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by

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A NOTE ON THE DERIVATION OF HOMOGENIZED BENDING PLATE MODEL

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Abstract. We derive, via simultaneous homogenization and dimension reduction, the Γ-limit for thin elastic plates of thickness h whose energy density oscillates on a scale $\varepsilon(h)$ such that $\varepsilon(h)^2 \ll h \ll \varepsilon(h)$. We consider the energy scaling that corresponds to Kirchhoff's nonlinear bending theory of plates.

Keywords: elasticity, dimension reduction, homogenization, nonlinear plate theory, two-scale convergence.

1. Introduction

In this paper we derive a model for homogenized bending plate, by means of Γ-convergence, from $3d$ nonlinear elasticity. The pioneering papers in derivation of lower dimensional models by means of Γ convergence are [ABP91] where the equations for elastic string are derived and [LDR95] where membrane plate equations are derived. It was well known that the obtained models depend on the assumption on the order of external loads with respect to the thickness of the body (see [Cia97, Cia00] for the approach via formal asymptotic expansion). The pioneering works in deriving higher dimensional models (e.g. bending and von-Kármán) via Γ-convergence are [FJM02, FJM06] where the key indegredient is the theorem on geometric rigidity.

Recently, models of homogenized bending plate were derived in the special case when the relation between the thickness of the body h and the oscillations of the material $\varepsilon(h)$ satisfy the condition $h \sim \varepsilon(h)$ or $\varepsilon(h) \ll h$ i.e. the situations when is such that $\lim_{h\to 0} \frac{h}{\varepsilon(h)} =: \gamma \in (0, \infty]$, see [HNV]. Here we partially cover the case $\gamma = 0$, by assuming additionally $\varepsilon(h)^2 \ll h \ll \varepsilon(h)$. The von-Kármán case of plate and shells are discussed in [NV, HV]. In the case of von-Kármán plate all the cases for γ are obtained; the case $\gamma = 0$ corresponds to the situation where dimensional reduction prevails and the case $\gamma = \infty$ corresponds to the situation where homogenization prevails. Both of these cases can be obtained as limit cases from the intermediate thin films that arises when $\gamma \to 0$ i.e. $\gamma \to \infty$. In the case of shells the surprising fact was there are different scenarios for $\gamma = 0$; one scenario for $\varepsilon(h)^2 \ll h \ll \varepsilon(h)$, the other for $h \sim \varepsilon(h)^2$.

For the relation of the model obtained here with the ones obtained in [HNV], see Remark 6 below. The recovery sequence for this model is significantly

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different then the one defined in [HNV] and its gradient includes the terms of order $\varepsilon(h) \gg h$, which then has to be of the specific form (RA, where R is the rotation matrix and A skew symmetric matrix), in order to obtain the energy of order h^2 (see the expression (51) below). The compactness result, given in [FJM02], which gives the lower bound of Γ-limit, forces us to work with piecewise constant map with values in SO(3) which creates some additional technical difficulties in the compactness lemmas that are needed for recognizing the oscillatory part of two scale limit (see Lemma 3.9 and Lemma 3.11 below). The situation $\varepsilon(h)^2 \sim h$ seems to be more involving, since in that case we have lack of compactness result, partially due to the possible occurrence of oscillations of order different than $\varepsilon(h)$, see Remark 3 and Remark 7 below.

By so(3) we denote the space of skew symmetric matrices in $\mathbb{M}^3$. For a matrix A, $\operatorname{sym} A$ denotes its symmetric part while $\operatorname{skw} A$ denotes its skew symmetric part i.e. $\operatorname{sym} A = \frac{1}{2}(A + A^T)$, $\operatorname{skw} A = \frac{1}{2}(A - A^T)$. For a vector $v \in \mathbb{R}^3$ by A_v we denote the antisymmetric tensor given by $A_v x = v \times x$. We call v the axial vector of A_v. One easily obtains

$$A_v = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \tag{1}$$

For $v, w \in \mathbb{R}^3$ we have

$$A_v \cdot A_w = 2v \cdot w. \tag{2}$$

By δ_{ij} we denote the Kronecker delta. $A \lesssim B$ means that the inequality is valid up to a multiplicative constant $C > 0$ on the right hand side.

We need to introduce some function spaces of periodic functions. From now on, $Y = [0,1)^2$, and we denote by $\mathcal{Y}$ the set Y endowed with the torus topology, so that functions on $\mathcal{Y}$ will be Y-periodic.
We write $C(\mathcal{Y})$, $C^k(\mathcal{Y})$ and $C^\infty(\mathcal{Y})$ for the Banach spaces of Y-periodic functions on $\mathbb{R}^2$ that are continuous, k-times continuously differentiable and smooth, respectively. Moreover, $H^k(I \times \mathcal{Y})$ denotes the closure of $C^\infty(I, C^\infty(\mathcal{Y}))$ with respect to the norm in $H^k(I \times Y)$ and we write $\mathring{H}^k(\mathcal{Y})$ for the subspace of functions $f \in H^k(\mathcal{Y})$ with $\int_Y f = 0$. In the analogous way we define $\mathring{H}^1(I \times \mathcal{Y})$, $\mathring{C}^k(\mathcal{Y})$. The definitions extend in the obvious way to vector-valued functions.

2. General framework and main result

By $S \subset \mathbb{R}^2$ we denote a bounded Lipschitz domain whose boundary is piecewise C^1. For the proof of the lower bound we need only Lipschitz domain. The piecewise C^1-condition is necessary only for the proof of the upper bound, cf. Section 4.2 for the details.
By $\Omega_h := S \times hI$, where $h > 0$ and $I := (-\frac{1}{2}, \frac{1}{2})$, we denote the reference configuration of the thin plate of thickness h. The elastic energy per unit volume associated with a deformation $v^h : \Omega_h \to \mathbb{R}^3$ is given by

$$\frac{1}{h} \int_{\Omega_h} W(\frac{z_3}{h}, \frac{z'}{\varepsilon}, \nabla v^h(z))\, dz. \tag{3}$$

We denote by $z' = (x_1, x_2)$ in-plane coordinates of a generic element $z = (x_1, x_2, x_3) \in \Omega_h$. By W we denote a energy density that models the elastic properties of a periodic composite.

Assumption 2.1. We assume that

$$W : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{M}^3 \to [0, \infty], \qquad (x_3, y, F) \mapsto W(x_3, y, F)$$

is measurable and $[0, 1)^2$-periodic in y for all F. Furthermore, we assume that for almost every $(x_3, y) \in \mathbb{R} \times \mathbb{R}^2$, the map $\mathbb{M}^3 \ni F \mapsto W(x_3, y, F) \in [0, \infty]$ is continuous and satisfies the following properties:

(FI)

(frame indifference)

$$W(x_3, y, RF) = W(x_3, y, F) \quad \text{ for all } F \in \mathbb{M}^3,\ R \in \mathrm{SO}(3);$$

(ND)

(non degeneracy)

$$W(x_3, y, F) \geq c_1 \operatorname{dist}^2(F, \mathrm{SO}(3)) \quad \text{ for all } F \in \mathbb{M}^3;$$
$$W(x_3, y, F) \leq c_2 \operatorname{dist}^2(F, \mathrm{SO}(3)) \quad \text{ for all } F \in \mathbb{M}^3 \text{ with } \operatorname{dist}^2(F, \mathrm{SO}(3)) \leq \rho;$$

(QE)

(quadratic expansion at identity)

$$\lim_{G \to 0} \operatorname*{ess\,sup}_{y \in \mathcal{Y}} \frac{|W(x_3, y, I + G) - Q(x_3, y, G)|}{|G|^2} = 0$$

for some quadratic form $Q(x_3, y, \cdot)$ on $\mathbb{M}^3$.

Here c_1, c_2 and ρ are positive constants which are fixed from now on.

We define $\Omega := S \times I$. The standard procedure in deriving lower dimensional models is to rescale the out-of-plane coordinate: for $x = (x', x_3) \in \Omega$ we consider the scaled deformation $u^h(x', x_3) := v^h(x', hx_3)$. Then (13) becomes

$$\mathcal{E}^{h,\varepsilon}(u^h) := \int_\Omega W(x_3, \frac{x'}{\varepsilon}, \nabla_h u^h(x))\, dx. \tag{4}$$

By $\nabla_h u^h := \left(\nabla' u^h,\ \frac{1}{h} \partial_3 u^h \right)$ we have denoted the scaled gradient, and by $\nabla' u^h := \left(\partial_1 u^h,\ \partial_2 u^h \right)$ we have denoted the gradient in the plane.

As already mentioned in the introduction, it is well known that different models for thin bodies can be obtained from three dimensional elasticity equations by the method of Γ-convergence. The main assumption that influence the derivation of the model is the assumption on the relation of the order of the energy and the thickness of the body (also the assumption on the boundary conditions can influence the model). The plate behavior, due to its more complex geometry, is much more complex than the behavior of rods. We recall some known results on dimension reduction in the homogeneous case when $W(x_3, y, F) = W(F)$. In [FJM06] a hierarchy of plate models is derived from $\mathcal{E}^h := \mathcal{E}^{h,1}$ in the zero-thickness limit $h \to 0$. The case $\mathcal{E}^h \sim 1$ leads to a membrane model (see [LDR95]), which is a fully nonlinear plate model for plates without resistance to compression. The reason for that is that the compression enables the plate to preserve the metric on the mid-plane and thus these deformations have lower order energy. The obtained equations are of the same type as the original $3D$ equations i.e. quasilinear and of the second order. In the regime $\mathcal{E}^h \sim h^4$ finite energy deformations converge to rigid deformations and to obtain the limit energy we need to introduce the correctors from the rigid deformation. In [FJM06] it is shown that $h^{-4}\mathcal{E}^h$ converges to a plate model of "von-Kármán"-type. In the case of higher scalings one obtains the linear plate model.

In this article we study the *bending regime* $\mathcal{E}^h \sim h^2$, which, as shown in [FJM02], the Γ-limit as $h \to 0$ of the functionals $h^{-2}\mathcal{E}^h$ is the functional

$$\int_S Q_2(\mathrm{II}(x'))\,dx', \tag{5}$$

with $Q_2 : \mathbb{M}^2 \to \mathbb{R}$ is defined by the relaxation formula

$$Q_2(A) = \min_{d\in\mathbb{R}^3} Q\left(\iota(A) + d \otimes e_3\right); \tag{6}$$

here, Q is the quadratic form introduced in (QE) and ι denotes the natural embedding of $\mathbb{M}^2$ into $\mathbb{M}^3$.

Denoting the standard basis of $\mathbb{R}^3$ by (e_1, e_2, e_3) it is given by

$$\iota(A) = \sum_{\alpha,\beta=1}^{2} A_{\alpha\beta}(e_\alpha \otimes e_\beta).$$

The special case of layered material is considered in [Sch07]. Dependence on x_3 variable produces non-trivial effects on the relaxation formula. Namely the limit functional is then given by

$$\int_S \bar{Q}_2(\mathrm{II}(x'))\,dx', \tag{7}$$

where

$$\bar{Q}_2(A) = \min_{B\in\mathbb{M}^2_{\mathrm{sym}}} \int_I Q_2\left(\iota(x_3 A + B)\right)\,dx_3. \tag{8}$$

In [HNV] it was shown that in the non-homogeneous case the effective quadratic form Q_2 is determined by a relaxation formula that is more complicated. For construction of the recovery sequence it was also helpful to understand the behavior of layered materials. The obtained depended on the relative scaling between the thickness h and the material period ε. Here we will make the assumption that ε and h are coupled as follows:

Assumption 2.2. We assume that $\varepsilon = \varepsilon(h)$ is a nondecreasing function from $(0, \infty)$ to $(0, \infty)$ such that $\varepsilon(h) \to 0$ and $\varepsilon(h)^2 \ll h \ll \varepsilon(h)$ i.e. $\lim_{h\to 0} \frac{h}{\varepsilon(h)} = \lim_{h\to 0} \frac{\varepsilon(h)^2}{h} = 0$.

The energy density of the homogenized plate we derive here is given by means of a relaxation formula that we introduce next.

Definition 2.3 (Relaxation formula)**.** Let Q be as in Assumption 2.1. We define $Q_2^{\mathrm{rel}} : \mathbb{M}^2_{\mathrm{sym}} \to [0,\infty)$ by

$$Q_2^{\mathrm{rel}}(A) \quad := \quad \inf_{B,U} \iint_{I\times Y} Q\left(x_3, y,\ \iota(x_3 A + B) + U\right)\, dy\, dx_3,$$

taking the infimum over all $B \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ and $U \in L_0(I\times\mathcal{Y}, \mathbb{M}^3_{\mathrm{sym}})$, where

$$L_0(I\times\mathcal{Y}, \mathbb{M}^3_{\mathrm{sym}}) \quad := \quad \Bigg\{ \begin{pmatrix} \operatorname{sym}\nabla_y\zeta + x_3\nabla_y^2\varphi & \begin{matrix} g_1 \\ g_2 \end{matrix} \\ (g_1,\ g_2) & g_3 \end{pmatrix} : \zeta \in \mathring{H}^1(\mathcal{Y},\mathbb{R}^2),$$
$$\varphi \in \mathring{H}^2(\mathcal{Y}),\ \ g \in L^2(I\times\mathcal{Y},\mathbb{R}^3) \Bigg\}.$$

We also define the mapping $\mathcal{U} : \mathring{H}^1(\mathcal{Y},\mathbb{R}^2) \times \mathring{H}^2(\mathcal{Y}) \times L^2(I\times\mathcal{Y},\mathbb{R}^3) \to L_0(I\times\mathcal{Y}, \mathbb{M}^3_{\mathrm{sym}})$ by

$$\mathcal{U}(\zeta,\varphi,g) = \begin{pmatrix} \operatorname{sym}\nabla_y\zeta + x_3\nabla_y^2\varphi & \begin{matrix} g_1 \\ g_2 \end{matrix} \\ (g_1,\ g_2) & g_3 \end{pmatrix}.$$

For a simpler definition of Q_2^{rel}, see Remark 5.

Remark 1. It can be easily seen by using Korn's inequality that for $\zeta \in \mathring{H}^1(\mathcal{Y},\mathbb{R}^2)$, $\varphi \in \mathring{H}^2(\mathcal{Y})$, $g \in L^2(I\times\mathcal{Y},\mathbb{R}^3)$ we have

$$\|\zeta\|_{H^1} + \|\varphi\|_{H^2} + \|g\|_{L^2} \lesssim \|\mathcal{U}(\zeta,\varphi,g)\|_{L^2}. \tag{9}$$

We define the constraint on the deformation to be an isometry:

$$\partial_\alpha u \cdot \partial_\beta u = \delta_{\alpha\beta}, \qquad \alpha,\beta \in \{1,2\}. \tag{10}$$

We define the set of isometries of S into $\mathbb{R}^3$

$$H^2_\delta(S,\mathbb{R}^3) \quad := \quad \Big\{ u \in H^2(S,\mathbb{R}^3) \ :\ u \text{ satisfies (10) a.e. in } S \Big\}. \tag{11}$$

For given map $u \in H^2_\delta(S)$ we define the normal as $n := \partial_1 u \wedge \partial_2 u$, and we define its second fundamental form $\mathrm{II} : S \to \mathbb{M}^2_{\mathrm{sym}}$ by defining its entries as

$$\mathrm{II}_{\alpha\beta} = \partial_\alpha u \cdot \partial_\beta n = -\partial_\alpha\partial_\beta u \cdot n. \tag{12}$$

We write II^h and n^h for the second fundamental form and normal associated with some $u^h \in H^2_\delta(S,\mathbb{R}^3)$. The Γ-limit is a functional of the form (5) trivially extended to $L^2(\Omega,\mathbb{R}^3)$ by infinity: we define $\mathcal{E} : L^2(\Omega,\mathbb{R}^3) \to [0,\infty]$,

$$\mathcal{E}(u) := \begin{cases} \displaystyle\int_S Q_2^{\mathrm{rel}}(\mathrm{II}(x'))\, dx' & \text{if } u \in H^2_\delta(S,\mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

We identify functions on S with their trivial extension to $\Omega = S\times I$: above $u \in H^2_\delta(S,\mathbb{R}^3)$ means that $u(x',x_3) = \overline{u}(x') := \int_I u(x',z)\, dz$ for almost every $x_3 \in I$, and $\overline{u} \in H^2_\delta(S,\mathbb{R}^3)$. Our main result is the following:

Theorem 2.4. *Suppose that Assumptions 2.1 and 2.2 are satisfied. Then:*

(i) *(Lower bound). If $\{u^h\}_{h>0}$ is a sequence with $u^h - \fint_\Omega u^h\,dx \to u$ in $L^2(\Omega,\mathbb{R}^3)$, then*

$$\liminf_{h\to 0} h^{-2}\mathcal{E}^{h,\varepsilon(h)}(u^h) \geq \mathcal{E}(u).$$

(ii) *(Upper bound). For every $u \in H^2_\delta(S,\mathbb{R}^3)$ there exists a sequence $\{u^h\}_{h>0}$ with $u^h \to u$ strongly in $H^1(\Omega,\mathbb{R}^3)$ such that*

$$\lim_{h\to 0} h^{-2}\mathcal{E}^{h,\varepsilon(h)}(u^h) = \mathcal{E}(u).$$

The first step for identifying the Γ-limit is the compactness result. It gives us the information on the limit deformations that can be concluded from the smallness of energy. The result is given in [FJM02] and relies on the theorem on geometric rigidity which is the key mathematical indegredient for deriving lower dimensional models

Theorem 2.5 ([FJM02, Theorem 4.1])**.** *Let $(u^h)_{h>0} \subset H^1(\Omega,\mathbb{R}^3)$ be such that*

$$\limsup_{h\to 0} \frac{1}{h^2}\int_\Omega \operatorname{dist}^2(\nabla_h u^h(x), \mathrm{SO}(3))\,dx < \infty.$$

Then there exists a map $u \in H^2_\delta(S,\mathbb{R}^3)$ satisfying

$$\begin{aligned} u^h - \fint_\Omega u^h\,dx &\to u, && \text{strongly in } L^2(\Omega,\mathbb{R}^3),\\ \nabla_h u^h &\to (\,\nabla' u,\, n\,) && \text{strongly in } L^2(\Omega,\mathbb{R}^{3\times 3}), \end{aligned}$$

as $h \to 0$ after passing to subsequences and extending u and n trivially to Ω.

Remark 2. We could also consider the energy densities satisfying additionally on the macroscopic variable $x' \in \omega$ i.e. look the energy functionals of the form

$$\frac{1}{h}\int_{\Omega_h} W(z', \frac{z_3}{h}, \frac{z'}{\varepsilon}, \nabla v^h(z))\,dz. \tag{13}$$

This changes the relaxation formula in an obvious way, but creates some additional technical considerations, see [NV].

3. Two-scale limits of the nonlinear strain

Two-scale convergence was introduced in [Ngu89, All92] and has been extensively applied to various problems in homogenization. This is mainly related to convex energies for which it is known that the oscillation that relax the energy are of the same type as the oscillations of the material. In the non convex case more complex behavior is expected. In this article we work with the following variant of two-scale convergence which is adapted to dimension reduction.

Definition 3.1 (two-scale convergence)**.** We say a bounded sequence $\{f^h\}_{h>0}$ in $L^2(\Omega)$ two-scale converges to $f \in L^2(\Omega\times Y)$ and we write $f^h \xrightharpoonup{2,0} f$, if

$$\lim_{h\to 0}\int_\Omega f^h(x)\psi(x, \frac{x'}{\varepsilon(h)})\,dx = \iint_{\Omega\times Y} f(x,y)\psi(x,y)\,dy\,dx$$

for all $\psi \in C_0^\infty(\Omega, C(\mathcal{Y}))$. When $||f^h||_{L^2(\Omega)} \to ||f||_{L^2(\Omega\times Y)}$ in addition, we say that f^h strongly two-scale converges to f and write $f^h \xrightarrow{2,0} f$. For vector-valued functions, two-scale convergence is defined componentwise.

As it is seen in the definition we allow only oscillations in variable x'. It can be easily seen that this restriction does not influence the main results of two scale convergence. Moreover, since we identify functions on S with their trivial extension to Ω, the definition above contains the standard notion of two-scale convergence on $S \times Y$ as a special case.

Indeed, when $\{f^h\}_{h>0}$ is a bounded sequence in $L^2(S)$, then $f^h \xrightharpoonup{2,0} f$ is equivalent to

$$\lim_{h\to 0}\int_S f^h(x')\psi(x', \frac{x'}{\varepsilon(h)})\,dx' = \iint_{S\times Y} f(x',y)\psi(x',y)\,dy\,dx'$$

for all $\psi \in C_0^\infty(S, C(\mathcal{Y}))$.

Since the limit energy is expected to be convex in strain, one needs to identify the two scale limit of the strain. However, the strain itself is not a convex function of the gradient and it is not a-priori guaranteed that the two scale analysis will be enough to obtain the two scale limit of the strain itself. However in this regime, as well as in the regimes studied in [HNV], it is enough to include only the oscillations that follow the oscillations of the material to obtain the two scale limit of the strain. We have the following characterization of the possible two-scale limits of nonlinear strains.

Proposition 3.2. *Let $(u^h)_{h>0}$ be a sequence of deformations with finite bending energy, let $u \in H^2_\delta(S,\mathbb{R}^3)$ with the second fundamental form II, and assume that*

$$u^h - \fint_\Omega u^h\,dx \to u \qquad \textit{strongly in } L^2(\Omega,\mathbb{R}^3),$$

$$E^h := \frac{\sqrt{(\nabla_h u^h)^t\nabla_h u^h} - I}{h} \xrightharpoonup{2,0} E \qquad \textit{weakly two-scale}$$

for some $E \in L^2(\Omega\times Y;\mathbb{M}^3)$. Then there exist $B \in L^2(S,\mathbb{M}^2_{\mathrm{sym}})$, and $\zeta \in L^2(S, \mathring{H}^1(I\times\mathcal{Y},\mathbb{R}^2))$, $\varphi \in L^2(S,\mathring{H}^2(\mathcal{Y}))$ and $g \in L^2(S, L^2(I\times\mathcal{Y},\mathbb{R}^3))$ such that

$$E(x,y) = \iota\left(x_3\,\mathrm{II}(x') + B(x')\right) + \mathcal{U}\big(\zeta(x,\cdot),\varphi(x,\cdot),g(x,\cdot,\cdot)\big)(x_3,y). \tag{14}$$

The starting point of the proof of the previous Proposition is [FJM06, Theorem 6] (see also the proof of [FJM02, Theorem 4.1].

Lemma 3.3. *There exist constants $C, c > 0$, depending only on S, such that the following is true: if $u \in H^1(\Omega,\mathbb{R}^3)$ then there exists a map $R : S \to SO(3)$ which is piecewise constant on each cube $x + hY$ with $x \in h\mathbb{Z}^2$ and there exists $\widetilde{R} \in H^1(S,\mathbb{M}^3)$ such that for each $\xi \in \mathbb{R}^2$ which satisfy $|\xi|_\infty = \max\{|\xi\cdot e_1|, |\xi\cdot e_2|\} < h$ and for each $S' \subset S$ which satisfy $\mathrm{dist}(S',\partial S) > ch$ we have:*

$$\|\nabla_h u - R\|^2_{L^2(\Omega)} + \|R-\widetilde{R}\|^2_{L^2(S)} + h^2\|R-\widetilde{R}\|^2_{L^\infty(S)} + h^2\|\nabla'\widetilde{R}\|^2_{L^2(S)}$$
$$+\|\tau_\xi R - R\|^2_{L^2(S')} \le C\|\,\mathrm{dist}(\nabla_h u, \mathrm{SO}(3))\|^2_{L^2(\Omega)},$$

where $(\tau_\xi R)(x') := R(x' + \xi)$.

Let us recall some well-known properties of two-scale convergence. The following lemma is already stated in [HNV] . We also refer to [All92, Vis06, MT07] for proofs in the standard two-scale setting and to [Neu10] for the adaption to the notion of two-scale convergence considered here.

Lemma 3.4. *(i) Any sequence that is bounded in* $L^2(\Omega)$ *admits a two-scale convergent subsequence.*

(ii) Let $\widetilde{f} \in L^2(\Omega \times Y)$ *and let* $(f^h)_{h>0} \subset L^2(\Omega)$ *be such that* $f^h \xrightharpoonup{2,0} \widetilde{f}$. *Then* $f^h \rightharpoonup \int_Y \widetilde{f}(\cdot, y)\, dy$ *weakly in* $L^2(\Omega)$.

(iii) Let $f^0 \in L^2(\Omega)$ *and* $(f^h)_{h>0} \subset L^2(\Omega)$ *be such that* $f^h \rightharpoonup f^0$ *weakly in* $L^2(\Omega)$. *Then (after passing to subsequences) we have* $f^h \xrightharpoonup{2,0} f^0(x) + \widetilde{f}$ *for some* $\widetilde{f} \in L^2(\Omega \times Y)$ *with* $\int_Y \widetilde{f}(\cdot, y)\, dy = 0$ *almost everywhere in* S. $\widetilde{f}$ *is uniquely characterized by the fact that* $\int_S f^h \psi(x, \frac{x}{\varepsilon(h)})\, \mathrm{d}x \to \int_S \int_{\mathcal{Y}} \widetilde{f}(x, y) \psi(x, y)\, \mathrm{d}y\, \mathrm{d}x$, *for every* $\psi \in C_0^\infty(S, \mathring{C}^\infty(\mathcal{Y}))$.

(iv) Let $f^0 \in H^1(\Omega)$ *and* $(f^h)_{h>0} \subset H^1(\Omega)$ *be such that* $f^h \to f^0$ *strongly in* $L^2(\Omega)$. *Then* $f^h \xrightarrow{2,0} f^0$, *where we extend* f^0 *trivially to* $\Omega \times Y$.

(v) Let f^0 *and* $f^h \in H^1(S)$ *be such that* $f^h \rightharpoonup f^0$ *weakly in* $H^1(S)$. *Then (after passing to subsequences)*

$$\nabla' f^h \xrightharpoonup{2,0} \nabla' f^0 + \nabla_y \phi$$

for some $\phi \in L^2(S, H^1(\mathcal{Y}))$.

For the proof of the following lemma see [Neu10, Theorem 6.3.3].

Lemma 3.5. *Let* $u^0 \in H^1(\Omega, \mathbb{R}^3)$ *and* $(u^h)_{h>0} \subset H^1(\Omega, \mathbb{R}^3)$ *be such that* $u^h \rightharpoonup u^0$ *weakly in* $H^1(\Omega, \mathbb{R}^3)$ *and* $\liminf_{h\to 0} \|\nabla_h u^h\|_{L^2} < \infty$. *Under the assumption* $\lim_{h\to 0} \frac{h}{\varepsilon(h)} = 0$ *there exist* $\phi \in L^2(S, \mathring{H}^1(\mathcal{Y}, \mathbb{R}^3))$, $d \in L^2(S, L^2(I \times \mathcal{Y}, \mathbb{R}^3))$ *such that (after passing to subsequences)*

$$\nabla_h u^h \xrightharpoonup{2,0} (\nabla' u^0\, ,\, 0) + (\nabla_y \phi\, ,\, d). \tag{15}$$

Here u^0 *is the weak limit of* u^h *i.e.* $\int_I u^h$ *in* $H^1(\Omega, \mathbb{R}^3)$ *i.e.* $H^1(S, \mathbb{R}^3)$.

The following two lemmas are proved in [Vel].

Lemma 3.6. *Let* $(u^h)_{h>0}$ *be a bounded sequence in* $L^2(\Omega)$ *which two-scale converges to* $u_0(x, y) \in L^2(\Omega \times \mathcal{Y})$. *Let* $(v^h)_{h>0}$ *be a sequence bounded in* $L^\infty(\Omega)$ *which converges in measure to* $v_0 \in L^\infty(\Omega)$. *Then* $v^h u^h \xrightharpoonup{2,0} v_0(x) u_0(x, y)$.

Lemma 3.7. *Let* $(u^h)_{h>0}$ *be a sequence which converges strongly to* u *in* $H^1(\Omega)$ *and* $(v^h)_{h>0}$ *be a sequence which is bounded in* $H^1(\Omega, \mathbb{R}^n)$ *such that for each* $h > 0$

$$\|\nabla u^h - v^h\|_{L^2} \leq C \eta(h), \tag{16}$$

for some $C > 0$ *and* $\eta(h)$ *which satisfies* $\lim_{h\to 0} \frac{\eta(h)}{\varepsilon(h)} = 0$. *Then for any subsequence of* $(\nabla v^h)_{h>0}$ *which converges two scale there exists a unique* $v \in L^2(\Omega, \mathring{H}^2(\mathcal{Y}))$ *such that* $\nabla v^h \xrightharpoonup{2,0} \nabla^2 u(x) + \nabla_y^2 v(x, y)$.

We prove the following lemma.

Lemma 3.8. *Let* $(R^h)_{h>0} \subset L^\infty(S, \mathrm{SO}(3))$ *and* $(\widetilde{R}^h)_{h>0} \subset H^1(S, \mathbb{M}^3)$ *satisfy for each* $h > 0$

$$\|\widetilde{R}^h - R^h\|_{L^2} \leq C\eta(h), \ \ \|\nabla' \widetilde{R}^h\|_{L^2} \leq C, \|\widetilde{R}^h\|_{L^\infty} \leq C \tag{17}$$

where $C > 0$ *is independent of* h *and* $\eta(h)$ *satisfies* $\lim_{h\to 0} \frac{\eta(h)}{\varepsilon(h)} = 0$. *Then for any subsequence of* $(\nabla' \widetilde{R}^h)_{h>0}$ *which converges two scale there exists a unique* $w \in L^2(S, \mathring{H}^1(\mathcal{Y}, \mathbb{R}^3))$ *such that* $\partial_\alpha \widetilde{R}^h \xrightarrow{2,0} \partial_\alpha R + RA_{\partial_{y_\alpha} w(x,y)}$, *where* $R \in H^1(S, \mathrm{SO}(3))$ *is the weak limit of* $\widetilde{R}^h$.

Proof. We have to prove only that

$$M^h_\alpha := \mathrm{sym}[(\widetilde{R}^h)^T \partial_\alpha \widetilde{R}^h] \xrightarrow{2,0} \mathrm{sym}(R^T \partial_\alpha R) = 0, \ \text{for } \alpha = 1, 2. \tag{18}$$

The rest is a direct consequence of (v) in Lemma 3.4. Namely, let us assume (18). Then we conclude that for $\alpha = 1, 2$ there exists $\widetilde{w}_\alpha \in L^2(S, \mathring{H}^1(\mathcal{Y}, \mathbb{R}^3))$ such that

$$(\widetilde{R}^h)^T \partial_\alpha \widetilde{R}^h \xrightarrow{2,0} R^T \partial_\alpha R + A_{\widetilde{w}_\alpha}. \tag{19}$$

Using the fact that $\widetilde{R}^h \to R$, boundedly in measure and Lemma 3.6 we conclude from (19) that

$$\partial_\alpha \widetilde{R}^h \xrightarrow{2,0} \partial_\alpha R + RA_{\widetilde{w}_\alpha(x,y)}. \tag{20}$$

By using (v) of Lemma 3.4 we can also conclude that there exists $M \in L^2(S, \mathring{H}^1(\mathcal{Y}, \mathbb{M}^3))$ such that

$$\partial_\alpha \widetilde{R}^h \xrightarrow{2,0} \partial_\alpha R + \partial_{y_\alpha} M. \tag{21}$$

From (20) and (21) we conclude that

$$A_{\widetilde{w}_\alpha(x,y)} = \partial_{y_\alpha}(R^T M),$$

which implies that $\widetilde{w}_\alpha = \partial_\alpha w$, where w is the axial vector of $\mathrm{skw}(R^T M)$.

It remains to prove (18). Notice that

$$M^h_{\alpha,ij} = \tfrac{1}{2}\partial_\alpha(\widetilde{R}^h_i \cdot \widetilde{R}^h_j). \tag{22}$$

Take $\psi \in C^\infty_0(S, \mathring{C}^\infty(\mathcal{Y}; \mathbb{R}^3))$ and calculate

$$\begin{aligned}
\lim_{h\to 0} \int_S M^h_{\alpha,ij} \psi(\cdot, \tfrac{\cdot}{\varepsilon(h)})\, dx &= \lim_{h\to 0} \int_S \tfrac{1}{2}(\widetilde{R}^h_i \cdot \widetilde{R}^h_j)\partial_{x_\alpha}\psi(\cdot, \tfrac{\cdot}{\varepsilon(h)}) \\
&\quad + \lim_{h\to 0} \int_S \tfrac{1}{2\varepsilon(h)}(\widetilde{R}^h_i \cdot \widetilde{R}^h_j)\partial_{y_\alpha}\psi(\cdot, \tfrac{\cdot}{\varepsilon(h)}) \\
\begin{array}{c}\text{using (17) and}\\ \text{Cauchy inequality}\end{array} &= \lim_{h\to 0} \int_S \tfrac{1}{2}(R^h_i \cdot R^h_j)\partial_{x_\alpha}\psi(\cdot, \tfrac{\cdot}{\varepsilon(h)}) \\
&\quad + \lim_{h\to 0} \int_S \tfrac{1}{2\varepsilon(h)}(R^h_i \cdot R^h_j)\partial_{y_\alpha}\psi(\cdot, \tfrac{\cdot}{\varepsilon(h)}) \\
&= \int_S \partial_\alpha(\delta_{ij})\psi(\cdot, \tfrac{\cdot}{\varepsilon(h)}) \\
&= 0.
\end{aligned}$$

This together with (iii) of Lemma 3.4 implies (18). $\square$

Lemma 3.9. *Let $(\overline{u}^h)_{h>0} \subset H^1(S,\mathbb{R}^3)$, $(R^h)_{h>0} \subset L^\infty(S,\mathrm{SO}(3))$ and $(\widetilde{R}^h)_{h>0} \subset H^1(S,\mathbb{M}^3)$ satisfy for each $h>0$:*

$$\begin{aligned} &\|\nabla'\overline{u}^h - (R^h e_1, R^h e_2)\|_{L^2} + \|R^h - \widetilde{R}^h\|_{L^2} + \|\nabla'\widetilde{R}^h\|_{L^2} \leq C\eta(h), \\ &\|\widetilde{R}^h\|_{L^\infty} \leq C \end{aligned} \tag{23}$$

where $C>0$ is independent of h and $\eta(h)$ satisfies $\lim_{h\to 0}\frac{\eta(h)}{\varepsilon(h)} = 0$. Then for any subsequence of $(\nabla'\widetilde{R}^h)_{h>0}$ which converges two scale there exists a unique $w \in L^2(S,\mathring{H}^2(\mathcal{Y}))$ such that

$$\partial_\alpha \widetilde{R}^h \xrightharpoonup{2,0} \partial_\alpha R + R \begin{pmatrix} 0 & 0 & -\partial_{y_1 y_\alpha} w \\ 0 & 0 & -\partial_{y_2 y_\alpha} w \\ \partial_{y_1 y_\alpha} w & \partial_{y_2 y_\alpha} w & 0 \end{pmatrix}, \tag{24}$$

where R is the weak limit of $\widetilde{R}^h$ in H^1.

Proof. From (23) we conclude that there exists $R: S \to \mathrm{SO}(3)$ and $u \in H^2(S,\mathbb{R}^3)$ such that $Re_\alpha = \partial_\alpha u$, for $\alpha = 1,2$, $\widetilde{R}^h \rightharpoonup R$ weakly in H^1, $\partial_\alpha \overline{u}^h \to Re_\alpha$, $R^h \to R$ strongly in L^2. Also from (23) and Lemma 3.8 we have that there exists $\widetilde{w} \in L^2(S,\mathring{H}^1(\mathcal{Y},\mathbb{R}^3))$ such that for $\alpha = 1,2$

$$\partial_\alpha \widetilde{R}^h \xrightharpoonup{2,0} \partial_\alpha R + RA_{\partial_{y_\alpha}\widetilde{w}}. \tag{25}$$

Using Lemma 3.7 we conclude that there exists $v \in L^2(S,\mathring{H}^2(\mathcal{Y},\mathbb{R}^3))$ such that

$$R \begin{pmatrix} 0 & -\partial_{y_\alpha}\widetilde{w}_3 \\ \partial_{y_\alpha}\widetilde{w}_3 & 0 \\ -\partial_{y_\alpha}\widetilde{w}_2 & \partial_{y_\alpha}\widetilde{w}_1 \end{pmatrix} = \begin{pmatrix} \partial_{y_1 y_\alpha} v_1 & \partial_{y_2 y_\alpha} v_1 \\ \partial_{y_1 y_\alpha} v_2 & \partial_{y_2 y_\alpha} v_2 \\ \partial_{y_1 y_\alpha} v_3 & \partial_{y_2 y_\alpha} v_3 \end{pmatrix}. \tag{26}$$

By putting $\widetilde{v} = R^T v$ we have that for $\alpha = 1,2$

$$\begin{pmatrix} 0 & -\partial_{y_\alpha}\widetilde{w}_3 \\ \partial_{y_\alpha}\widetilde{w}_3 & 0 \\ -\partial_{y_\alpha}\widetilde{w}_2 & \partial_{y_\alpha}\widetilde{w}_1 \end{pmatrix} = \begin{pmatrix} \partial_{y_1 y_\alpha} \widetilde{v}_1 & \partial_{y_2 y_\alpha} \widetilde{v}_1 \\ \partial_{y_1 y_\alpha} \widetilde{v}_2 & \partial_{y_2 y_\alpha} \widetilde{v}_2 \\ \partial_{y_1 y_\alpha} \widetilde{v}_3 & \partial_{y_2 y_\alpha} \widetilde{v}_3 \end{pmatrix}. \tag{27}$$

From this one easily concludes that $\widetilde{w}_3 = 0$ which implies the claim by defining $w = \widetilde{v}_3$. $\square$

The following lemma was already used in [HV] and can be easily proved by e.g. Fourier transform.

Lemma 3.10. *Let $M \in L^2(S; L^2(\mathcal{Y},\mathbb{M}^2_{\mathrm{sym}})$ such that for every*

$$\Psi \in C_0^\infty(S, C^\infty(\mathcal{Y};\mathbb{M}^2_{\mathrm{sym}}))$$

which satisfies

$$\Psi(\cdot,y) = (\mathrm{cof}\,\nabla^2 F)(y)\psi(\cdot) \tag{28}$$

for some $\psi \in C_0^\infty(S)$, $F \in \mathring{C}^\infty(\mathcal{Y})$, we have that

$$\iint_{S\times\mathcal{Y}} M(\cdot,y) : \Psi(\cdot,y) = 0.$$

Then there exist unique $M_0 \in L^2(S, \mathbb{M}^2_{\rm sym})$ *and* $\zeta \in L^2(S, \mathring{H}^1(\mathcal{Y}; \mathbb{R}^2))$ *such that*

$$M = M_0 + \operatorname{sym} \nabla_y \zeta.$$

The following lemma is crucial for the proof of the Proposition 3.2.

Lemma 3.11. *Let Assumption 2.2 be satisfied. Let* $(\widetilde{u}^h)_{h>0} \subset H^2(S, \mathbb{R}^3)$, $(\widetilde{R}^h)_{h>0} \subset H^1(S, \mathbb{M}^3)$ *and* $(R^h)_{h>0} \subset L^\infty(S, \mathrm{SO}(3))$ *such that for each* $h > 0$ R^h *is piecewise constant on each cube* $x + hY$ *with* $x \in h\mathbb{Z}^2$ *and for each* $\xi \in \mathbb{R}^2$ *which satisfy* $|\xi|_\infty = \max\{|\xi \cdot e_1|, |\xi \cdot e_2|\} < h$ *we have*

$$\begin{aligned}(29)\quad & h^2\|\nabla'^2\widetilde{u}^h\|^2_{L^2} + \|\nabla'\widetilde{u}^h - (R^h e_1, R^h e_2)\|^2_{L^2(\Omega)} + \|R^h - \widetilde{R}^h\|^2_{L^2(S)} \\ & + h^2\|R^h - \widetilde{R}^h\|^2_{L^\infty(S)} + h^2\|\nabla'\widetilde{R}^h\|^2_{L^2(S)} + \|\tau_\xi R^h - R^h\|^2_{L^2(S^h)} \le Ch^2,\end{aligned}$$

for some $C > 0$ *and for each sequence of subdomains* $S^h \subset S$ *which satisfy* $\operatorname{dist}(S^h, \partial S) \ge ch$ *for some* $c > 0$. *Then there exist* $M_0 \in L^2(S, \mathbb{M}^2_{\rm sym})$ *and* $\zeta \in L^2(S, \mathring{H}^1(\mathcal{Y}, \mathbb{R}^2))$ *such that for* $\alpha, \beta = 1, 2$ *(on a subsequence) we have:*

$$\begin{aligned} M^h_{\alpha\beta} &:= \tfrac{1}{2h}[(R^h e_\alpha) \cdot \partial_\beta \widetilde{u}^h + (R^h e_\beta) \cdot \partial_\alpha \widetilde{u}^h] - \delta_{\alpha\beta} \\ &\xrightarrow{2,0} M_{0,\alpha\beta} + \tfrac{1}{2}(\partial_{y_\alpha}\zeta_\beta + \partial_{y_\beta}\zeta_\alpha). \end{aligned}$$

Proof. From (29) we can assume that there exists $u \in H^2(S, \mathbb{R}^3)$ and $R \in H^1(S, \mathrm{SO}(3))$ such that $\partial_\alpha u = R e_\alpha$, $\partial_\alpha \widetilde{u}^h \rightharpoonup \partial_\alpha u$ weakly in H^1 for $\alpha = 1, 2$, $\widetilde{R}^h \rightharpoonup R$ weakly in H^1 and $R^h \to R$ strongly in L^2. Let us suppose that $M^h \xrightarrow{2,0} M$, for some $M \in L^2(S \times \mathcal{Y}, \mathbb{M}^2)$. Using Lemma 3.10 it is enough to see that

$$\iint_{S \times \mathcal{Y}} M(\cdot, y) : \Psi(\cdot, y) = 0,$$

where Ψ is defined by (28). Let us observe

$$\begin{aligned}
&\iint_{S \times \mathcal{Y}} M(\cdot, y) : \Psi(\cdot, y) \\
&= \lim_{h \to 0} \int_S M^h : (\operatorname{cof} \nabla^2 F)\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi \\
&= \lim_{h \to 0} \tfrac{\varepsilon(h)^2}{h} \int_S hM^h : \operatorname{cof}\left[\nabla^2\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right) - 2\nabla\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\nabla\psi\right) + F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\nabla^2\psi\right] \\
&= \lim_{h \to 0} \tfrac{\varepsilon(h)^2}{h} \int_S hM^h : \operatorname{cof}\left[\nabla^2\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right) - 2\nabla\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\nabla\psi\right)\right].
\end{aligned}$$

It is easy to conclude that

$$\lim_{h \to 0} \tfrac{\varepsilon(h)^2}{h} \int_S hM^h : \operatorname{cof}\left[\nabla\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\nabla\psi\right)\right] = 0.$$

Namely, it is enough to conclude that the sequence

$$I^h := \int_S hM^h : \operatorname{cof}\left[\nabla\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\nabla\psi\right)\right],$$

is bounded. To see this notice that, because of (29) we have $|I^h - \widetilde{I}^h| \to 0$, where

$$\widetilde{I}^h := \int_S M_c^h : \mathrm{cof}\left[\nabla\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\nabla\psi\right)\right], \tag{30}$$

and

$$M_c^h := \tfrac{1}{2}[(\widetilde{R}^h e_\alpha)\cdot\partial_\beta \widetilde{u}^h + (\widetilde{R}^h e_\beta)\cdot\partial_\alpha \widetilde{u}^h].$$

By partial integration in (30) and the fact that $\|\nabla M_c^h\|_{L^1(S)}$ is bounded we easily obtain the boundedness of $\widetilde{I}^h$. From this it follows the boundedness of I^h. It remains to prove that

$$\lim_{h\to 0} \tfrac{\varepsilon(h)^2}{h}\int_S hM^h : \mathrm{cof}\,\nabla^2\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right) = 0. \tag{31}$$

By partial integration we obtain for h small enough:

(32)

$$\begin{aligned}
&\int_S hM^h : \mathrm{cof}\,\nabla^2\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right) = \\
&\int_S (R^h e_2)\cdot\partial_{11}\widetilde{u}^h\partial_2\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right) - \int_S (R^h e_2)\cdot\partial_{12}\widetilde{u}^h\partial_1\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right)\\
&+\sum_{x'\in h\mathbb{Z}^2\cap S^h}\Big((R^h e_1)(x') - (R^h e_1)(x'+h(0,1))\Big)\cdot\left(\int_{\Gamma^{2,h}_{x'}}\partial_1\widetilde{u}^h\partial_2\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right)\right)\\
&-\sum_{x'\in h\mathbb{Z}^2\cap S^h}\Big((R^h e_1)(x') - (R^h e_1)(x'+h(1,0))\Big)\cdot\left(\int_{\Gamma^{1,h}_{x'}}\partial_2\widetilde{u}^h\partial_2\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right)\right)\\
&-\sum_{x'\in h\mathbb{Z}^2\cap S^h}\Big((R^h e_2)(x') - (R^h e_2)(x'+h(1,0))\Big)\cdot\left(\int_{\Gamma^{1,h}_{x'}}\partial_1\widetilde{u}^h\partial_2\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right)\right)\\
&+\sum_{x'\in h\mathbb{Z}^2\cap S^h}\Big((R^h e_2)(x') - (R^h e_2)(x'+h(1,0))\Big)\cdot\left(\int_{\Gamma^{1,h}_{x'}}\partial_2\widetilde{u}^h\partial_1\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right)\right),
\end{aligned}$$

where $\Gamma^{1,h}_{x'}$ is the segment $[x'+h(1,0), x'+h(1,1)]$, $\Gamma^{2,h}_{x'}$ is the segment $[x'+h(0,1), x'+h(1,1)]$ and S^h is a compact subset of S such that $\mathrm{spt}\,\psi\subset S^h$. First we will prove that $\lim_{h\to 0}\frac{\varepsilon(h)^2}{h} I_1^h = 0$, where

$$I_1^h = \int_S (R^h e_2)\cdot\partial_{11}\widetilde{u}^h\partial_2\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right) - \int_S (R^h e_2)\cdot\partial_{12}\widetilde{u}^h\partial_1\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right).$$

To prove this it is enough to prove the boundedness of the sequence I_1^h. Notice, as before, using (29) and Cauchy inequality that $|I_1^h - \widetilde{I}_1^h| \to 0$, where

$$\widetilde{I}_1^h = \int_S (\widetilde{R}^h e_2)\cdot\partial_{11}\widetilde{u}^h\partial_2\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right) - \int_S (\widetilde{R}^h e_2)\cdot\partial_{12}\widetilde{u}^h\partial_1\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right).$$

By replacing $\widetilde{u}^h$ with a smooth function $\widetilde{u}_c^h \in C^3(S)$ such that $\|\widetilde{u}^h - \widetilde{u}_c^h\|_{H^2} \ll \varepsilon(h)$ we obtain, after partial integration, that $|\widetilde{I}_1^h - \widetilde{I}_{1,c}^h| \to 0$, where

$$\widetilde{I}_{1,c}^h = -\int_S \partial_2(\widetilde{R}^h e_2)\cdot\partial_{11}\widetilde{u}_c^h F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi + \int_S \partial_1(\widetilde{R}^h e_2)\cdot\partial_{12}\widetilde{u}_c^h F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi.$$

Now we easily obtains the boundedness of $\widetilde{I}^h_{1,c}$ which implies the boundedness of I^h_1. We want to show that $\lim_{h\to 0}\frac{\varepsilon(h)^2}{h} I^h_2 = 0$, where

$$I^h_2 := \sum_{x'\in h\mathbb{Z}^2\cap S^h} \Big((R^h e_1)(x') - (R^h e_1)(x'+h(0,1))\Big)\cdot \left(\int_{\Gamma^{2,h}_{x'}} \partial_1 \widetilde{u}^h \partial_2 \left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right)\right).$$

We will prove even more that $I^h_2 \to 0$. Since $\widetilde{u}^h \in H^2(\Omega^h_{x'},\mathbb{R}^3)$ we have that

$$\int_{\Gamma^{2,h}_{x'}} \left|\partial_1 \widetilde{u}^h - \tfrac{1}{h}\int_{\Gamma^{1,h}_{\cdot}} \partial_1 \widetilde{u}^h\right|^2 \leq \tfrac{h}{3}\int_{\Omega^h_{x'}} |\partial_{12}\widetilde{u}^h|^2, \tag{33}$$

where for $x \in \Gamma^{2,h}_{x'}$ we put $\Gamma^{1,h}_x = [x - h(0,1), x]$ and $\Omega^h_{x'}$ is the square of side h whose left corner is x'. From (29) we easily conclude that for $\alpha = 1,2$ and $\xi \in \mathbb{R}^2$, $|\xi|_\infty = 1$ we have

$$\sum_{x'\in h\mathbb{Z}^2\cap S^h} \Big((R^h e_\alpha)(x') - (R^h e_\alpha)(x'+h\xi)\Big)^2 \leq C, \tag{34}$$

Using Cauchy inequality and (33), (34) we conclude that $|I^h_2 - \widetilde{I}^h_2| \to 0$, where

$$\widetilde{I}^h_2 := \sum_{x'\in h\mathbb{Z}^2\cap S^h} \Big((R^h e_1)(x') - (R^h e_1)(x'+h(0,1))\Big)\cdot \left(\frac{1}{h}\int_{\Omega^h_{x'}} \partial_1 \widetilde{u}^h \partial_2 \left(F\left(\frac{P^h_{x'}(\cdot)}{\varepsilon(h)}\right)\psi\left(P^h_{x'}(\cdot)\right)\right)\right),$$

and $P^h_{x'}:\Omega^h_{x'} \to \Gamma^{2,h}_{x'}$ is the projection. From (29) and Cauchy inequality we can easily conclude that $|\widetilde{I}^h_2 - \widetilde{I}^h_{2,c}| \to 0$, where

$$\widetilde{I}^h_{2,c} := \sum_{x'\in h\mathbb{Z}^2\cap S^h} \Big((R^h e_1)(x') - (R^h e_1)(x'+h(0,1))\Big)\cdot (R^h e_1)(x')\int_{\Gamma^{2,h}_{x'}} \partial_2\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right).$$

By using (34) we easily obtain that $|\widetilde{I}^h_{2,c} - \widetilde{I}^h_{2,cc}| \to 0$, where

$$\widetilde{I}^h_{2,cc} := \sum_{x'\in h\mathbb{Z}^2\cap S^h} \Big((R^h e_1)(x') - (R^h e_1)(x'+h(0,1))\Big)\cdot \tfrac{1}{2}\Big((R^h e_1)(x') + (R^h e_1)(x'+h(0,1))\Big)\int_{\Gamma^{2,h}_{x'}} \partial_2\left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right) = 0.$$

This implies that $I^h_2 \to 0$. In the same way we can conclude that $I^h_4 \to 0$, where

$$I^h_4 := \sum_{x'\in h\mathbb{Z}^2\cap S^h} \Big((R^h e_2)(x') - (R^h e_2)(x'+h(1,0))\Big)\cdot \left(\int_{\Gamma^{1,h}_{x'}} \partial_2 \widetilde{u}^h \partial_1 \left(F\left(\tfrac{\cdot}{\varepsilon(h)}\right)\psi\right)\right)$$

It remains to check the part

$$
\begin{aligned}
I_3^h := \sum_{x' \in h\mathbb{Z}^2 \cap S^h} & \Big((R^h e_1)(x') - (R^h e_1)(x' + h(1,0)) \Big) \cdot \\
& \left(\int_{\Gamma^{1,h}_{x'}} \partial_2 \widetilde{u}^h \partial_2 \left(F \left(\tfrac{\cdot}{\varepsilon(h)} \right) \psi \right) \right) \\
- \sum_{x' \in h\mathbb{Z}^2 \cap S^h} & \Big((R^h e_2)(x') - (R^h e_2)(x' + h(1,0)) \Big) \cdot \\
& \left(\int_{\Gamma^{1,h}_{x'}} \partial_1 \widetilde{u}^h \partial_2 \left(F \left(\tfrac{\cdot}{\varepsilon(h)} \right) \psi \right) \right)
\end{aligned}
$$

We follow the same pattern to replace $\partial_1 \widetilde{u}^h$ i.e. $\partial_2 \widetilde{u}^h$ by $R^h e_1$ i.e. $R^h e_2$ and obtain that $|I_3^h - I_{3,c}^h| \to 0$, where

$$
\begin{aligned}
I_{3,c}^h := \sum_{x' \in h\mathbb{Z}^2 \cap S^h} & \Big((R^h e_1)(x') - (R^h e_1)(x' + h(1,0)) \Big) \cdot \\
& (R^h e_2)(x') \int_{\Gamma^{1,h}_{x'}} \partial_2 \left(F \left(\tfrac{\cdot}{\varepsilon(h)} \right) \psi \right) \\
+ \sum_{x' \in h\mathbb{Z}^2 \cap S^h} & \Big((R^h e_2)(x') - (R^h e_2)(x' + h(1,0)) \Big) \cdot \\
& (R^h e_1)(x') \int_{\Gamma^{1,h}_{x'}} \partial_2 \left(F \left(\tfrac{\cdot}{\varepsilon(h)} \right) \psi \right).
\end{aligned}
$$

Using again (34) we easily obtain that $|I_{3,c}^h - I_{3,cc}^h| \to 0$, where

$$
\begin{aligned}
I_{3,cc}^h \quad := \quad & \sum_{x' \in h\mathbb{Z}^2 \cap S^h} \Big((R^h e_1)(x') - (R^h e_1)(x' + h(1,0)) \Big) \cdot \\
& \quad \tfrac{1}{2} \Big((R^h e_2)(x') + (R^h e_2)(x' + h(1,0)) \Big) \int_{\Gamma^{1,h}_{x'}} \partial_2 \left(F \left(\tfrac{\cdot}{\varepsilon(h)} \right) \psi \right) \\
& + \sum_{x' \in h\mathbb{Z}^2 \cap S^h} \Big((R^h e_2)(x') - (R^h e_2)(x' + h(1,0)) \Big) \cdot \\
& \quad \tfrac{1}{2} \Big((R^h e_1)(x') + (R^h e_1)(x' + h(1,0)) \Big) \int_{\Gamma^{1,h}_{x'}} \partial_2 \left(F \left(\tfrac{\cdot}{\varepsilon(h)} \right) \psi \right) \\
= \quad & 0.
\end{aligned}
$$

This finishes the proof of the claim. □

Remark 3. In the case $\varepsilon(h)^2 \sim h$ the major difficulty is to deal with the expression $\widetilde{I}^h_{1,c}$, since under the integral we have the product of two sequences which are only bounded in L^2.

For proving the Proposition 3.2 we need the following remark, the same as we needed in the proof of [NV, Proposition 3.1].

Remark 4. Assume that $\widetilde{S} \subset \mathbb{R}^2$ a bounded domain. Let us look the following minimization problem

$$\min_{\substack{v \in H^1(\widetilde{S}) \\ \int_{\widetilde{S}} v = 0}} \int_{\widetilde{S}} |\nabla v - p|^2 \, dx',$$

where $p \in H^1(\widetilde{S}, \mathbb{R}^2)$ is a given field. The associated Euler-Lagrange equation reads

$$\begin{cases} -\triangle v = \; - \nabla \cdot p & \text{in } \widetilde{S} \\ \partial_\nu v = p \cdot \nu & \text{on } \partial \widetilde{S}, \end{cases}$$

subject to $\int_{\widetilde{S}} v \, dx = 0$. Above, ν denotes the normal on $\partial \widetilde{S}$. Since $\nabla \cdot p \in L^2$, we obtain by standard regularity estimates that $v \in H^2(\widetilde{S})$ under the assumption that $\partial \widetilde{S}$ is $C^{1,1}$. In this case it holds $\|v\|_{H^2(\widetilde{S})} \lesssim \|\nabla \cdot p\|_{L^2(\widetilde{S})} + \|p\|_{L^2(\widetilde{S})}$.

Proof of Proposition 3.2. Without loss of generality we assume that all u^h have average zero. Let R^h, $\widetilde{R}^h$ be the maps obtained by applying Lemma 3.3 to u^h. Due to the uniform bound on $\nabla' \widetilde{R}^h$ given by Lemma 3.3, R^h and $\widetilde{R}^h$ are precompact in $L^2(S, \mathbb{M}^3)$. Hence, R^h and $\widetilde{R}^h$ strongly converge in $L^2(S, \mathbb{M}^3)$ to $R \in H^1(S, \mathrm{SO}(3))$ on a subsequence. Also we can conclude that $u^h \to u$ strongly in $H^1(\Omega, \mathbb{R}^3)$ and $\nabla_h u^h \to R = (\nabla' u, \ n)$ strongly in $L^2(\Omega, \mathbb{M}^3)$, for $u \in H^2(S, \mathbb{R}^3)$. Take $\widetilde{S} \subset S$, open, such that $\partial \widetilde{S}$ is of class $C^{1,1}$ and define $\widetilde{\Omega} := \widetilde{S} \times I$, $\overline{u}^h = \int_I u^h(x', x_3) \, \mathrm{d}x_3$ and notice that

$$\|\nabla' \overline{u}^h - (\widetilde{R}^h e_1, \widetilde{R}^h e_2)\|_{L^2}^2 \leq C h^2, \tag{35}$$

for some $C > 0$. Define $\widetilde{u}^h \in H^2(\widetilde{S}, \mathbb{R}^3)$ such that $\widetilde{u}^h$ minimizes the problem

$$\min_{\substack{v \in H^1(\widetilde{S}, \mathbb{R}^3) \\ \int_{\widetilde{S}} v = 0}} \int_{\widetilde{S}} |\nabla v - (\widetilde{R}^h e_1, \widetilde{R}^h e_2)|^2 \, dx'.$$

From Remark 4 we conclude that there exists $C > 0$ such that

$$\|\widetilde{u}^h\|_{H^2(\widetilde{S})} \leq C, \quad \|\nabla' \widetilde{u}^h - (\widetilde{R}^h e_1, \widetilde{R}^h e_2)\|_{L^2(\widetilde{S})}^2 \leq C h^2,$$
$$\|\nabla' \widetilde{u}^h - \nabla' \overline{u}^h\|_{L^2(\widetilde{S})}^2 \leq C h^2.$$

Let us suppose that on a subsequence

$$\tfrac{1}{h} (R^h)^t (\nabla' \overline{u}^h - \nabla' \widetilde{u}^h) \xrightarrow{2,0} \theta(x') + \nabla_y v(x', y), \tag{36}$$

where $\theta \in L^2(\widetilde{S}, \mathbb{M}^{3 \times 2})$ and $v \in L^2(\widetilde{S}, \mathring{H}^1(\mathcal{Y}, \mathbb{R}^3))$. This can be done without loss of generality by using Lemma 3.4, Lemma 3.6 and Poincare inequality. Following [FJM02], we introduce the approximate strain

$$G^h(x) = \frac{(R^h)^t \nabla_h u^h(x) - I}{h}. \tag{37}$$

Since G^h is bounded in $L^2(\Omega)$ we can assume that $G^h \xrightarrow{2,0} G \in L^2(\Omega \times \mathcal{Y}, \mathbb{M}^3)$. First notice that it suffices to identify the symmetric part of the two-scale limit G of the sequence G^h. Indeed, since $\sqrt{(I + hF)^t (I + hF)} = I + h \, \mathrm{sym}\, F$ up to terms of higher order, the convergence $G^h \xrightarrow{2,0} G$ implies

$E = \operatorname{sym} G$ (see e.g. [Neu12, Lemma 4.4] for a proof). We define $z^h \in H^1(\Omega, \mathbb{R}^3)$ via

$$u^h(x', x_3) = \overline{u}^h(x') + h x_3 \widetilde{R}^h(x') e_3 + h z^h(x', x_3). \tag{38}$$

Then clearly $\int_I z^h(x', x_3) dx_3 = 0$ and we compute

(39)

$$\begin{aligned} G^h &= \frac{\iota\big((R^h e_1\,,\, R^h e_2)^t \nabla' \widetilde{u}^h - I'\big)}{h} + \frac{1}{h} \sum_{\alpha=1,2} (R^h e_3 \cdot \partial_\alpha \widetilde{u}^h) e_3 \otimes e_\alpha \\ &\quad + \frac{1}{h} (R^h)^t (\nabla' \overline{u}^h - \nabla' \widetilde{u}^h\,,\, 0) + \frac{1}{h} \Big((R^h)^t \widetilde{R}^h e_3 \otimes e_3 - (0\,,0\,,e_3) \Big) \\ &\quad + x_3 (R^h)^t (\nabla' \widetilde{R}^h e_3\,,\, 0) + (R^h)^t \nabla_h z^h, \end{aligned}$$

where by I' we have denoted the unit matrix in $\mathbb{M}^2$. By using Lemma 3.11 we conclude that there exist $\widetilde{B} \in L^2(\widetilde{S}, \mathbb{M}^2_{\mathrm{sym}})$, $\widetilde{\zeta} \in L^2(\widetilde{S}, \mathring{H}^1(\mathcal{Y}, \mathbb{R}^2))$ such that on a subsequence

$$\operatorname{sym} \frac{\big((R^h e_1\,,\ R^h e_2)^t \nabla' \widetilde{u}^h - I'\big)}{h} \xrightharpoonup{2,0} \widetilde{B}(x') + \operatorname{sym} \nabla_y \widetilde{\zeta}(x', y). \tag{40}$$

Using Lemma 3.6 and Lemma 3.9 we conclude that there exists $\varphi \in L^2(S, \mathring{H}^2(\mathcal{Y}))$ such that

$$x_3 (R^h)^t (\nabla' \widetilde{R}^h e_3) \xrightharpoonup{2,0} x_3 \nabla_y^2 \varphi(x', y). \tag{41}$$

Using Lemma 3.5 and Lemma 3.6 and the fact that $\int_I z^h = 0$ we conclude that there exists $\phi \in L^2(S, \mathring{H}^1(\mathcal{Y}, \mathbb{R}^3))$, $d \in L^2(S, L^2(I \times \mathcal{Y}, \mathbb{R}^3))$ such that on a subsequence

$$(R^h)^t \nabla_h z^h \xrightharpoonup{2,0} (\nabla_y \phi\,,\, d) \tag{42}$$

Without loss of generality we can assume that there exist $p \in L^2(\widetilde{S} \times \mathcal{Y}, \mathbb{R}^2)$ and $z \in L^2(S \times \mathcal{Y}; \mathbb{R}^3)$ such that

$$R^h e_3 \cdot \partial_\alpha \widetilde{u}^h \xrightharpoonup{2,0} p_\alpha(x', y), \quad \text{for } \alpha = 1, 2, \tag{43}$$

$$\frac{1}{h} \Big((R^h)^t \widetilde{R}^h e_3 - e_3 \Big) \xrightharpoonup{2,0} z(x', y) \tag{44}$$

Using (36) as well as (40)-(43) we conclude that there exists $\zeta \in L^2(\widetilde{\Omega}, \mathring{H}^1(\mathcal{Y}, \mathbb{R}^2))$, $\varphi \in L^2(\widetilde{S}, \mathring{H}^2(\mathcal{Y}))$ and $g \in L^2(\widetilde{\Omega} \times \mathcal{Y}, \mathbb{R}^3)$ such that

$$\begin{aligned} E(x, y) &= \iota\left(x_3 \,\mathrm{II}(x') + B(x')\right) + \mathcal{U}\big(\zeta(x, \cdot), \varphi(x, \cdot), g(x, \cdot, \cdot)\big)(x_3, y), \\ &\quad \forall (x, y) \in \widetilde{\Omega} \times \mathcal{Y}, \end{aligned} \tag{45}$$

where

$$\begin{aligned} B &= \widetilde{B} + \operatorname{sym} \sum_{\alpha,\beta=1,2} \theta_{\alpha\beta} e_\alpha \otimes e_\beta \\ \zeta &= \widetilde{\zeta} + \sum_{\alpha=1,2} (\phi_\alpha + v_\alpha) e_\alpha \\ g &= \tfrac{1}{2} \sum_{\alpha=1,2} (\partial_{y_\alpha} \phi_3 + \partial_{y_\alpha} v_3 + d_\alpha + \theta_{3\alpha} + p_\alpha + z_\alpha) e_\alpha + (d_3 + z_3) e_3 \end{aligned}$$

To obtain the representation (45) for all $(x,y) \in \Omega \times \mathcal{Y}$ and some $\zeta \in L^2(\Omega, \mathring{H}^1(\mathcal{Y},\mathbb{R}^2))$, $\varphi \in L^2(S, \mathring{H}^2(\mathcal{Y}))$ and $g \in L^2(\Omega \times \mathcal{Y}, \mathbb{R}^3)$ it is enough to use that $E \in L^2(\Omega \times \mathcal{Y}, \mathbb{R}^{3\times 3})$ and to exhaust S by an increasing sequence $(\widetilde{S}_n)_{n\in\mathbb{N}}$ of the sets with $C^{1,1}$ boundary. □

4. Proof of Theorem 2.4

4.1. **Lower bound.** As a preliminary step we need to establish some continuity properties of the quadratic form appearing in (QE) and its relaxed version introduced in Definition 2.3. For the proof we refer to [Neu12, Lemma 2.7].

Lemma 4.1. *Let W be as in Assumption 2.1 and let Q be the quadratic form associated with W via (QE). Then*

(Q1) $Q(\cdot, G)$ is Y-periodic and measurable for all $G \in \mathbb{M}^3$,
(Q2) for almost every $(x_3, y) \in \mathbb{R} \times \mathbb{R}^2$ the map $Q(x_3,y,\cdot)$ is quadratic and satisfies

(46)
$$c_1|\operatorname{sym} G|^2 \leq Q(x_3,y,G) = Q(x_3,y,\operatorname{sym} G) \leq c_2|\operatorname{sym} G|^2 \ \forall G \in \mathbb{R}^{3\times 3}.$$

Furthermore, there exists a monotone function $r : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$, such that $r(\delta) \to 0$ as $\delta \to 0$ and

$$\forall G \in \mathbb{M}^3 \ : \ |W(x_3,y,I+G) - Q(x_3,y,G)| \leq |G|^2 r(|G|) \tag{47}$$

for almost every $y \in \mathbb{R}^2$.

Lemma 4.2. *For all $A \in \mathbb{M}^2_{\text{sym}}$ there exist a unique quadraple (B,ζ,φ,g) with $B \in \mathbb{M}^2_{\text{sym}}$ and $\zeta \in \mathring{H}^1(\mathcal{Y},\mathbb{R}^2)$, $\varphi \in \mathring{H}^2(\mathcal{Y})$, $g \in L^2(I \times \mathcal{Y}, \mathbb{R}^3)$ such that:*

$$Q_2^{rel}(A) = \iint_{I\times Y} Q\left(x_3, y, \ \iota(x_3 A + B) + \mathcal{U}(\zeta,\varphi,g)\right) dy dx_3$$

The induced mapping $\mathbb{M}^2_{\text{sym}} \ni A \mapsto (B,\zeta,\varphi,g) \in \mathbb{M}^2_{\text{sym}} \times \mathring{H}^1(\mathcal{Y},\mathbb{R}^2) \times \mathring{H}^2(\mathcal{Y}) \times L^2(I \times \mathcal{Y}, \mathbb{R}^3)$ is bounded and linear and thus $A \mapsto Q_2^{rel}(A)$ is quadratic.

Proof. By (46) and by Remark 1 for each $A \in \mathbb{M}^2_{\text{sym}}$ the bilinear form associated with the quadratic functional

$$G \mapsto \int_{Y\times I} Q(x_3, y, x_3 A + G) \, dy dx_3$$

is elliptic on the closed linear subspace of $L^2(I \times \mathcal{Y}, \mathbb{M}^3_{\text{sym}})$ given by

$$X := \iota(\mathbb{M}^2_{\text{sym}}) + L_0(I \times \mathcal{Y}, \mathbb{M}^3_{\text{sym}}).$$

Hence it admits a unique minimizer $G_0 \in X$ by Riesz representation theorem. Linearity of G_0 in A follows immediately from that. □

Remark 5. It can be easily seen that

$$Q_2^{\mathrm{rel}}(A) \quad := \quad \inf_{B,\zeta,\varphi} \iint_{I\times Y} Q_2\left(x_3, y,\ \iota(x_3 A + B) + \mathrm{sym}(\nabla_y \zeta + x_3 \nabla_y^2 \varphi)\right)\, dy\, dx_3,$$

where the infimum is taken over all $B \in \mathbb{M}^2_{\mathrm{sym}}$, $\zeta \in \mathring{H}^1(\mathcal{Y}, \mathbb{R}^2)$ and $\varphi \in \mathring{H}^2(\mathcal{Y})$. In the case when W and consequently Q are independent of x_3, but still dependent on y, the relaxation formula looks significantly simpler i.e. the minimizing B and ζ in the above expression are easily shown to be 0. Thus the homogenization of the layered materials is more complex than the materials that have energy density independent of x_3.

Remark 6. The following observation is already made in [HNV]. Under the assumption that $\frac{h}{\varepsilon(h)} \to \gamma \in (0,\infty)$ the quadratic functional associated with the Γ-limit is given by $Q_{2,\gamma} : \mathbb{M}^2_{\mathrm{sym}} \to [0,\infty)$ by

$$Q_{2,\gamma}(A) \quad := \quad \inf_{B,\phi} \iint_{I\times Y} Q\left(x_3, y,\ \ \iota(x_3 A + B) + (\nabla_y \phi\, ,\ \tfrac{1}{\gamma}\partial_3 \phi)\right)\, dy\, dx_3$$

where the infimum is taken over all $B \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ and $\phi \in \mathring{H}^1(I \times \mathcal{Y}, \mathbb{R}^3)$. Using [NV][Lemma 5.2] one can easily obtain that for $A \in \mathbb{M}^2_{\mathrm{sym}}$ we have

$$Q_2^{\mathrm{rel}}(A) = \lim_{\gamma \to 0} Q_{2,\gamma}(A).$$

The continuity in γ, for all $\gamma \in [0,\infty]$, of the quadratic functional associated with the Γ-limit was already observed in the case of von-Kármán plate (see [NV]). The case of von-Kármán shell resembles the case of bending plate since we obtain that the continuity holds under the assumption that $\varepsilon(h)^2 \ll h \ll \varepsilon(h)$ as already commented in the introduction.

Proof of Theorem 2.4 (lower bound). Without loss of generality we may assume that $\fint_\Omega u^h\, dx = 0$ and $\limsup_{h\to 0} h^{-2} \mathcal{E}^{h,\varepsilon(h)}(u^h) < \infty$. In view of (ND), the sequence u^h has finite bending energy and the sequence E^h is bounded in $L^2(\Omega, \mathbb{M}^3)$ by using the inequality $|\sqrt{F^T F} - I|^2 \lesssim \mathrm{dist}^2(F, \mathrm{SO}(3))$, valid for an arbitrary $F \in \mathbb{M}^3$. Hence, from Theorem 2.5 we deduce that $u \in H^2_\delta(S, \mathbb{R}^3)$. By Lemma 3.4 (i) and Proposition 3.2 (i) we can pass to a subsequence such that, for some $E \in L^2(\Omega \times Y; \mathbb{R}^{3\times 3})$,

$$E^h \xrightharpoonup{2,0} E,$$

where E can be written in the form of (14). As explained in [Neu12] (cf. [FJM02] for the corresponding argument in the homogeneous case), a careful Taylor expansion of $W(\frac{x'}{\varepsilon(h)}, I + hE^h(x))$ combined with the lower semicontinuity of convex integral functionals with respect to weak two-scale convergence (see e.g. [Vis07, Proposition 1.3]) yields the lower bound

$$\liminf_{h\to 0} \frac{1}{h^2} \mathcal{E}^{h,\varepsilon(h)}(u^h) \geq \iint_{\Omega\times Y} Q(x_3, y, E(x,y))\, dy\, dx =$$

$$\iint_{\Omega\times Y} Q\Big(x_3, y, \iota(x_3\, \mathrm{II}(x') + B(x')) + \mathcal{U}\big(\zeta(x,\cdot), \varphi(x,\cdot), g(x,\cdot,\cdot)\big)(x_3,y)\Big)\, dy\, dx,$$

where we have used (14). Minimization over $B \in L^2(S, \mathbb{M}^2)$ and $\zeta \in L^2(S, \mathring{H}^1(\mathcal{Y}, \mathbb{R}^2))$, $\varphi \in L^2(S, \mathring{H}^2(\mathcal{Y}))$, $g \in L^2(\Omega \times \mathcal{Y}, \mathbb{R}^3)$ yields

$$\liminf_{h\to 0} \frac{1}{h^2} \mathcal{E}^{h,\varepsilon(h)}(u^h) \geq \int_S Q_2^{\mathrm{rel}}(\Pi(x'))\, dx' = \mathcal{E}(u).$$

□

4.2. **Upper bound.** It remains to prove the upper bound. We modify the argumentation given in [HNV] by adding additional oscillations. To recover the matrix B in the relaxation formula 2.3 we use the same ansatz as in [HNV]. Since for Γ-limit it is enough to do the construction for dense subsets, first we will say which dense subset of $H^2_\delta(S, \mathbb{R}^3)$ is of interest. The density of smooth isometries in H^2 isometric immersions is established in [Hor11b, Hor11a] (cf. also [Pak04] for an earlier result in this direction). The results in [Hor11a] forces us to consider domains S which are not only Lipschitz but also piecewise C^1. More precisely, we can require only that the outer unit normal be continuous away from a subset of ∂S with length zero.

As in [HNV] for a given $u \in H^2_\delta(S, \mathbb{R}^3)$ and for a displacement $V \in H^2(S, \mathbb{R}^3)$ we denote by q_V the quadratic form

$$q_V = \mathrm{sym}\left((\nabla u)^T(\nabla V)\right).$$

This quadratic form can be seen as symmetrized gradients on developable shell given by parametrization u. We denote by $\mathcal{A}(S)$ the set of all $u \in H^2_\delta(S, \mathbb{R}^3) \cap C^\infty(\overline{S}, \mathbb{R}^3)$ with the property that

$$\Big\{B \in C^\infty(\overline{S}, \mathbb{M}^2_{\mathrm{sym}}) : B = 0 \text{ in a neighborhood of } \{x' \in S : \Pi(x') = 0\}\Big\} \subset \{q_V : V \in C^\infty(\overline{S}; \mathbb{R}^3)\}.$$

In other words, if $u \in \mathcal{A}(S)$ and $B \in C^\infty(\overline{S}, \mathbb{M}^2_{\mathrm{sym}})$ is a matrix field which vanishes in a neighborhood of $\{\Pi = 0\}$, then there exists a displacement $V \in C^\infty(\overline{S}; \mathbb{R}^3)$ such that $q_V = B$. To recover the matrix B in the cell formula we use the following lemma. The short argument, which relies on [Sch07, Lemma 3.3] is given in [HNV]. The claim of it is connected to the fact that on developable surface without the flat regions the space of symmetric gradients is equal to the space of all symmetric matrices (see also [HoLePa]).

Lemma 4.3. *The set $\mathcal{A}(S)$ is dense in $H^2_\delta(S, \mathbb{R}^3)$ with respect to the strong H^2 topology.*

Thanks to Lemma 4.3 it will be enough to construct recovery sequences for limiting deformations belonging to $\mathcal{A}(S)$. First we present a construction assuming additional information about the limit. Then we use the standard diagonalizing argument for the general case. The meaning of Lemma 4.3 is that we can recover the arbitrary matrix field out of the flat parts of the isometry. On the flat parts, however, the matrix B that relaxes the cell formula 2.3 is equal to 0.

Lemma 4.4. *Let $u \in H^2_\delta(S,\mathbb{R}^3) \cap W^{2,\infty}(S,\mathbb{R}^3)$ and let $V \in W^{2,\infty}(S,\mathbb{R}^3)$. Let $\zeta \in C_c^\infty(S, \mathring{C}^\infty(\mathcal{Y},\mathbb{R}^2))$, $\varphi \in C_c^\infty(S,\mathring{C}^\infty(\mathcal{Y}))$, $g \in C_c^\infty(S,\mathring{C}^\infty(I\times\mathcal{Y},\mathbb{R}^3))$. Then there exists a sequence $(u^h) \subset H^1(\Omega,\mathbb{R}^3)$ such that $u^h \to u$ and $\nabla_h u^h \to (\nabla' u,\, n)$ uniformly in Ω and*

$$\lim_{h\to 0}\frac{1}{h^2}\mathcal{E}^{h,\varepsilon(h)}(u^h) = \tag{48}$$
$$\iint_{\Omega\times Y} Q\left(x_3, y,\ \iota(x_3\,\mathrm{II}(x') + q_V(x')) + \mathcal{U}\big(\zeta(x,\cdot),\varphi(x,\cdot), g(x,\cdot,\cdot)\big)(x_3,y)\,\right)\, dy\, dx_3\, dx'.$$

Proof. First we start with the following Kirchhoff-Love type ansatz to which we add its linearization induced by the displacement V:

$$v^h(x) \ := \ u(x') + hx_3 n(x') + h\left(V(x') + hx_3\mu(x')\right),$$

where μ is given by

$$\mu = (I - n\otimes n)(\partial_1 V \wedge \partial_2 u + \partial_1 u \wedge \partial_2 V).$$

We set $R(x') = (\nabla' u(x'),\ n(x'))$. A straighforward computation shows that

$$\nabla_h v^h = R + h\Big((\nabla' V,\mu) + x_3(\nabla' n, 0)\Big) + h^2 x_3 (\nabla'\mu, 0). \tag{49}$$

The actual recovery sequence u^h is obtained by adding to v^h the oscillating corrections given by ζ, φ, g:

(50)
$$\begin{aligned} u^h(x) \ &:= \ v^h(x) - \varepsilon(h)^2 n(x')\varphi(x', \frac{x'}{\varepsilon(h)}) \\ &\quad + h\varepsilon(h)^2 x_3 R(x') \begin{pmatrix} \partial_{x_1}\varphi(x,\frac{x'}{\varepsilon(h)}) + \frac{1}{\varepsilon(h)}\partial_{y_1}\varphi(x,\frac{x'}{\varepsilon(h)}) \\ \partial_{x_2}\varphi(x,\frac{x'}{\varepsilon(h)}) + \frac{1}{\varepsilon(h)}\partial_{y_2}\varphi(x,\frac{x'}{\varepsilon(h)}) \\ 0 \end{pmatrix} \\ &\quad + h\varepsilon(h) R(x') \begin{pmatrix} \zeta(x',\frac{x'}{\varepsilon(h)}) \\ 0 \end{pmatrix} + h^2\int_{-1/2}^{x_3} R(x') g(x', t, \frac{x'}{\varepsilon(h)})\, dt. \end{aligned}$$

By the regularity of V, the uniform convergence of u^h and $\nabla_h u^h$ is immediate. Equation (49) implies

(51)
$$
\begin{aligned}
R^t \nabla_h u^h = I &+ \varepsilon(h) \begin{pmatrix} 0 & 0 & \partial_{y_1}\varphi \\ 0 & 0 & \partial_{y_2}\varphi \\ -\partial_{y_1}\varphi & -\partial_{y_2}\varphi & 0 \end{pmatrix} \\
&+ h\iota\left((\nabla' u)^t(\nabla' V) + x_3 \,\mathrm{II}\right) \\
&+ h\left((\mu \cdot \nabla' u) \otimes e_3 + e_3 \otimes (n \cdot \nabla' V)\right) \\
&+ h\iota\left(x_3 \nabla_y^2 \varphi + \nabla_y \zeta\right) + h \begin{pmatrix} 0 & 0 & g_1 \\ 0 & 0 & g_2 \\ 0 & 0 & g_3 \end{pmatrix} \\
&- \varepsilon(h)^2 \iota(\varphi \,\mathrm{II}) + \varepsilon(h)^2 \begin{pmatrix} 0 & 0 & \partial_{x_1}\varphi \\ 0 & 0 & \partial_{x_2}\varphi \\ -\partial_{x_1}\varphi & -\partial_{x_2}\varphi & 0 \end{pmatrix} \\
&+ h\varepsilon(h)^2 x_3 \iota \begin{pmatrix} \partial_{x_1x_1}\varphi + \frac{1}{\varepsilon(h)}\partial_{x_1y_1}\varphi & \partial_{x_1x_2}\varphi + \frac{1}{\varepsilon(h)}\partial_{x_2y_1} \\ \partial_{x_1x_2}\varphi + \frac{1}{\varepsilon(h)}\partial_{x_1y_2}\varphi & \partial_{x_2x_2}\varphi + \frac{1}{\varepsilon(h)}\partial_{x_2y_2} \end{pmatrix} \\
&+ h^2 x_3 R^t(\nabla'\mu, 0) + h\varepsilon(h)^2 x_3 (R^t \nabla' R) \begin{pmatrix} \partial_{x_1}\varphi + \frac{1}{\varepsilon(h)}\partial_{y_1}\varphi \\ \partial_{x_2}\varphi + \frac{1}{\varepsilon(h)}\partial_{y_2}\varphi \\ 0 \end{pmatrix} \\
&+ h\varepsilon(h)\iota(\nabla_{x'}\zeta) + h\varepsilon(h)(R^t\nabla' R)\begin{pmatrix} \zeta \\ 0 \end{pmatrix} + \frac{h^2}{\varepsilon(h)}\Big(\int_{-1/2}^{x_3} \nabla_y g \, , 0\Big) \\
&+ h^2\Big(\int_{-1/2}^{x_3} \nabla_{x'} g \, , 0\Big) + h^2 (R^t \nabla' R) \int_{-1/2}^{x_3} g;
\end{aligned}
$$

the argument of the functions ζ, φ, g and their derivatives is $(x, x'/\varepsilon(h))$. Let us define

$$E^h = \frac{\sqrt{(\nabla_h u^h)^t \nabla_h u^h} - I}{h}.$$

Using $n \cdot \nabla' V + \mu \cdot \nabla' u = 0$, the Assumption 2.2 and Taylor expansion we deduce from (51) that

$$E^h \xrightarrow{2,0} E := \iota\left(q_V + x_3 \,\mathrm{II}\right) + \mathcal{U}(\zeta(x', \cdot), \varphi(x', \cdot), g(x', \cdot))(x_3, y),$$

Properties (FI), (QE) and (47) yield

$$\limsup_{h \to 0} \left| \frac{1}{h^2} \mathcal{E}^{h, \varepsilon(h)}(u^h) - \int_\Omega Q(x_3, \tfrac{x'}{\varepsilon(h)}, E^h(x))\, dx \right| = 0. \tag{52}$$

Hence, by (46) and by strong two-scale convergence of E^h, we can pass to the limit in the second term in (52). $\square$

Proof of Theorem 2.4 (Upper bound). The following is the standard argument and is already given in [HNV]. We may assume that $\mathcal{E}(u) < \infty$, so $u \in H^2_\delta(S, \mathbb{R}^3)$. Moreover, since Q_2^{rel} is quadratic (cf. Lemma 4.2), it suffices to prove the statement for a dense subset of $H^2_\delta(S, \mathbb{R}^3)$. Hence, by Lemma 4.3, we may assume without loss of generality that $u \in \mathcal{A}(S)$.

By Lemma 4.2 there exist $B \in L^2(S, \mathbb{M}^2_{\mathrm{sym}})$ and $\zeta \in L^2(S, \mathring{H}^1(\mathcal{Y}, \mathbb{R}^2))$, $\varphi \in L^2(S, \mathring{H}^2(\mathcal{Y}))$, $g \in L^2(S, L^2(I \times \mathcal{Y}, \mathbb{R}^3))$ such that:

$$\mathcal{E}(u) = \iint_{\Omega \times Y} Q(x_3, y, \iota(x_3 \,\mathrm{II} + B) + \mathcal{U}(\zeta, \varphi, g))\, dy dx. \tag{53}$$

Since $B(x')$ depends linearly on $\mathrm{II}(x')$, we know in addition that $B(x') = 0$ for almost every $x' \in \{\, \mathrm{II} = 0 \,\}$.

By a density argument it suffices to show the following: There exists a doubly indexed sequence $u^{h,\delta} \in H^1(\Omega, \mathbb{R}^3)$ such that

$$\limsup_{\delta \to 0} \limsup_{h \to 0} \| u^{h,\delta} - u \|_{H^1} = 0, \tag{54}$$

$$\limsup_{h \to 0} \left| \frac{1}{h^2} \mathcal{E}^{h,\varepsilon(h)}(u^{h,\delta}) - \mathcal{E}(u) \right| \lesssim \delta. \tag{55}$$

If we establish this, then we can obtain the desired sequence by diagonalizing argument (e. g. by appealing to [Att84, Corollary 1.16]).

We construct $u^{h,\delta}$ as follows: By density, for each $\delta > 0$ there exist maps $B^\delta \in C^\infty(\overline{S}, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ and $\zeta^\delta \in C_c^\infty(S, \mathring{C}^\infty(\mathcal{Y}, \mathbb{R}^2))$, $\varphi^\delta \in C_c^\infty(S, \mathring{C}^\infty(\mathcal{Y}))$, $g^\delta \in C_c^\infty(S, \mathring{C}^\infty(I \times \mathcal{Y}, \mathbb{R}^3))$

such that

$$\| B^\delta - B \|_{L^2(S)} + \| \mathcal{U}(\zeta^\delta, \varphi^\delta, g^\delta) - \mathcal{U}(\zeta, \varphi, g) \|_{L^2(\Omega \times Y)} \le \delta^2, \tag{56}$$

$$B^\delta = 0 \text{ in a neighborhood of } \{\, \mathrm{II} = 0 \,\}. \tag{57}$$

Since $u \in \mathcal{A}(S, \mathbb{R}^3)$ and due to (57), for each $\delta > 0$ there exists a smooth displacement V_δ such that

$$B^\delta = q_{V_\delta}.$$

We apply Lemma 4.4 to u and V_δ to obtain a sequence $u^{h,\delta}$ that converges uniformly to u as $h \to 0$. Hence (54) is satisfied. Lemma 4.4 also ensures that

$$\lim_{h \to 0} \frac{1}{h^2} \mathcal{E}^h(u^{h,\delta}) = \iint_{\Omega \times Y} Q\left(x_3, y, \iota(x_3 \,\mathrm{II} + B^\delta) + \mathcal{U}(\zeta^\delta, \varphi^\delta, g^\delta)\right) dy dx.$$

By continuity of the functional on the right-hand side, combined with (56) and (53), the bound (55) follows. $\square$

Remark 7. In the case $\varepsilon(h)^2 \sim h$ one can define the sequence, instead of the expression (50), which includes the oscillations which are of order different than $\varepsilon(h)$ and which possibly additionally relax the energy. This can be achieved e.g. by putting in (50) instead of $\varphi \in C_c^\infty(S, \mathring{C}^\infty(\mathcal{Y}))$ the function $\varphi_k \in C_c^\infty(S, \mathring{C}^\infty(k\mathcal{Y}))$. It can be easily seen that this is allowable recovery sequence which could have less energy than the original one. Namely, in the expression for the strain then would also appear the matrix $A^T A$, where

$$A = \begin{pmatrix} 0 & 0 & \partial_{y_1} \varphi_k \\ 0 & 0 & \partial_{y_2} \varphi_k \\ -\partial_{y_1} \varphi_k & -\partial_{y_2} \varphi_k & 0 \end{pmatrix}.$$

This nonlinear term would cause nonconvexity of the energy in $\nabla\varphi$ which has the consequence that the oscillations which are not of the order $\varepsilon(h)$ could also be energetically convenient. This partially explains the lack of compactness, which is commented in Remark 3.

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