

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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on Sub-quantum State Decomposition

by

Xue-Na Zhu, Ming-Jing Zhao, and Shao-Ming Fei

Preprint no.: 55

2012



Lower Bound of Multipartite Concurrence Based on Sub-quantum State Decomposition

Xue-Na Zhu¹, Ming-Jing Zhao², and Shao-Ming Fei^{2,3}

¹*Department of Mathematics, School of Science, South China University of Technology, Guangzhou 510640, China*

²*Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany*

³*School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*

We study the entanglement of tripartite quantum states and provide analytical lower bound of concurrence in terms of the concurrence of sub-states. The lower bound may improve all the existing lower bounds of concurrence. The approach is generalized to arbitrary dimensional multipartite systems.

PACS numbers: 03.67.Mn, 03.65.Ud

I. INTRODUCTION

Quantum entanglement [1] is considered to be the most nonclassical manifestation of quantum mechanics and plays an important role not only in quantum information sciences [2–5] but also in condensed-matter physics [6]. The operational measure of entanglement for arbitrary mixed states is not known yet, but the concurrence [7–10] is one of the well accepted entanglement measures [11–16]. Nevertheless, calculation of the concurrence is a formidable task for higher dimensional case.

To estimate the concurrence for general mixed states, efforts have been made toward the analytical lower bounds of concurrence. Therefore some nice algorithms and progresses have been concentrated on possible lower bounds of the concurrence for three quantum systems [17–19]. For arbitrary bipartite quantum states, Ref. [20] and Ref. [21] provide a detailed proof of an analytical lower bound of concurrence by decomposing the joint Hilbert space into many $2 \otimes 2$ and $s \otimes t$ -dimensional subspaces, which may be used to improve all the known lower bounds of concurrence. A natural problem is whether the arbitrary dimensional tripartite quantum states can be dealt with this.

In this paper we provide a detailed proof of an analytical lower bound of concurrence for tripartite quantum states by decomposing the joint Hilbert space into any

lower dimensional subspaces. Moreover, the generalized lower bound of concurrence can be generalized to the multipartite case.

II. LOWER BOUND OF CONCURRENCE FOR TRIPARTITE QUANTUM SYSTEMS

Let H_{A_1} , H_{A_2} , H_{A_3} be three N -dimensional Hilbert spaces associated with the systems A_1 , A_2 and A_3 . A pure state $|\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes H_{A_3}$ has the form

$$|\psi\rangle = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N a_{ijk} |ijk\rangle, \quad (1)$$

where $a_{ijk} \in \mathbb{C}$, $\sum_{ijk} |a_{ijk}|^2 = 1$, $\{|ijk\rangle\}$ is the basis of $H_{A_1} \otimes H_{A_2} \otimes H_{A_3}$.

The concurrence of state $|\psi\rangle$ is defined by, up to an N dependent factor $\sqrt{N/6(N-1)}$,

$$C(|\psi\rangle) = \sqrt{6 - 2 \text{Tr}(\rho_{A_1}^2 + \rho_{A_2}^2 + \rho_{A_3}^2)}, \quad (2)$$

where the reduced density matrix ρ_{A_1} (resp. ρ_{A_2} , ρ_{A_3}) is obtained by tracing over the subsystems A_2 and A_3 (resp. A_1 and A_3 , A_1 and A_2). $C(|\psi\rangle)$ can be equivalently written as [10]

$$C(|\psi\rangle) = \sqrt{\sum (|a_{ijk}a_{pqm} - a_{ijm}a_{pqk}|^2 + |a_{ijk}a_{pqm} - a_{iqk}a_{pjm}|^2 + |a_{ijk}a_{pqm} - a_{pjk}a_{iqm}|^2)}. \quad (3)$$

The concurrence for a tripartite mixed state ρ is defined by the convex roof,

$$C(\rho) \equiv \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \quad (4)$$

for all possible pure state decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $|\psi_i\rangle \in H_{A_1} \otimes H_{A_2} \otimes H_{A_3}$, $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$.

To evaluate $C(\rho)$, we project high dimensional states to “lower dimensional” sub-states. For a given $N \otimes N \otimes N$ pure state, we define its “ $m \otimes m \otimes m$ ”, $m \leq N$, pure state $|\psi\rangle_{m \otimes m \otimes m} = \sum_{i=i_1}^{i_m} \sum_{j=j_1}^{j_m} \sum_{k=k_1}^{k_m} a_{ijk} |ijk\rangle = B_1 \otimes B_2 \otimes B_3 |\psi\rangle$, where $B_1 = \sum_{i=i_1}^{i_m} |i\rangle\langle i|$, $B_2 = \sum_{j=j_1}^{j_m} |j\rangle\langle j|$, $B_3 = \sum_{k=k_1}^{k_m} |k\rangle\langle k|$. Its concurrence $C(|\psi\rangle_{m \otimes m \otimes m})$ is similarly given by Eq.(3), with the subindices of a , i (resp. j , k)

associated with the system A_1 (resp. A_2, A_3) running from i_1 (resp. j_1, k_1) to i_m (resp. j_m, k_m). In fact, for any $N \otimes N \otimes N$ pure state $|\psi\rangle$, there are $\binom{N}{m}^3$ different $m \otimes m \otimes m$ sub-states with respect to $|\psi\rangle$. Without causing confusion, in the following we simply use $|\psi\rangle_{m \otimes m \otimes m}$ to denote one of such states, as these substates will always be considered together.

Correspondingly for a mixed state ρ , we define its “ $m \otimes$

$m \otimes m$ ” mixed (unnormalized) sub-states $\rho_{m \otimes m \otimes m} = B_1 \otimes B_2 \otimes B_3 \rho B_1^\dagger \otimes B_2^\dagger \otimes B_3^\dagger$. The concurrence of $\rho_{m \otimes m \otimes m}$ is defined by $C(\rho_{m \otimes m \otimes m}) \equiv \min \sum_i p_i C(|\phi_i\rangle)$, minimized over all possible $m \otimes m \otimes m$ pure state decompositions of $\rho_{m \otimes m \otimes m} = \sum_i p_i |\phi_i\rangle\langle\phi_i|$, with $\sum_i p_i = \text{tr}(\rho_{m \otimes m \otimes m})$. The $m \otimes m \otimes m$ submatrices $\rho_{m \otimes m \otimes m}$ have the following form,

$$\rho_{m \otimes m \otimes m} = \begin{pmatrix} \rho_{i_1 j_1 k_1, i_1 j_1 k_1} & \cdots & \rho_{i_1 j_1 k_1, i_1 j_1 k_m} & \rho_{i_1 j_1 k_1, i_1 j_2 k_1} & \cdots & \rho_{i_1 j_1 k_1, i_m j_m k_m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{i_1 j_1 k_m, i_1 j_1 k_1} & \cdots & \rho_{i_1 j_1 k_m, i_1 j_1 k_m} & \rho_{i_1 j_1 k_m, i_1 j_2 k_1} & \cdots & \rho_{i_1 j_1 k_m, i_m j_m k_m} \\ \rho_{i_1 j_2 k_1, i_1 j_1 k_1} & \cdots & \rho_{i_1 j_2 k_1, i_1 j_1 k_m} & \rho_{i_1 j_2 k_1, i_1 j_2 k_1} & \cdots & \rho_{i_1 j_2 k_1, i_m j_m k_m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{i_m j_m k_1, i_1 j_1 k_1} & \cdots & \rho_{i_m j_m k_1, i_1 j_1 k_m} & \rho_{i_m j_m k_1, i_1 j_2 k_1} & \cdots & \rho_{i_m j_m k_1, i_m j_m k_m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{i_m j_m k_m, i_1 j_1 k_1} & \cdots & \rho_{i_m j_m k_m, i_1 j_1 k_m} & \rho_{i_m j_m k_m, i_1 j_2 k_1} & \cdots & \rho_{i_m j_m k_m, i_m j_m k_m} \end{pmatrix}, \quad (5)$$

where $i_1 < \dots < i_m, j_1 < \dots < j_m$ and $k_1 < \dots < k_m$ with subindices i_1, \dots, i_m associated with the space H_{A_1} , j_1, \dots, j_m with the space H_{A_2} and k_1, \dots, k_m with the space H_{A_3} .

Theorem 1 For any $N \otimes N \otimes N$ ($N \geq 2$) tripartite mixed quantum state ρ , the concurrence $C(\rho)$ satisfies

$$C^2(\rho) \geq c_{m \otimes m \otimes m} \sum_{P_{m \otimes m \otimes m}} C^2(\rho_{m \otimes m \otimes m}), \quad (6)$$

where $m \geq 2$, $c_{m \otimes m \otimes m} = \left[\binom{N-1}{m-1} \right]^{-1} \left[\binom{N-2}{m-2} \right]^{-2}$, and $\sum_{P_{m \otimes m \otimes m}}$ stands for summing over all possible $m \otimes m \otimes m$ mixed sub-states.

[Proof]. For any $N \otimes N \otimes N$ tripartite pure state $|\psi\rangle = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N a_{ijk} |ijk\rangle$, and any given term

$$|a_{i_0 j_0 k_0} a_{p_0 q_0 m_0} - a_{i_0 j_0 m_0} a_{p_0 q_0 k_0}|^2, \quad k_0 \neq m_0, \quad (7)$$

in Eq. (3), if $i_0 \neq p_0$ and $j_0 \neq q_0$, then there are $\left[\binom{N-2}{m-2} \right]^3$ different $m \otimes m \otimes m$ sub-states $|\psi\rangle_{m \otimes m \otimes m} = B_1 \otimes B_2 \otimes B_3 |\psi\rangle$, with $B_1 = |i_0\rangle\langle i_0| + |p_0\rangle\langle p_0| + \sum_{i=i_3}^{i_m} |i\rangle\langle i|$, $B_2 = |j_0\rangle\langle j_0| + |q_0\rangle\langle q_0| + \sum_{j=j_3}^{j_m} |j\rangle\langle j|$, $B_3 = |k_0\rangle\langle k_0| + |m_0\rangle\langle m_0| + \sum_{k=k_3}^{k_m} |k\rangle\langle k|$, where $\{i\}_{i=i_3}^{i_m} \subseteq \{i\}_{i=1}^N$, $\{j\}_{j=j_3}^{j_m} \subseteq \{j\}_{j=1}^N$, $\{k\}_{k=k_3}^{k_m} \subseteq \{k\}_{k=1}^N$, such that the term (7) appears in the concurrence of $|\psi\rangle_{m \otimes m \otimes m} = B_1 \otimes B_2 \otimes B_3 |\psi\rangle$. If $i_0 = p_0$ and $j_0 \neq q_0$, then there are $\binom{N-1}{m-1} \left[\binom{N-2}{m-2} \right]^2$ different $m \otimes m \otimes m$ sub-states $|\psi\rangle_{m \otimes m \otimes m} = D_1 \otimes B_2 \otimes B_3 |\psi\rangle$, with $D_1 = |i_0\rangle\langle i_0| + \sum_{i=i_2}^{i_m} |i\rangle\langle i|$, $\{i\}_{i=i_2}^{i_m} \subseteq \{i\}_{i=1}^N$, such that the

term (7) appears in the concurrence of $|\psi\rangle_{m \otimes m \otimes m} = D_1 \otimes B_2 \otimes B_3 |\psi\rangle$. Otherwise, if $i_0 \neq p_0$ and $j_0 = q_0$, then there are $\binom{N-1}{m-1} \left[\binom{N-2}{m-2} \right]^2$ different $m \otimes m \otimes m$ sub-states $|\psi\rangle_{m \otimes m \otimes m} = B_1 \otimes D_2 \otimes B_3 |\psi\rangle$ with $D_2 = |j_0\rangle\langle j_0| + \sum_{j=j_2}^{j_m} |j\rangle\langle j|$, $\{j\}_{j=j_2}^{j_m} \subseteq \{j\}_{j=1}^N$, such that the term (7) appears in the concurrence of $|\psi\rangle_{m \otimes m \otimes m} = B_1 \otimes D_2 \otimes B_3 |\psi\rangle$. Since $\binom{N-2}{m-2} \geq \binom{N-1}{m-1}$ generally, the concurrences of the pure state $|\psi\rangle$ and the sum of the concurrence of all the sub-states $|\psi\rangle_{m \otimes m \otimes m}$ with respect to $|\psi\rangle$ have the following relation,

$$C^2(|\psi\rangle) \geq c_{m \otimes m \otimes m} \sum_{P_{m \otimes m \otimes m}} C^2(|\psi\rangle_{m \otimes m \otimes m}). \quad (8)$$

Therefore for mixed state $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$, we have

$$\begin{aligned} C(\rho) &= \min \sum_i p_i C(|\psi_i\rangle) \\ &\geq \sqrt{c_{m \otimes m \otimes m}} \min \sum_i p_i \left(\sum_{P_{m \otimes m \otimes m}} C^2(|\psi_i\rangle_{m \otimes m \otimes m}) \right)^{\frac{1}{2}} \\ &\geq \sqrt{c_{m \otimes m \otimes m}} \min \left[\sum_{P_{m \otimes m \otimes m}} \left(\sum_i p_i C(|\psi_i\rangle_{m \otimes m \otimes m}) \right)^2 \right]^{\frac{1}{2}} \\ &\geq \sqrt{c_{m \otimes m \otimes m}} \left[\sum_{P_{m \otimes m \otimes m}} \left(\min \sum_i p_i C(|\psi_i\rangle_{m \otimes m \otimes m}) \right)^2 \right]^{\frac{1}{2}} \\ &= \sqrt{c_{m \otimes m \otimes m}} \left[\sum_{P_{m \otimes m \otimes m}} C^2(\rho_{m \otimes m \otimes m}) \right]^{\frac{1}{2}}, \end{aligned}$$

where the relation $(\sum_j (\sum_i x_{ij})^2)^{\frac{1}{2}} \leq \sum_i (\sum_j x_{ij}^2)^{\frac{1}{2}}$ has been used in the second inequality, the first three minimizations run over all possible pure state decompositions of the mixed state ρ , while the last minimization runs over all $m \otimes m \otimes m$ pure state decompositions of $\rho_{m \otimes m \otimes m} = \sum_i p_i |\phi_i\rangle\langle\phi_i|$ associated with ρ . \square

(6) gives a lower bound of $C(\rho)$. One can estimate $C(\rho)$ by calculating the concurrence of the sub-states $\rho_{m \otimes m \otimes m}$, $2 \leq m \leq N-1$. Different choices of m may give rise to different lower bounds. A convex combination of these lower bounds is still a lower bound. Hence generally we have

Corollary 1 For any $N \otimes N \otimes N$ tripartite mixed quantum state ρ , the concurrence $C(\rho)$ satisfies

$$C^2(\rho) \geq \sum_{m=2}^N p_m c_{m \otimes m \otimes m} \sum_{P_{m \otimes m \otimes m}} C^2(\rho_{m \otimes m \otimes m}), \quad (9)$$

where $0 \leq p_m \leq 1$ and $\sum_{m=2}^N p_m = 1$.

The lower bound (6) is in general not operationally computable, as we still have no analytical results for concurrence of lower dimensional states. Nevertheless, we have already some analytical lower bounds for three-qubit mixed quantum states [17]. If we replace the computation of concurrence of lower dimensional sub-states " $\rho_{m \otimes m \otimes m}$ " by that of the lower bounds of these sub-states, (6) gives an operational lower bound based on known lower bounds. The lower bound obtained in this way should be the same or better than the previous known lower bounds. Hence (6) may be used to improve the existing lower bounds in this sense. We first present an operational analytical lower bound for three-qubit mixed quantum states.

Theorem 2 The concurrence $C(\rho)$ of three-qubit mixed quantum state ρ satisfies

$$C^2(\rho) \geq \sum_{j=1}^3 \max[(\|\rho^{\mathcal{T}_j}\| - 1)^2, (\|R_{j,\bar{j}}(\rho)\| - 1)^2], \quad (10)$$

where $\rho^{\mathcal{T}_j}$ stands for the partial transposition of ρ with respect to the j -th subsystem A_j , $R_{j,\bar{j}}(\rho)$ is the realignment of ρ with respect to the bipartite partition between j -th and the rest systems, $\|A\| = \text{Tr}\sqrt{AA^\dagger}$ is the trace norm of a matrix.

[Proof]. For three-qubit state $|\psi\rangle$, one has,

$$\begin{aligned} & 1 - \text{Tr}\rho_{A_j}^2 \\ &= \frac{1}{2} (\|(|\psi\rangle\langle\psi|)^{\mathcal{T}_j}\| - 1)^2 \\ &= \frac{1}{2} (\|R_{j,\bar{j}}(|\psi\rangle\langle\psi|)\| - 1)^2. \end{aligned}$$

According to (2) we obtain

$$\begin{aligned} & C^2(|\psi\rangle) \\ &= \sum_{j=1}^3 (\|(|\psi\rangle\langle\psi|)^{\mathcal{T}_j}\| - 1)^2 \\ &= \sum_{j=1}^3 (\|R_{j,\bar{j}}(|\psi\rangle\langle\psi|)\| - 1)^2. \end{aligned}$$

Assume that $\sum_i p_i |\psi_i\rangle\langle\psi_i|$ is the optimal decomposition of ρ achieving the infimum of $C(\rho)$. Then

$$\begin{aligned} C^2(\rho) &= \left(\sum_i p_i C(|\psi_i\rangle) \right)^2 \\ &= \left(\sum_i p_i \sqrt{\sum_j (\|(|\psi_i\rangle\langle\psi_i|)^{\mathcal{T}_j}\| - 1)^2} \right)^2 \\ &\geq \sum_{j=1}^3 \left(\sum_i p_i (\|(|\psi_i\rangle\langle\psi_i|)^{\mathcal{T}_j}\| - 1) \right)^2 \\ &\geq \sum_{j=1}^3 (\|\rho^{\mathcal{T}_j}\| - 1)^2. \end{aligned} \quad (11)$$

Similarly one can prove that $C^2(\rho) \geq \sum_{j=1}^3 (\|R_{j,\bar{j}}(\rho)\| - 1)^2$. \square

To see the tightness of the inequality (10), we consider the following example.

Example 1. Let us consider the Dür-Cirac-Trarrach state [22],

$$\begin{aligned} \rho_{DCT} &= \sum_{\sigma=\pm} \lambda_0^\sigma |\psi_0^\sigma\rangle\langle\psi_0^\sigma| + \sum_{j=1}^3 \lambda_j (|\psi_j^+\rangle\langle\psi_j^+| \\ &\quad + |\psi_j^-\rangle\langle\psi_j^-|), \end{aligned} \quad (12)$$

where $|\psi_0^\pm\rangle = \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle)$,

$$|\psi_j^\pm\rangle = \frac{1}{\sqrt{2}}(|j\rangle_{AB}|0\rangle_C \pm |(3-j)\rangle_{AB}|1\rangle_C),$$

$|j\rangle_{AB} = |j_1\rangle_A |j_2\rangle_B$ with $j = j_1 j_2$ in binary notation. From our lower bound (10), we obtain $C(\rho_{DCT}) \geq \sqrt{\frac{4}{27}} \approx 0.385$ for $\lambda_0^+ = \frac{1}{6}$, $\lambda_0^- = \frac{1}{2}$ and $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{18}$.

Another lower bound of concurrence for three-qubit mixed quantum states had been given in Ref. [17]. Form Ref. [17] the lower bound of concurrence for ρ_{DCT} is $C(\rho_{DCT}) \geq 0.314$, where the difference of a constant factor $\sqrt{2}$ in defining the concurrence for pure states has been already taken into account. Therefore the lower bound (10) is better than the lower bound in Ref. [17] in detecting entanglement of the three-qubit mixed state ρ_{DCT} .

By using the analytical lower bounds (10) for three-qubit quantum states, from (6) we have

Corollary 2 For any $N \otimes N \otimes N$ tripartite mixed quantum state ρ , the concurrence $C(\rho)$ satisfies

$$C^2(\rho) \geq \frac{1}{N-1} \sum_{P_{2 \otimes 2 \otimes 2}} \max \left[\sum_{j=1}^3 (\|\rho_{2 \otimes 2 \otimes 2}^{\mathcal{T}_j}\| - 1)^2, \sum_{j=1}^3 (\|R_{j,\bar{j}}(\rho_{2 \otimes 2 \otimes 2})\| - 1)^2 \right]. \quad (13)$$

(6) presents a lower bound of concurrence for $N \otimes N \otimes N$ tripartite mixed quantum states. Generally it is not operational, while (10) gives an operational lower bound of concurrence. The combination of these two results gives rise to operational lower bounds for general $N \otimes N \otimes N$ states.

Example 2. We consider the $3 \otimes 3 \otimes 3$ state $\rho = \frac{1-x}{27} I_{27} + x|\psi^+\rangle\langle\psi^+|$, where $0 \leq x \leq 1$ represents the degree of the depolarization, $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |222\rangle)$. As $\rho^{\mathcal{T}_{y_i}} = (\rho^{\mathcal{T}_{y_i}})^\dagger$, the square root of the eigenvalues of $\rho^{\mathcal{T}_{y_i}}(\rho^{\mathcal{T}_{y_i}})^\dagger$ is the absolute value of the eigenvalues of $\rho^{\mathcal{T}_{y_i}}$. According to the above Corollary, our result can detect the entanglement of ρ when $\frac{2}{29} \leq x \leq 1$, see Fig.1.

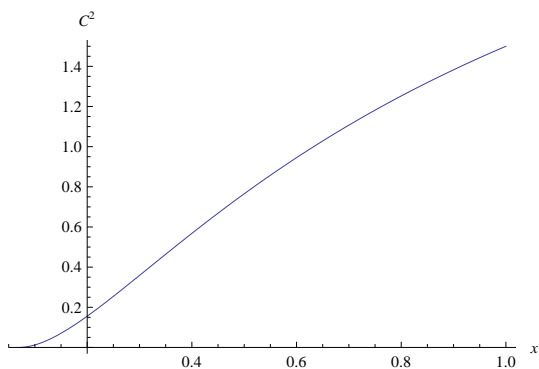


Fig. 1. The lower bound concurrence of ρ for $\frac{2}{29} \leq x \leq 1$.

Now we generalize our results to arbitrary dimensional n -partite systems. Let $H_{A_1}, H_{A_2}, \dots, H_{A_n}$ be N dimensional vector spaces respectively. A pure state $|\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_n}$ has the form,

$$|\psi\rangle = \sum_{i_1=1}^N \sum_{i_2=1}^N \dots \sum_{i_n=1}^N a_{i_1 i_2 \dots i_n} |i_1 i_2 \dots i_n\rangle, \quad (14)$$

where $a_{i_1 i_2 \dots i_n} \in \mathbb{C}$, $\sum_{i_1 i_2 \dots i_n} |a_{i_1 i_2 \dots i_n}|^2 = 1$, $\{|i_1 i_2 \dots i_n\rangle\}$ is the basis of $H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_n}$. The concurrence of $|\psi\rangle$ has the form [10],

$$C(|\psi\rangle) = \sqrt{\sum_p \sum_{\alpha, \dot{\alpha}, \beta, \dot{\beta}} |a_{\alpha\beta} a_{\dot{\alpha}\dot{\beta}} - a_{\alpha\dot{\beta}} a_{\dot{\alpha}\beta}|}, \quad (15)$$

where \sum_p stands for the summation over all possible combinations of the indices of α, β . α (or $\dot{\alpha}$) and β (or $\dot{\beta}$)

$\dot{\beta}$) span the whole space of a given subindex of a . The concurrence is extended to mixed state ρ by the convex roof $C(\rho) = \min \sum_i p_i C(|\psi_i\rangle)$ for all possible ensemble realizations $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$.

For a given $N \otimes N \otimes \dots \otimes N$ pure state, we define its “ $m \otimes m \otimes \dots \otimes m$ ” pure states

$$\begin{aligned} |\psi\rangle_{m \otimes m \otimes \dots \otimes m} &= \sum_{i_1=j_1}^{j_m} \sum_{i_2=k_1}^{k_m} \dots \sum_{i_n=l_1}^{l_m} a_{i_1 i_2 \dots i_n} |i_1 i_2 \dots i_n\rangle \\ &= B_1 \otimes B_2 \otimes \dots \otimes B_n |\psi\rangle, \end{aligned}$$

where $B_1 = \sum_{i_1=j_1}^{j_m} |i_1\rangle\langle i_1|$, $B_2 = \sum_{i_2=k_1}^{k_m} |i_2\rangle\langle i_2|$, \dots , $B_n = \sum_{i_n=l_1}^{l_m} |i_n\rangle\langle i_n|$, $\{j_1, \dots, j_m\} \subseteq \{1, \dots, N\}$, $\{k_1, \dots, k_m\} \subseteq \{1, \dots, N\}$, \dots , and $\{l_1, \dots, l_m\} \subseteq \{1, \dots, N\}$. For a mixed state ρ , correspondingly we define its “ $m \otimes m \otimes \dots \otimes m$ ” sub-states

$$\rho_{m \otimes m \otimes \dots \otimes m} = B_1 \otimes B_2 \otimes \dots \otimes B_n \rho B_1^\dagger \otimes B_2^\dagger \otimes \dots \otimes B_n^\dagger.$$

The concurrence of $\rho_{m \otimes m \otimes \dots \otimes m}$ is defined by $C(\rho_{m \otimes m \otimes \dots \otimes m}) \equiv \min \sum_i p_i C(|\phi_i\rangle)$, minimized over all possible $m \otimes m \otimes \dots \otimes m$ pure state decompositions of $\rho_{m \otimes m \otimes \dots \otimes m} = \sum_i p_i |\phi_i\rangle\langle\phi_i|$, with $\sum_i p_i = \text{tr}(\rho_{m \otimes m \otimes \dots \otimes m})$. Similar to the tripartite case, we can prove the following theorem:

Theorem 3 For any n -partite N dimensional mixed state $\rho \in H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_n}$,

$$C^2(\rho) \geq c_{m \otimes m \otimes \dots \otimes m} \sum_{P_{m \otimes m \otimes \dots \otimes m}} C^2(\rho_{m \otimes m \otimes \dots \otimes m}), \quad (16)$$

where $c_{m \otimes m \otimes \dots \otimes m}$ is a fixed number depending on m , $\sum_{P_{m \otimes m \otimes \dots \otimes m}}$ stands for summing over all possible $m \otimes m \otimes \dots \otimes m$ mixed states.

III. CONCLUSIONS

In summary, we have proposed a method in constructing hierarchy lower bounds of concurrence for tripartite mixed states, in terms of the concurrences of all the lower dimensional mixed sub-states. The lower bounds may be used to improve all the existing lower bounds of concurrence. The approach can be readily generalized to arbitrary dimensional multipartite systems.

Acknowledgments This work is supported by PHR201007107.

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