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by

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Improved lower and upper bounds for entanglement of formation

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We provide analytical lower and upper bounds for entanglement of formation for bipartite systems, which give a direct relation between the bounds of entanglement of formation and concurrence, and improve the previous results. Detailed examples are presented.

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Quantum entanglement [1] is of special importance in quantum-information processing and is responsible for many quantum tasks such as quantum teleportation [2, 3], dense coding [4], swapping [5, 6], error correction [7, 8] and remote state preparation [9, 10]. The entanglement of formation (EoF) [11, 12] is a well-defined important measure of entanglement for bipartite systems.

Let H_A and H_B be m- and n-dimensional ($m \leq n$) vector spaces, respectively. A pure state $|\psi\rangle \in H_A \otimes H_B$ has a Schmidt decomposition $|\psi\rangle = \sum_{i=1}^{m} \sqrt{\mu_i}|ii\rangle$, where $\mu_i \geq 0$ and $\sum_{i=1}^{m} \mu_i = 1$. The entanglement of formation is given by the entropy of the reduced density matrix $\rho_A = Tr_B(|\psi\rangle\langle\psi|)$,

$$E(|\psi\rangle) = S(\rho_A) = -\sum_{i=1}^{m} \mu_i \log \mu_i \equiv H(\vec{\mu}), \qquad (1)$$

where log stands for the natural logarithm throughout the paper, μ_i $(i = 1, 2, ..., m)$ are the non-zero eigenvalues of ρ_A and $\vec{\mu}$ is the Schmidt vector $(\mu_1, \mu_2, ..., \mu_m)$. For a bipartite mixed state ρ, the entanglement of formation is given by the minimum average marginal entropy of the ensemble decompositions of ρ,

$$E(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \qquad (2)$$

for all possible ensemble realizations $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, where $p_i \geq 0$ and $\sum_i p_i = 1$.

Another significant measure of quantum entanglement is the concurrence. The concurrence of a pure bipartite state $|\psi\rangle$ is given by

$$C(|\psi\rangle) = \sqrt{2[1 - Tr(\rho_A^2)]} = \sqrt{2(1 - \sum_{i=1}^{m} \mu_i^2)}. \qquad (3)$$

It is extended to mixed states by the convex roof construction

$$C(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \qquad (4)$$

for all possible ensemble realizations $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$.

Considerable effort has been made to estimate the entanglement of formation and concurrence for bipartite quantum states, and their lower and upper bounds via analytical and numerical approaches. For the two-qubit case, EoF is a monotonically increasing function of the concurrence, and an analytical formula of concurrence has been derived [13]. For the general high-dimensional case, due to the extremizations involved in the computation, only a few analytic formulas have been obtained for isotropic states [14] and Werner states [15] for EoF, and for some special symmetric states [16–18] for concurrence.

Instead of analytic formulas, some progress has been made toward the lower and upper bounds of EoF and concurrence for any $m \otimes n$ ($m \leq n$) mixed quantum state ρ. In [19–23], explicit analytical lower and upper bounds of concurrence have been presented. In Ref.[19], a simple analytical lower bound of EoF has been derived. Recently new results related to the bounds of EoF have been further derived in [24, 25]. In this article, we give new lower and upper bounds of EoF based on the concurrence. Detailed examples are presented, showing that our bounds improve the bounds in [24, 25].

In Ref.[24], the authors defined $X(\lambda)$ and $Y(\lambda)$ to derive measurable lower and upper bounds of EoF. We give an improved definition of $X(\lambda)$ and $Y(\lambda)$ in this paper. For a given pure state $|\psi\rangle = \sum_{i=1}^{m} \sqrt{\mu_i}|ii\rangle$, the concurrence of $|\psi\rangle$ is given by $c = \sqrt{2(1 - \sum_i^m \mu_i^2)}$. However, the pure states with the same value of concurrence c are not unique. Namely, different sets of the Schmidt coefficients $\{\mu_i\}$ may give rise to the same concurrence c. The entanglement of formation $H(\vec{\mu})$ defined in Eq.(1) for a pure state depends on the Schmidt coefficients $\{\mu_i\}$. We define the maximum and minimum of $H(\vec{\mu})$ to be $X(c)$ and $Y(c)$ for fixed c,

$$X(c) = \max \left\{ H(\vec{\mu}) \,\middle|\, \sqrt{2(1 - \sum_{i=1}^{m} \mu_i^2)} \equiv c \right\}, \qquad (5)$$

and

$$Y(c) = \min \left\{ H(\vec{\mu}) \,\middle|\, \sqrt{2(1 - \sum_{i=1}^{m} \mu_i^2)} \equiv c \right\}, \qquad (6)$$

respectively, where the maximum and minimum are taken over all possible Schmidt coefficient distributions $\{\mu_i\}$ such that the value of concurrence $c = \sqrt{2(1 - \sum_i^m \mu_i^2)}$ is fixed.

Let $\varepsilon(c)$ be the largest monotonically increasing convex function that is bounded above by $Y(c)$, and $\eta(c)$ be the smallest monotonically increasing concave function that is bounded below by $X(c)$.

Theorem 1 *For any $m\otimes n$ ($m\le n$) quantum state ρ, the entanglement of formation $E(\rho)$ satisfies*

$$\varepsilon(C(\rho))\le E(\rho)\le \eta(C(\rho)). \tag{7}$$

Proof. We assume that $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$ is the optimal decomposition of $E(\rho)$. Therefore

$$\begin{aligned} E(\rho) &= \textstyle\sum_i p_i E(|\psi_i\rangle)=\sum_i p_i H(\vec{\mu}_i)\\ &\ge \textstyle\sum_i p_i\varepsilon(c_i)\ge\varepsilon(\sum_i p_i c_i)\\ &\ge \varepsilon(C(\rho)).\end{aligned} \tag{8}$$

We have used the definition of $\varepsilon(c)$ to obtain the first inequality. The second inequality is due to the convex property of $\varepsilon(c)$, and the last one is derived from the definition of concurrence. On the other hand, as $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$ is the optimal decomposition of $C(\rho)$, we have

$$\begin{aligned} E(\rho) &\le \textstyle\sum_i p_i E(|\psi_i\rangle)=\sum_i p_i H(\vec{\mu}_i)\\ &\le \textstyle\sum_i p_i\eta(c_i)\le\eta(\sum_i p_i c_i)\\ &= \eta(C(\rho)),\end{aligned} \tag{9}$$

where we have used the definition of $E(\rho)$ to obtain the first inequality. The second and the third inequalities are due to the definition of $\eta(c)$. □

Our analytical bounds (7) give an explicit relations between the EoF and the concurrence. In fact, if we denote $\underline{c}$ and $\overline{c}$ the analytical lower and upper bounds of concurrence, respectively, according to Theorem 1, we have the following corollary:

Corollary 1 *For any $m\otimes n(m\le n)$ quantum state ρ, the entanglement of formation $E(\rho)$ satisfies*

$$\varepsilon(\underline{c})\le E(\rho)\le\eta(\overline{c}). \tag{10}$$

Here $\underline{c}$ and $\overline{c}$ could be any known analytical lower and upper bounds of concurrence. For example, from the known bounds of concurrence in [19, 24, 25], one may choose

$$\underline{c}=\max\left\{0,\sqrt{\tfrac{2}{m(m-1)}}(||\rho^{T_A}||-1),\sqrt{\tfrac{2}{m(m-1)}}\left[R(\rho)-1\right],\sqrt{2[Tr(\rho^2)-Tr(\rho_A^2)]},\sqrt{2[Tr(\rho^2)-Tr(\rho_B^2)]}\right\}$$

and $\overline{c}=\min\left\{\sqrt{2[1-Tr(\rho_A^2)]},\sqrt{2[1-Tr(\rho_B^2)]}\right\}$, where ρ^{T_A} stands for the partial transpose with respect to the subsystem A, $R(\rho)$ is the realigned matrix of ρ, $||\cdot||$ stands for the trace norm, and ρ_A and ρ_B are the reduced density matrices with respect to the subsystems A and B respectively.

The maximal admissible $H(\vec{\mu})$ and the minimal admissible $H(\vec{\mu})$ in Eqs.(5) and (6) for a given c can be estimated following the approach in Ref.[24]. Let n_1 be the number of entries such that $\mu_i=\alpha$ and let n_2 be the number of entries such that $\mu_i=\beta$. The maximal admissible $H(\vec{\mu})$ and the minimal admissible $H(\vec{\mu})$ for a given c become, for fixed $n_1,n_2,n_1+n_2\le m$, one of maximizing or minimizing the function

$$F_{n_1,n_2}=n_1h(\alpha_{n_1n_2})+n_2h(\beta_{n_1n_2}), \tag{11}$$

where $h(x)=-x\log x$,

$$\alpha_{n_1n_2}=\frac{n_1+\sqrt{n_1^2-n_1(n_1+n_2)[1-n_2(1-\frac{c^2}{2})]}}{n_1(n_1+n_2)},$$

and $\beta_{n_1n_2}=(1-n_1\alpha_{n_1n_2})/n_2$.

When $m=3$, to find the expressions of upper and lower bounds in Eqs.(7) and (10) is to obtain the maximization and minimization over the three functions $F_{11}(c)$, $F_{12}(c)$ and $F_{21}(c)$. From Eq.(11), for $m=3$, we have

$$X(c)=\begin{cases}F_{11}(c), & 0<c\le 1,\\ F_{12}(c), & 1<c\le\dfrac{2}{\sqrt{3}},\end{cases} \tag{12}$$

and

$$Y(c)=\begin{cases}F_{11}(c), & 0<c\le 1,\\ F_{21}(c), & 1<c\le\dfrac{2}{\sqrt{3}}.\end{cases} \tag{13}$$

To determine $\varepsilon(c)$ and $\eta(c)$, we study the concavity and convexity of functions $F_{11}(c),F_{12}(c)$ and $F_{21}(c)$. Since $F_{11}^{''}\ge 0$ and $F_{21}^{''}\le 0$, where $F_{ij}^{''}$ is the second derivative of F_{ij}, we have that $\varepsilon(c)$ is the curve consisting of F_{11} for $c\in(0,1]$ and the line connecting the points $[1,F_{21}(1)]$ and $[\frac{2}{\sqrt{3}},F_{21}(\frac{2}{\sqrt{3}})]$ for $c\in(1,\frac{2}{\sqrt{3}}]$, that is,

$$\varepsilon(c)=\begin{cases}F_{11}(c), & 0<c\le 1,\\ \dfrac{\sqrt{3}\log 3/2}{2-\sqrt{3}}(c-1)+\log 2, & 1<c\le\dfrac{2}{\sqrt{3}}.\end{cases} \tag{14}$$

Similarly, since $F_{11}^{''}\ge 0$ and $F_{12}^{''}\ge 0$, we have that $\eta(c)$ is the curve connecting the points $[0,0]$ and $[\frac{2}{\sqrt{3}},F_{12}(\frac{2}{\sqrt{3}})]$

for $c \in (0,1]$, and the line connecting the points $[1, F_{12}(1)]$ and $[\frac{2}{\sqrt{3}}, F_{12}(\frac{2}{\sqrt{3}})]$ for $c \in (1, \frac{2}{\sqrt{3}}]$, that is,

$$\eta(c) = \begin{cases} \log 2(c), \quad 0 < c \leq 1, \\ \dfrac{2\log 3/2 + \log 6 - 3\log 3}{3(\sqrt{3}-2)}(\sqrt{3}\,c - 2) + \log 3, \quad 1 < c \leq \dfrac{2}{\sqrt{3}}. \end{cases} \tag{15}$$

See Fig. 1.

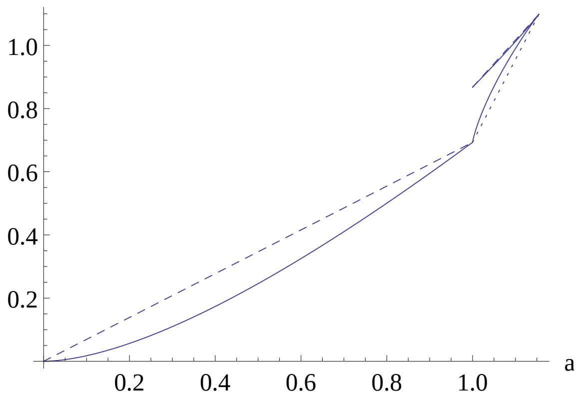

FIG. 1. Upper and lower bounds of $E(\rho)$ (dished lines and dotted lines), and F_{11}, F_{12} and F_{21} (solid lines).

Similarly, for any m, we can get the expressions of $X(c)$ and $Y(c)$,

$$X(c) = \begin{cases} F_{11}(c), \quad 0 < c \leq 1, \\ F_{12}(c), \quad 1 < c \leq \dfrac{2}{\sqrt{3}}, \\ \ldots. \\ F_{1(m-1)}(c), \quad \sqrt{\dfrac{2(m-2)}{m-1}} < c \leq \sqrt{\dfrac{2(m-1)}{m}}, \end{cases} \tag{16}$$

and

$$Y(c) = \begin{cases} F_{11}(c), \quad 0 < c \leq 1, \\ F_{21}(c), \quad 1 < c \leq \dfrac{2}{\sqrt{3}}, \\ \ldots \\ F_{(m-1)1}(c), \quad \sqrt{\dfrac{2(m-2)}{m-1}} < c \leq \sqrt{\dfrac{2(m-1)}{m}}. \end{cases} \tag{17}$$

The representations of $\varepsilon(c)$ and $\eta(c)$ can be also similarly calculated analytically in accordance with the following principles ($2 \leq t \leq m-1$):

If $F_{1t}^{''}(c) \geq 0$, $c \in (\sqrt{\frac{2(t-1)}{t}}, \sqrt{\frac{2t}{t+1}}]$, then $\eta(c) = [F_{1t}(\sqrt{\frac{2t}{t+1}}) - F_{1t}(\sqrt{\frac{2(t-1)}{t}})](c - \sqrt{\frac{2t}{t+1}})/[\sqrt{\frac{2t}{t+1}} - \sqrt{\frac{2(t-1)}{t}}] + F_{1t}(\sqrt{\frac{2t}{t+1}})$; If $F_{1t}^{''}(c) \leq 0$, $c \in (\sqrt{\frac{2(t-1)}{t}}, \sqrt{\frac{2t}{t+1}}]$, then $\eta(c) = F_{1t}(c)$; If $F_{t1}^{''}(c) \geq 0$, $c \in (\sqrt{\frac{2(t-1)}{t}}, \sqrt{\frac{2t}{t+1}}]$, then $\varepsilon(c) = F_{t1}(c)$; If $F_{t1}^{''}(c) \leq 0$, $c \in (\sqrt{\frac{2(t-1)}{t}}, \sqrt{\frac{2t}{t+1}}]$, then $\varepsilon(c) = [F_{t1}(\sqrt{\frac{2t}{t+1}}) - F_{t1}(\sqrt{\frac{2(t-1)}{t}})](c - \sqrt{\frac{2t}{t+1}})/[\sqrt{\frac{2t}{t+1}} - \sqrt{\frac{2(t-1)}{t}}] + F_{t1}(\sqrt{\frac{2t}{t+1}})$.

The bounds given in Theorem 1 and Corollary 1 can be used to improve the bounds of EoF presented in [24] and [25]. In fact, the lower bound obtained in Ref.[25] is better than the lower bound from Ref.[24], while the upper bounds are the same. Our bounds are obtained from the improved bounding functions (5) and (6). They are directly given by the concurrence. From the concurrence, or the lower and upper bounds of the concurrence of a given mixed state, one can get analytical lower and upper bounds of EoF of the state. To see the tightness of inequalities (7) and (10), let us consider the following examples.

Example. 1 Let us consider the well-known Werner states, which are a class of mixed states for $d \otimes d$ systems that are invariant under the transformations $U \otimes U$, for any unitary transformation U [15, 26]. The density matrix of the Werner states can be expressed as

$$\rho_f = \frac{1}{d^3 - d}[(d-f)I + (df-1)\mathcal{F}], \tag{18}$$

where $\mathcal{F}$ is the flip operator defined by $\mathcal{F}(\phi \otimes \psi) = \psi \otimes \phi$. Consider the case $d = 3$. We have $(\rho_f)_A = (\rho_f)_B = \frac{1}{3}(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|)$ and $1 - Tr[(\rho_f)_A^2] = \frac{2}{3}$. By Refs.[24, 25], the upper bound of EoF is given by $E(\rho_f) \leq 1.099$. From Eq.(15), we get the upper bound of ρ_f,

$$E(\rho_f) \leq -f\log 2, \quad -1 \leq f < 0, \tag{19}$$

It is obvious that $-f\log 2 < 1.099$ for $-1 \leq f < 0$. Hence the upper bound (15) is better than the upper bound in Refs.[24, 25].

Example. 2 Consider the $3 \otimes 3$ mixed state $\rho = \frac{x}{9}I + (1-x)|\psi\rangle\langle\psi|$, where the column vector $|\psi\rangle = (a, 0, 0, 0, \frac{1}{\sqrt{3}}, 0, 0, 0, \frac{1}{\sqrt{3}})^t/\sqrt{a^2 + 2/3}$ with $a \in [0,1]$, t stands for vector transposition. For this state we have $Tr(\rho^2) - Tr(\rho_A^2) = Tr(\rho^2) - Tr(\rho_B^2) = 2[9 - 26x + 9a^4(-2+x)x + 13x^2 + 6a^2(9 - 22x + 11x^2)]/9(2+3a^2)^2$ and $1 - Tr(\rho_A^2) = 1 - Tr(\rho_B^2) = [6 + 4x - 18a^4(-2+x)x - 2x^2 + 12a^2(3 - 2x + x^2)]/3(2+3a^2)^2$.

For fixed $x = 0.1$, one has

$$\begin{aligned} \underline{c} &= \sqrt{2[Tr(\rho^2) - Tr(\rho_A^2)]} \\ &= \frac{2\sqrt{6.53 + 41.46a^2 - 1.71a^4}}{3(2+3a^2)}, \end{aligned} \tag{20}$$

$$\begin{aligned} \overline{c} &= \sqrt{2[1 - Tr(\rho_A^2)]} \\ &= \frac{1}{2+3a^2}\sqrt{\frac{2(6.38 + 33.72a^2 + 3.42a^4)}{3}}. \end{aligned} \tag{21}$$

Substituting $\underline{c}$ and $\overline{c}$ into Eqs.(14) and (15), we have the upper and lower bounds for $E(\rho)$, see Fig. 2.

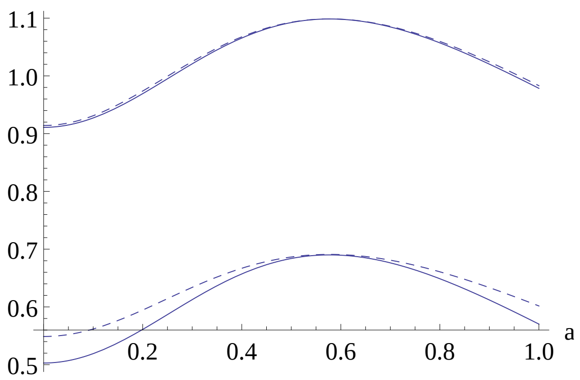

FIG. 2. Upper and lower bounds of $E(\rho)$ when x=0.1. Dished lines are given by Eqs.(14) and (15), solid lines are given by Ref. [24].

Similarly for $x = 0.001$, one has $\underline{c}$ and $\overline{c}$, and the upper and lower bounds for $E(\rho)$, see Fig. 3. Form Figs. 2 and 3, we see that the lower bound of EoF (14) is better than the one from [24].

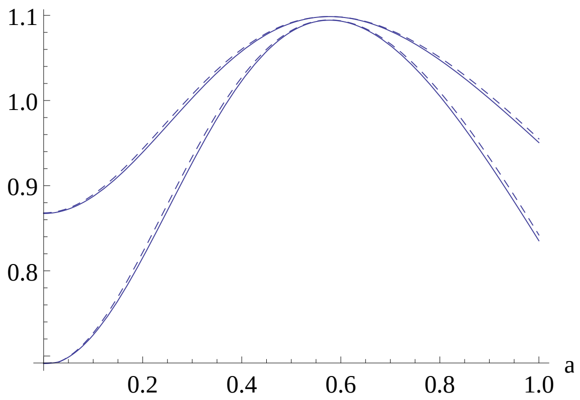

FIG. 3. Upper and lower bounds of $E(\rho)$ when x=0.001. Dished lines are given by Eqs.(14) and (15), solid lines are given by Ref. [24].

In fact, we can make the bounds better by choosing suitable $\underline{c}$ and $\overline{c}$. For instance, to find a better lower bound, we may choose $\underline{c} = \sqrt{\frac{1}{3}(||\rho^{T_A}|| - 1)}$. Then for $x = 0.1$ and $a \in [0.5, 0.66]$, one has

$$||\rho^{T_A}|| - 1 = \frac{2[5 + 6.9a^2 - 0.9a^4 + 9.353a(2+3a^2)]}{3(2+3a^2)^2}. \tag{22}$$

When $x = 0.001$ and $a \in [0.57, 0.59]$, one has

$$||\rho^{T_A}|| - 1 = \frac{2[5.99 + 8.978a^2 - 0.009a^4 + 10.382a(2+3a^2)]}{3(2+3a^2)^2}. \tag{23}$$

Substitute Eqs.(22) and (23) into (14), respectively, we get another lower bound of EoF for the state in example 2, see Figs. 4 and 5.

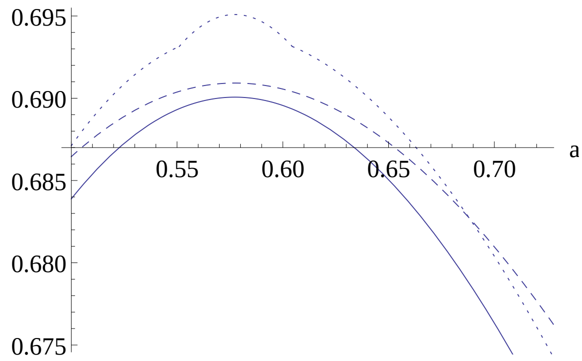

FIG. 4. Lower bounds of $E(\rho)$ when x=0.1. Dashed line is obtained by Ref.[25], dotted line is obtained by Eq.(14) and solid line is obtained by Ref.[24].

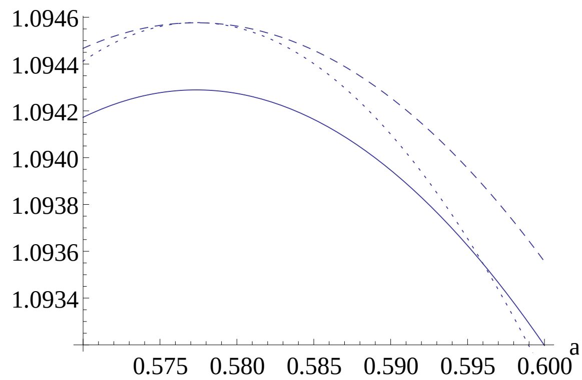

FIG. 5. Lower bounds of $E(\rho)$ when x=0.001. Dashed line is obtained by Ref.[25], dotted line is obtained by Eq.(14) and solid line is obtained by Ref.[24].

From Figs. 4 and 5, it is obvious that $\varepsilon(\sqrt{\frac{1}{3}(||\rho^{T_A}|| - 1)}) > \varepsilon_1 > \varepsilon_0$ when $x = 0.1$ and $a \in [0.5, 0.66]$, and $\varepsilon_1 = \varepsilon(\sqrt{2[Tr(\rho^2) - Tr(\rho_A^2)]}) > \varepsilon(\sqrt{\frac{1}{3}(||\rho^{T_A}|| - 1)}) > \varepsilon_0$ when $x = 0.001$ and $a \in [0.57, 0.59]$, where ε_0 and ε_1 are obtained by Ref.[24] and Ref.[25], respectively . The lower bounds are improved in the particular interval of a.

The density matrix in example 2 is close to being separable (pure) when x is close to 1 (0). To show better the advantage of our results, we now take $x = 0.6$. We have then $Tr(\rho^2) - Tr(\rho_A^2) = Tr(\rho^2) - Tr(\rho_B^2) < 0$, and hence the lower bounds from Refs.[24, 25] are $\varepsilon_0 = \varepsilon_1 = 0$. However, by choosing

$$\underline{c} = \frac{||\rho^{T_A}|| - 1}{\sqrt{3}} = \frac{2(6 + a^2(9 - 21x) + 6\sqrt{3}a(2 + 3a^2)(x - 1) - 10x - 9a^4 x)}{3\sqrt{3}(2 + 3a^2)^2}, \tag{24}$$

we have $\underline{c} > 0$ when $a \in (0.205, 1)$. From Corollary 1 we get that $E(\rho) \geq \varepsilon(\underline{c}) > 0$, see Fig.6 for the lower bound $\varepsilon(\underline{c})$,

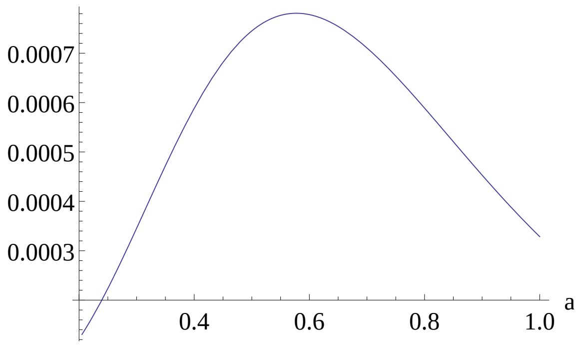

FIG. 6. Lower bounds of $E(\rho)$ when x=0.6.

In summary, we have presented analytic lower and upper bounds of EoF for arbitrary bipartite mixed states. The bounds can be used to improve the previous results on bounds of EoF. Although the EoF is a monotonically increasing function of the concurrence only in the two-qubit case, it turns out that for higher dimensional cases, the bounds of EoF have a tight relation to the concurrence or the bounds of concurrence.

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