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The convex support of the k-star model

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This paper describes the polytope $\mathbf{P}_{k;N}$ of i-star counts, for all $i \leq k$, for graphs on N vertices. The vertices correspond to graphs that are regular or as regular as possible. For even N the polytope is a cyclic polytope, and for odd N the polytope is well-approximated by a cyclic polytope. As N goes to infinity, $\mathbf{P}_{k;N}$ approaches the convex hull of the moment curve. The affine symmetry group of $\mathbf{P}_{k;N}$ contains just a single non-trivial element, which corresponds to forming the complement of a graph.

The results generalize to the polytope $\mathbf{P}_{I;N}$ of i-star counts, for i in some set I of non-consecutive integers. In this case, $\mathbf{P}_{I;N}$ can still be approximated by a cyclic polytope, but it is usually not a cyclic polytope itself.

1. Introduction

In this work all graphs are assumed to be undirected and simple, i.e. there are no multiple edges between the same nodes, and there are no loops. Let G be a graph with vertex set V of cardinality N. For each $x \in V$ denote by d_x the degree of x, i.e. the number of edges of G containing x. A *k-star* is a graph with one node of degree k and k nodes of degree one. The *number of k-stars* in G is defined as

$$N_k(G) := \sum_x \binom{d_x}{k} = \frac{1}{k!} \sum_x [d_x]_k,$$

where $[d_x]_k = d_x(d_x - 1) \cdots (d_x - k + 1)$. It is the number of subgraphs of G that are isomorphic to a k-star. Alternatively, it is the number of graph homomorphisms from a k-star to G. The *k-star density* is

$$n_k(G) = \frac{1}{N\binom{N-1}{k}} N_k(G) = \frac{1}{N[N-1]_k} \sum_x [d_x]_k.$$

$n_k(G)$ is normalized such that $0 \leq n_k(G) \leq 1$, with equality for the empty graph and the full graph, respectively. When the graph G is understood, n_k and N_k will be written instead of $n_k(G)$ and $N_k(G)$.

The object of study of this work is the convex hull of the set of possible values of the vector $\vec{n}_k := (n_1, \ldots, n_k)$ of i-star statistics:

$$\mathbf{P}_{k;N} := \operatorname{conv}\{\vec{n}_k(G) : G \text{ a graph on } N \text{ vertices}\}.$$

See Figures 1, 2 and 3 for examples with $k = 2$ and $k = 3$. As shown in [5], $\mathbf{P}_{k;N+1} \subset \mathbf{P}_{k;N}$. Hence there is a well-defined limit object $\mathbf{P}_{k;\infty} := \bigcap_N \mathbf{P}_{k;N}$.

The polytope $\mathbf{P}_{k;N}$ has the following interpretation: If $\hat{G}$ is a random graph on N vertices, then the expectation value of $\vec{n}_k(\hat{G})$ lies in $\mathbf{P}_{k;N}$. Conversely, any element of $\mathbf{P}_{k;N}$ arises in this way. Such polytopes of expectation values appear as convex supports of exponential families. See [1] for an introduction to exponential families, and see [11, 2] for overviews on exponential random graph models. The polytope $\mathbf{P}_{k;N}$ is the convex support of the exponential random graph model known as the k-star model. The main results in this paper are independent of this interpretation, and hence this connection will not explained here. Some remarks in this direction will be made in Section 6.

The numbers $n_1, \ldots, n_{N-1}$ are related to the degree distribution. Denote by $\tilde{N}_d$ the number of nodes of degree d in G, and let $p_d = \frac{1}{N}\tilde{N}_d$. The numbers $(p_0, \ldots, p_{N-1})$ form a probability distribution on $\{0, \ldots, N-1\}$, called *degree distribution.* Then

$$n_k = \frac{1}{[N-1]_k} \sum_{d=0}^{N-1} p_d [d]_k. \tag{1}$$

Therefore, $\mathbf{P}_{k;N}$ is a projection of the polytope of degree distributions $\mathbf{D}_N$, defined as the convex hull of all probability distributions on $\{0, \ldots, N-1\}$ that arise as degree distributions of graphs with N nodes. For even N and arbitrary $d \in \{0, \ldots, N-1\}$ there exists a d-regular graph with N vertices. Remember that a graph is *d-regular*, if $d_x = d$ for all $x \in V$. Hence $\mathbf{D}_N$ is a simplex if N is even. For odd N the polytope $\mathbf{D}_N$ is described in Section 3.

Expression (1) can be further transformed using the expansion

$$[d]_k = \sum_{i=0}^{k} s_{k,i} d^i.$$

The coefficients $s_{k,i}$ are called *Stirling numbers of the first kind.* Denote by $\mu_i = \sum_x \frac{d_x^i}{N} = \sum_{d=0}^{N-1} p_d d^i$ the i-th moment of the degree distribution. As observed in [10], the Stirling numbers relate the k-star densities with the moments of the degree distribution:

$$n_k = \frac{1}{[N-1]_k} \sum_{i=0}^{k} s_{k,i} \sum_x \frac{d_x^i}{N} = \frac{1}{[N-1]_k} \sum_{i=0}^{k} s_{k,i} \mu_i. \tag{2}$$

Hence the polytope $\mathbf{P}_{k;N}$ is affinely equivalent to the polytope of the first k moments of the degree distribution.

This paper is organized as follows: The limit polytope is described in Section 2. Moreover, a cyclic polytope $\tilde{\mathbf{P}}_{k;N}$ is found that can serve as an approximation of $\mathbf{P}_{k;N}$, with error $O(1/N)$. For even N, the two polytopes $\mathbf{P}_{k;N}$ and $\tilde{\mathbf{P}}_{k;N}$ agree. For odd N, the vertices of $\mathbf{P}_{k;N}$ are described in Section 3, and a coarse classification of the facets is found. Section 4 discusses the related polytope $\mathbf{P}_{I;N}$ obtained when dropping some coordinates of the vector $\vec{n}_k$. The resulting polytope is usually not a cyclic polytope, but can be approximated by cyclic polytopes with error $O(1/N)$. The affine symmetry group of $\mathbf{P}_{k;N}$ is computed in Section 5. For $N > k+1$ the only non-trivial symmetry is the involution induced from the permutation that replaces each graph by its complement graph. Section 6 discusses some implications for the corresponding exponential families.

2. The limit polytope

By (1), the vector $\vec{n} := (n_1, \cdots, n_{N-1})$ equals the expectation value of the vector

$$\left(\frac{d}{N-1}, \frac{[d]_2}{[N-1]_2}, \dots, \frac{[d]_{N-1}}{(N-1)!}\right)$$

under the degree distribution. Denote by D the $N \times k$-matrix with entries $D_{i,d} = \frac{[d]_i}{[N-1]_i}$ for $i = 1, \dots, k$ and $d = 0, \dots, N-1$, and let $\tilde{\mathbf{P}}_{k;N}$ be the convex hull of the columns of D. When multiplied by the degree distribution $p(G)$ of a graph G (considered as a column vector), the matrix D computes the vector $\vec{n}(G)$. Therefore, $\vec{n}(G) \in \tilde{\mathbf{P}}_{k;N}$ for any graph G on N vertices, and so $\mathbf{P}_{k;N} \subseteq \tilde{\mathbf{P}}_{k;N}$. If N is even, then $\tilde{\mathbf{P}}_{k;N} = \mathbf{P}_{k;N}$. This follows from the fact that if N is even and $0 \leq d \leq N-1$, then there exists a d-regular graph on N vertices. On the other hand, if N is odd and if d is odd, then there is no d-regular graph on N vertices. Nevertheless, $\tilde{\mathbf{P}}_{k;N}$ is a valuable relaxation of the polytope $\mathbf{P}_{k;N}$, and many properties of $\mathbf{P}_{k;N}$ can be deduced from $\tilde{\mathbf{P}}_{k;N}$. The difference between $\mathbf{P}_{k;N}$ and $\tilde{\mathbf{P}}_{k;N}$ is characterized in Section 3; see also Figures 1 and 3.

To describe $\mathbf{P}_{k;\infty}$ the following definition is needed: The *moment curve* is the curve

$$s^k : [0,1] \to \mathbb{R}^k, p \mapsto (p^1, p^2, \dots, p^k).$$

The moment curve corresponds to the *k-spine* in [5]. It contains the expectation values of $\vec{n}_k$ for the *Erdős-Rényi random graphs* $G(N,p)$: Under $G(N,p)$, for each $x, y \in \{1, \dots, N\}$, $x < y$, an independent Bernoulli variable $\theta_{x,y}$ with parameter p is drawn, and (x,y) is an edge in $G(N,p)$ if and only if $\theta_{x,y} = 1$. Since $s^k(p)$ equals the expectation value of $\vec{n}_k$ under $G(N,p)$, the moment curve is a subset of $\mathbf{P}_{k;N}$ for all N.

The convex hull of the moment curve has a nice interpretation: It consists of all vectors $(\mu_1, \dots, \mu_k) \in \mathbb{R}^k$ such that there exists a probability distribution p on the unit interval $[0,1]$ such that $\mu_1, \dots, \mu_k$ are the first k moments of p. In [9] the convex hull of s^k is studied from this point of view.

Theorem 1. $\mathbf{P}_{k;\infty}$ *equals the convex hull of the moment curve.*

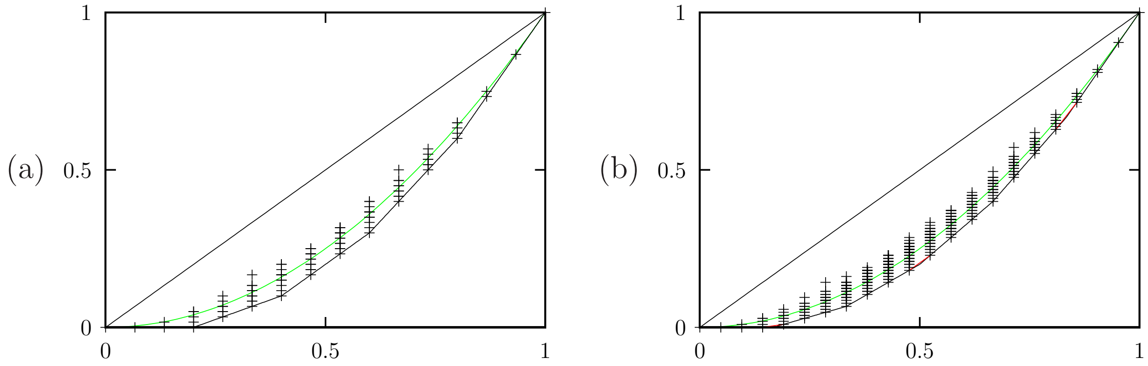

Figure 1: (a) The polytope $\mathbf{P}_{2,6}$. (b) The polytopes $\tilde{\mathbf{P}}_{2,7}$ and $\mathbf{P}_{2,7}$. In both figures, each cross is a possible value of $\vec{n}_2$ for some graph with six or seven nodes, and the moment curve is marked in green. In (b), the polytope $\tilde{\mathbf{P}}_{2,7}$ is bounded by the black edges, while $\mathbf{P}_{2,7}$ is bounded by the black and red edges. The red edges bound the special facets defined in Section 3.

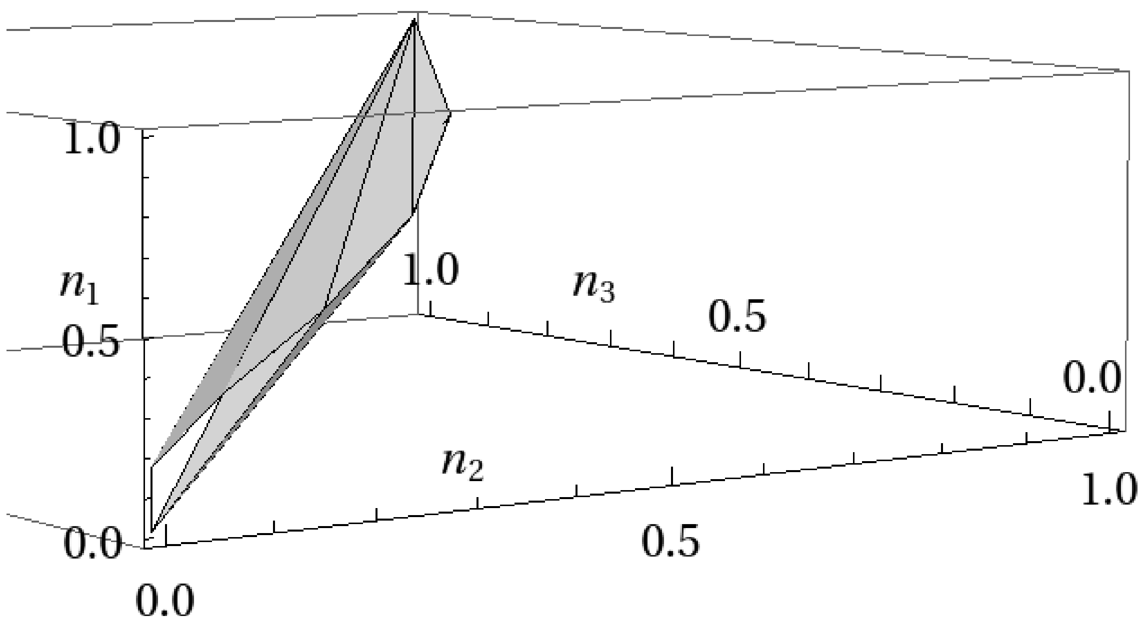

Figure 2: The polytope $\tilde{\mathbf{P}}_{3,7}$, plotted by Mathematica [13]. The volume of $\tilde{\mathbf{P}}_{3,7}$, according to Polymake [6], is 7/675, while the diameter is $\sqrt{3}$, hence the polytope is stretched along the diagonal of the unit cube. This makes it difficult to obtain a good perspective of $\tilde{\mathbf{P}}_{3,7}$.

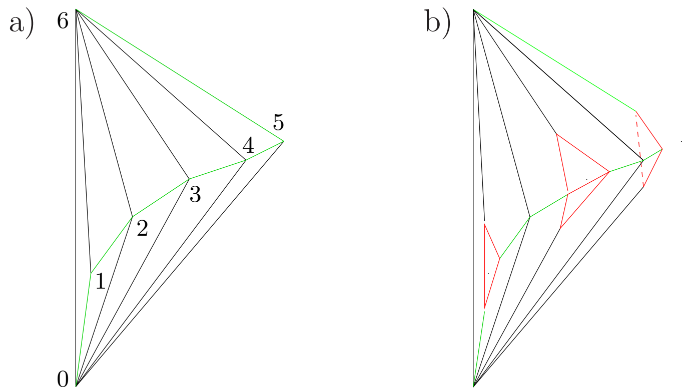

Figure 3: Sketches of the polytopes $\tilde{\mathbf{P}}_{3,7}$ (a) and $\mathbf{P}_{3,7}$ (b), motivated by the pictures in [8]. The number d labels the vertex D_d of $\tilde{\mathbf{P}}_{3,7}$. The green edges in $\tilde{\mathbf{P}}_{3,7}$ are the edges from D_d to D_{d+1}. Hence, the green line corresponds to the moment curve. The red edges in $\mathbf{P}_{3,7}$ mark the difference between $\mathbf{P}_{3,7}$ and $\tilde{\mathbf{P}}_{3,7}$.

Proof. From $\mathbf{P}_{k;2l} = \tilde{\mathbf{P}}_{k;2l} \subset \mathbf{P}_{k;2l-1} \subset \tilde{\mathbf{P}}_{k;2l-1}$ follows $\mathbf{P}_{k;\infty} = \bigcap_N \tilde{\mathbf{P}}_{k;N}$. Now,

$$\begin{aligned} D_{j,d} = \frac{[d]_j}{[N-1]_j} &= \sum_{i=0}^{j} s_{j,i} \frac{d^i}{[N-1]_j} \\ &= \left(\frac{d}{N-1}\right)^j + \frac{d^j}{N-1}\left(\frac{1}{[N-2]_{j-1}} - \frac{1}{(N-1)^{j-1}}\right) + \sum_{i=0}^{j-1} s_{j,i}\frac{d^i}{[N-1]_j} \\ &= \left(\frac{d}{N-1}\right)^j + \frac{d^j}{N-1}\left(\frac{g_j(N)}{[N-2]_{j-1}(N-1)^{j-1}}\right) + \sum_{i=0}^{j-1} s_{j,i}\frac{d^i}{[N-1]_j}, \end{aligned}$$

where $g_j(N)$ is a polynomial in N of degree at most $j-2$. Therefore, since $d/(N-1) \leq 1$ and $s_{j,j} = 1$,

$$D_{j,d} - \left(\frac{d}{N-1}\right)^j = O(\frac{1}{N}).$$

Therefore, the columns of D lie close to the point $s^k(\frac{d}{N-1})$. Since the vertices lie in an $O(1/N)$-neighbourhood of the moment curve, it follows that any point in $\mathbf{P}_{k;N}$ lies in an $O(1/N)$-neighbourhood of the convex hull of the moment curve; hence $\mathbf{P}_{k;\infty}$ is contained in the convex hull of the moment curve. Conversely, since $\mathbf{P}_{k;N}$ contains the moment curve for all N, it is clear that $\mathbf{P}_{k;\infty}$ contains the convex hull of the moment curve. $\square$

Remark 2. Theorem 1 disproves the following Conjecture 6.3 from [5]: *For any finite collection of subgraphs there exists an integer m such that for any $\epsilon > 0$ the limit object of the polytopes of subgraph statistics as $N \to \infty$ can be approximated by a polytope by intersecting the limit object with at most m closed half-spaces that cut away at most a volume of ϵ.* It is not possible to approximate the convex hull of the moment curve arbitrarily close in this way with a fixed number of hyperplanes.

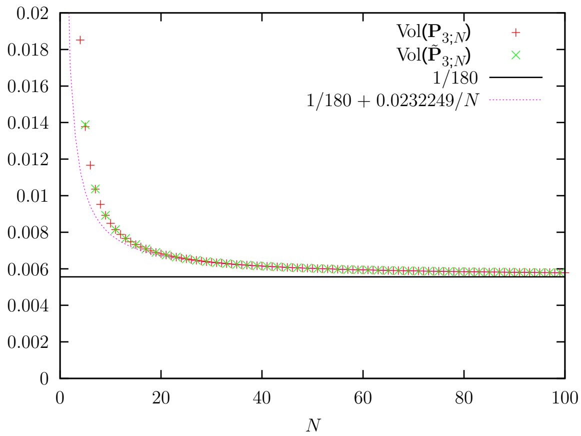

Figure 4: The volumes of $\mathbf{P}_{3;N}$, $\tilde{\mathbf{P}}_{3;N}$ and $\mathbf{P}_{3;\infty}$ (solid line). The dashed line is the function $\mathrm{Vol}(\mathbf{P}_{3;\infty}) + c/N$, with c fit by gnuplot [12].

The moment curve is a part of the *algebraic moment curve* $\gamma^k : t \mapsto (t^1, t^2, \ldots, t^k)$. The convex hull of N distinct points on the algebraic moment curve is a *cyclic polytope*. In a sense, $\mathbf{P}_{k;\infty}$ equals a cyclic polytope with an infinite number of vertices.

The face lattice of a cyclic polytope is independent of the choice of the N points. A polytope is $C(k; N)$ if it is combinatorially equivalent to the convex hull of N distinct points on the algebraic moment curve, i.e. if it has the same face lattice as the convex hull of N distinct points on the algebraic moment curve.

Cyclic polytopes appear not only in the limit $N \to \infty$:

Theorem 3. *$\tilde{\mathbf{P}}_{k;N}$ is a $C(k, N)$ cyclic polytope. Therefore, if N is even, then $\mathbf{P}_{k;N}$ is a $C(k, N)$ cyclic polytope.*

Proof. The affine map $\varphi : \mathbb{R}^k \to \mathbb{R}^k$, defined via

$$\varphi(m_1, \ldots, m_k)_j = \frac{1}{[N-1]_j} \left(\sum_{i=1}^{j} s_{j,i} m_i + s_{j,0} \right)$$

maps the point $(d, d^2, \ldots, d^k)$ on the algebraic moment curve to a vertex of $\tilde{\mathbf{P}}_{k;N}$. $\square$

The facet description of the cyclic polytopes is known, and the facet description of $\tilde{\mathbf{P}}_{k;N}$ can be deduced from this:

Gale's evenness condition. *Let $t_0 < t_1 < \cdots < t_{N-1}$, and let $\mathbf{C}(t_0, \ldots, t_{N-1})$ be the convex hull of the points $\{\gamma^k(t_i) : i = 0, \ldots, N-1\}$ on the algebraic moment curve. Let $0 \le d_1 < d_2 < \cdots < d_k \le N-1$ be integers. Then the vertices $\gamma^k(t_{d_1}), \ldots, \gamma^k(t_{d_k})$ define a facet of $\mathbf{C}(t_0, \ldots, t_{N-1})$ if and only if for any two integers $d, d' \in \{0, \ldots, N-1\} \setminus \{d_1, \ldots, d_k\}$ the set $\{i : t_d < t_{d_i} < t_{d'}\}$ has even cardinality.*

Figure 4 shows a plot of the volumes of $\mathbf{P}_{3;N}$ and $\tilde{\mathbf{P}}_{3;N}$. It can be seen that the volume of $\tilde{\mathbf{P}}_{3;N} \setminus \tilde{\mathbf{P}}_{3;N}$ is negligible, and the volume of $\mathbf{P}_{3;N} \setminus \mathbf{P}_{3;\infty}$ decreases as c/N, with

$c \approx 0.0232249$. According to [9], the volume of the convex hull of the moment curve is

$$\mathrm{Vol}(\mathbf{P}_{k;\infty}) = \prod_{i=0}^{k-1} \frac{(i!)^2}{(2i+1)!}.$$

In particular, $\mathrm{Vol}(\mathbf{P}_{3;\infty}) = 1/180$. Note that the diameter of $\mathbf{P}_{k;\infty}$ equals the distance of the two points $(0, \ldots, 0)$ and $(1, \ldots, 1)$ corresponding to the empty and the full graph, respectively; and so $\mathrm{diam}(\mathbf{P}_{k;\infty}) = \sqrt{k}$.

3. The vertices and facets of $\mathbf{P}_{k;N}$

In this section the polytope $\mathbf{P}_{k;N}$ for odd N is described. First a description of the polytope $\mathbf{D}_N$ of degree distributions of graphs with N vertices is needed.

A sequence $(d_i)_{i=1}^N$ of integers is called *graphical* if there exists a graph with vertex set $\{1, \ldots, N\}$ and degree sequence $(d_i)_{i=1}^N$. Similarly, a distribution p on $\{0, \ldots, N-1\}$ is called *graphical*, if there exists a graph with N vertices and degree distribution p. The following theorem, due to Erdős and Gallai, characterizes graphical sequences.

Erdős-Gallai theorem. *Assume that $d_1 \geq d_2 \geq \cdots \geq d_N$. The sequence is graphical if and only if $\sum_{i=1}^N d_x$ is even and, for all $k = 1, \ldots, N$,*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\}. \tag{3}$$

In particular, the constant degree sequence $d_i = d$ is graphical, provided that N is even or d is even. Moreover, one can show that, if N and d are odd but d' is even, then the degree sequence with $d_1 = \cdots = d_{N-1} = d$ and $d_N = d'$ is graphical.

Let δ^d be the distribution on $\{0, \ldots, N-1\}$ concentrated on the degree d, and let $\delta^{d,d'}$ be the distribution on $\{0, \ldots, N-1\}$ with $\delta^{d,d'}_d = \frac{N-1}{N}$ and $\delta^{d,d'}_{d'} = \frac{1}{N}$. Assume that N is odd. By what was said above, δ^d is graphical if d is even, and $\delta^{d,d'}$ is graphical if d is odd and d' is even. Any graphical degree distribution is a convex combination of degree distributions of this form. Denote by $\mathbf{D}_N$ the convex hull of the set of graphical degree distributions of graphs on N nodes. It is easy to see that δ^d for even d and $\delta^{d,d'}$ for odd d and even d' are not convex combinations of other graphical distributions; and hence they are the vertices of $\mathbf{D}_N$.

The polytope $\mathbf{P}_{k;N}$ agrees with the convex hull of the image of the set of all degree distributions under the matrix D, and the vertices of $\mathbf{P}_{k;N}$ are among the images of the vertices of the set of graphical degree distributions. It remains to decide, which vertices of $\mathbf{D}_N$ yield vertices of $\mathbf{P}_{k;N}$. Note that $D_d = D\delta^d$. Let $n^{d,d'} = D\delta^{d,d'}$.

Theorem 4. *Let N be odd. The set of vertices of $\mathbf{P}_{k;N}$ consists of the points D_d for even d and the points $n^{d,d'}$ for odd d and all d' satisfying the following conditions:*

- *If $k = 2$, then $d' \in \{d-1, d+1\}$.*

- *If* $k = 3$*, then* $d' \in \{0, d-1, d+1, N-1\}$.
- *If* $k = 4$*, then* d' *is any even number satisfying* $0 \leq d' \leq N-1$.

For an illustration see Figure 3.

Proof. If d is even, then D_d is a vertex of $\mathbf{P}_{k;N}$, because it is a vertex of $\tilde{\mathbf{P}}_{k;N}$. To see when $n^{d,d'}$ is a vertex, one needs to know the edges of $\tilde{\mathbf{P}}_{k;N}$ containing the vertex D_d for odd d. Gale's evenness condition implies:

- If $k = 2$, then all edges are of the form $(d, d+1)$ or $(0, N-1)$ (here, the integer d is identified with the vertex D_d).
- If $k = 3$, then the facets are of the form $(0, d, d+1)$ or $(d, d+1, N-1)$. Hence, the edges are of the form $(d, d+1)$, $(0, d)$ and $(d, N-1)$.
- If $k \geq 4$, then any pair (d, d') is an edge, since any such pair is contained in some facet.

Hence it suffices to show that, if d is odd and d' is even, then $n^{d,d'}$ is a vertex of $\mathbf{P}_{k;N}$ if and only if (d, d') is an edge of $\tilde{\mathbf{P}}_{k;N}$.

If (d, d') is an edge, then $n^{d,d'}$ lies on an edge of $\tilde{\mathbf{P}}_{k;N}$. Therefore, there is one unique way of expressing $n^{d,d'}$ as a convex combination of the vertices of $\tilde{\mathbf{P}}_{k;N}$, namely $n^{d,d'} = \frac{N-1}{N} D_d + \frac{1}{N} D_{d'}$. Since only $D_{d'}$ belongs to $\mathbf{P}_{k;N}$, but not D_d, it follows that $n^{d,d'}$ is an extreme point, or vertex, of $\mathbf{P}_{k;N}$. In particular, if $k \geq 4$, all points $n^{d,d'}$ are vertices.

To show that $n^{d,d'}$ is not a vertex of $\mathbf{P}_{k;N}$ if (d, d') is not an edge of $\tilde{\mathbf{P}}_{k;N}$, it suffices to write $n^{d,d'}$ as a convex combination of other points of $\mathbf{P}_{k;N}$. The calculations are deferred to the appendix, see Lemmas 17 and 18. $\square$

Assume that N is odd. The two-dimensional polytope $\mathbf{P}_{2;N}$ is a polygon with $\frac{3N-1}{2}$ facets: $\mathbf{P}_{2;N}$ is obtained from $\tilde{\mathbf{P}}_{2;N}$ by replacing every vertex D_d of $\tilde{\mathbf{P}}_{2;N}$ with odd d by the two vertices $n^{d,d-1}$ and $n^{d,d+1}$, and hence $\mathbf{P}_{2;N}$ has $N + \frac{N-1}{2} = \frac{3N-1}{2}$ vertices and the same number of facets. For $k \geq 3$, the facet description is more complicated. On the other hand, the facets of $\mathbf{D}_N$ have an easy description:

Theorem 5. *Let* N *be odd. The polytope* $\mathbf{D}_N$ *has* $N+1$ *facets:*

- *The inequality* $\sum_{i \text{ odd}} p_i \leq 1 - \frac{1}{N}$ *defines a facet* H *of* $\mathbf{D}_N$ *containing the vertices* $\delta^{d,d'}$ *with odd* d *and even* d'*. The facet* H *is a Cartesian product of two simplices of dimension* $\frac{N-3}{2}$ *and* $\frac{N-1}{2}$*.*
- *All other facets are of the form* $F \cap \mathbf{D}_N$*, where* F *is a facet of the simplex of probability distributions on* $\{0, \ldots, N-1\}$*. They correspond to the inequalities* $p_i \geq 0$ *for* $i = 0, \ldots, N-1$*.*

Proof. $\mathbf{D}_N$ arises from the simplex Δ_{N-1} of all probability distributions on $\{0,\dots,N-1\}$ by replacing the points δ^d with odd d by the vertices $\delta^{d,d'}$ with even d'. All vertices of $\mathbf{D}_N$ satisfy $\sum_{i \text{ odd}} p_i \leq 1-\frac{1}{N}$. Conversely, all probability distributions on $\{0,\dots,N-1\}$ satisfying this inequality can be written as a convex combination of the vertices of $\mathbf{D}_N$. Therefore, the inequalities in the statement of the theorem are the defining inequalities of $\mathbf{D}_N$.

The probability simplex over $\{0,\dots,N-1\}$ can be identified with the join of the simplex $\Delta_{N-1,\mathrm{even}}$ over $\{0,2,\dots,N-1\}$ and the simplex $\Delta_{N-1,\mathrm{odd}}$ over $\{1,3,\dots,N-2\}$. The facet H corresponds to a slice of this join, and so it is affinely equivalent to the Cartesian product $\Delta_{N-1,\mathrm{even}} \times \Delta_{N-1,\mathrm{odd}}$. □

Lemma 6. *Let $k \geq 3$ and let N be odd. For any facet F of $\tilde{\mathbf{P}}_{k;N}$ the intersection $\hat{F} := F \cap \mathbf{P}_{k;N}$ is a facet of $\mathbf{P}_{k;N}$, and $\hat{F}$ is neither a simplex nor a non-trivial Minkowski sum. Any other facet of $\mathbf{P}_{k;N}$ has vertices of the form $n^{d_1,d'_1},\dots,n^{d_l,d'_l}$ and is either a simplex or a (non-trivial) Minkowski sum.*

Proof. If F is a facet of $\tilde{\mathbf{P}}_{k;N}$, then $\hat{F} = F \cap \mathbf{P}_{k;N}$ is a face of $\mathbf{P}_{k;N}$. By Gale's evenness condition, F contains a vertex of the form D_d with even d, and D_d is also a vertex of $\hat{F}$. The intersection of any edge of F containing D_d with $\mathbf{P}_{k;N}$ is an edge of $\mathbf{P}_{k;N}$ (possibly shorter), and therefore $\hat{F}$ has the same dimension as F and is a facet of $\mathbf{P}_{k;N}$. By Gale's evenness condition, F contains a vertex of the form $D_{d'}$ with odd d'. Therefore, $\hat{F}$ has more vertices than the simplex F, and thus it is not a simplex. Moreover, F contains an edge with vertices D_{d_1} and D_{d_2}, where both d_1 and d_2 are even. The line segment (D_{d_1}, D_{d_2}) is an edge of $\mathbf{P}_{k;N}$, and any other edge of $\mathbf{P}_{k;N}$ that is parallel to it has a shorter length. Therefore, $\hat{F}$ is not a Minkowski sum.

Any facet of $\mathbf{P}_{k;N}$ that contains a vertex D_d must be of the form $\hat{F}$. This is because the edges of $\mathbf{P}_{k;N}$ containing D_d are in one-to-one correspondence with the edges of $\tilde{\mathbf{P}}_{k;N}$ containing D_d, and so the vertex figures of D_d in $\mathbf{P}_{k;N}$ and $\tilde{\mathbf{P}}_{k;N}$ agree.

Let F' be a facet of $\mathbf{P}_{k;N}$ with vertices $n^{d_1,d'_1},\dots,n^{d_l,d'_l}$. Since preimages of faces are faces, the distributions $\delta^{d_1,d'_1},\dots,\delta^{d_l,d'_l}$ define a face H' of $\mathbf{D}_N$, and H' must be a face of the facet H from Theorem 5. Hence H' can be identified with the Cartesian product $H'_1 \times H'_2$, where H'_1 is the subsimplex generated by $\delta^{d_1},\dots,\delta^{d_l}$, and H'_2 is the subsimplex generated by $\delta^{d'_1},\dots,\delta^{d'_l}$. If both H'_1 and H'_2 have dimension larger than zero, then H' is a non-trivial Minkowski sum. Hence F' is also a Minkowski sum, and it cannot be a trivial Minkowski sum, since F' has the same number of vertices as H'. If H'_1 consists of a single vertex, then $d_1 = d_2 = \cdots = d_l$. The statement now follows from the fact that the points $n^{d_1,d'_1}, n^{d_1,d'_2},\dots,n^{d_1,d'_l}$ define a cyclic polytope, and F' must be a facet of this cyclic polytope, hence F' is a simplex. If H'_2 consists of a single vertex, a similar argument shows that F' is a simplex. □

The facets not described by Lemma 6 are called *special facets* in the following.

Lemma 7. *If $k = 3$, then any special facet is a simplex with vertices $n^{d,d'_1}, n^{d,d'_2}, n^{d,d'_3}$, with $d'_1, d'_2, d'_3 \in \{0, d-1, d+1, N-1\}$.*

Proof. Let $d_1 < d_2$ be two odd integers and d' be even such that $n^{d_1,d'}$ and $n^{d_2,d'}$ are two vertices. Then either $d' \in \{0, N-1\}$, or $d_2 - 1 = d' = d_1 + 1$. In any case, the line segment L between D_{d_1} and D_{d_2} intersects the convex hull $\mathbf{C}$ of D_0, D_{d_1+1} and D_{N-1}. The line segment L' between $n^{d_1,d'}$ and $n^{d_2,d'}$ is parallel to L, with both end points shifted in the direction of the same vertex of $\mathbf{C}$; hence L' also intersects $\mathbf{C}$. Therefore, L' is not an edge. This shows that all vertices $n^{d,d'}$ of F' have the same d. The statement now follows from the fact that the points $n^{d,0}, n^{d,d-1}, n^{d,d+1}, n^{d,N-1}$ define a cyclic polytope, and F' must be a facet of this cyclic polytope, hence F' is a simplex. □

Remark 8. By (2), the polytope $\mathbf{P}_{k;N}$ is affinely equivalent to the polytope of the first k moments of the degree distribution. For $k \geq N-1$ the map that maps a probability distribution on $\{0, \ldots, N-1\}$ to its first k moments is an injective linear map (it is described by a Vandermonde matrix). This shows that, for $k \geq N-1$, the polytope $\mathbf{P}_{k;N}$ is affinely equivalent to the polytope $\mathbf{D}_N$ of all graphical degree distributions. For $k < N-1$, $\mathbf{P}_{k;N}$ is a projection of $\mathbf{D}_N$, and Theorem 4 shows that this projection preserves the number of vertices if $k > 3$.

4. Subsets of k-stars

Some of the above results generalize to the situation where one is interested not in the complete vector $\vec{n}_k$, but in some subvector $(n_i)_{i\in I}$ for some set $I \subseteq \{1, \ldots, k\}$. Denote the corresponding polytopes by $\mathbf{P}_{I;N}$, and let s^I be the curve $[0,1] \to \mathbb{R}^I$ defined via $s_i^I(t) = t^i$ for all $i \in I$. Then $\mathbf{P}_{I;N}$ and s^I are orthogonal projections of $\mathbf{P}_{k;N}$ and s^k into a coordinate hyperplane. Hence the limit $\mathbf{P}_{I;\infty} = \bigcap_N \mathbf{P}_{I;N}$ equals the convex hull of s^I. The convex hull of a finite number of N points on the curve s^I is also a cyclic polytope. This can be seen as follows:

Let $\mathbb{I} \subseteq \mathbb{R}$ be an interval. A curve γ is *of order* k if for any hyperplane H in $\mathbb{R}^k$ the cardinality of $\gamma(\mathbb{I}) \cap H$ is at most k. It is known that the convex hull of N points on a curve of order k in $\mathbb{R}^k$ is a cyclic polytope $C(k, N)$, see [4].

Lemma 9. *Let $I = \{i_1, \ldots, i_l\}$ be a set of positive integers. The curve $\gamma^I : [0, \infty) \to \mathbb{R}^l$ defined by $\gamma^I(t)_j = t^{i_j}$ is of order l.*

Proof. Any hyperplane $H \subseteq \mathbb{R}^l$ is defined as the vanishing set of some affine function $f(x) = \sum_{j=1}^{l} a_j x_j - b$. The intersection points of γ^I and H correspond to the zeros of the polynomial

$$f \circ \gamma^I(t) = \sum_{j=1}^{l} a_j t^{i_j} - b.$$

According to Décartes' rule of signs, there are at most l such zeros. □

Corollary 10. *The convex hull of a finite number of N points on the curve s^I is a cyclic polytope.*

The corollary shows that, if N is large, then $\mathbf{P}_{I;N}$ is almost a cyclic polytope. To be precise, the proof of Theorem 1 shows the following: There is a $C(|I|,N)$ cyclic polytope $\mathbf{C}$ such that for any vertex v of $\mathbf{P}_{I;N}$ there is a vertex v' of $\mathbf{C}$ with $\|v-v'\| = O(1/N)$. However, in general $\mathbf{P}_{I;N}$ is not a $C(|I|,N)$ cyclic polytope, even if N is even, as the following lemma shows:

Lemma 11. *1. If $|I| > 2$ and $2 \notin I$ and $N > 3$ is even, then $\mathbf{P}_{I;N}$ has less than $|N|$ vertices.*

2. If I is not a sequence of consecutive integers, then let j be the smallest positive integer not in I, and let k be the smallest integer in I that is larger than j. If $N > j$, then $\mathbf{P}_{I;N}$ is not a cyclic polytope.

Proof. Let $\tilde{\mathbf{P}}_{I;N}$ be the convex hull of the N columns of the $(|I| \times N)$-matrix D^I with entries $D^I_{i,d} = \frac{[d]_i}{[N-1]_i}$ for $i \in I$ and $d = 0, \ldots, N-1$. If N is even, then $\mathbf{P}_{I;N} = \tilde{\mathbf{P}}_{I;N}$.

1. If $1 \notin I$, then the first two columns of D^I are zero vectors. If $1 \in I$, then the first three columns of D^I are

$$\begin{pmatrix} 0 & \frac{1}{N-1} & \frac{2}{N-1} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix},$$

and so the second column lies in the convex hull of the first and the third column.

2. If $i \in I$ is larger than j, then $D^I_{i,d} = 0$ for $d \leq i$ and $D^I_{i,d} > 0$ for $d \geq i$. The equations $n_i = 0$ for $i \in I$, $i > j$, define a proper face F of $\tilde{\mathbf{P}}_{I;N}$. The face F is affinely equivalent to the convex hull of the columns of the first k columns of the matrix $D^{\{1,\ldots,j-1\}}$; hence F is a $C(j-1,k)$ cyclic polytope. In particular, F is not a simplex. However, all proper faces of cyclic polytopes are simplices. Hence $\tilde{\mathbf{P}}_{I;N}$ is not a cyclic polytope.

If N is odd, then $\mathbf{P}_{I;N}$ is a strict subset of $\tilde{\mathbf{P}}_{I;N}$, obtained by cutting off some of the vertices. Since $\tilde{\mathbf{P}}_{I;N}$ contains a face which is not a simplex, the same is true for $\mathbf{P}_{I;N}$. $\square$

Remark 12. Lemma 11 shows that the main argument in the proof of Proposition 6.1 in [5] is false. In fact, the idea of the proof can be used to construct a counter-example to [5, Proposition 6.1].

5. The complement symmetry

The objects of study of this section are the affine symmetries of the polytopes $\mathbf{P}_{k;N}$ and $\tilde{\mathbf{P}}_{k;N}$, i.e. affine maps $\mathbb{R}^k \to \mathbb{R}^k$ that restrict to bijections of the corresponding polytopes. It turns out that, if $N > k+1$, then there is just a single non-trivial affine symmetry,

The automorphism group of the face lattice of $C(k;N)$ cyclic polytopes is known, see Theorem 8.3 in [8]. The automorphism group of the "standard cyclic polytope," corresponding to N points on the moment curve s^k (or on the algebraic moment curve) with equidistant values of t is probably also known, but difficult to find in the literature.

Theorem 14 will show that these polytopes have only one non-trivial symmetry for $N > k + 1$.

The complement $\tilde{G} = (V, \tilde{E})$ of a graph $G = (V, E)$ is a graph with the same vertex set V as G and with edge set $\tilde{E} = \{(x, y) \in V^2 : x \neq y, (x, y) \notin E\}$. The complement $\tilde{G}$ satisfies $\tilde{d}_i = N - 1 - d_i$. The operation $\tilde{G} \mapsto G$ is an involution of the set of graphs with N vertices. It induces an involution on the polytopes $\mathbf{D}_N$, $\mathbf{P}_{k;N}$ and $\tilde{\mathbf{P}}_{k;N}$:

Theorem 13. *The map $\phi : \mathbb{R}^k \to \mathbb{R}^k$ defined by $\phi(x)_j = \sum_{i=0}^{j} (-1)^i \binom{j}{i} x_i$ is an involution that satisfies $\phi(\vec{n}(G)) = \vec{n}(\tilde{G})$, whenever $\tilde{G}$ is the complement of the graph G. Both polytopes $\mathbf{P}_{k;N}$ and $\tilde{\mathbf{P}}_{k;N}$ are invariant under ϕ.*

Proof. The following formula

$$\binom{a}{b} = \sum_{i=0}^{b} (-1)^i \binom{c}{i} \binom{a + c - i}{b - i}$$

is needed, which follows from induction and $\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$. For any $0 \leq d \leq N - 1$,

$$\frac{[N - 1 - d]_k}{[N - 1]_k} = \frac{\binom{N-1-d}{k}}{\binom{N-1}{k}} = \frac{1}{\binom{N-1}{k}} \sum_{i=0}^{k} (-1)^i \binom{d}{i} \binom{N - 1 - i}{k - i}$$

$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{[d]_i}{[N - 1]_i}. \quad (4)$$

Therefore, ϕ permutes the vertices of $\tilde{\mathbf{P}}_{k;N}$. The equality $\phi(\vec{n}(G)) = \vec{n}(\tilde{G})$ and the statement about $\mathbf{P}_{k;N}$ follow by taking expectation values of (4) under the degree distribution of G. Since ϕ is a linear map that restricts to an involution on the full-dimensional polytope $\tilde{\mathbf{P}}_{k;N}$, it follows that ϕ is an involution on all of $\mathbb{R}^k$. □

Theorem 14.
- *If $N \leq k + 1$, then $\tilde{\mathbf{P}}_{k;N}$ is a simplex. The set of affine automorphisms of $\tilde{\mathbf{P}}_{k;N}$ is the symmetric group $\mathbb{S}_N$.*
- *If $N > k + 1$, then ϕ is the only non-trivial affine symmetry of $\tilde{\mathbf{P}}_{k;N}$.*

Proof. The case $N \leq k + 1$ is trivial, so assume $N > k + 1$. The polytope $\tilde{\mathbf{P}}_{k;N}$ is a $C(k; N)$ cyclic polytope.

Suppose $\phi' : x \mapsto Ax + b$ is an affine symmetry of $\tilde{\mathbf{P}}_{k;N}$, where $A \in \mathbb{R}^{k \times k}$ and $b \in \mathbb{R}^k$. Since ϕ' preserves the volume of the full-dimensional polytope $\tilde{\mathbf{P}}_{k;N}$, it follows that $|\det(A)| = 1$. For any subset $S \subset \{0, \ldots, N - 1\}$ of cardinality $k + 1$ let D_S be the submatrix of D consisting of those columns of D indexed by S, and let $\tilde{D}_S$ be the same matrix with an additional row of ones added on top. Then $\tilde{D}_S$ is a Vandermonde matrix with determinant $\det(\tilde{D}_S) = \prod_{i \in S} \prod_{i < j \in S} (j - i)$. In particular, $|\det(\tilde{D}_S)|$ is minimal if and only if S consists of consecutive integers.

The map ϕ' corresponds to a permutation σ of the column indices $0, \ldots, N-1$ of D in such a way that $\phi'(D_i) = D_{\sigma(i)}$. Since ϕ' preserves volumes, $|\det(\tilde{D}_{\sigma(S)}))| = |\det(\tilde{D}_S)|$.

In particular, σ maps consecutive integers to consecutive integers, and therefore, either σ equals the identity, in which case ϕ' equals the identity of $\mathbb{R}^k$, or σ inverts the order on $\{0,\dots,N-1\}$, in which case $\phi' = \phi$. □

Theorem 15. *Let N be odd.*

- *If $N \leq k+1$, then $\mathbf{P}_{k;N}$ is affinely equivalent to $\mathbf{D}_N$, and the group of affine symmetries of $\mathbf{P}_{k;N}$ is the subgroup $\mathbb{S}_{\lceil \frac{n}{2} \rceil} \times \mathbb{S}_{\lfloor \frac{n}{2} \rfloor} \subseteq \mathbb{S}_n$ of the symmetry group of $\tilde{\mathbf{P}}_{k;N}$, where the first factor permutes the vertices D_d with even d, and the second factor permutes the points D_d with odd d.*
- *If $N > k+1$, then ϕ is the only non-trivial symmetry of $\mathbf{P}_{k;N}$.*

The proof makes use of the following result:

Lemma 16. *Any affine symmetry of $\mathbf{P}_{k;N}$ extends to an affine symmetry of $\tilde{\mathbf{P}}_{k;N}$.*

Proof. The statement is trivial if N is even and if $N = 1$, so assume that $N \geq 3$ is odd. Let $k = 2$. Then $\mathbf{P}_{k;N}$ is a $(\frac{3N-1}{2})$-gon The symmetry group of the face lattice of this polygon is the dihedral group $\mathbb{D}_{\frac{3N-1}{2}}$. The edge $(0, N-1)$ is characterized by the following property: Consider the lines defined by all edges of $\mathbf{P}_{k;N}$. All intersection points of these lines lie in one closed hyperplane supporting the edge $(0, N-1)$. This property is invariant under affine transformations. Hence, the edge $(0, N-1)$ is mapped to itself under any affine symmetry. There are only two elements of the dihedral group with this property, the identity and the permutation corresponding to the complement symmetry ϕ.

Now assume that $k \geq 3$, and let d be odd. By Gale's evenness condition, there is a face F of $\tilde{\mathbf{P}}_{k;N}$ with vertices D_0, D_d and D_{d+1}. Then D_0, $n^{d,0}$, $n^{d,d+1}$ and D_{d+1} are vertices of $\hat{F} = F \cap \mathbf{P}_{k;N}$.

Let ϕ' be an affine symmetry of $\mathbf{P}_{k;N}$. Then ϕ' permutes the facets. Under this action, simplices are mapped to simplices, and non-trivial Minkowski sums are mapped to non-trivial Minkowski sums. By Lemma 6, ϕ' maps $\hat{F}$ to another face $\hat{F}' = F' \cap \mathbf{P}_{k;N}$, where F' is a face of $\tilde{\mathbf{P}}_{k;N}$. The vertices of the form $D_{d'}$ for even d' are distinguished by the fact that they are not contained in any special face. Hence ϕ' permutes these vertices among themselves. Moreover, the line connecting $\phi'(D_0)$ and $\phi'(n^{d,0})$ intersects the line connecting $\phi'(n^{d,d+1})$ and $\phi'(D_{d+1})$ in a unique point, which must be of the form $D_{\hat{d}}$ for some odd $\hat{d}$. This shows that ϕ' maps $\tilde{\mathbf{P}}_{k;N}$ into $\tilde{\mathbf{P}}_{k;N}$, and hence ϕ' is a symmetry of $\tilde{\mathbf{P}}_{k;N}$. □

Proof of Theorem 15. The statement for $N > k+1$ follows from Lemma 16 and Theorem 14. Assume $N \leq k+1$. By Lemma 16, the group of affine symmetries of $\mathbf{P}_{k;N}$ is a subgroup of the group of affine symmetries of $\tilde{\mathbf{P}}_{k;N}$. Moreover, any affine symmetry of $\mathbf{P}_{k;N}$ has to preserve the sets $\{D_d : d \text{ odd}\}$ and $\{D_d : d \text{ even}\}$. Conversely, any affine symmetry of $\tilde{\mathbf{P}}_{k;N}$ that preserves these sets also permutes the vertices $n^{d,d'}$ of $\mathbf{P}_{k;N}$ and thus restricts to a symmetry of $\mathbf{P}_{k;N}$. □

6. Exponential random graphs

Let $\mathcal{G}_N$ be the set of graphs with vertex set $V = \{1,\dots,N\}$, and fix an integer k. For all real numbers $\beta_1,\dots,\beta_k$ let $\psi_{\beta_1,\dots,\beta_k}(G) = \exp(\sum_{i=1}^k \beta_i n_i(G))$ and $Z_{\beta_1,\dots,\beta_k} = \sum_{G\in\mathcal{G}_N} \psi_{\beta_1,\dots,\beta_k}(G)$. Then $P_{\beta_1,\dots,\beta_k}(G) = \psi_{\beta_1,\dots,\beta_k}(G)/Z_{\beta_1,\dots,\beta_k}$ defines a probability distribution on $\mathcal{G}_N$. The family $\mathcal{E}_{k;N} := (P_{\beta_1,\dots,\beta_k})_{\beta_1,\dots,\beta_k}$ is called the *k-star model*. It is an example of an exponential random graph model, and hence a particular example of an exponential family.

It follows from the general theory of exponential families that the map $(\beta_1,\dots,\beta_k) \mapsto \mathbb{E}_{P_{\beta_1,\dots,\beta_k}}(\vec{n})$ that maps a set of parameters of $\mathcal{E}_{k;N}$ to the expectation value of $\vec{n}$ under $P_{\beta_1,\dots,\beta_k}$ is a homeomorphism of $\mathbb{R}^k$ to the interior of the polytope $\mathbf{P}_{k;N}$. It induces a homeomorphism from $\mathcal{E}_{k;N}$ to the interior of $\mathbf{P}_{k;N}$ that extends to a homeomorphism $\overline{\mathcal{E}_{k;N}} \cong \mathbf{P}_{k;N}$, where $\overline{\mathcal{E}_{k;N}}$ denotes the closure of $\mathcal{E}_{k;N}$ with respect to the induced topology when considering a probability distribution P on the finite set $\mathcal{G}_N$ as a real vector with $|\mathcal{G}_N|$ components.

For $k < l$, $\mathcal{E}_{k;N}$ is a submodel of $\mathcal{E}_{l;N}$, namely the submodel defined by $\beta_{k+1} = \dots = \beta_l = 0$. The location of $\mathcal{E}_{k;N}$ within $\mathcal{E}_{l;N}$ can be visualized by considering the image of $\mathcal{E}_{k;N}$ under the map $P \mapsto \mathbb{E}_P(\vec{n}_l)$. The closer $\mathbb{E}_P(\vec{n}_l)$ lies to the image of $\mathcal{E}_{k;N}$, the closer P lies to $\mathcal{E}_{k;N}$.

In particular, $\mathcal{E}_{1;N} \subset \mathcal{E}_{k;N}$ for all k. The random graph $P_{\beta_1} \in \mathcal{E}_{1;N}$ is identical to the Erdős-Rényi random graph $G(N,p)$ with parameter $p = \frac{e^{\beta_1}}{1+e^{\beta_1}}$. As β_1 goes from $-\infty$ to $+\infty$, the Erdős-Rényi parameter p goes from zero to one, and hence the expectation values of $\vec{n}_k$ retrace the complete moment curve.

The fact that $\mathbf{P}_{k;N}$ converges to the convex hull of the moment curve can be interpreted as follows: The vertices of $\mathbf{P}_{k;N}$ correspond to graphs that are (almost) regular. By the law of large numbers, when N is large, the degree distribution of the Erdős-Rényi graph is concentrated around its mean value. In other words, for large N the Erdős-Rényi graph is almost regular, and the degree can be tuned by varying the parameter p.

When all parameter but one, say β_i, are fixed, then $\mathbb{E}_{P_{\beta_1,\dots,\beta_k}}(n_i)$ is a monotone function of β_i. More generally, when the parameter vector $\beta = (\beta_1,\dots,\beta_k)$ moves in a fixed direction, then the relative weights of those graphs whose expectation values lie in the same direction increase. In the limit, as $\theta \to +\infty$, the measures $P_{\beta_0+\theta\beta}$ converge to a probability distribution supported on those graphs (G) that maximize $\sum_{i=1}^k \beta_i n_i(G)$. This function $\sum_{i=1}^k \beta_i n_i(G)$ is maximized on a face of the polytope $\mathbf{P}_{k;N}$. If the face happens to be a vertex, say, the vertex corresponding to d-regular graphs, then the limit distribution will be the uniform distribution on the d-regular graphs, since every element $P \in \mathcal{E}_{k;N}$ assigns the same probability to all graphs having the same degree distribution.

It has been observed that for large N and certain parameter ranges the k-star models become ill-behaved in the sense that the exponential random graph gives a large probability mass to graphs that are almost empty or almost complete, see for example [7]. This fact, which was empirically well-known, was recently proven (for a different model) in some cases in a manuscript by Chatterjee and Diaconis that studies the limit of exponential random graphs in the language of graph limits [2]. This fact can also be partially understood from properties of the polytopes $\mathbf{P}_{k;N}$: Note that the two vertices $(0,\dots,0)$ and $(1,\dots,1)$, corresponding to the empty graph and the complete graph, are the "most exposed" vertices, i.e. there are "many" parameter values $\beta_1,\dots,\beta_k$ such that the maximum of the function $\sum_{i=1}^k \beta_i n_i(G)$ lies either at the empty or the complete graph. This is a consequence of the elongated shape of the polytope. For example, if $\beta_i \geq 0$ for all $i = 2,\dots,k$, then either the complete graph (if $\beta_1 > 0$) or the empty graph (if $\beta_1 \ll 0$) maximize the function $\sum_{i=1}^k \beta_i n_i(G)$.

The form of $\mathbf{P}_{k;N}$ also sheds light on another phenomenon: There are large parameter ranges such that the k-star model is very close to an Erdős-Rényi random graph. This phenomenon is also studied by Chatterjee and Diaconis in [2]. The shape of the polytope $\mathbf{P}_{k;N}$ allows the following heuristic argument:

Fix $\beta_1,\dots,\beta_k$, and consider the measures $P_\theta = P_{\beta_1,\theta\beta_2,\dots,\theta\beta_k} \in \mathcal{E}_{k;N}$ for $\theta > 0$. Note that P_0 belongs to the Erdős-Rényi model. Moreover, when $\theta \to \infty$, then P_θ converges to a probability measure P_∞ supported on those graphs that maximize $\sum_{i=1}^k \beta_i n_i(G)$. These graphs correspond to a face of the polytope $\mathbf{P}_{k;N}$. For a generic choice of the parameters β_i, this face will be a vertex, and therefore, the random graph will be uniformly distributed on the set of graphs that are d-regular or almost d-regular, where d depends on the parameters β_i.

If N is large, then $\mathbf{P}_{k;N}$ can be approximated by the convex hull of the moment curve. Therefore, P_0 lies on the moment curve, and P_∞ lies close to the moment curve. If the vector $(0,\beta_2,\dots,\beta_k)$ points away from the convex hull of the moment curve (for example, if the parameters β_i for $i \geq 2$ are negative), then P_0 and P_∞ will be close to each other. In this case, for any non-negative value of θ, the measure P_θ will be well-approximated by the Erdős-Rényi graph P_0. The influence of the parameters $\beta_2,\dots,\beta_k$ is only small.

While these considerations give a nice geometric picture to think about exponential random graphs, they are not sufficient to give a precise description of what happens for intermediate values of β_i. Quite generally, for exponential families, the corresponding polytope, called convex support, can only tell what happens in asymptotic parameter regimes. For finite values of the parameters, a more fine-grained analysis is needed. The tool used in [2] is a large deviation principle for the Erdős-Rényi graph, due to [3].

A qualitative version of this large deviation principle can be obtained from the description of the polytope as follows: When N is large, the expectation value $\vec{n}_k(G(N,p))$ of the vector of i-star statistics for the Erdős-Rényi graph $G(N,p)$ is very close to the boundary of $\mathbf{P}_{k;N}$. Note that $\vec{n}_k(G(N,p))$ is a convex combination of points in $\mathbf{P}_{k;N}$. Since this convex combination is very close to the boundary of $\mathbf{P}_{k;N}$, it follows that most of the points that contribute to the convex combination $\vec{n}_k(G(N,p))$ must lie close to $\vec{n}_k(G(N,p))$. Of course, this argument only shows that the probability mass of $\vec{n}_k(G(N,P))$ is concentrated within a radius of $O(1/N)$ around its mean value. The large deviation principle is much stronger, since it yields exponential concentration and also applies to other graph observables that are not a function of $\vec{n}_k$.

A. Proof of Theorem 4

Lemma 17. *Let N be odd. If d is odd, d' is even and $|d' - d| \neq 1$, then $n^{d,d'}$ is an interior point of $\mathbf{P}_{2;N}$.*

Proof. Using the symmetry ϕ it suffices to consider the case $d' < d - 1$. By Gale's evenness condition, D_d and $D_{d'}$ do not define an edge in the cyclic polytope with vertices $D_{d'}$, D_{d-1}, D_d and D_{d+1}. Therefore, the line segment $L_{d,d'}$ between D_d and $D_{d'}$ intersects the line segment between D_{d-1} and D_{d+1}. Hence the line segment between D_d and $n^{d,d'}$ intersects the line segment between $n^{d,d-1}$ and $n^{d,d+1}$ in a point x. Since x lies on $L_{d,d'}$, it follows that $n^{d,d'}$ is a convex combination of $n^{d,d-1}$, $n^{d,d+1}$ and $D_{d'}$. $\square$

Lemma 18. *Let N be odd. If d is odd, d' is even and $d' \notin \{0, d-1, d+1, N-1\}$, then $n^{d,d'}$ is an interior point of $\mathbf{P}_{3;N}$.*

Proof. Using the symmetry ϕ it suffices to consider the case $d' < d-1$. Let $L_{d,d'}$ be the line segment between D_d and $D_{d'}$. By Gale's evenness condition, $L_{d,d'}$ is not an edge of the cyclic polytope with vertices D_0, $D_{d'}$, D_{d-1}, D_d and D_{d+1}. Therefore, $L_{d,d'}$ intersects the convex hull of D_0, D_{d-1} and D_{d+1}. Hence, the line segment between D_d and $n^{d,d'}$ also intersects the convex hull C of $n^{d,0}$, $n^{d,d-1}$ and $n^{d,d+1}$ in a unique point x. Since x lies on L, the point $n^{d,d'}$ lies in the convex hull of x and $D_{d'}$. Therefore, $n^{d,d'}$ is a convex combination of $n^{d,0}$, $n^{d,d-1}$, $n^{d,d+1}$ and $D_{d'}$. $\square$

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