_{C^\alpha_s(\bar{B}^+_{R_1}(z_0))} \leq C\left(\|f\|_{W^{k+m,p}(B^+_{R_0}(z_0),y^{\beta-1})} + \|u\|_{L^2(B^+_{R_0}(z_0),y^{\beta-1})}\right). \tag{5.12}$$

Proof. For arbitrary $\ell \geq 1$ and $f \in W^{\ell,p}(B^+_{R_0}(z_0), y^{\beta-1})$, Proposition 5.5 already implies that

$$\|D^\ell_x u\|_{C^\alpha_s(\bar{B}^+_{R_3}(z_0))} \leq C\left(\|f\|_{W^{\ell,p}(B^+_{R_0}(z_0),y^{\beta-1})} + \|u\|_{L^2(B^+_{R_0}(z_0),y^{\beta-1})}\right),$$

for some $R_2 = R_2(R_0)$, and hence that (5.12) holds when $m = 0$, so we may assume without loss of generality that $m \geq 1$ in our proof of Proposition 5.6. Therefore, to establish (5.12), it suffices to consider the inductive step $(k, m-1) \implies (k, m)$ (one extra derivative with respect to y), assuming

$$\begin{gathered}\|D^{\ell-n}_x D^n_y u\|_{C^\alpha_s(\bar{B}^+_{R_2}(z_0))} \leq C\left(\|f\|_{W^{\ell+n,p}(B^+_{R_0}(z_0),y^{\beta-1})} + \|u\|_{L^2(B^+_{R_0}(z_0),y^{\beta-1})}\right),\\ \text{for all } \ell \geq 1, \quad \text{all } n \text{ such that } 0 \leq n \leq m-1, \quad \text{and } 1 \leq m \leq k,\end{gathered} \tag{5.13}$$

where $R_2 = R_2(m-1, R_0)$ (we point out the origin of the dependence on m further along in the proof). The proof of this inductive step follows the pattern of proof of Proposition 5.2.

By our hypotheses on f, Theorem 1.2 implies that

$$u \in \mathscr{H}^{k+2+m}(B^+_{R_2}(z_0), \mathfrak{w}) \subset \mathscr{H}^{k+3}(B^+_{R_2}(z_0), \mathfrak{w})$$

(since we assume $m \geq 1$ for the inductive step), for any R_2 in the range $0 < R_2 < R_0$ to be determined, and that

$$\|u\|_{\mathscr{H}^{k+3}(B^+_{R_2}(z_0),\mathfrak{w})} \leq C\left(\|f\|_{W^{k+1,2}(B^+_{R_0}(z_0),\mathfrak{w})} + \|u\|_{L^2(B^+_{R_0}(z_0),\mathfrak{w})}\right). \tag{5.14}$$

We have $D^{k-m}_x D^m_y u \in H^1(B^+_{R_2}(z_0), \mathfrak{w}_m)$ by Definition 4.3 of $\mathscr{H}^{k+3}(B^+_{R_2}(z_0), \mathfrak{w})$, since $u \in \mathscr{H}^{k+3}(B^+_{R_2}(z_0), \mathfrak{w})$ implies

$$\begin{gathered}D^{k+1-m}_x D^m_y u, \quad D^{k-m}_x D^m_y u \in \begin{cases} L^2(B^+_{R_2}(z_0), \mathfrak{w}_{m-1}), & 2 \leq m \leq k,\\ L^2(B^+_{R_2}(z_0), \mathfrak{w}), & m = 1,\end{cases}\\ D^{k-m}_x D^{m+1}_y u \in L^2(B^+_{R_2}(z_0), \mathfrak{w}_m), \quad 1 \leq m \leq k,\end{gathered}$$

and we have $D_x^k u \in H^1(B^+_{R_2}(z_0), \mathfrak{w})$ by Proposition 4.10. Proposition 4.15 then ensures that $D_x^{k-m}D_y^m u$ obeys the variational equation on $B^+_{R_2}(z_0)$,

$$\begin{aligned}\mathfrak{a}_m(D_x^{k-m}D_y^m u, v) &= (D_x^{k-m}D_y^m f, v)_{L^2(B^+_{R_2}(z_0),\mathfrak{w}_m)}\\ &\quad - m(BD_x^{k-m}D_y^{m-1}u, v)_{L^2(B^+_{R_2}(z_0),\mathfrak{w}_m)},\end{aligned}$$

for all $v \in H^1_0(\underline{B}^+_{R_2}(z_0), \mathfrak{w}_m)$. By hypothesis, $D_x^{k-m}D_y^m f \in L^p(B^+_{R_2}(z_0), y^{\beta+m-1})$ and *provided* we also know $BD_x^{k-m}D_y^{m-1} \in L^p(B^+_{R_2}(z_0), y^{\beta+m-1})$, that is,

$$\begin{aligned}&D_x^{k+1-m}D_y^{m-1}u \in L^p(B^+_{R_2}(z_0), y^{\beta+m-1}),\\ &D_x^{k+2-m}D_y^{m-1}u \in L^p(B^+_{R_2}(z_0), y^{\beta+m-1}), \quad 1 \le m \le k,\end{aligned} \tag{5.15}$$

we can apply Theorem 2.12 to the variational equation for $D_x^{k-m}D_y^m u$ and conclude that $D_x^{k-m}D_y^m u \in C^\alpha_s(\bar{B}^+_{R_1}(z_0))$ for some[22] $R_1 = R_1(R_2)$ obeying $R_1 < R_2$, and

$$\begin{aligned}\|D_x^{k-m}D_y^m u\|_{C^\alpha_s(\bar{B}^+_{R_1}(z_0))} \le C\Bigg(&\|D_x^{k-m}D_y^m f\|_{L^p(B^+_{R_2}(z_0),y^{\beta+m-1})}\\ &+ \sum_{j=1}^2 \|D_x^{k+j-m}D_y^{m-1}u\|_{L^p(B^+_{R_2}(z_0),y^{\beta+m-1})}\\ &+ \|D_x^{k-m}D_y^m u\|_{L^2(B^+_{R_2}(z_0),y^{\beta+m-1})}\Bigg), \quad 1 \le m \le k.\end{aligned} \tag{5.16}$$

It is important to note that the Hölder exponent, α, in (5.16) depends on the coefficients defining the bilinear map, $\mathfrak{a}_m$, that is, on the coefficients of A_m and thus on the coefficients of A and on m, the number of derivatives with respect to y, and this is why we write $\alpha = \alpha(A, m, p, R_0)$ in the statement of Proposition 5.6. The integrability conditions (5.15) are implied by

$$\begin{aligned}&D_x^{k+1-m}D_y^{m-1}u \in L^\infty(B^+_{R_2}(z_0)),\\ &D_x^{k+2-m}D_y^{m-1}u \in L^\infty(B^+_{R_2}(z_0)), \quad 1 \le m \le k.\end{aligned} \tag{5.17}$$

Note that $D_x^{k+1-m}D_y^{m-1}u = D_x^{k-(m-1)}D_y^{m-1}u$ and $D_x^{k+2-m}D_y^{m-1}u = D_x^{k+1-(m-1)}D_y^{m-1}u$ and the properties (5.17) hold by the inductive hypothesis (5.13). Therefore, the inductive hypothesis (5.13) gives

$$\begin{aligned}&\|D_x^{k+1-m}D_y^{m-1}u\|_{L^\infty(B^+_{R_2}(z_0))}\\ &\quad \le C\left(\|f\|_{W^{k+m-1,p}(B^+_{R_0}(z_0),y^{\beta-1})} + \|u\|_{L^2(B^+_{R_0}(z_0),y^{\beta-1})}\right),\end{aligned} \tag{5.18}$$

$$\begin{aligned}&\|D_x^{k+2-m}D_y^{m-1}u\|_{L^\infty(B^+_{R_2}(z_0))}\\ &\quad \le C\left(\|f\|_{W^{k+m,p}(B^+_{R_0}(z_0),y^{\beta-1})} + \|u\|_{L^2(B^+_{R_0}(z_0),y^{\beta-1})}\right),\end{aligned} \tag{5.19}$$

for $1 \le m \le k$. Hence, the integrability conditions for the derivatives of u in (5.15) are satisfied and the estimate (5.16) holds.

[22] The dependence on m appears in this step.

Applying the Definition 4.3 of $\mathscr{H}^{k+3}(B^+_{R_2}(z_0),\mathfrak{w})$ and the L^∞ estimates (5.18) and (5.19) for the derivatives of u, the inequality (5.16) yields

$$\|D_x^{k-m}D_y^m u\|_{C^\alpha_s(\bar B^+_{R_1}(z_0))} \leq C\left(\|f\|_{W^{k+m,p}(B^+_{R_0}(z_0),y^{\beta-1})} + \|u\|_{\mathscr{H}^{k+3}(B^+_{R_2}(z_0),\mathfrak{w})}\right). \tag{5.20}$$

Combining inequalities (5.14) and (5.20) completes the proof. ☐

Theorem 1.6 now follows easily, extending Theorem 5.4 from the case $k=1$ to any $k\geq 1$.

Proof of Theorem 1.6. When $k=0$ or 1, then the conclusion follows from Theorems 2.12 or 5.4, respectively, so we may assume that $k\geq 2$ and, by induction, that the conclusion holds for $k-1$ in place of k. Since

$$\|u\|_{C^{k,\alpha}_s(\bar B^+_{R_1}(z_0))} = \|D^k u\|_{C^\alpha_s(\bar B^+_{R_1}(z_0))} + \|u\|_{C^{k-1,\alpha}_s(\bar B^+_{R_1}(z_0))}, \tag{5.21}$$

by Definition 2.14, it suffices to show that $D_x^{k-m}D_y^m u\in C^\alpha_s(\bar B^+_{R_1}(z_0))$, for $0\leq m\leq k$, but this inclusion and estimate are given by Proposition 5.6. ☐

We may combine Theorem 1.6 with standard results from [18] for linear, second-order, elliptic differential equations to give a *weak* version of Theorem 1.8 which will, nonetheless, provide a useful stepping stone to the proof of Theorem 1.8 itself. Although their statements appear similar, Proposition 5.7 is nevertheless strictly weaker than Theorem 1.8, despite the more relaxed hypothesis on f because, in the former case, $\alpha=\alpha(A,d_1,k,p)$ depends on the choice of precompact subdomain, $\mathscr{O}''\Subset\underline{\mathscr{O}}$, through the constant d_1 whereas in the latter case, $\alpha=\alpha(A,k,p)$ is *independent* of the choice of precompact subdomain, $\mathscr{O}''\Subset\underline{\mathscr{O}}$.

Proposition 5.7 (Interior $C^{k,\alpha}_s$ regularity on subdomains)**.** *Let $k\geq 0$ be an integer, let $d_1<\Lambda$ be positive constants, and let $p>\max\{4,2+k+\beta\}$. Then there are positive constants $\alpha=\alpha(A,d_1,k,p)\in(0,1)$ and $C=C(A,k,d_1,\Lambda,p)$ such that the following holds. If $f\in L^2(\mathscr{O},\mathfrak{w})$ and $u\in H^1(\mathscr{O},\mathfrak{w})$ is a solution to the variational equation* (2.11)*, and $f\in W^{2k,p}_{\text{loc}}(\underline{\mathscr{O}},\mathfrak{w})$, and $\mathscr{O}''\subset\mathscr{O}$ is a subdomain such that $\mathscr{O}''\Subset\underline{\mathscr{O}}$ with $\mathscr{O}''\subset(-\Lambda,\Lambda)\times(0,\Lambda)$ and $\operatorname{dist}(\partial_1\mathscr{O}'',\partial_1\mathscr{O})\geq d_1$, then*

$$u\in C^{k,\alpha}_s(\underline{\mathscr{O}''})\cap C^\alpha_s(\bar{\mathscr{O}}'').$$

Moreover, u solves (1.1) *on $\mathscr{O}''$ and if $\mathscr{O}'\subset\mathscr{O}''$ is a subdomain with $\mathscr{O}'\Subset\underline{\mathscr{O}''}$ and $\operatorname{dist}(\partial_1\mathscr{O}',\partial_1\mathscr{O}'')\geq d_1$, then*

$$\|u\|_{C^{k,\alpha}_s(\bar{\mathscr{O}}')} \leq C\left(\|f\|_{W^{2k,p}(\mathscr{O}'',\mathfrak{w})} + \|u\|_{L^2(\mathscr{O}'',\mathfrak{w})}\right). \tag{5.22}$$

Proof. Choose $R_0=d_1/2$ and let $R_1=R_1(k,R_0)<R_0$ be defined by Theorem 1.6. Since $\mathscr{O}''\subset(-\Lambda,\Lambda)\times(0,\Lambda)$ and the rectangle is covered by balls, $B_{R_1}(z_l)\subset\mathbb{R}^2$, with a finite sequence of centers $\{z_l\}\subset[-\Lambda,\Lambda]\times[0,\Lambda]$ on rectangular grid with square cells of width R_1. We may now choose finite subsequences of points, $\{z_{0,i}\}\subset\{z_l\}\cap\partial_0\mathscr{O}$ and $\{z_{1,j}\}\subset\{z_l\}\cap\mathscr{O}$, such that

$$\bar{\mathscr{O}}''\subset\bigcup_{i,j}B^+_{R_1}(z_{0,i})\cup B_{R_1}(z_{1,j}),$$

where $\mathbb{H}\cap B_{R_0}(z_{0,i})\subset\mathscr{O}$ for all i and $B_{R_0}(z_{1,j})\Subset\mathscr{O}$ for all j. Let $\alpha=\alpha(A,k,p,R_0)=\alpha(A,d_1,k,p)\in(0,1)$ be the constant defined by Theorem 1.6.

According to [18, Theorem 9.19], for each $r>0$ and ball $B_r\Subset\mathscr{O}$ of radius r and integer $k\geq 0$, we have $u\in W^{k+2,p}(B_r)$, since $f\in W^{2k,p}_{\text{loc}}(\underline{\mathscr{O}})$ by hypothesis and, in particular, $f\in W^{k,p}_{\text{loc}}(\mathscr{O})$.

By the Sobolev embedding [1, Theorem 5.4 (C′)], since $p > 2$, there is a continuous embedding, $W^{k+2,p}(B_r) \hookrightarrow C^{k+1}(\bar{B}_r)$. Thus, $u \in C^{k,\alpha}(\mathscr{O})$ because, a fortiori,

$$u \in C^{k+1}(\bar{B}_r), \quad \forall B_r \Subset \mathscr{O}. \tag{5.23}$$

Moreover, letting $B_{r/2} \subset B_{3r/4} \subset B_r$ denote concentric balls,

$$\|u\|_{C^{k+1}(\bar{B}_{r/2})} \leq C \left(\|f\|_{W^{k,p}(B_r)} + \|u\|_{L^2(B_r)} \right), \tag{5.24}$$

where $C = C(A, k, p, r)$, since[23]

$$\begin{aligned} \|u\|_{C^{k+1}(\bar{B}_{r/2})} &\leq C\|u\|_{W^{k+2,p}(B_{r/2})} \quad \text{(by } p > 2 \text{ and [1, Theorem 5.4 (C}'\text{)])} \\ &\leq C \left(\|f\|_{W^{k,p}(B_{3r/4})} + \|u\|_{L^p(B_{3r/4})} \right) \quad \text{(by [18, Theorems 9.11 \& 9.19])} \\ &\leq C \left(\|f\|_{W^{k,p}(B_{3r/4})} + \|u\|_{W^{1,2}(B_{3r/4})} \right) \quad \text{(by } p > 2 \text{ and [1, Theorem 5.4 (B)])} \\ &\leq C \left(\|f\|_{W^{k,p}(B_r)} + \|u\|_{L^2(B_r)} \right) \quad \text{(by } p > 2 \text{ and [18, Exercise 8.2]).} \end{aligned}$$

The conclusion $u \in C^{k,\alpha}_s(\underline{\mathscr{O}}'') \cap C(\bar{\mathscr{O}}'')$ now follows from Theorem 1.6.

For the estimate (5.22) of u over $\mathscr{O}' \Subset \underline{\mathscr{O}}''$, observe that, since $\operatorname{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O}'') \geq d_1$, the closure $\bar{\mathscr{O}}'$ is covered by finitely many half-balls, $B^+_{R_0}(z_{0,i})$ with $\mathbb{H} \cap B_{R_0}(z_{0,i}) \Subset \underline{\mathscr{O}}''$, and balls, $B_{R_0}(z_{1,j}) \Subset \underline{\mathscr{O}}''$, where the total number of balls and half-balls of radius $R_0 = R_0(d_1)$ is determined by d_1 and Λ. We obtain (5.22) by applying (1.7) to each half-ball $B^+_{R_0}(z_{0,i}) \subset \mathscr{O}$ and applying (5.24) to each ball $B_{R_0}(z_{1,j}) \Subset \mathscr{O}$, noting that, by definition (2.5) of $\mathfrak{w}$,

$$\|f\|_{W^{k,p}(B_r)} \leq e^{\gamma\sqrt{1+\Lambda^2}} \|f\|_{W^{k,p}(B_r,\mathfrak{w})}, \tag{5.25}$$

for each ball $B_r \Subset \mathscr{O}''$, together with $\|f\|_{W^{k,p}(B_r,\mathfrak{w})} \leq \|f\|_{W^{2k,p}(\mathscr{O}'',\mathfrak{w})}$. This completes the proof. □

5.3. **Proofs of Corollary 1.7, Theorem 1.8, Corollary 1.9, Theorem 1.11, and Corollaries 1.15 and 1.16.** We first have the easy

Proof of Corollary 1.7. For any $z_0 \in \mathscr{O}$, there is a constant $R_0 > 0$ and a ball $B_{R_0}(z_0)$ such that $B_{R_0}(z_0) \subset \mathscr{O}$ and [18, Theorem 6.17] implies that $u \in C^\infty(B_{R_0/2}(z_0))$. If $z_0 \in \partial_0\mathscr{O}$, there is a constant $R_0 > 0$ such that $\mathbb{H} \cap B_{R_0}(z_0) \subset \mathscr{O}$ and Theorem 1.6 implies that there is a positive constant $R_1 = R_1(k, R_0) < R_0$ such that $u \in C^{k,\alpha}_s(B^+_{R_1}(z_0))$ for any integer $k \geq 1$. Hence, by combining these observations, $u \in C^\infty(\underline{\mathscr{O}})$. □

In order to prove Theorem 1.8 and obtain $u \in C^{k,\alpha}_s(\underline{\mathscr{O}})$ with an a priori interior estimate (1.8) on each pair of subdomains $\mathscr{O}' \subset \mathscr{O}'' \subset \mathscr{O}$ with $\mathscr{O}' \Subset \underline{\mathscr{O}}''$ and $\mathscr{O}'' \subset (-\Lambda, \Lambda) \times (0, \Lambda)$, we shall need to examine u near the "corner points", $z_0 \in \overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$, as well as u near "interior" points, $z_0 \in \partial_0\mathscr{O}$, and u near points $z_0 \in \mathscr{O}$ away from $\partial_0\mathscr{O}$ (where classical results from [18] apply). Otherwise, as noted prior to the statement of Proposition 5.7, we would not obtain a Hölder exponent, $\alpha \in (0, 1)$, which is independent of $\mathscr{O}'$ and $\mathscr{O}''$.

Proof of Theorem 1.8. Choose $R_0 = 1$ and let $R_1 = R_1(k) < 1$ and $\alpha = \alpha(A, k, p) \in (0, 1)$ be the constants defined by Theorem 1.6. Since $f \in W^{k,p}_{\mathrm{loc}}(\underline{\mathscr{O}})$ because, a fortiori, $f \in W^{2k+2,p}_{\mathrm{loc}}(\underline{\mathscr{O}})$ by hypothesis, we know from the proof of Proposition 5.7 that $u \in C^{k+1}(\mathscr{O}) \subset C^{k,\alpha}(\mathscr{O})$ because of (5.23) and that the estimate (5.24) holds for any ball $B_r \Subset \mathscr{O}$.

[23]The estimate (5.24) also follows from [18, Corollary 6.3 & Theorem 6.17].

To complete the proof that $u \in C_s^{k,\alpha}(\underline{\mathscr{O}})$, it remains to check that for every point $z_0 \in \partial_0\mathscr{O}$, there is an open ball $B_r(z_0)$, for some $r > 0$, such that $\mathbb{H} \cap B_r(z_0) \Subset \underline{\mathscr{O}}$ and $u \in C_s^{k,\alpha}(\bar{B}^+_{r/2}(z_0))$, *with* $\alpha \in (0,1)$ *as fixed at the beginning of the proof*. According to Theorem 1.6, for each point $z_0 \in \partial_0\mathscr{O}$ such that $\mathbb{H} \cap B_{R_0}(z_0) \subset \mathscr{O}$, we have $u \in C_s^{k,\alpha}(\bar{B}^+_{R_1}(z_0))$.

It remains to consider points $z_0 \in \partial_0\mathscr{O}$ such that $\mathbb{H} \cap B_{R_0}(z_0) \not\subset \mathscr{O}$; in fact, our analysis of this case is valid regardless of whether $\mathbb{H} \cap B_{R_0}(z_0) \subset \mathscr{O}$ or $\mathbb{H} \cap B_{R_0}(z_0) \not\subset \mathscr{O}$. Choose $r > 0$ small enough that $r \leq R_1$ and $\mathbb{H} \cap B_r(z_0) \Subset \underline{\mathscr{O}}$. Let $\zeta \in C_0^\infty(\bar{\mathbb{H}})$ be a cutoff function such that $0 \leq \zeta \leq 1$ on $\mathbb{H}$ and $\zeta = 1$ on $B^+_{r/2}(z_0)$ and $\operatorname{supp}\zeta \subset \underline{B}^+_r(z_0)$. To prove $u \in C_s^{k,\alpha}(\bar{B}^+_{r/2}(z_0))$, it suffices to show that

$$D_x^{\ell-m} D_y^m u \in C_s^\alpha(\bar{B}^+_{r/2}(z_0)), \quad 0 \leq \ell \leq k, \quad 0 \leq m \leq \ell, \tag{5.26}$$

and since the argument will be similar for any $0 \leq \ell \leq k$, it is enough to consider $\ell = k$.

We have $f \in W^{k,p}_{\text{loc}}(\underline{\mathscr{O}})$ since, a fortiori, $f \in W^{2k+2,p}_{\text{loc}}(\underline{\mathscr{O}})$ by hypothesis, and therefore $u \in \mathscr{H}^{k+2}(B^+_r(z_0), \mathfrak{w})$, by Theorem 1.2, and so $D_x^{k-m}D_y^m u \in H^1(B^+_r(z_0), \mathfrak{w}_m)$ and $D_x^k u \in H^1(B^+_r(z_0), \mathfrak{w})$, by Definition 4.3 of $\mathscr{H}^{k+2}(B^+_r(z_0), \mathfrak{w})$. Thus, Proposition 4.15 implies that $D_x^{k-m}D_y^m u$ obeys the variational equation on $B^+_r(z_0)$,

$$\mathfrak{a}_m(D_x^{k-m}D_y^m u, v) = (f_{k,m,u}, v)_{L^2(B^+_r(z_0), \mathfrak{w}_m)}, \quad \forall v \in H^1_0(\underline{B}^+_r(z_0), \mathfrak{w}_m),$$

where

$$f_{k,m,u} := D_x^{k-m}D_y^m f - mBD_x^{k-m}D_y^{m-1}u.$$

Consequently, since $\operatorname{supp}\zeta \subset \underline{\mathscr{O}}$, Lemma 3.5 implies that $\zeta D_x^{k-m}D_y^m u$ obeys the variational equation on $\mathbb{H}$,

$$\mathfrak{a}_m(\zeta D_x^{k-m}D_y^m u, v) = (\zeta f_{k,m,u} + [A,\zeta]D_x^{k-m}D_y^m u, v)_{L^2(\mathbb{H}, \mathfrak{w}_m)}, \quad \forall v \in H^1_0(\mathbb{H}, \mathfrak{w}_m).$$

Moreover, $\zeta D_x^{k-m}D_y^m u \in H^1(\mathbb{H}, \mathfrak{w})$ and, *provided*

$$\zeta f_{k,m,u} + [A,\zeta]D_x^{k-m}D_y^m u \in L^p(B^+_r(z_0), y^{\beta-1}), \tag{5.27}$$

noting that $\operatorname{supp}\zeta \subset \underline{B}^+_r(z_0)$ and $r \leq R_0$ (in fact, $r \leq R_1 < R_0$), Theorem 1.6 will apply to give

$$\zeta D_x^{k-m}D_y^m u \in C_s^\alpha(\bar{B}^+_r(z_0)),$$

where α was fixed at the beginning of the proof, and a positive constant, $C = C(A,k,p)$, noting that $R_0 = 1$ and $\operatorname{supp}\zeta \subset \underline{B}^+_r(z_0)$, such that

$$\begin{aligned}
&\|\zeta D_x^{k-m}D_y^m u\|_{C_s^\alpha(\bar{B}^+_r(z_0))} \\
&\quad \leq C\Big(\|\zeta f_{k,m,u}\|_{L^p(B^+_r(z_0),y^{\beta-1})} + \|[A,\zeta]D_x^{k-m}D_y^m u\|_{L^p(B^+_r(z_0),y^{\beta-1})} \\
&\qquad + \|\zeta D_x^{k-m}D_y^m u\|_{L^2(B^+_r(z_0),y^{\beta-1})}\Big).
\end{aligned} \tag{5.28}$$

We observe that

$$\begin{aligned}
&\|\zeta f_{k,m,u}\|_{L^p(B^+_r(z_0),y^{\beta-1})} \\
&\quad \leq \|D_x^{k-m}D_y^m f\|_{L^p(B^+_r(z_0),y^{\beta-1})} + \frac{m}{2}\|D_x^{k+1-m}D_y^{m-1}u\|_{L^p(B^+_r(z_0),y^{\beta-1})} \\
&\qquad + \frac{m}{2}\|D_x^{k+2-m}D_y^{m-1}u\|_{L^p(B^+_r(z_0),y^{\beta-1})} \\
&\quad \leq C\Big(\|D_x^{k-m}D_y^m f\|_{L^p(B^+_r(z_0),y^{\beta-1})} + m\|D_x^{k+1-m}D_y^{m-1}u\|_{L^\infty(B^+_r(z_0))} + m\|D_x^{k+2-m}D_y^{m-1}u\|_{L^\infty(B^+_r(z_0))}\Big),
\end{aligned}$$

for $C = C(r,p)$, while Lemma 3.6 implies that

$$\begin{aligned}
&\|[A,\zeta]D_x^{k-m}D_y^m u\|_{L^p(B_r^+(z_0),y^{\beta-1})}\\
&\quad\leq C\left(\|yD_x^{k+1-m}D_y^m u\|_{L^p(B_r^+(z_0),y^{\beta-1})} + \|yD_x^{k-m}D_y^{m+1}u\|_{L^p(B_r^+(z_0),y^{\beta-1})}\right.\\
&\qquad\left. + \|(1+y)D_x^{k-m}D_y^m u\|_{L^p(B_r^+(z_0),y^{\beta-1})}\right)\\
&\quad\leq C\left(\|D_x^{k+1-m}D_y^m u\|_{L^\infty(B_r^+(z_0))} + \|D_x^{k-m}D_y^{m+1}u\|_{L^\infty(B_r^+(z_0))} + \|D_x^{k-m}D_y^m u\|_{L^\infty(B_r^+(z_0))}\right),
\end{aligned}$$

where $C = C(A,r)$ and ζ is chosen such that $\|\zeta\|_{C^2(\bar{B}_r^+(z_0))} \leq Mr^{-2}$, where $M > 0$ is a universal constant. But $f \in W^{2k+2,p}_{\mathrm{loc}}(\underline{\mathscr{O}})$ by hypothesis and from Proposition 5.7 (applied with k replaced by $k+1$ and $p > \max\{4, 2+(k+1)+\beta\} = \max\{4, 3+k+\beta\}$), we know that

$$\begin{gathered}
D_x^{k+1-m}D_y^{m-1}u,\quad D_x^{k+2-m}D_y^{m-1}u \in C(\bar{B}_r^+(z_0)),\quad 1\leq m\leq k,\\
D_x^{k+1-m}D_y^{m}u,\quad D_x^{k-m}D_y^{m+1}u,\quad D_x^{k-m}D_y^{m}u \in C(\bar{B}_r^+(z_0)),\quad 0\leq m\leq k,
\end{gathered}$$

since $B_r^+(z_0) \Subset \underline{\mathscr{O}}$ and $u \in C^{k+1}(\bar{B}_r^+(z_0))$. The preceding inequalities and boundedness conditions ensure that the integrability condition (5.27) holds.

In particular, $\zeta D_x^{k-m}D_y^m u \in C_s^\alpha(\bar{B}_{R_1}^+(z_0))$ and, because $\zeta = 1$ on $B_{r/2}^+(z_0)$, we obtain $D_x^{k-m}D_y^m u \in C_s^\alpha(\bar{B}_{r/2}^+(z_0))$, as desired. This completes the proof of (5.26) (when $\ell = k$) and hence that $u \in C_s^{k,\alpha}(\underline{\mathscr{O}})$.

Finally, we prove the a priori estimate (1.8). Since $\mathscr{O}' \Subset \underline{\mathscr{O}}''$ and $\operatorname{dist}(\partial_1\mathscr{O}', \partial_1\mathscr{O}'') \geq d_1$ and $\mathscr{O}'' \subset (-\Lambda,\Lambda)\times(0,\Lambda)$ by hypothesis, there are a finite number of balls of radius $r := \min\{d_1/4, R_1\}$ (the number of balls is determined by $r = r(d_1,R_1) = r(d_1,k)$ and Λ) such that

$$\mathscr{O}' \subset \bigcup_{i,j} B_{r/2}^+(z_{0,i}) \cup B_{r/2}(z_{1,j}), \tag{5.29}$$

with

$$\mathbb{H}\cap B_{4r}(z_{0,i}) \subset \mathscr{O}'' \quad\text{and}\quad B_{4r}(z_{1,j}) \subset \mathscr{O}''.$$

For each i, we let $\zeta_i \in C_0^\infty(\bar{\mathbb{H}})$ be a cutoff function such that $0 \leq \zeta_i \leq 1$ on $\mathbb{H}$, and $\zeta_i = 1$ on $B_{r/2}^+(z_{0,i})$, and $\operatorname{supp}\zeta_i \subset \underline{B}_r^+(z_{0,i})$, and $\|\zeta_i\|_{C^2(\bar{B}_r^+(z_{0,i}))} \leq Mr^{-2}$. The preceding L^p estimate for $\zeta f_{k,m,u}$ yields

$$\|\zeta_i f_{k,m,u}\|_{L^p(B_r^+(z_{0,i}))} \leq C\left(\|f\|_{W^{k,p}(B_r^+(z_{0,i})),y^{\beta-1})} + \|u\|_{C_s^{k+1}(\bar{B}_r^+(z_{0,i}))}\right),\quad 0\leq m\leq k,$$

with $C = C(k,p,r)$. Thus, Proposition 5.7 (noting that $B_r^+(z_0) \Subset \underline{B}_{2r}^+(z_0)$ and $B_{2r}^+(z_0) \Subset \underline{\mathscr{O}}''$ with $\operatorname{dist}(\partial_1 B_{2r}^+(z_0), \partial_1\mathscr{O}'') \geq 2r$ since $B_{4r}^+(z_0) \subset \mathscr{O}''$) and the definition of $\mathfrak{w} = y^{\beta-1}e^{-\mu y-\gamma\sqrt{1+x^2}}$ in (2.5) gives

$$\|\zeta_i f_{k,m,u}\|_{L^p(B_r^+(z_{0,i}))} \leq C\left(\|f\|_{W^{2k+2,p}(B_{2r}^+(z_{0,i})),\mathfrak{w})} + \|u\|_{L^2(B_{2r}^+(z_{0,i})),\mathfrak{w})}\right),\quad 0\leq m\leq k, \tag{5.30}$$

with $C = C(A,d_1,k,p,r) = C(A,d_1,k,p)$. Similarly, the preceding L^p estimate for $[A,\zeta]D_x^{k-m}D_y^m u$ yields

$$\|[A,\zeta_i]D_x^{k-m}D_y^m u\|_{L^p(B_r^+(z_{0,i}))} \leq C\left(\|f\|_{W^{k,p}(B_r^+(z_{0,i}),y^{\beta-1})} + \|u\|_{C_s^{k+1}(\bar{B}_r^+(z_{0,i}))}\right),\quad 0\leq m\leq k,$$

with $C = C(A,p,r)$. Thus, Proposition 5.7 now gives, for $0 \leq m \leq k$,

$$\|[A,\zeta_i]D_x^{k-m}D_y^m u\|_{L^p(B_r^+(z_{0,i}))} \leq C\left(\|f\|_{W^{2k+2,p}(B_{2r}^+(z_{0,i})),\mathfrak{w})} + \|u\|_{L^2(B_{2r}^+(z_{0,i})),\mathfrak{w})}\right), \tag{5.31}$$

with $C = C(A, d_1, k, p)$. By combining (5.28), (5.30), and (5.31), and recalling that $\zeta_i = 1$ on $B^+_{r/2}(z_{0,i})$ and $\operatorname{supp}\zeta_i \subset \underline{B}^+_r(z_{0,i})$, we obtain, for $0 \le m \le k$,

$$\|D_x^{k-m}D_y^m u\|_{C_s^\alpha(B^+_{r/2}(z_{0,i}))} \le C\left(\|f\|_{W^{2k+2,p}(B^+_{2r}(z_{0,i}),\mathfrak{w})} + \|u\|_{L^2(B^+_{2r}(z_{0,i}),\mathfrak{w})}\right),$$

with $C = C(A, d_1, k, p)$. Therefore, by the same argument, for any $0 \le m \le \ell \le k$, we have

$$\|D_x^{\ell-m}D_y^m u\|_{C_s^\alpha(B^+_{r/2}(z_{0,i}))} \le C\left(\|f\|_{W^{2k+2,p}(B^+_{2r}(z_{0,i}),\mathfrak{w})} + \|u\|_{L^2(B^+_{2r}(z_{0,i}),\mathfrak{w})}\right), \tag{5.32}$$

with $C = C(A, d_1, k, p)$.

On the other hand, by applying (5.24) to the balls $B_r(z_{1,j})$, we obtain

$$\|u\|_{C^{k+1}(\bar{B}_{r/2}(z_{1,j}))} \le C\left(\|f\|_{W^{2k,p}(B_r(z_{1,j}))} + \|u\|_{L^2(B_r(z_{1,j}))}\right), \tag{5.33}$$

with $C = C(A, k, p, r) = C(A, d_1, k, p)$. The desired a priori estimate (1.8) now follows by combining the a priori estimates (5.32) and (5.33), noting that $C^{k+1}(\bar{B}^+_{r/2}(z_{1,j})) \hookrightarrow C^{k,\alpha}(\bar{B}^+_{r/2}(z_{1,j}))$ and using (5.25) to give

$$\|f\|_{W^{2k,p}(B_r(z_{1,j}))} \le e^{\gamma\sqrt{1+\Lambda^2}}\|f\|_{W^{2k,p}(B_r(z_{1,j}),\mathfrak{w})}.$$

and $\|f\|_{W^{2k,p}(B_r(z_{1,j}),\mathfrak{w})} \le \|f\|_{W^{2k,p}(\mathscr{O}'',\mathfrak{w})}$. This completes the proof. $\square$

Next, we have the

Proof of Corollary 1.9. As in the proof of Theorem 1.3, it suffices to choose a cover (5.29) of $\mathscr{O}'$ by open balls $B_{r/2}(z_{1,j})$ or half-balls $B^+_{r/2}(z_{0,j})$ contained in $\mathscr{O}''$ which is *uniformly locally finite.* Again using the definition of $\mathfrak{w} = y^{\beta-1}e^{-\mu y-\gamma\sqrt{1+x^2}}$ in (2.5) to replace the integral weights $\mathfrak{w}$ in (5.32) and 1 in (5.33), respectively, by $y^{\beta-1}$ on the right-hand side and arguing just as in the proof of Proposition 5.1 to eliminate factors such as $e^{\frac{\gamma}{2}|x_{0,i}|}$ or $e^{\frac{\gamma}{2}|x_{1,j}|}$, we see that

$$\|u\|_{C_s^{k,\alpha}(\bar{B}^+_{r/2}(z_{0,i}))} \le C\left(\|f\|_{W^{2k+2,p}(B^+_{2r}(z_{0,i}),y^{\beta-1})} + \|u\|_{L^2(B^+_{2r}(z_{0,i}),y^{\beta-1})}\right) \quad \text{(by (5.32))},$$

$$\|u\|_{C^{k+1}(\bar{B}_{r/2}(z_{1,j}))} \le C\left(\|f\|_{W^{2k,p}(B_r(z_{1,j}),y^{\beta-1})} + \|u\|_{L^2(B_r(z_{1,j}),y^{\beta-1})}\right) \quad \text{(by (5.33))},$$

with $C = C(A, d_1, k, p)$. Therefore, using $C^{k+1}(\bar{B}_{r/2}(z_{1,j})) \hookrightarrow C_s^{k,\alpha}(\bar{B}_{r/2}(z_{1,j}))$, we obtain

$$\sup_i \|u\|_{C_s^{k,\alpha}(\bar{B}^+_{r/2}(z_{0,i})} + \sup_j \|u\|_{C_s^{k,\alpha}(\bar{B}_{r/2}(z_{1,j})} \le C\left(\|f\|_{W^{2k+2,p}(\mathscr{O}'',y^{\beta-1})} + \|u\|_{L^2(\mathscr{O}'',y^{\beta-1})}\right),$$

with $C = C(A, d_1, k, p)$. Since

$$\|u\|_{C^k(\bar{\mathscr{O}}')} \le \sup_i \|u\|_{C^k(\bar{B}^+_{R_1}(z_{0,i}))} + \sup_j \|u\|_{C^k(\bar{B}_{R_1}(z_{1,j}))},$$

and, denoting

$$[D^k u]_{C_s^\alpha(\bar{U})} := \max_{0\le m\le k}[D_x^{k-m}D_y^m u]_{C_s^\alpha(\bar{U})},$$

for any $U \subset \mathbb{H}$, we see that

$$\begin{aligned}\|u\|_{C_s^{k,\alpha}(\bar{\mathscr{O}}')} &\equiv \|u\|_{C^k(\bar{\mathscr{O}}')} + [D^k u]_{C_s^\alpha(\bar{\mathscr{O}}')} \quad \text{(Definition 2.14)}\\ &\le C(r)\|u\|_{C^k(\bar{\mathscr{O}}')} + \sup_i [D^k u]_{C_s^\alpha(\bar{B}^+_{r/2}(z_{0,i}))} + \sup_j [D^k u]_{C_s^\alpha(\bar{B}_{r/2}(z_{1,j}))}.\end{aligned}$$

Combining the preceding estimates and recalling that $r \equiv \min\{d_1/4, R_1\}$ in (5.29), so $r = r(d_1, k)$, yields the desired a priori bound (1.9) for $\|u\|_{C_s^{k,\alpha}(\bar{\mathscr{O}}')}$. $\square$

Lastly, we turn to the

Proof of Theorem 1.11. By hypothesis, we have $f \in C(\bar{\mathscr{O}})$ and $(1+y)g \in W^{2,\infty}(\mathscr{O})$, since $g \in C(\bar{\mathscr{O}})$ with $(1+y)g \in C^2(\bar{\mathscr{O}})$. Therefore, Theorem 1.5 (with $k=0$) implies that there exists a unique solution $u \in H^1(\mathscr{O}, \mathfrak{w})$ to the variational equation (2.11), with boundary condition $u - g \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w})$.

By [18, Corollary 8.28], we must have $u \in C(\mathscr{O} \cup \partial_1\mathscr{O})$, since $u = g$ on $\partial\mathscr{O}$ in the sense of $H^1(\mathscr{O}, \mathfrak{w})$ and $g \in C(\overline{\partial_1\mathscr{O}})$, so $u = g$ on $\partial_1\mathscr{O}$, that is, u obeys the boundary condition (1.2). Because $f \in C(\bar{\mathscr{O}})$ by hypothesis, the maximum principle [11, Theorem 8.15] implies that u is bounded, that is, $u \in L^\infty(\mathscr{O})$.

By hypothesis, we also have $f \in C^{2k+6,\alpha}_s(\underline{\mathscr{O}})$. For any $1 \leq p \leq \infty$, there is a continuous embedding $C^{2k+6,\alpha}_s(\bar{U}) \hookrightarrow W^{2k+6,p}(U, \mathfrak{w})$ for any $U \Subset \bar{\mathbb{H}}$, by Definition 2.14 of $C^{\ell,\alpha}_s(\bar{U})$ for $\ell \geq 0$, and hence $f \in W^{2k+6,p}_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w}) = W^{2(k+2)+2,p}_{\mathrm{loc}}(\underline{\mathscr{O}}, \mathfrak{w})$. Choose $p = \max\{4, 3+(k+2)+\beta\}+1 = 6+k+\beta$ and observe that Theorem 1.8 implies that $u \in C^{k+2,\alpha}_s(\underline{\mathscr{O}})$. From the Definition 2.15 of $C^{k,2+\alpha}_s(\bar{U})$, it follows that there is a continuous embedding $C^{k+2,\alpha}_s(\bar{U}) \hookrightarrow C^{k,2+\alpha}_s(\bar{U})$, for any $U \Subset \bar{\mathbb{H}}$. Hence, $u \in C^{k,2+\alpha}_s(\underline{\mathscr{O}})$.

Since the equation (1.1) and the desired Schauder a priori estimate (1.10) are invariant under translations with respect to x and $\operatorname{diam}(\mathscr{O}'') \leq \Lambda$ by hypothesis, we may assume without loss of generality that $\mathscr{O}'' \subset (-\Lambda, \Lambda) \times (0, \Lambda)$. Therefore, the desired Schauder estimate (1.10) follows from Theorem 1.8 and the a priori estimate (1.8) and the fact that

$$\|u\|_{C^{k,2+\alpha}_s(\bar{\mathscr{O}}')} \leq C\|u\|_{C^{k+2,\alpha}_s(\bar{\mathscr{O}}')},$$

where $C = C(\Lambda)$. □

Proof of Corollary 1.15. Theorem 2.11 implies that $u \in C^\alpha_s(\bar{B}^+_{R_1}(z_0))$, for each corner point $z_0 \in \overline{\partial_0\mathscr{O}} \cap \overline{\partial_1\mathscr{O}}$. Thus, $u \in C^\alpha_{s,\mathrm{loc}}(\bar{\mathscr{O}})$ and, because u is bounded, we obtain $u \in C(\bar{\mathscr{O}})$. Moreover, because u is uniformly $C^\alpha_s(\bar{B}^+)$-Hölder continuous for all balls $B \subset \mathbb{R}^2$, where $\alpha = \alpha(A, k, K) \in (0,1)$ is the smallest Hölder exponent in Theorems 1.8 and 2.11 and in [18, Theorem 8.29], we have $u \in C^\alpha_{\mathrm{loc},s}(\bar{\mathscr{O}})$. □

Proof of Corollary 1.16. The desired Schauder estimate (1.11) follows by replacing the role of the a priori estimate (1.8) with that of the a priori estimate (1.9) in the proof of Theorem 1.11. □

Appendix A. Appendix

For the convenience of the reader, we collect here some useful facts from some our earlier articles for easier reference, together with some technical proofs of results used in the body of this article. In §A.1, we describe approximation results for the weighted Sobolev spaces appearing in this article and which are used, for example, to prove integration-by-parts formulae, as illustrated in §A.2. The relationship between "cycloidal" and Euclidean balls and half-balls is discussed in §A.3. Section A.4 describes how to translate [12, Theorems 1.7 & 1.11] into forms more suitable for application in this article, namely Theorems 2.5, 2.10, and 2.12, though we include Theorems 2.6, 2.11, and 2.13 as well, even though not needed for the present article. Finally, in §A.5, we explain the need for some of the technical hypotheses in Proposition 4.15.

A.1. **Approximation by smooth functions.** We begin with the

Definition A.1 (C^1-orthogonal curves in the upper half-space)**.** We say that a curve $T \subset \mathbb{H}$ is *uniformly* C^1*-orthogonal to* $\partial\mathbb{H}$ if T is a relatively open C^1-curve and there is a positive constant, δ, such that for each point $z_0 = (x_0, 0) \in \bar{T} \cap \partial\mathbb{H}$ we have

$$\bar{T} \cap B_\delta(z_0) \subset \{(x_0, y) \in \mathbb{R}^2 : y \geq 0\}.$$

The next approximation result[24] follows from [3, Corollary A.12].

Theorem A.2 (Density of smooth functions)**.** *Let $\mathscr{O} \subset \mathbb{H}$ be a domain such that $\partial_1\mathscr{O}$ is uniformly C^1-orthogonal to $\partial\mathbb{H}$. Then $C^\infty_0(\bar{\mathscr{O}})$ is a dense subset of $H^k(\mathscr{O}, \mathfrak{w})$, and $\mathscr{H}^k(\mathscr{O}, \mathfrak{w})$, and $W^k(\mathscr{O}, \mathfrak{w})$ for all integers $k \geq 0$.*

Proof. When $k = 0, 1, 2$, the conclusion for $H^k(\mathscr{O}, \mathfrak{w}) = \mathscr{H}^k(\mathscr{O}, \mathfrak{w})$ is given by [3, Corollary A.12], which asserts that $C^\infty_0(\bar{\mathscr{O}})$ is a dense subset of $H^k(\mathscr{O}, \mathfrak{w})$. The proof of [3, Corollary A.12] extends easily to include the remaining cases. □

A.2. **Integration by parts.** We recall the special case of [3, Lemma 2.23]; no hypothesis on $\partial_1\mathscr{O}$ is required here because we assume $v \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w})$ rather than allow any $v \in H^1(\mathscr{O}, \mathfrak{w})$.

Lemma A.3 (Integration by parts for the Heston operator)**.** *Let $\mathscr{O} \subseteqq \mathbb{H}$ be a domain. If $u \in H^2(\mathscr{O}, \mathfrak{w})$ and $v \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w})$, then $Au \in L^2(\mathscr{O}, \mathfrak{w})$ and*

$$(Au, v)_{L^2(\mathscr{O},\mathfrak{w})} = \mathfrak{a}(u, v). \tag{A.1}$$

Proof. When $\tilde{u} \in C^\infty_0(\bar{\mathscr{O}})$ and $\tilde{v} \in C^\infty_0(\underline{\mathscr{O}})$, we obtain

$$(A\tilde{u}, \tilde{v})_{L^2(\mathscr{O},\mathfrak{w})} = \mathfrak{a}(\tilde{u}, \tilde{v})$$

by direct calculation, as in the proof of [3, Lemma 2.23]. Because $\operatorname{supp}\tilde{v} \subset \underline{\mathscr{O}}$ is compact, we may choose a subdomain $\mathscr{O}' \Subset \underline{\mathscr{O}}$ such that $\partial_1\mathscr{O}'$ is uniformly C^1-orthogonal to $\partial\mathbb{H}$ and $\operatorname{supp}\tilde{v} \subset \underline{\mathscr{O}}'$. If $u \in H^2(\mathscr{O}, \mathfrak{w})$, Theorem A.2 implies that there is a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty_0(\bar{\mathscr{O}}')$ such that $u_n \to u$ strongly in $H^2(\mathscr{O}, \mathfrak{w})$ as $n \to \infty$ and thus

$$(Au, \tilde{v})_{L^2(\mathscr{O},\mathfrak{w})} = (Au, \tilde{v})_{L^2(\mathscr{O}',\mathfrak{w})} = \lim_{n\to\infty}(Au_n, \tilde{v})_{L^2(\mathscr{O}',\mathfrak{w})} = \lim_{n\to\infty}\mathfrak{a}(u_n, \tilde{v}) = \mathfrak{a}(u, \tilde{v}),$$

because $A : H^2(\mathscr{O}, \mathfrak{w}) \to L^2(\mathscr{O}, \mathfrak{w})$ is a continuous linear operator and $\mathfrak{a} : H^1(\mathscr{O}, \mathfrak{w}) \times H^1(\mathscr{O}, \mathfrak{w}) \to \mathbb{R}$ is a continuous bilinear map. . Since $v \in H^1_0(\underline{\mathscr{O}}, \mathfrak{w})$, there is a sequence $\{v_n\}_{n\in\mathbb{N}} \subset C^\infty_0(\underline{\mathscr{O}})$ such that $v_n \to v$ strongly in $H^1(\mathscr{O}, \mathfrak{w})$ as $n \to \infty$ and thus

$$(Au, v)_{L^2(\mathscr{O},\mathfrak{w})} = \lim_{n\to\infty}(Au, v_n)_{L^2(\mathscr{O},\mathfrak{w})} = \lim_{n\to\infty}\mathfrak{a}(u, v_n) = \mathfrak{a}(u, v).$$

This completes the proof. □

A.3. **Relationship between cycloidal and Euclidean balls.** The relationships (2.19) between the cycloidal and Euclidean distance functions and (2.20) between the cycloidal and Euclidean balls are easily generalized from the case $y_0 = 0$ to $y_0 \geq 0$. Denoting $S = s(z, z_0)$ and $D = |z - z_0|$, and using $y \leq y_0 + D$, then

$$D = S\sqrt{y + y_0 + D} \leq S\sqrt{2y_0 + 2D},$$

and one finds that

$$\left|D - S^2\right| \leq S\sqrt{2y_0 + S^2} \leq S\left(\sqrt{2y_0} + S\right),$$

[24] While the conclusion holds for weaker hypotheses on $\partial_1\mathscr{O}$, the result suffices for applications in this article and counterexamples show that some conditions on the regularity of $\partial_1\mathscr{O}$ and the geometry of its intersection with $\partial\mathbb{H}$ are required.

and hence

$$D \leq 2S^2 + \sqrt{2y_0}S.$$

Therefore,

$$|z - z_0| \leq 2s(z, z_0)^2 + s(z, z_0)\sqrt{2y_0}, \tag{A.2}$$

and so

$$\mathscr{B}_r(z_0) \subset \mathbb{H} \cap B_{2r^2+r\sqrt{2y_0}}(z_0), \quad \forall z_0 \in \bar{\mathbb{H}},\ r > 0, \tag{A.3}$$

as desired.

A.4. **Proofs of Theorems 2.5, 2.6, 2.10, 2.11, 2.12, and 2.13.** The original statement of [12, Theorem 1.7] combines Theorem 2.5 (supremum estimate away from corner points) and Theorem 2.6 (supremum estimate near corner points). Similarly, the original statement of [12, Theorem 1.11] combines Theorem 2.10 (Hölder regularity and a priori estimate away from corner points) and Theorem 2.11 (Hölder regularity and a priori estimate near corner points). The theorem statements in [12] use balls defined by the cycloidal distance function, do not make the dependencies of the constants as explicit as we do here and, in the case of [12, Theorem 1.11], state estimates in terms of L^∞ rather than L^2 norms of u. In this appendix, we describe how to translate [12, Theorems 1.7 & 1.11] into the forms used in this article, noting that in [12] the dimension of the upper half-space is denoted by $n = 2$.

Proof of Theorem 2.5. A close inspection of the proof of [12, Theorem 1.7] reveals that the supremum estimate holds for *any* positive constant $\bar{R}$ with the property that $\mathscr{B}^+_{\bar{R}}(z_0) \subset \mathscr{O}$. In particular, we may choose $\bar{R} := \sqrt{R_0/2}$, where $R_0 > 0$ is the constant in the hypothesis of Theorem 2.5 such that $f \in L^p(B^+_{R_0}(z_0), y^{\beta-1})$. Since $R_0 = 2\bar{R}^2$, we see by (2.20) that $\mathscr{B}^+_{\bar{R}}(z_0) \subset B^+_{R_0}(z_0)$, and so the hypothesis[25] on f in [12, Theorem 1.7] (with $s = p$) is satisfied when

$$f \in L^p(B^+_{\bar{R}}(z_0), y^{\beta-1}), \tag{A.4}$$

since $f \in L^p(B^+_{R_0}(z_0), y^{\beta-1})$ and $L^p(B^+_{R_0}(z_0), y^{\beta-1}) \subset L^p(\mathscr{B}^+_{\bar{R}}(z_0), y^{\beta-1})$.

For any $R > 0$, we have by (2.18) that $B^+_R(z_0) \subset \mathscr{B}^+_{\sqrt{R}}(z_0)$ and therefore we obtain, for all $R > 0$ obeying $2\sqrt{R} \leq \bar{R}$ or, equivalently, $8R \leq R_0$,

$$\begin{aligned}
\|u\|_{L^\infty(B^+_R(z_0))} &\leq \|u\|_{L^\infty(\mathscr{B}^+_{\sqrt{R}}(z_0))} \quad \text{(by (2.18))} \\
&\leq C\left(\|u\|_{L^2(\mathscr{B}^+_{2\sqrt{R}}(z_0), y^{\beta-1})} + \|f\|_{L^p(\mathscr{B}^+_{2\sqrt{R}}(z_0), y^{\beta-1})}\right) \quad \text{(by [12, Equation (1.21)])} \\
&\leq C\left(\|u\|_{L^2(B^+_{8R}(z_0), y^{\beta-1})} + \|f\|_{L^p(B^+_{8R}(z_0), y^{\beta-1})}\right),
\end{aligned}$$

and thus,

$$\|u\|_{L^\infty(B^+_R(z_0))} \leq C\left(\|u\|_{L^2(B^+_{8R}(z_0), y^{\beta-1})} + \|f\|_{L^p(B^+_{8R}(z_0), y^{\beta-1})}\right), \tag{A.5}$$

where the last inequality follows from the fact that $\mathscr{B}^+_{2\sqrt{R}}(z_0) \subset B^+_{8R}(z_0)$ by (2.20) and $C = C(A, p, R)$. We obtain the desired inequality (2.14) by choosing $R_1 := R_0/8$ and setting $R = R_1$ in (A.5). □

[25] In [12], the ball $\mathscr{B}^+_r(z_0)$ was denoted by $B_r(z_0)$ for $r > 0$.

Proof of Theorem 2.10. As in the proof of Theorem 2.5, we choose $\bar{R}_0 := \sqrt{R_0/2}$, which implies by (2.20) that $\mathscr{B}^+_{\bar{R}_0}(z_0) \subset B^+_{R_0}(z_0)$ since $R_0 = 2\bar{R}_0^2$, and so the hypothesis [25] on f in [12, Equation (1.26)] (with $s = p$) is satisfied when f obeys (A.4).

A closer examination of the proof of [12, Theorem 1.11] shows us that there are positive constants $C = C(A, p, \bar{R}_0)$, and $\alpha = \alpha(A, p, R_0) \in (0, 1)$, and $\bar{R}_1 = \bar{R}_1(\bar{R}_0)$ such that

$$[u]_{C^\alpha_s(\mathscr{B}^+_{\bar{R}_1}(z_0))} \leq C \left(\|f\|_{L^p(\mathscr{B}^+_{\bar{R}_0}(z_0), y^{\beta-1})} + \|u\|_{L^\infty(\mathscr{B}^+_{\bar{R}_0}(z_0))} \right).$$

If we choose $R_1 := \bar{R}_1^2$, then (2.18) yields $B^+_{R_1}(z_0) \subset \mathscr{B}^+_{\bar{R}_1}(z_0)$, and thus

$$\begin{aligned} [u]_{C^\alpha_s(\bar{B}^+_{R_1}(z_0))} &\leq [u]_{C^\alpha_s(\mathscr{B}^+_{\bar{R}_1}(z_0))} \\ &\leq C \left(\|f\|_{L^p(\mathscr{B}^+_{\bar{R}_0}(z_0), y^{\beta-1})} + \|u\|_{L^\infty(\mathscr{B}^+_{\bar{R}_0}(z_0))} \right), \\ &\leq C \left(\|f\|_{L^p(B^+_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^\infty(B^+_{R_0}(z_0))} \right), \end{aligned}$$

where we used the inclusion $\mathscr{B}^+_{\bar{R}_0}(z_0) \subset B^+_{R_0}(z_0)$ to obtain the final inequality. This concludes the proof of Theorem 2.10. □

Proof of Theorem 2.12. Let $R_2 = R_2(R_0) < R_0$ be the constant produced by Theorem 2.5, given $R_0 > 0$, so (2.14) gives

$$\|u\|_{L^\infty(B^+_{R_2}(z_0))} \leq C \left(\|f\|_{L^p(B^+_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(B^+_{R_0}(z_0), y^{\beta-1})} \right),$$

and let $R_1 = R_1(R_2) < R_2$ (and recall that $R_2 = R_2(R_0)$) be the constant produced by Theorem 2.10, given $R_2 > 0$, so (2.23) gives

$$[u]_{C^\alpha_s(\bar{B}^+_{R_1}(z_0))} \leq C \left(\|f\|_{L^p(B^+_{R_2}(z_0), y^{\beta-1})} + \|u\|_{L^\infty(B^+_{R_2}(z_0))} \right).$$

Finally, noting that $\|u\|_{C^\alpha_s(\bar{B}^+_{R_1}(z_0))} = \|u\|_{C(\bar{B}^+_{R_1}(z_0))} + [u]_{C^\alpha_s(\bar{B}^+_{R_1}(z_0))}$, we obtain the desired inequality (2.24) by combining the preceding two estimates. □

Proofs of Theorems 2.6, 2.11, and 2.13. The proofs of Theorems 2.6, 2.11, and 2.13 follow exactly in the same way as the proofs of Theorems 2.5, 2.10, and 2.12, with the only observation that all constants now also depend on the cone, K. □

A.5. **Need for the auxiliary regularity condition in Proposition 4.15.** We explain the role of the hypothesis, $D^k_x u \in H^1(\mathscr{O}, \mathfrak{w})$, in the statement of Proposition 4.15.

First, we explain the role of the auxiliary regularity condition when $m = 0$ in (4.23). If $k = 1, m = 0$ and $u \in H^2(\mathscr{O}, \mathfrak{w})$, then we recall from (3.9) that while this ensures $(1+y)^{1/2} u_x$ belongs to $L^2(\mathscr{O}, \mathfrak{w})$, it does not imply that $y^{1/2} u_{xx}$, $y^{1/2} u_{xy}$ belong to $L^2(\mathscr{O}, \mathfrak{w})$, and so $u \in H^2(\mathscr{O}, \mathfrak{w})$ does not imply $u_x \in H^1(\mathscr{O}, \mathfrak{w})$. However, when $k = 1, m = 1$, we have seen that $u \in H^2(\mathscr{O}, \mathfrak{w})$ does imply $u_y \in H^1(\mathscr{O}, \mathfrak{w}_1)$.

If $u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w})$ and $k \geq 2$, then we recall from Definition 4.3 that

$$yD_x^{k+1-m}D_y^m u \in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-2}), & 3 \leq m \leq k, \\ L^2(\mathscr{O},\mathfrak{w}), & m = 1,2, \end{cases}$$

$$yD_x^{k-m}D_y^{m+1} u \in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-1}), & 2 \leq m \leq k, \\ L^2(\mathscr{O},\mathfrak{w}), & m = 1, \end{cases}$$

that is,

$$y^{1/2}D_x^{k+1-m}D_y^m u \in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-1}), & 3 \leq m \leq k, \\ L^2(\mathscr{O},\mathfrak{w}_1), & m = 1,2, \end{cases}$$

$$y^{1/2}D_x^{k-m}D_y^{m+1} u \in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m}), & 2 \leq m \leq k, \\ L^2(\mathscr{O},\mathfrak{w}_1), & m = 1. \end{cases}$$

Note that $L^2(\mathscr{O},\mathfrak{w}_{m-1}) \subset L^2(\mathscr{O},\mathfrak{w}_m)$ for all $m \geq 1$. Moreover, $u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w})$ with $k \geq 2$ implies $(1+y)D_x^{k-m}D_y^m u \in L^2(\mathscr{O},\mathfrak{w}_{m-2}) \subset L^2(\mathscr{O},\mathfrak{w}_m)$ when $3 \leq m \leq k$ and $(1+y)D_x^{k-m}D_y^m u \in L^2(\mathscr{O},\mathfrak{w}) \subset L^2(\mathscr{O},\mathfrak{w}_m)$ when $m = 0,1,2$. Thus, for $k \geq 2$,

$$u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w}) \implies D_x^{k-m}D_y^m u \in H^1(\mathscr{O},\mathfrak{w}_m), \quad k \geq 2,\ 1 \leq m \leq k.$$

However, when $k \geq 2$ and $m = 0$, the auxiliary condition $D_x^k u \in H^1(\mathscr{O},\mathfrak{w})$ required for the *left-hand side* of (4.23) to be well-defined is not implied by the hypothesis $u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w})$, since the latter condition implies $yD_x^{k+1}u,\ yD_x^kD_y u \in L^2(\mathscr{O},\mathfrak{w})$ but not $y^{1/2}D_x^{k+1}u,\ y^{1/2}D_x^kD_y u \in L^2(\mathscr{O},\mathfrak{w})$.

Second, we explain the role of the auxiliary regularity condition when $m = 1$ in (4.23). If $u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w})$ and $k \geq 2$, then we recall from Definition 4.3 that

$$yD_x^{k+2-m}D_y^{m-1} u \in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-3}), & m \geq 4, \\ L^2(\mathscr{O},\mathfrak{w}), & m = 1,2,3, \end{cases}$$

that is,

$$D_x^{k+2-m}D_y^{m-1} u \in \begin{cases} L^2(\mathscr{O},\mathfrak{w}_{m-1}), & m \geq 4, \\ L^2(\mathscr{O},\mathfrak{w}_2), & m = 1,2,3. \end{cases}$$

Hence, when $k \geq 2$ and $m = 1$, the auxiliary condition $D_x^{k+1}u \in L^2(\mathscr{O},\mathfrak{w}_1)$ required for the *right-hand side* of (4.23) to be well-defined is not implied by the hypothesis $u \in \mathscr{H}^{k+1}(\mathscr{O},\mathfrak{w})$. However, the condition $D_x^k u \in H^1(\mathscr{O},\mathfrak{w})$ ensures, by definition (2.2) of $H^1(\mathscr{O},\mathfrak{w})$, that $y^{1/2}D_x^{k+1}u \in L^2(\mathscr{O},\mathfrak{w})$ or, equivalently, $D_x^{k+1}u \in L^2(\mathscr{O},\mathfrak{w}_1)$.

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