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The Steiner tree problem revisited through
rectifiable G-currents

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THE STEINER TREE PROBLEM REVISITED THROUGH RECTIFIABLE G -CURRENTS

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ABSTRACT. The Steiner tree problem seeks a connected set of minimal length containing a given set of finitely many points. We show how to formulate it as a mass-minimization problem for 1-dimensional currents with coefficients in a suitable normed group. The representation used for these currents allows to state a calibration principle for this problem. We also exhibit calibrations in some examples.

INTRODUCTION

The classical *Steiner tree problem* consists in finding the shortest connected set containing n given distinct points p_1, \dots, p_n in \mathbb{R}^d . Some very well-known examples are shown in Figure 1.

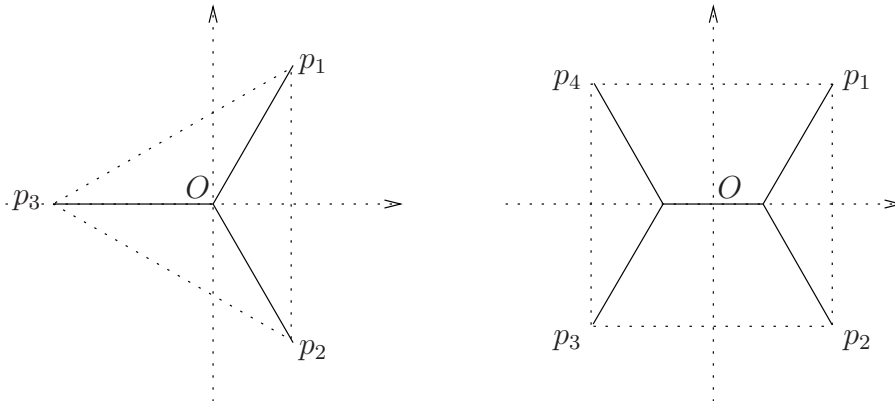


FIGURE 1. Solutions for the vertices of an equilateral triangle and a square

The problem is completely solved in \mathbb{R}^2 and there exists a wide literature on the subject, mainly devoted to improving the efficiency of algorithms for the construction of solutions: see, for instance, [GP] and [IT] for a survey of the problem. The recent papers [PS2] and [PU] witness the current studies on the problem and its generalizations.

Our aim is to rephrase the Steiner tree problem with an equivalent mass minimization problem by replacing connected sets with 1-currents with coefficients in a more suitable group than \mathbb{Z} , in such a way that solutions of one problem correspond to solutions of the other, and vice-versa. The use of currents allows to exploit techniques and tools from the Calculus of Variations and the Geometric Measure Theory.

Let us briefly point out a few facts suggesting that classical polyhedral chains with integer coefficients might not be the correct environment for our problem. First,

one should make the given points p_1, \dots, p_n in the Steiner problem correspond to some integral polyhedral 0-chain supported on p_1, \dots, p_n , with suitable multiplicities m_1, \dots, m_n . One has to impose that $m_1 + \dots + m_n = 0$ in order for this 0-chain to be the boundary of a compactly supported 1-chain. In the example of the equilateral triangle, see Figure 1, the condition $m_3 = -(m_1 + m_2)$ forces to break symmetry, leading to the minimizer in Figure 2. The desired solution is instead depicted in Figure 1. In the second example from Figure 1, we get the “wrong” non-connected minimizer even though all boundary multiplicities have modulus 1; see Figure 2.

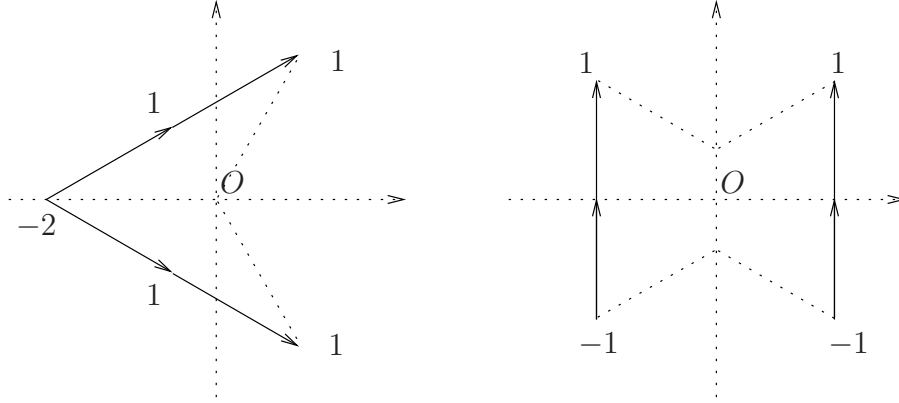


FIGURE 2. Solutions for the mass minimization problems among polyhedral chains with integer coefficients

These examples show that \mathbb{Z} is not the right group of coefficients.

Our framework will be that of currents with coefficients in a normed abelian group G (briefly: G -currents), which we will introduce in Section 1.

Currents with coefficients in a group were introduced by W. Fleming. There is a vast literature on the subject: let us mention only the seminal paper [Fl], the work of B. White [W2, W3], and the more recent papers by T. De Pauw and R. Hardt [DH] and by L. Ambrosio and M. G. Katz [AKa]. A Closure Theorem holds for these flat G -chains, see [Fl] and [W3].

In Section 2 we recast Steiner problem in terms of a mass minimization problem over currents with coefficients in a discrete group G , chosen only on the basis of the number of boundary points. As we already said, this construction provides a way to pass from a mass minimizer to a Steiner solution and vice-versa.

This new formulation permits to initiate a study of calibrations as a sufficient condition for minimality; this is the subject of Section 3. Classically a calibration ω associated with a given oriented k -submanifold $S \subset \mathbb{R}^d$ is a unit closed k -form taking value 1 on the tangent space of S . The existence of a calibration guarantees the minimality of S among oriented submanifolds with the same boundary ∂S . Indeed, Stokes Theorem and the assumptions on ω imply that

$$\text{vol}(S) = \int_S \omega = \int_{S'} \omega \leq \text{vol}(S')$$

for any submanifold S' sharing the same boundary of S .

In order to define calibrations in the framework of G -currents, it is convenient to view currents as linear functionals on forms, which is not always possible in the usual

setting of currents with coefficients in groups. This motivates the preliminary work in Section 1, where we embed the group G in a normed linear space E and we construct the currents with coefficients in E in a classical way. In Definition 3.5, the notion of calibration is slightly weakened in order to include piecewise smooth forms, which appear in Examples 3.10 and 3.11, where we exhibit calibrations for the problem on the right of Figure 1 and for the Steiner tree problem on the vertices of a regular hexagon plus the center. It is worthwhile to note that our theory works for the Steiner tree problem in \mathbb{R}^d and for currents supported in \mathbb{R}^d ; we made explicit computations only on 2-dimensional configurations for simplicity reasons. We conclude Section 3 with some remarks concerning the use of calibrations in similar contexts, see for instance [M1].

The existence of a calibration is a sufficient condition for a manifold to be a minimizer; one could wonder whether this condition is necessary as well. In general, a smooth (or piecewise smooth, according to Definition 3.7) calibration might not exist; nevertheless, one can still search for some *weak* calibration, for instance a differential form with bounded measurable coefficients. In Section 4 we discuss a strategy in order to get the existence of such a weak calibration. A duality argument due to H. Federer [Fe2] ensures that a weak calibration exists for mass-minimizing normal currents; the same argument works for mass-minimizing normal currents with coefficients in the normed vector space E . Therefore an equivalence principle between minima among normal and integral 1-currents with coefficients in E and G , respectively, is sufficient to conclude that a calibration exists. Proposition 4.4 guarantees the equivalence between minima in the case of integral 1-currents; hence a weak calibration always exists. The proof of this result is subject to the validity of a homogeneity property for the candidate minimizer stated in Remark 4.5. Example 4.6 shows that for 1-dimensional G -currents an interesting new phenomenon occurs, since (at least in a non-Euclidean setting) this homogeneity property might not hold; the validity of the homogeneity property may be related to the ambient space. The problem of the existence of a calibration in the Euclidean space is still open.

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1. RECTIFIABLE CURRENTS OVER A COEFFICIENT GROUP

In this section we provide definitions for currents over a coefficient group, with some basic examples.

Fix an open set $U \subset \mathbb{R}^d$ and a normed vector space $(E, \|\cdot\|_E)$ with finite dimension $m \geq 1$. We will denote by $(E^*, \|\cdot\|_{E^*})$ its dual space endowed with the dual norm

$$\|f\|_{E^*} := \sup_{\|v\|_E \leq 1} \langle f; v \rangle .$$

Definition 1.1. We say that a map

$$\omega : \Lambda_k(\mathbb{R}^d) \times E \rightarrow \mathbb{R}$$

is an E^* -valued k -covector in \mathbb{R}^d if

- (i) $\forall \tau \in \Lambda_k(\mathbb{R}^d), \quad \omega(\tau, \cdot) \in E^*$, that is $\omega(\tau, \cdot) : E \rightarrow \mathbb{R}$ is a linear function.
- (ii) $\forall v \in E, \quad \omega(\cdot, v) : \Lambda_k(\mathbb{R}^d) \rightarrow \mathbb{R}$ is a (classical) k -covector.

Sometimes we will use $\langle \omega; \tau, v \rangle$ instead of $\omega(\tau, v)$, in order to simplify the notation. The space of E^* -valued k -covectors in \mathbb{R}^d is denoted by $\Lambda_E^k(\mathbb{R}^d)$ and it is endowed with the comass norm

$$(1.1) \quad \|\omega\| := \sup \{ \|\omega(\tau, \cdot)\|_{E^*} : |\tau| \leq 1, \tau \text{ simple} \} .$$

Remark 1.2. Fix an orthonormal system of coordinates in \mathbb{R}^d , $(\mathbf{e}_1, \dots, \mathbf{e}_d)$; the corresponding dual base in $(\mathbb{R}^d)^*$ is (dx_1, \dots, dx_d) . Consider a complete biorthonormal system for E , i.e., a pair

$$(v_1, \dots, v_m) \in E^m; (w_1, \dots, w_m) \subset (E^*)^m$$

such that $\|v_i\|_E = 1$, $\|w_i\|_{E^*} = 1$ and $\langle w_i; v_j \rangle = \delta_{ij}$. Given an E^* -valued k -covector ω , we denote

$$\omega^j := \omega(\cdot, v_j).$$

For each $j \in \{1, \dots, m\}$, ω^j is a k -covector in the usual sense. Hence the biorthonormal system $(v_1, \dots, v_m), (w_1, \dots, w_m)$ allows to write ω in “components”

$$\omega = (\omega^1, \dots, \omega^m),$$

in fact we have

$$\omega(\tau, v) = \sum_{j=1}^m \langle \omega^j; \tau \rangle \langle w_j; v \rangle .$$

In particular ω^j admits the usual representation

$$\omega^j = \sum_{1 \leq i_1 < \dots < i_k \leq d} a_{i_1 \dots i_k}^j dx_{i_1} \wedge \dots \wedge dx_{i_k}, \quad j = 1, \dots, m.$$

Definition 1.3. An E^* -valued differential k -form in $U \subset \mathbb{R}^d$, or just a k -form when it is clear which group we are referring to, is a map

$$\omega : U \rightarrow \Lambda_E^k(\mathbb{R}^d);$$

we say that ω is \mathcal{C}^∞ -regular if every component ω^j is so (see Remark 1.2). We denote by $\mathcal{C}_c^\infty(U, \Lambda_E^k(\mathbb{R}^d))$ the vector space of \mathcal{C}^∞ -regular E^* -valued k -forms with compact support in U .

We are mainly interested in E^* -valued 1-forms, nevertheless we analyze k -forms in wider generality, in order to ease other definitions, such as the differential of an E^* -valued form and the boundary of an E -current.

Definition 1.4. We define the differential $d\omega$ of a \mathcal{C}^∞ -regular E^* -valued k -form ω by components:

$$d\omega^j = d(\omega^j) : U \rightarrow \Lambda^{k+1}(\mathbb{R}^d), \quad j = 1, \dots, m ,$$

Moreover, $\mathcal{C}_c^\infty(U, \Lambda_E^1(\mathbb{R}^d))$ has a norm, denoted by $\|\cdot\|$, given by the supremum of the comass norm of the form defined in (1.1). Hence we mean

$$(1.2) \quad \|\omega\| := \sup_{x \in U} \|\omega(x)\| .$$

Definition 1.5. A k -dimensional current T in $U \subset \mathbb{R}^d$, with coefficients in E , or just an E -current when there is no doubt on the dimension, is a linear and continuous function

$$T : \mathcal{C}_c^\infty(U, \Lambda_E^k(\mathbb{R}^d)) \longrightarrow \mathbb{R} ,$$

where the continuity is meant with respect to the locally convex topology on the space $\mathcal{C}_c^\infty(U, \Lambda_E^k(\mathbb{R}^d))$, built in analogy with the topology on $\mathcal{C}_c^\infty(\mathbb{R}^n)$, with respect to which distributions are dual. This defines the weak* topology on the space of k -dimensional E -currents. Convergence in this topology is equivalent to the convergence of all the “components” in the space of classical¹ k -currents, by which we mean the following. We define for every k -dimensional E -current T its components T^j , for $j = 1, \dots, m$, and we write

$$T = (T^1, \dots, T^m),$$

denoting

$$\langle T^j; \varphi \rangle := \langle T; \tilde{\varphi}_j \rangle ,$$

for every (classical) compactly supported differential k -form φ on \mathbb{R}^d . Here $\tilde{\varphi}_j$ denotes the E^* -valued differential k -form on \mathbb{R}^d such that

$$(1.3) \quad \tilde{\varphi}_j(\cdot, v_j) = \varphi,$$

$$(1.4) \quad \tilde{\varphi}_j(\cdot, v_i) = 0 \quad \text{for } i \neq j .$$

It turns out that a sequence of k -dimensional E -currents T_h weakly* converges to an E -current T (in this case we write $T_h \xrightarrow{*} T$) if and only if the sequence of the components T_h^j converge to T^j in the space of classical k -currents, for $j = 1, \dots, m$.

Definition 1.6. For a k -current T over E we define the boundary operator

$$\langle \partial T; \varphi \rangle := \langle T; d\varphi \rangle \quad \forall \varphi = (\varphi^1, \dots, \varphi^m) \in \mathcal{C}_c^\infty(U, \Lambda_E^{k-1}(\mathbb{R}^d))$$

and the mass

$$\mathbf{M}(T) := \sup_{\|\omega\| \leq 1} \langle T; \omega \rangle .$$

As one can expect, the boundary $\partial(T^j)$ of every component T^j is the relative component $(\partial T)^j$ of the boundary ∂T .

Definition 1.7. A k -dimensional normal E -current in $U \subset \mathbb{R}^d$ is an E -current T with $\mathbf{M}(T) < +\infty$ and $\mathbf{M}(\partial T) < +\infty$. Thanks to the Riesz Theorem, T admits the following representation:

$$\langle T; \omega \rangle = \int_U \langle \omega(x); \tau(x), v(x) \rangle d\mu_T(x) , \quad \forall \omega \in \mathcal{C}_c^\infty(U, \Lambda_E^k(\mathbb{R}^d)) .$$

where μ_T is a Radon measure on U and $v : U \rightarrow E$ is summable with respect to μ_T and $|\tau| = 1$, μ_T -a.e. A similar representation holds for the boundary ∂T .

Definition 1.8. A rectifiable k -current T in $U \subset \mathbb{R}^d$, over E , or a rectifiable E -current is an E -current admitting the following representation:

$$\langle T; \omega \rangle := \int_\Sigma \langle \omega(x); \tau(x), \theta(x) \rangle d\mathcal{H}^k(x), \quad \forall \omega \in \mathcal{C}_c^\infty(\mathbb{R}^d, \Lambda_E^k(U))$$

¹In the sequel we will use “classical” to refer to the usual currents, with coefficients in \mathbb{R} or possibly in \mathbb{Z} .

where Σ is an \mathcal{H}^k -rectifiable set contained in U , $\tau(x) \in T_x \Sigma$ with $|\tau(x)| = 1$ for \mathcal{H}^k -a.e. $x \in \Sigma$ and $\theta \in L^1(U; E)$. We will refer to such a current as $T = T(\Sigma, \tau, \theta)$. If B is a Borel set and $T(\Sigma, \tau, \theta)$ is a rectifiable E -current, we denote by $T \llcorner B$ the current $T(\Sigma \cap B, \tau, \theta)$.

Consider now a discrete subgroup $G < E$, endowed with the restriction of the norm $\|\cdot\|_E$. If the multiplicity θ takes only values in G , and if the same representation holds for ∂T , we call T a rectifiable G -current. Pay attention to the fact that, in the framework of currents over the coefficient group E , rectifiable E -currents play the role of (classical) rectifiable current, while rectifiable G -currents correspond to (classical) integral currents. Actually this correspondence is an equality, when E is the group \mathbb{R} (with the euclidean norm) and G is \mathbb{Z} .

Example 1.9. Let $E = \mathbb{R}^d$ and let G be the additive subgroup generated by m elements g_1, \dots, g_m . Given $m+1$ points $p_1, \dots, p_m, p_{m+1} \in \mathbb{R}^2$, consider the cone C over (p_1, \dots, p_m) with respect to p_{m+1} : if Σ_r is the oriented segment from p_r to p_{m+1} , $r = 1, \dots, m$, then

$$C = \bigcup_{r=1}^m \Sigma_r .$$

We can define a rectifiable G -current supported on C as

$$\langle T; \omega \rangle := - \sum_{r=1}^m \int_{\Sigma_r} \langle \omega(x); \tau_r(x), g_r \rangle d\mathcal{H}^1(x),$$

where τ_r is the unit tangent vector to Σ_r , pointing towards p_{m+1} . It is easy to see that, denoting $g_{m+1} = -(g_1 + \dots + g_m)$ we can represent the 0-dimensional rectifiable G -current ∂T with the points p_1, \dots, p_{m+1} with multiplicities g_1, \dots, g_{m+1} , respectively. From now on we will denote such a current as $g_1 \delta_{p_1} + \dots + g_{m+1} \delta_{p_{m+1}}$.

Proposition 1.10. *Let $T = T(\Sigma, \tau, \theta)$ be a rectifiable E -current, then*

$$\mathbb{M}(T) = \int_{\Sigma} \|\theta(x)\|_G d\mathcal{H}^1(x) .$$

Since the mass is lower semicontinuous, we can apply the direct method of the Calculus of Variations for the existence of minimizers with given boundary, once we provide the following compactness result. Here we assume for simplicity that G is the subgroup of E generated by v_1, \dots, v_m . A similar argument works for every discrete subgroup G .

Theorem 1.11. *Let $(T_h)_{h \geq 1}$ be a sequence of rectifiable G -currents such that there exists a positive finite constant C satisfying*

$$\mathbb{M}(T_h) + \mathbb{M}(\partial T_h) \leq C \quad \text{for every } h \geq 1 .$$

Then there exists a subsequence $(T_{h_i})_{i \geq 1}$ and a rectifiable G -current T such that

$$T_{h_i} \xrightarrow{*} T .$$

Proof. The statement of the theorem can be proved component by component. In fact, let T_h^1, \dots, T_h^m be the components of T_h . Since $(v_1, \dots, v_m), (w_1, \dots, w_m)$ is a biorthonormal sistem, we have

$$\mathbb{M}(T_h^j) + \mathbb{M}(\partial T_h^j) \leq \mathbb{M}(T_h) + \mathbb{M}(\partial T_h) \leq C ,$$

hence, after a diagonal procedure, we can find a subsequence $(T_{h_i})_{i \geq 1}$ such that $(T_{h_i}^j)_{i \geq 1}$ weakly* converges to some integral current T^j , for every $j = 1, \dots, m$. Denoting by T the rectifiable G -current, whose components are T^1, \dots, T^m , we have

$$T_{h_i} \xrightarrow{*} T.$$

□

2. STEINER TREE PROBLEM REVISITED

In this section we establish the equivalence between the Steiner tree problem and a mass minimization problem in a family of G -currents. We first need to choose the right group of coefficients G . Once we fix the n points in the Steiner problem, we look for a subgroup $(G, \|\cdot\|_G)$, of a normed vector space $(E, \|\cdot\|_E)$, (where $\|\cdot\|_G$ is the restriction to G of the norm $\|\cdot\|_E$) satisfying the following properties:

- (P1) there exist $g_1, \dots, g_{n-1} \in G$ and $h_1, \dots, h_{n-1} \in E^*$ such that (g_1, \dots, g_{n-1}) with (h_1, \dots, h_{n-1}) is a complete biorthonormal system for E and G is additively generated by g_1, \dots, g_{n-1} ;
- (P2) $\|g_{i_1} + \dots + g_{i_k}\|_G = 1$ whenever $1 \leq i_1 < \dots < i_k \leq n-1$ and $k \leq n-1$;
- (P3) $\|g\|_G \geq 1$ for every $g \in G \setminus \{0\}$.

For the moment we will assume the existence of G and E . The proof of their existence and an explicit representation, useful for the computations, will be given later in this section.

The next lemma has a fundamental role: through it, we can give a nice structure of 1-dimensional rectifiable G -current to every suitable competitor for the Steiner tree problem. From now on we will denote $g_n := -(g_1 + \dots + g_{n-1})$.

Lemma 2.1. *Let B be a connected 1-rectifiable set with finite length in \mathbb{R}^d , containing p_1, \dots, p_n . Then there exists a connected set $B' \subset B$ containing p_1, \dots, p_n and a 1-dimensional rectifiable G -current $T_{B'} = T(B', \tau, \theta)$, such that*

- (i) $\|\theta(x)\|_E = 1$ for a.e. $x \in B'$,
- (ii) $\partial T_{B'}$ is the 0-dimensional G -current $g_1 \delta_{p_1} + \dots + g_n \delta_{p_n}$

Proof. Since B is a connected set of finite length, then B is connected by paths of finite length (see Lemma 3.12 of [Fa]). Consider a simple path B_1 contained in B going from p_1 to p_n . In analogy with Example 1.9, associate it with a current T_1 with constant multiplicity $-g_1$ and orientation going from p_1 to p_n . Repeat this procedure keeping the ending point p_n and replacing at each step p_1 with p_2, \dots, p_{n-1} . To be precise, in this procedure, as soon as a new path B_i intersects an other path B_j ($i > j$), then the remaining part of B_j must coincide with the remaining part of B_i . The set $B' = B_1 \cup \dots \cup B_{n-1} \subset B$ is a connected set containing p_1, \dots, p_n and the 1-dimensional rectifiable G -current $T = T_1 + \dots + T_{n-1}$ satisfies the requirements of the lemma, in particular condition (i) comes from (P2). □

Via the next lemma, we can say that mass minimizers for our problem have connected supports. In its proof we will need the following theorem on the structure of classical integral 1-currents. This theorem has been firstly stated as a corollary of Theorem 4.2.25 in [Fe1]. As we said in the Introduction, it allows us to consider an

integral 1-current as a countable sum of oriented simple Lipschitz curves with integer multiplicities.

Theorem 2.2. *Let T be an integral 1-current in \mathbb{R}^d , then*

$$(2.1) \quad T = \sum_{k=1}^K T_k + \sum_{\ell \geq 1} C_\ell,$$

with

- (i) T_k and C_ℓ are integral 1-currents associated to oriented simple Lipschitz curves with finite length, for $k = 1, \dots, K$ and $\ell \geq 1$;
- (ii) $\partial C_\ell = 0$ for every $\ell \geq 1$.

Moreover

$$(2.2) \quad \mathbb{M}(T) = \sum_{k=1}^K \mathbb{M}(T_k) + \sum_{\ell \geq 1} \mathbb{M}(C_\ell)$$

and

$$(2.3) \quad \mathbb{M}(\partial T) = \sum_{k=1}^K \mathbb{M}(\partial T_k).$$

Lemma 2.3. *Let T be a 1-dimensional rectifiable G -current, such that ∂T is the 0-current $g_1 \delta_{p_1} + \dots + g_n \delta_{p_n}$. Then there exists a rectifiable G -current $\tilde{T} = T(\tilde{\Sigma}, \tilde{\tau}, \tilde{\theta})$ such that*

- (i) $\partial \tilde{T} = \partial T = g_1 \delta_{p_1} + \dots + g_n \delta_{p_n}$;
- (ii) $\mathbb{M}(\tilde{T}) \leq \mathbb{M}(T)$ and the equality holds only if $\tilde{T} = T$;
- (iii) the support of \tilde{T} is a connected 1-rectifiable set containing $\{p_1, \dots, p_n\}$ and it is contained in the support of T ;
- (iv) $\mathcal{H}^1(\text{supp}(\tilde{T}) \setminus \tilde{\Sigma}) = 0$.

Proof. Let $T^j = T(\Sigma^j, \tau^j, \theta^j)$ be the components of T , for $j = 1, \dots, n-1$ (with respect to the biorthonormal system $(g_1, \dots, g_{n-1}), (h_1, \dots, h_{n-1})$).

For every j , we can use Theorem 2.2 and write

$$T^j = \sum_{k=1}^{K_j} T_k^j + \sum_{\ell \geq 1} C_\ell^j.$$

Notice that, for every $j = 1, \dots, n-1$, if θ_k^j denotes the multiplicity of T_k^j , then, by (2.2), we have

$$(2.4) \quad \sum_{k=1}^{K_j} |\theta_k^j| \leq |\theta^j| \quad \mathcal{H}^1\text{-a.e. on } \text{supp}(T^j).$$

We choose \tilde{T} the rectifiable G -current whose components are

$$\tilde{T}^j := \sum_{k=1}^{K_j} T_k^j.$$

Again, because of (2.2), we have $\text{supp}(\tilde{T}) \subset \text{supp}(T)$ (the cyclic part of T^j never cancels the acyclic one).

Property (i) is easy to check. Property (ii) is a consequence of (2.4) and of the following property of the norm $\|\cdot\|_G$: if $\theta = \sum_{j=1}^{n-1} \theta^j g_j$ and $\tilde{\theta} = \sum_{j=1}^{n-1} \tilde{\theta}^j g_j$ (with $0 \leq \tilde{\theta}^j \leq \theta^j$ when $\theta^j \geq 0$ and $0 \geq \tilde{\theta}^j \geq \theta^j$ otherwise), then $\|\tilde{\theta}\|_G \leq \|\theta\|_G$ (this property follows from the fact that (g_1, \dots, g_{n-1}) , (h_1, \dots, h_{n-1}) is a complete biorthonormal system for E).

Property (iv) is also easy to check, because the corresponding property holds for every T_k^j and therefore for every component \tilde{T}^j .

It remains to prove property (iii). By construction \tilde{T} is a finite sum of oriented curves with multiplicities; since we are considering curves with ending points (closed sets), $\text{supp}(\tilde{T})$ has a finite number of (closed) connected components far apart: consider S a connected component of $\text{supp}(\tilde{T})$ and the related restriction $\tilde{T} \llcorner S$. Notice that S has positive distance from any other connected component of $\text{supp}(\tilde{T})$. We want to prove that either S contains all the p_i 's or none of them. Assume by contradiction that S contains a proper subset of $\{p_1, \dots, p_n\}$, let us relabel the points such that $S \supset \{p_1, \dots, p_{\tilde{n}}\}$, with $1 \leq \tilde{n} < n$, and $p_j \notin S$ if $j > \tilde{n}$. Thus $\partial(\tilde{T} \llcorner S)$ is the 0-current associated with $p_1, \dots, p_{\tilde{n}}$ with multiplicities $g_1, \dots, g_{\tilde{n}}$. Then we can choose an element $w \in E^*$ such that $w(g_j) = 1$ for $j = 1, \dots, \tilde{n}$ and take $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d, \Lambda_E^1(\mathbb{R}^d))$ a smooth E^* -valued 1-form such that

$$\begin{aligned} \varphi &\equiv w && \text{on } S \\ \varphi &\equiv 0 && \text{on } \text{supp}(\tilde{T}) \setminus S. \end{aligned}$$

Then $0 = \tilde{T} \llcorner S(d\varphi) = \partial(\tilde{T} \llcorner S)(\varphi) = \tilde{n}$, which is clearly a contradiction. Therefore there is no boundary for the restriction of \tilde{T} to every connected component of its support, but one. Possibly replacing \tilde{T} by its restriction to this non-trivial connected component, we get the thesis. \square

Before stating the main theorem, let us point out that the existence of a solution to the mass minimization problem is a consequence of the direct method of the Calculus of Variations.

Theorem 2.4. *Assume that $T_0 = T(\Sigma_0, \tau_0, \theta_0)$ is a mass-minimizer among rectifiable 1-dimensional G -currents with boundary*

$$B = g_1 \delta_{p_1} + \dots + g_n \delta_{p_n}.$$

Then $S_0 := \text{supp}(T_0)$ is a solution of the Steiner tree problem. Conversely, given a set C which is a solution of the Steiner problem for the points p_1, \dots, p_n , there exists a canonical 1-dimensional G -current, supported on C , minimizing the mass among the currents with boundary B .

Proof. The existence of T_0 is a direct consequence of Theorem 1.11. Moreover, since T_0 is a mass minimizer, then it must coincide with the current \tilde{T}_0 given by Lemma 2.3. In particular, Lemma 2.3 guarantees that S_0 is a connected set.

Let S be a competitor for the Steiner tree problem and let S' and $T_{S'}$ be the connected set and the rectifiable 1-current given by Lemma 2.1, respectively. Hence

we have

$$\mathcal{H}^1(S) \geq \mathcal{H}^1(S') \stackrel{(i)}{=} \mathbb{M}(T_{S'}) \stackrel{(ii)}{\geq} \mathbb{M}(T_0) \stackrel{(iii)}{\geq} \mathcal{H}^1(\Sigma_0) \stackrel{(iv)}{=} \mathcal{H}^1(S_0) ,$$

in fact

(i) thanks to the second property of Lemma 2.1 and Proposition 1.10, we obtain

$$\mathbb{M}(T_{S'}) = \int_{S'} \|\theta_{S'}(x)\|_G d\mathcal{H}^1(x) = \mathcal{H}^1(S') ;$$

(ii) we assumed that T_0 is a mass-minimizer;

(iii) from property (P3), we get

$$\mathbb{M}(T_0) = \int_{\Sigma_0} \|\theta_0(x)\|_G d\mathcal{H}^1(x) \geq \int_{\Sigma_0} 1 d\mathcal{H}^1(x) = \mathcal{H}^1(\Sigma_0) ;$$

(iv) is property (iv) in Lemma 2.3.

To prove the second part of the Theorem, apply Lemma 2.1 to the set C . Notice that with the procedure described in the lemma, the rectifiable G -current $T_{C'}$ is uniquely determined, because for every point p_i , C contains exactly one path from p_i to p_n , in fact it is well-known that solutions of the Steiner tree problem cannot contain cycles. By Lemma 2.3 $T_{C'}$ is a solution of the mass minimization problem. \square

Eventually, we give an explicit representation for G and E . Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis of \mathbb{R}^n ; we consider on \mathbb{R}^n the seminorm

$$\|u\|_{\star} := \max_{i=1, \dots, n} u \cdot \mathbf{e}_i - \min_{i=1, \dots, n} u \cdot \mathbf{e}_i .$$

We now take the quotient

$$E := \frac{\mathbb{R}^n}{\text{Span}\{\mathbf{e}_1 + \dots + \mathbf{e}_n\}}$$

and denote by π the projection of \mathbb{R}^n onto E . According to the relation in the quotient, we get $[(u_1, \dots, u_n)] = [(u_1 + c, \dots, u_n + c)]$, for every $c \in \mathbb{R}$ and for every $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ (here $[u]$ denotes the element of the quotient associated with the vector $u \in \mathbb{R}^n$). Since $\|u\|_{\star} = \|u + v\|_{\star}$ for every $u \in \mathbb{R}^n, v \in \text{Span}\{\mathbf{e}_1 + \dots + \mathbf{e}_n\}$, then it is well defined the corresponding seminorm $\|\cdot\|_E$ induced on E and it is actually a norm. Moreover $\|\cdot\|_{\star}$ is constant on every fibre. For the sake of completeness, we remark that, with this notation, the dual space E^* can be represented as $E^* = \{(z_1, \dots, z_n) \in \mathbb{R}^n : \sum_{i=1}^n z_i = 0\}$ and its dual norm $\|\cdot\|_{E^*}$ coincides with $\frac{1}{2}\|\cdot\|_1$. In fact, for every $[u] \in E$ with $\|[u]\|_E = 1$ we can choose a representative u , such that $|u_i| \leq \frac{1}{2}$, $i = 1, \dots, n$ and then

$$\|z\|_{E^*} = \sup_{\|[u]\|_E=1} \sum_{i=1}^n z_i u_i = \frac{1}{2} \sum_{i=1}^n |z_i| .$$

The choice of E as a quotient is motivated by the idea that the sum of the coefficients \mathbf{e}_i must be zero, for boundary reasons. Anyway, we find that a slightly different representation of E , would ease computations later and we would rather introduce G with this new representation. Consider

$$F := \{v \in \mathbb{R}^n : v \cdot \mathbf{e}_n = 0\} \subset \mathbb{R}^n$$

and the homomorphism $\phi : \mathbb{R}^n \rightarrow F$ such that

$$(2.5) \quad \phi(u_1, \dots, u_n) := (u_1 - u_n, \dots, u_{n-1} - u_n, 0) ;$$

the seminorm $\|\cdot\|_*$ is a norm on F .

The homomorphism ϕ in (2.5) induces an isometrical isomorphism $\tilde{\phi} : E \rightarrow F$ defined by the relation $\tilde{\phi} \circ \pi = \phi$: in fact, if $v \in E$ and $u \in \pi^{-1}(v)$, then $\|v\|_E = \|u\|_* = \|\phi(u)\|_* = \|\tilde{\phi}(v)\|_*$. For every $i = 1, \dots, n-1$, define $g_i = \tilde{\phi}^{-1}(\mathbf{e}_i)$ and define $g_n = -(g_1 + \dots + g_{n-1})$. Let G be the subgroup of E generated by g_1, \dots, g_{n-1} . For every $i = 1, \dots, n-1$ denote by h_i the element of E^* satisfying $h_i(g_j) = \delta_{ij}$. The pair $(g_1, \dots, g_{n-1}), (h_1, \dots, h_{n-1})$ is a biorthonormal system. With these coordinates, an element $v \in E$ has unit norm $\|v\|_E = 1$ if and only if

$$(2.6) \quad \|v\|_E = \|\tilde{\phi}(v)\|_* = \max_{i=1, \dots, n-1} (v_i \vee 0) - \min_{i=1, \dots, n-1} (v_i \wedge 0) = 1 .$$

The norm $\|\cdot\|_{E^*}$ of an element $w = w_1 h_1 + \dots + w_{n-1} h_{n-1} \in E^*$ can be characterized in the following way: let us abbreviate $w^P := \sum_{i=1}^{n-1} (w_i \vee 0)$ and $w^N := -\sum_{i=1}^{n-1} (w_i \wedge 0)$ and $\lambda(v) = \max_{i=1, \dots, n-1} (v_i \vee 0) \in [0, 1]$, then

$$(2.7) \quad \begin{aligned} \|w\|_{E^*} &= \sup_{\|v\|_E=1} \sum_{i=1}^{n-1} w_i v_i = \sup_{\|v\|_E=1} [\lambda(v) w^P + (1 - \lambda(v)) w^N] \\ &= \sup_{\lambda \in [0,1]} [(\lambda w^P + (1 - \lambda) w^N)] = w^P \vee w^N . \end{aligned}$$

Notice that, recalling the notation of Section 1, $m = n - 1$. Properties (P1), (P2) and (P3) are easy to check.

In the sequel, we will fix both the normed space E and the group G , where n is the number of points in the corresponding Steiner tree problem that we want to solve.

Remark 2.5. We already know that the elements g_1, \dots, g_n are the multiplicities of the n points in the boundary, for the Steiner tree problem. The definition we just gave does not seem to be “symmetric”, in fact g_n has, in a certain sense, a privileged role, while the n points in the Steiner tree problem have of course all the same importance. To restore this lost symmetry, one may note that the group E is represented in \mathbb{R}^n as the hyperplane $P := \{x_1 + \dots + x_n = 0\}$ with a norm which is a multiple of the norm induced on P by the seminorm $\|\cdot\|_*$ on \mathbb{R}^n . Here g_1, \dots, g_n are the orthogonal projections on P of $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}$ and $-(\mathbf{e}_1 + \dots + \mathbf{e}_{n-1})$ respectively. It is easy to see that these points of π are the vertices of an $(n-1)$ -dimensional regular tetrahedron. In particular the unit elements of G are the vertices of a convex $(n-1)$ -dimensional polyhedron which is symmetric with respect to the origin. The vertices of the polyhedron are all the points of the form $g_{i_1} + \dots + g_{i_k}$ with $1 \leq i_1 < \dots < i_k \leq n-1$ and their inverses. It is clear that in this representation the role of the p_i 's is perfectly symmetric.

3. CALIBRATIONS

As we recalled in the Introduction, our interest in calibrations is the reason why we have chosen to provide an integral representation for E -currents, in fact the existence of a calibration guarantees the minimality of the associated current, as we will see in Proposition 3.2.

Definition 3.1. A smooth calibration associated with a k -dimensional rectifiable G -current $T(\Sigma, \tau, \theta)$ is a smooth compactly supported E^* -valued differential k -form ω , with the following properties:

- (i) $\langle \omega(x); \tau(x), \theta(x) \rangle = \|\theta(x)\|_G$ for \mathcal{H}^k -a.e. $x \in \Sigma$;
- (ii) $d\omega = 0$;
- (iii) $\|\omega\| \leq 1$, i.e., $\|\langle \omega; \tau \rangle\|_{E^*} \leq 1$, for every simple k -vector τ with $|\tau| = 1$.

Proposition 3.2. A rectifiable G -current T which admits a smooth calibration ω is a minimizer for the mass among the normal E -currents with boundary ∂T .

Proof. Fix a competitor T' which is a normal E -current associated with the vectorfield τ' , the multiplicity θ' and the measure $\mu_{T'}$, with $\partial T' = \partial T$. Since $\partial(T - T') = 0$, then $T - T'$ is a boundary of some current S in \mathbb{R}^d , and then

$$(3.1) \quad \mathbb{M}(T) = \int_{\Sigma} \|\theta\|_G d\mathcal{H}^k$$

$$(3.2) \quad \stackrel{(i)}{=} \int_{\Sigma} \langle \omega(x); \tau(x), \theta(x) \rangle d\mathcal{H}^k = \langle T; \omega \rangle$$

$$(3.3) \quad \stackrel{(ii)}{=} \langle T'; \omega \rangle = \int_{\mathbb{R}^d} \langle \omega(x); \tau'(x), \theta'(x) \rangle d\mu_{T'}$$

$$(3.4) \quad \stackrel{(iii)}{\leq} \int_{\mathbb{R}^d} \|\theta'\|_G d\mu_{T'} = \mathbb{M}(T'),$$

where each equality (respectively inequality) holds because of the corresponding property of ω , as established in Definition 3.1. In particular, equality in (ii) follows from

$$\langle T - T'; \omega \rangle = \langle \partial S; \omega \rangle = \langle S; d\omega \rangle = 0.$$

□

Remark 3.3. If T is a rectifiable G -current calibrated by ω , then every mass minimizer with boundary ∂T is calibrated by the same form ω . In fact, choose a mass minimizer $T' = T(\Sigma', \tau', \theta')$ with boundary $\partial T' = \partial T$: obviously we have $\mathbb{M}(T) = \mathbb{M}(T')$, then equality holds in (3.4), which means

$$\langle \omega(x); \tau'(x), \theta'(x) \rangle = \|\theta'(x)\|_G \quad \text{for } \mathcal{H}^k - \text{a.e. } x \in \Sigma'.$$

At this point we need a short digression on the representation of a E^* -valued 1-form ω ; we will consider $d = 2$, all our examples being for the Steiner tree problem in \mathbb{R}^2 . Remember that in Section 2 we fixed a basis (h_1, \dots, h_{n-1}) for E^* , dual to the basis (g_1, \dots, g_{n-1}) for E . We will represent

$$\omega = \begin{pmatrix} \omega_{1,1} dx_1 + \omega_{1,2} dx_2 \\ \vdots \\ \omega_{n-1,1} dx_1 + \omega_{n-1,2} dx_2 \end{pmatrix},$$

so that, if $\tau = \tau_1 \mathbf{e}_1 + \tau_2 \mathbf{e}_2 \in \Lambda_1(\mathbb{R}^2)$ and $v = v_1 g_1 + \dots + v_{n-1} g_{n-1} \in E$, then

$$\langle \omega; \tau, v \rangle = \sum_{i=1}^{n-1} v_i (\omega_{i,1} \tau_1 + \omega_{i,2} \tau_2).$$

Example 3.4. Consider the vector space E and the group G defined in Section 2 with $n = 3$; let

$$p_0 = (0, 0), p_1 = (1/2, \sqrt{3}/2), p_2 = (1/2, -\sqrt{3}/2), p_3 = (-1, 0)$$

(see Figure 1). Consider the rectifiable G -current T supported in the cone over (p_1, p_2, p_3) , with respect to p_0 , with piecewise constant weights $g_1, g_2, g_3 = -(g_1 + g_2)$ on $\Sigma_1, \Sigma_2, \Sigma_3$ respectively (recall Example 1.9 for notation and orientation). This current T is a minimizer for the mass. In fact, a constant G -calibration ω associated with T can be represented as

$$\omega := \begin{pmatrix} \frac{1}{2} dx_1 + \frac{\sqrt{3}}{2} dx_2 \\ \frac{1}{2} dx_1 - \frac{\sqrt{3}}{2} dx_2 \end{pmatrix}.$$

Condition (i) is easy to check and condition (ii) is trivially verified because ω is constant. To check condition (iii) we note that, for the generical vector $\tau = \cos \alpha \mathbf{e}_1 + \sin \alpha \mathbf{e}_2$, we have

$$\langle \omega; \tau, \cdot \rangle = \begin{pmatrix} \frac{1}{2} \cos \alpha + \frac{\sqrt{3}}{2} \sin \alpha \\ \frac{1}{2} \cos \alpha - \frac{\sqrt{3}}{2} \sin \alpha \end{pmatrix}.$$

In order to calculate the comass norm of ω , we could stick to the method explained in Section 2, but for $n = 3$ computations are simpler. Since the unit ball of E is convex, and its extreme points are the unit points of G , then it is sufficient to evaluate $\langle \omega; \tau, \cdot \rangle$ on $\pm g_1, \pm g_2, \pm(g_1 + g_2)$ (remember that $\|g_1 - g_2\|_E = 2$). We have

$$\begin{aligned} |\langle \omega; \tau, g_1 \rangle| &= |\langle \omega; \tau, -g_1 \rangle| = \left| \sin \left(\alpha + \frac{\pi}{6} \right) \right| \leq 1, \\ |\langle \omega; \tau, g_2 \rangle| &= |\langle \omega; \tau, -g_2 \rangle| = \left| \sin \left(\alpha + \frac{5}{6}\pi \right) \right| \leq 1, \\ |\langle \omega; \tau, g_1 + g_2 \rangle| &= |\langle \omega; \tau, -(g_1 + g_2) \rangle| = |\cos \alpha| \leq 1. \end{aligned}$$

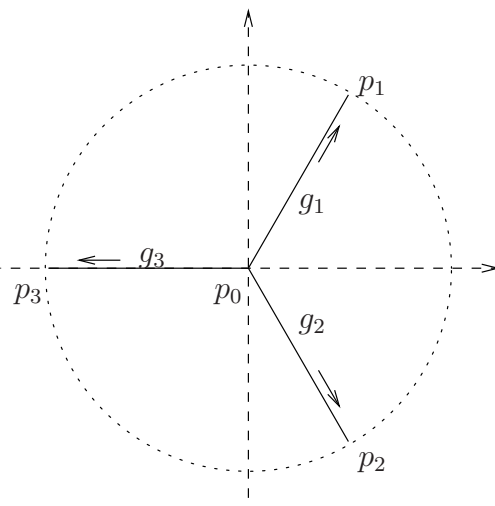


FIGURE 3. Solution for the problem with boundary on the vertex of an equilateral triangle

One may notice that, for every $x \in \mathbb{R}^2$, the E^* -covector $\omega(x)$ can be represented as a map from \mathbb{R}^2 to itself, since E^* and \mathbb{R}^2 coincide as vector spaces. Moreover, with a suitable choice of a basis for E^* this map is the identity. It turns out that the form ω has unit norm because the Euclidean unit ball is contained in the unit ball of E^* . One may be led to believe that for the same reason the cone on the vertices of a regular tetrahedron centered at the baricenter is the Steiner minimizer for the 4 vertices. This is not the case, and the analog of the form ω is not a calibration in this case, because its norm is bigger than one. In fact the Euclidean unit ball is not contained in the unit ball of E^* in dimension larger than 2.

An interesting way to generalize this result will be recalled in Remark 3.15.

In Definition 3.1 we intentionally kept vague the regularity of the form ω . Indeed ω has to be a compactly supported² smooth form, a priori, in order to fit Definition 1.5. Nevertheless, in some situations it will be useful to consider calibrations with lower regularity, for instance piecewise constant forms. As long as (3.2)-(3.4) remain valid, it is meaningful to do so; for this reason we introduce the following very general definition.

Definition 3.5. A *generalized calibration* associated with a k -dimensional normal E -current T is a linear and bounded functional ϕ on the space of normal E -currents satisfying the following conditions:

- (i) $\phi(T) = \mathbb{M}(T)$;
- (ii) $\phi(\partial R) = 0$ for any $(k+1)$ -dimensional normal E -current R ;
- (iii) $\|\phi\| \leq 1$.

Remark 3.6. The thesis in Proposition 3.2 is still true, since for every competitor T' with $\partial T = \partial T'$, there holds

$$\mathbb{M}(T) = \phi(T) = \phi(T') + \phi(\partial R) \leq \mathbb{M}(T') ,$$

where R is chosen such that $T - T' = \partial R$. Such R exists because T and T' are in the same homology class.

As examples, we present the calibrations for two well-known Steiner tree problems in \mathbb{R}^2 . Both calibrations in Example 3.10 and in Example 3.11 are piecewise constant 1-forms (with values in normed vector spaces of dimension 3 and 6, respectively), so first of all we have to establish a compatibility condition which brings piecewise constant forms back to Definition 3.5.

Definition 3.7. Fix a 1-dimensional rectifiable G -current T in \mathbb{R}^2 , $T = T(\Sigma, \tau, \theta)$. Assume we have a collection $\{C_r\}_{r \geq 1}$ which is a locally finite, Lipschitz partition of \mathbb{R}^2 , i.e., $\bigcup_{r \geq 1} C_r = \mathbb{R}^2$, the boundary of every set C_r is a Lipschitz curve and $C_r \cap C_s = \emptyset$ whenever $r \neq s$. Assume moreover that ∂C_r is a connected set for every r and that C_r contains the connected non-empty interior of its closure. Let us consider a compactly supported piecewise constant E^* -valued 1-form ω with

$$\omega \equiv \omega_r \quad \text{on } C_r$$

²Since we deal with currents that are compactly supported, we can easily drop the assumption that ω has compact support.

where $\omega_r \in \Lambda_E^1(\mathbb{R}^2)$ for every r . In particular $\omega \neq 0$ only on finitely many elements of the partition. Then we say that ω represents a *compatible calibration* for T if the following conditions hold:

- (i) for almost every $x \in \Sigma$, $\langle \omega(x); \tau(x), \theta(x) \rangle = \|\theta(x)\|_G$;
- (ii) for \mathcal{H}^1 -almost every point $x \in \partial C_r \cap \partial C_s$ we have

$$\langle \omega_r - \omega_s; \tau(x), \cdot \rangle = 0,$$

where τ is tangent to ∂C_r ;

- (iii) $\|\omega_r\| \leq 1$ for every r .

We will refer to condition (ii) with the expression of *compatibility condition* for a piecewise constant form.

Proposition 3.8. *Let ω be a compatible calibration for the rectifiable G -current T . Then T minimizes the mass among the normal E -currents with boundary ∂T .*

To prove this proposition we need the following result of decomposition of classical normal 1-currents, see [S] for the classical result and [PS1] for its generalization to metric spaces. Given a compact measure space (X, μ) and a family of k -currents $\{T_x\}_{x \in X}$ in \mathbb{R}^d , such that

$$\int_X \mathbb{M}(T_x) d\mu(x) < +\infty,$$

we denote by

$$T := \int_X T_x d\mu(x)$$

the k -current T satisfying

$$\langle T, \omega \rangle = \int_X \langle T_x, \omega \rangle d\mu(x),$$

for every smooth compactly supported k -form ω .

Proposition 3.9. *Every normal 1-current T in \mathbb{R}^d can be written as*

$$T = \int_0^M T_t dt,$$

where T_t is an integral current with $\mathbb{M}(T_t) \leq 2$ and $\mathbb{M}(\partial T_t) \leq 2$ for every t , and M is a positive number depending only on $\mathbb{M}(T)$ and $\mathbb{M}(\partial T)$. Moreover

$$\mathbb{M}(T) = \int_0^M \mathbb{M}(T_t) dt.$$

Proof of Proposition 3.8. Firstly we see that a suitable counterpart of Stokes Theorem holds. Namely, given a component ω^j of ω and a classical integral 1-current $T = T(\Sigma, \tau, 1)$ in \mathbb{R}^2 , without boundary, then the quantity

$$\langle \omega^j; T \rangle := \int_\Sigma \langle \omega^j(x); \tau(x) \rangle d\mathcal{H}^1(x)$$

is well defined, and we claim that it is equal to zero. The fact that it is well defined is a direct consequence of the compatibility condition (ii) in Definition 3.7. To prove that it is equal to zero, note that it is possible to find at most countably many unit

multiplicity integral 1-currents $T_i = T(\Sigma_i, \tau_i, 1)$ in \mathbb{R}^2 , without boundary, each one supported in a single set C_r , such that $\sum_i T_i = T$. Since

$$\int_{\Sigma_i} \langle \omega^j(x); \tau_i(x) \rangle d\mathcal{H}^1(x) = 0$$

for every i , then the claim follows from (ii). As a consequence we have that there exists a family of Lipschitz functions $\phi_j : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for every (classical) integral 1-current S with $\mathbb{M}(\partial S) \leq 2$ (in particular $\partial S = \delta_{x_S} - \delta_{y_S}$, and $x_S = y_S$ if and only if $\partial S = 0$) there holds:

$$\langle \omega^j; S \rangle = \phi_j(x_S) - \phi_j(y_S), \quad \text{for every } j.$$

In fact it is sufficient to choose $\phi_j(0) = 0$ and

$$\phi_j(x) = |x| \int_0^1 \langle \omega^j(tx); \frac{x}{|x|} \rangle dt.$$

Moreover it is easy to see that every ϕ_j is constant outside of the support of ω^j , so we can assume, possibly subtracting a constant, that ϕ_j is compactly supported.

Now, take a 2-dimensional normal E -current T . Let $\{T^j\}_j$ be the components of T . For every j , use Proposition 3.9 to write $S^j := \partial T^j = \int_0^{M_j} S_t^j dt$. Then we have

$$\langle \omega; \partial T \rangle = \sum_j \int_0^{M_j} \langle \omega^j; S_t^j \rangle dt = \sum_j \int_0^{M_j} \phi_j(x_{S_t^j}) - \phi_j(y_{S_t^j}) dt.$$

Since for every j we have

$$0 = \partial(\partial T^j) = \int_0^{M_j} \delta_{x_{S_t^j}} - \delta_{y_{S_t^j}} dt,$$

then, for every j , we must have

$$\int_0^{M_j} g(x_{S_t^j}) - g(y_{S_t^j}) dt = 0,$$

for every compactly supported Lipschitz function g , in particular for every ϕ_j . Hence we have $\langle \omega; \partial T \rangle = 0$. \square

Example 3.10. Consider the points

$$p_1 = (1, 1), p_2 = (1, -1), p_3 = (-1, -1), p_4 = (-1, 1) \in \mathbb{R}^2.$$

The length-minimizer graphs for the classical Steiner tree problem³ are those represented in Figure 1. We associate with each point p_j with $j = 1, \dots, 4$ the coefficients $g_j \in G$, where G has “dimension” $m = 3$: let us call

$$B := g_1 \delta_{p_1} + g_2 \delta_{p_2} + g_3 \delta_{p_3} + g_4 \delta_{p_4}.$$

This 0-dimensional current is our boundary. Intuitively our mass-minimizing candidates among 1-dimensional rectifiable G -currents are those represented in Figure 4:

³In dimension $d > 2$, an interesting question related to this problem is the following: is the cone over the $(d-2)$ -skeleton of the hypercube in \mathbb{R}^d area minimizing, among hypersurfaces separating the faces? The question has a positive answer if and only if $d \geq 4$ (see [B1] for the proof).

these currents $T_{\text{hor}}, T_{\text{ver}}$ are supported in the sets drawn, respectively, with continuous and dashed lines in Figure 4 and have piecewise constant coefficients intended to satisfy the boundary condition $\partial T_{\text{hor}} = B = \partial T_{\text{ver}}$.

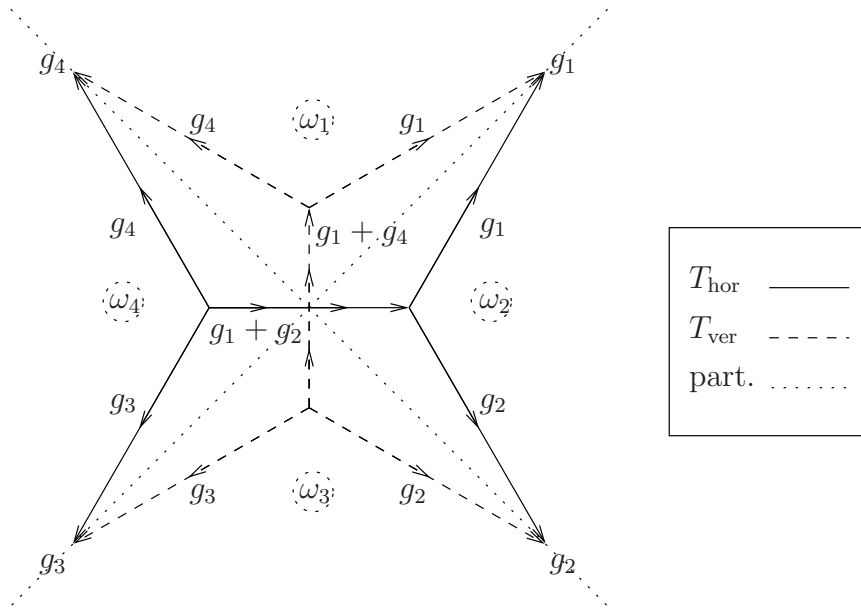


FIGURE 4. Solution for the mass minimization problem

In this case, a compatible calibration for both T_{hor} and T_{ver} is defined piecewise as follows (the notation is the same as in Example 3.4 and the partition is delimited by the dotted lines):

$$\begin{aligned} \omega_1 &\equiv \begin{pmatrix} \frac{\sqrt{3}}{2}dx_1 + \frac{1}{2}dx_2 \\ \left(1 - \frac{\sqrt{3}}{2}\right)dx_1 - \frac{1}{2}dx_2 \\ \left(-1 + \frac{\sqrt{3}}{2}\right)dx_1 - \frac{1}{2}dx_2 \end{pmatrix} & \omega_2 &\equiv \begin{pmatrix} \frac{1}{2}dx_1 + \frac{\sqrt{3}}{2}dx_2 \\ \frac{1}{2}dx_1 - \frac{\sqrt{3}}{2}dx_2 \\ -\frac{1}{2}dx_1 - \left(1 - \frac{\sqrt{3}}{2}\right)dx_2 \end{pmatrix} \\ \omega_3 &\equiv \begin{pmatrix} \left(1 - \frac{\sqrt{3}}{2}\right)dx_1 + \frac{1}{2}dx_2 \\ \frac{\sqrt{3}}{2}dx_1 - \frac{1}{2}dx_2 \\ -\frac{\sqrt{3}}{2}dx_1 - \frac{1}{2}dx_2 \end{pmatrix} & \omega_4 &\equiv \begin{pmatrix} \frac{1}{2}dx_1 + \left(1 - \frac{\sqrt{3}}{2}\right)dx_2 \\ \frac{1}{2}dx_1 - \left(1 - \frac{\sqrt{3}}{2}\right)dx_2 \\ -\frac{1}{2}dx_1 - \frac{\sqrt{3}}{2}dx_2 \end{pmatrix} \end{aligned}$$

It is easy to check that ω satisfies both condition (i) and the compatibility condition of Definition 3.7. To check that condition (iii) is satisfied, we can use formula (2.7).

Example 3.11. Consider the vertices of a regular hexagon plus the center, namely

$$\begin{aligned} p_1 &= (1/2, \sqrt{3}/2), & p_2 &= (1, 0), & p_3 &= (1/2, -\sqrt{3}/2), \\ p_4 &= (-1/2, -\sqrt{3}/2), & p_5 &= (-1, 0), & p_6 &= (-1/2, \sqrt{3}/2), & p_7 &= (0, 0) \end{aligned}$$

and associate with each point p_j the corresponding multiplicity $g_j \in G$, where G is the group with dimension $m = 6$. A mass-minimizer for the problem with boundary

$$B = \sum_{j=1}^7 g_j \delta_{p_j}$$

is illustrated in Figure 5, the other one can be obtained with a $\pi/3$ -rotation of the picture.

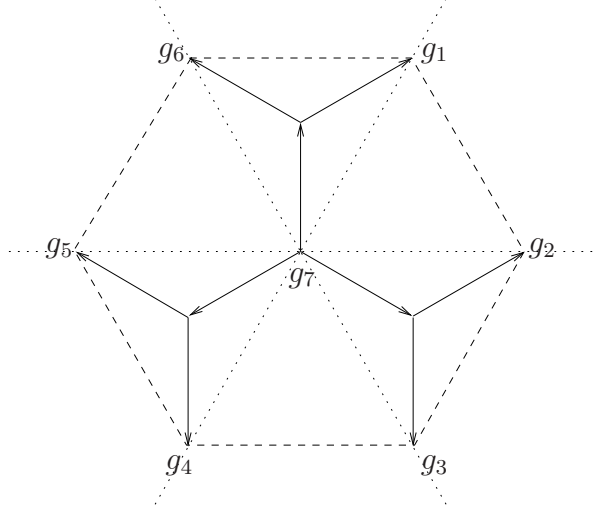


FIGURE 5. Solution for the mass minimization problem

Let us divide \mathbb{R}^2 in 6 cones of angle $\pi/3$, as in Figure 5; we will label each cone with a number from 1 to 6, starting from that containing $(0, 1)$ and moving clockwise. A compatible calibration for the two minimizers is the following

$$(3.5) \quad \begin{aligned} \omega_1 &= \begin{pmatrix} -\frac{\sqrt{3}}{2}dx_1 + \frac{1}{2}dx_2 \\ \frac{\sqrt{3}}{2}dx_1 + \frac{1}{2}dx_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \omega_2 &= \begin{pmatrix} 0 \\ dx_2 \\ \frac{\sqrt{3}}{2}dx_1 - \frac{1}{2}dx_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \omega_3 &= \begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{3}}{2}dx_1 + \frac{1}{2}dx_2 \\ -dx_2 \\ 0 \\ 0 \end{pmatrix} \\ \omega_4 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\sqrt{3}}{2}dx_1 - \frac{1}{2}dx_2 \\ -\frac{\sqrt{3}}{2}dx_1 - \frac{1}{2}dx_2 \\ 0 \end{pmatrix} & \omega_5 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -dx_2 \\ -\frac{\sqrt{3}}{2}dx_1 + \frac{1}{2}dx_2 \end{pmatrix} & \omega_6 &= \begin{pmatrix} dx_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{\sqrt{3}}{2}dx_1 - \frac{1}{2}dx_2 \end{pmatrix} \end{aligned}$$

Again, it is not difficult to check that ω satisfies both condition (i) and the compatibility condition of Definition 3.7. To check that condition (iii) is satisfied, we use formula (2.7).

Remark 3.12. We may wonder whether or not the calibration given in Example 3.11 can be adjusted so to work for the set of the vertices of the hexagon (without the seventh point in the center): the answer is negative, in fact the support of the current in Figure 5 is not a solution for the Steiner tree problem on the six points,

the perimeter of the hexagon minus one side being the shortest graph, as proved in [JK].

Remark 3.13. In both Examples 3.10 and 3.11, once we fixed the group G and we decided to look for a piecewise constant calibration for our candidates, the construction of ω was forced by both conditions (i) of Definition 3.1 and the compatibility condition of Definition 3.7. Notice that the calibration for the Example 3.11 has evident analogies with the one exhibited in the Example 3.4. Actually we obtained the first one simply pasting suitably “rotated” copies of the second one.

In the following remarks we intend to underline the analogies and the connections with calibrations in similar contexts. See Chapter 6 of [M2] for an overview on the subject of calibrations.

Remark 3.14 (Functionals defined on partitions and null lagrangians). There is an interesting and deep analogy between calibrations and null lagrangians, analogy that keeps unaltered in the group coefficients framework. Consider some points $\{\eta_1, \dots, \eta_n\} \subset \mathbb{R}^m$, with

$$(3.6) \quad |\eta_i - \eta_j| = 1 \quad \forall i \neq j ;$$

for instance, the vertices of the regular n -tetrahedron with unit edge in \mathbb{R}^{n-1} satisfy condition (3.6) (see Remark 2.5 to deepen the analogy with our group G in Section 2). We fix an open set with Lipschitz boundary $\Omega \subset \mathbb{R}^d$, for example $\Omega = B(0, 1)$ and consider a bounded variation map $u : \Omega \rightarrow \{\eta_1, \dots, \eta_n\}$. Let us call $S[u] \subset \Omega$ the jump set associated with u : if ν is the unit normal according to some orientation of $S[u]$, let us say that u^+ and u^- are the traces of the BV function from above and from below the jump set (with respect to ν) respectively. We are interested in BV maps because

$$\int_{\Omega} |Du(x)| dx = \int_{S[u]} |u^+(x) - u^-(x)| dx = \mathcal{H}^{d-1}(S[u]) ,$$

thanks to condition (3.6). Therefore it is natural to study the variational problem

$$(3.7) \quad \inf \left\{ \int_{\Omega} |Du| : u \in BV(\Omega; \{\eta_1, \dots, \eta_n\}), u|_{\partial\Omega} \equiv u_0 \right\} .$$

This problem concerns the theory of partitions of an open set Ω in a finite number of sets of finite perimeter. This theory was developed by Ambrosio and Braides in [AB1, AB2], which we refer to for a complete exposition.

Assume there exists a vector field $V : \Omega \times \{\eta_1, \dots, \eta_n\} \rightarrow \mathbb{R}^d$ such that the following conditions hold:

(i) for every $x \in S[u]$,

$$[V(x, u^+(x)) - V(x, u^-(x))] \cdot \nu(x) = 1 ;$$

(ii) marking $v_i(x) := V(x, \eta_i)$, $i = 1, \dots, n$,

$$\operatorname{div}_x V(x, \eta_i) = \operatorname{div} v_i(x) = 0 ;$$

(iii) for every $i, j = 1, \dots, n$,

$$|v_i(x) - v_j(x)| \leq 1 .$$

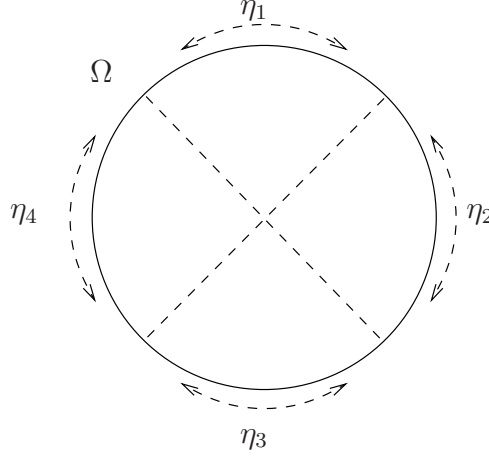


FIGURE 6. Boundary data

In this case we can say that the functional $u \mapsto \int_{\Omega} \operatorname{div}(V(x, u(x))) \, dx$ is a null lagrangian⁴, because it depends only on the boundary value u_0 . As it happens in Proposition 3.2, if u admits a vector field V with the previous properties, then u is a minimizer for the variational problem (3.7) with $u_0 = u|_{\partial\Omega}$, because

$$\begin{aligned}
 \int_{\Omega} |Du| \, dx &= \mathcal{H}^{d-1}(S[u]) \stackrel{(i)}{=} \int_{\Omega} \operatorname{div}(V(x, u(x))) \, dx = \int_{\Omega} \operatorname{div}(V(x, u'(x))) \, dx \\
 &\stackrel{(ii)}{=} \int_{\Omega} V_u(x, u'(x)) \cdot \nabla u'(x) \, dx \\
 &= \int_{S[u']} (V(x, (u')^+(x)) - V(x, (u')^-(x))) \cdot \nu(x) \, d\mathcal{H}^{d-1}(x) \\
 &\stackrel{(iii)}{\leq} \int_{S[u']} |(u')^+ - (u')^-| \, d\mathcal{H}^{d-1}(x) = \int_{\Omega} |Du'| \, dx .
 \end{aligned}$$

where u' is a competitor in $BV(\Omega; \{\eta_1, \dots, \eta_n\})$ with the same trace as u on $\partial\Omega$. In order to clarify the similarity of the null lagrangian problem with the Steiner tree problem, consider the trace u_0 in Figure 6. The minimizers of the problem (3.7) are showed in Figure 7. As a matter of fact, the minimizers $u_{\text{hor}}, u_{\text{ver}}$ admit a null lagrangian vector field, satisfying a compatibility condition and clearly related to the calibration ω defined above.

Remark 3.15 (Clusters with multiplicities). In [M1], F. Morgan applies flat chains with coefficients in a group G to soap bubble clusters and immiscible fluids, following the idea of B. White in [W1]. The model (in \mathbb{R}^d for m immiscible fluids) associates to each fluid a coefficient $f_i \in G$, where $G \cong \mathbb{Z}^m \subset \mathbb{R} \otimes G \cong \mathbb{R}^m$ throughout the paper. Naturally, we are looking for least-energy interfaces, that is a minimizing $(d-1)$ -dimensional flat chain with coefficient in G . The mass norm is induced by the largest norm in $\mathbb{R} \otimes G$ such that

$$\|f_i\|_G = a_i \quad \forall i \in \{1, \dots, m\}$$

⁴See [D] for an overview on null lagrangians.

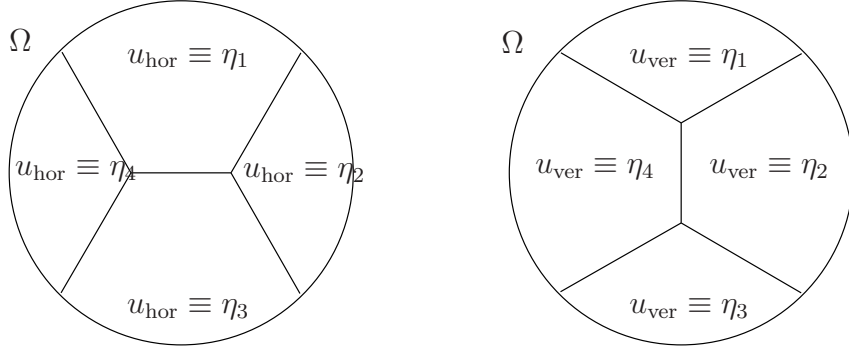


FIGURE 7. Minimizers

and

$$\|f_i - f_j\| = a_{ij} \quad \forall i, j.$$

Concerning soap bubble clusters, we choose $a_i = a_{ij} = 1$; hence, if $m = 2$, the unit ball is pictured in Figure 8.

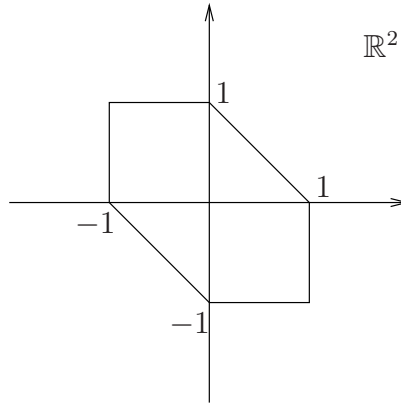


FIGURE 8. Unit ball in F. Morgan's model for soap bubble clusters

Following the idea in [M1], a calibration for a rectifiable m -chain T in \mathbb{R}^d is a homomorphism

$$\omega : G \rightarrow \mathcal{C}^\infty(\mathbb{R}^d, \Lambda^m(\mathbb{R}^d))$$

with the following properties:

- (i) $\langle \vec{T}(x); \omega(g)(x) \rangle = \|g\|_G$ for a.e. $x \in \text{supp}(T)$;
- (ii) $\omega(g)$ is a closed differential m -form for every $g \in G$;
- (iii) $\|\omega(g)\| \leq \|g\|_G$ for every $g \in G$.

These properties guarantee that T is a mass-minimizer among flat chains with the same boundary; the proof is by all means analogous to the one given in Proposition 3.2. Notice that this definition for the calibration works truly well in the case of a free abelian group \mathbb{Z}^m , because we are considering homomorphisms with values in a vector space and every finite order subgroup is trivialized by such a homomorphism. As F. Morgan shows in Proposition 4.5 of [M1], in this framework it is easy to prove

a generalization of Example 3.4: consider a cone $C = \sum_{i=1}^n g_i v_i$ in \mathbb{R}^d of unit vectors v_i with coefficients in $G = \text{span}\{g_i\}$ and assume that

$$\left| \sum_{i=1}^n \lambda_i \|g_i\|_G v_i \right| \leq \left\| \sum_{i=1}^n \lambda_i g_i \right\|_G \quad \forall \lambda_i \geq 0 ,$$

then C is a minimizer because it admits a calibration with constant coefficients.

Remark 3.16 (Paired calibrations for the tetrahedron). It is worth mentioning another analogy between the technique of calibrations (for currents with coefficients in a group) illustrated in this paper and the technique of paired calibrations in [LM]. We confine our attention on a specific example: consider the 1-skeleton of the tetrahedron in \mathbb{R}^3 , centered in the origin, then the truncated cone over the skeleton is the surface with least area among those separating the faces of the tetrahedron. In [LM] this is obtained through paired calibrations.

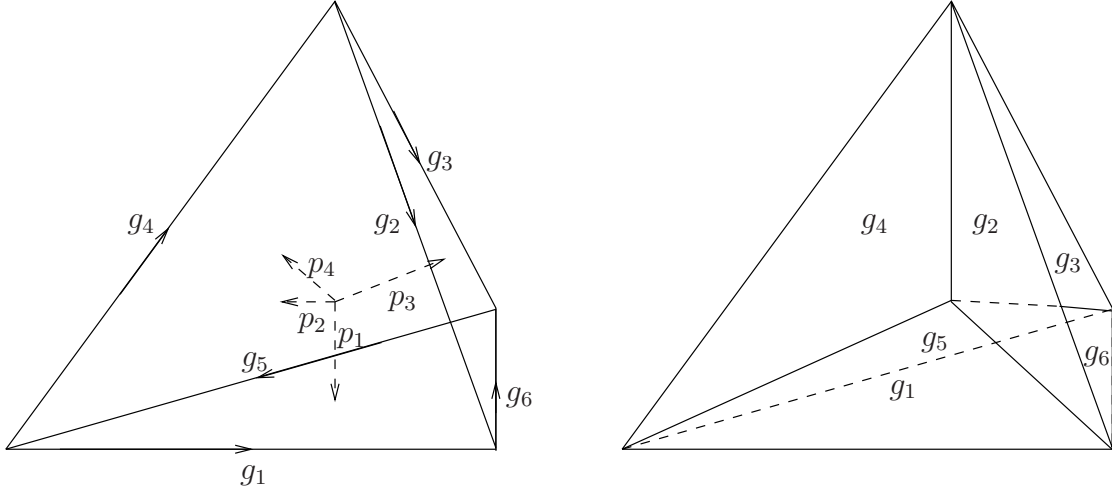


FIGURE 9. 1-skeleton of the tetrahedron and mass-minimizing current

We sketch here a way to get this result through currents with coefficients in a group⁵. Put

$$\begin{aligned} g_1 &:= p_2 - p_1 & g_2 &:= p_3 - p_2 & g_3 &:= p_4 - p_3 \\ g_4 &:= p_4 - p_2 & g_5 &:= p_4 - p_1 & g_6 &:= p_3 - p_1 , \end{aligned}$$

where $p_i \in \mathbb{R}^3$ are the unit vectors directed from the baricenter of the tetrahedron to the centers of the faces, with labels as in Figure 9. Notice that the Euclidean norm of $p_j - p_i$ is 1 for any $i \neq j$. This choice of g_i will be made clear in few lines, but let us remark that the coefficients g_i coincide with those of the paired calibration in [LM]. If $G \subset \mathbb{R}^3$ is the additive group generated by $\{p_1, p_2, p_3\}$ and it is endowed with the Euclidean norm in \mathbb{R}^3 , let us assign a (constant) coefficient g_i to each segment of the skeleton, as illustrated in Figure 9. Thus the identity is a calibration for the

⁵Notice that the theory of currents with coefficients in a group has been stated for every dimension k . Of course, the equivalence with the Steiner tree problem of Section 2 has no meaning in dimension $k \geq 2$.

2-current in the right side of Figure 9, that is the truncated cone on the 1-skeleton of the tetrahedron with coefficients g_1, \dots, g_6 on each piece of plane⁶.

Following an idea of Federer (see [Fe2]), in [M1] and [LM] (and in [B1] and [B2], as well) one can observe the exploitation of the duality between minimal surfaces and maximal flows through the same boundary. We will examine this duality in Section 4, but we conclude the present section with a remark closely related to this idea.

Remark 3.17 (Covering spaces and calibrations for soap films). In [B2] Brakke develops new tools in Geometric Measure Theory for the analysis of soap films: as the underlying physical problem suggests, one can represent a soap film as the superposition of two oppositely oriented currents. In order to avoid cancellations of multiplicities, the currents are defined in a covering space and, as stated in [B2], the calibration technique holds valid.

Let us remark that cancellations between multiplicities were a significant obstacle for the Steiner tree problem, too. The representation of currents in a covering space goes in the same direction of currents with coefficients in a group, though, as in Remark 3.16, a sort of Poincaré duality occurs in the formulation of the Steiner tree problem (1-dimensional currents in \mathbb{R}^d) with respect to the soap film problem (currents of codimension 1 in \mathbb{R}^d).

4. EXISTENCE OF THE CALIBRATION AND OPEN PROBLEMS

Once we established that the existence of a calibration is a sufficient condition for a rectifiable G -current to be a mass-minimizer, we may wonder if the converse is also true: does a calibration (of some sort) exist for every mass-minimizing rectifiable G -current?

Let us step backward: does it occur for classical integral currents? The answer is quite articulate, but we can briefly summarize the state of the art we will rely upon. See also [B2] for an overview of this problem of duality between minima and calibrations.

Remark 4.1. An actual calibration cannot exist for every minimizer. In fact there are currents which minimize the mass among integral currents with a fixed boundary, but not among normal currents (in some cases the two problems have different minima). This means that such integral currents cannot be calibrated, in fact the existence of a calibration proves the minimality among normal currents.

Remark 4.2. For every mass-minimizing classical normal k -current T , there exists a generalized calibration ϕ in the sense of Definition 3.5. Moreover, by means of the Riesz Representation Theorem, ϕ can be represented by a measurable map from U to $\Lambda^k(\mathbb{R}^d)$. This result is contained in [Fe2].

In particular, Remark 4.2 provides a positive answer to the question of the existence of a generalized calibration for mass-minimizing integral currents of dimension $k = 1$, because minima among both normal and integral currents coincide, as we prove in

⁶The orientation in each flat piece of the cone is determined by the orientation of the corresponding segment in the 1-skeleton of the tetrahedron, as shown in Figure 9. Let us finally point out that the boundary the truncated cone with coefficients g_1, \dots, g_6 has no support out of the 1-skeleton of the tetrahedron because the sum of the coefficients on the edges of the cone gives 0.

Proposition 4.4. It is possible to apply the same technique in the class of normal E -currents, therefore we have the following proposition.

Proposition 4.3. *For every mass minimizing normal E -current T , there exists a generalized calibration.*

In order to guarantee the existence of a generalized calibration also for 1-dimensional mass-minimizing rectifiable G -currents, we need an analog of Proposition 4.4 in the framework of G -currents. Namely, we need to prove that the minimum of the mass among 1-dimensional normal E currents with the same boundary coincides with the minimum calculated among rectifiable G -currents. Here the boundary is of course a 0-dimensional rectifiable G -current. This is a well-known issue for classical k -dimensional currents: for $k \geq 2$ it is not even known whether the two minima are commensurable, i.e., whether or not there exists a constant C such that, for every fixed $(k-1)$ -dimensional integral boundary B , the minimum of the mass among integral k -currents with boundary B is less than C times the minimum among normal k -currents with the same boundary. From the argument used in the proof of Proposition 4.4 we realize that the equality of the two minima in the framework of 1-dimensional E -currents is equivalent to the homogeneity property in Remark 4.5. This property, which is trivially verified for classical integral currents, seems to be an interesting issue in the class of rectifiable G -currents. In Example 4.6 we exhibit a subset $M \subset \mathbb{R}^2$ such that, if our currents are forced to be supported on M , then the homogeneity property does not hold. In other words, we can say that equality of the two minima does not hold in the framework of 1-dimensional E -currents on the metric space M . We can see the same phenomenon if we substitute the metric space M with the metric space \mathbb{R}^2 endowed with a density, which is unitary on the points of M and very high outside.

The following fact is probably in the folklore, unfortunately we were not able to find any literature on it. We give a proof here in order to put in evidence the problems arising in the case of currents with coefficients in a group.

Proposition 4.4. *Consider the boundary of an integral 1-current in \mathbb{R}^d , represented as*

$$(4.1) \quad B_0 = - \sum_{i=1}^{N_-} a_i \delta_{x_i} + \sum_{j=1}^{N_+} b_j \delta_{y_j}, \quad a_i, b_j \in \mathbb{N} .$$

If we denote

$$\mathcal{M}_N(B_0) := \min\{\mathbb{M}(T) : T \text{ is a normal current } , \partial T = B_0\}$$

and

$$\mathcal{M}_I(B_0) := \min\{\mathbb{M}(T) : T \text{ is an integral current } \partial T = B_0\} \geq \mathcal{M}_N(B_0) ,$$

then the minima of the mass of 1-currents with boundary B_0 among normal 1-currents and among integral 1-currents coincide, that is

$$\mathcal{M}_N(B_0) = \mathcal{M}_I(B_0) .$$

Proof. Let us assume that the minimum among normal currents is attained at some current T_0 , that is

$$\mathbb{M}(T_0) = \mathcal{M}_N(B_0) .$$

Let $\{T_h\}_{h \in \mathbb{N}}$ be an approximation of T_0 made by polyhedral 1-currents, such that

- $\mathbb{M}(T_h) \rightarrow \mathbb{M}(T_0)$ as $h \rightarrow \infty$,
- $\partial T_h = B_0$ for all $h \in \mathbb{N}$,
- the multiplicities allowed in T_h are only integer multiples of $\frac{1}{h}$.

The existence of such a sequence is a consequence of the Polyhedral Approximation Theorem (see Theorem 4.2.24 of [Fe1] or [KP] for the detailed statement and the proof). Thanks to Theorem 2.2, it is possible to decompose such a T_h as a sum of two addenda:

$$(4.2) \quad T_h = P_h + C_h ,$$

so that

$$\mathbb{M}(T_h) = \mathbb{M}(P_h) + \mathbb{M}(C_h) \quad \forall h \geq 1$$

and

- $\partial C_h = 0$, so C_h collects the cyclical part of T_h ;
- P_h does not admit any decomposition $P_h = A + B$ satisfying $\partial A = 0$ and $\mathbb{M}(P_h) = \mathbb{M}(A) + \mathbb{M}(B)$

It is clear that P_h is the sum of a certain number of polyhedral currents $P_h^{i,j}$ each one having boundary a non-negative multiple of $-\frac{1}{h}\delta_{x_i} + \frac{1}{h}\delta_{y_j}$ and satisfying

$$\mathbb{M}(P_h) = \sum_{i,j} \mathbb{M}(P_h^{i,j})$$

We replace each $P_h^{i,j}$ with the oriented segment $Q_h^{i,j}$, from x_i to y_j having the same boundary as $P_h^{i,j}$ (therefore having multiplicity a non-negative multiple of $\frac{1}{h}$). This replacement is represented in Figure 10

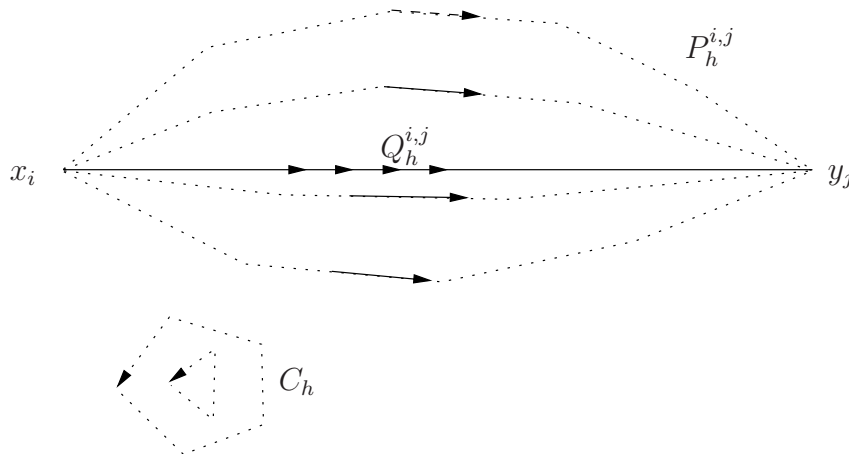


FIGURE 10. Replacement with a segment

Since this replacement obviously does not increase the mass, there holds $\mathbb{M}(P_h) \geq \mathbb{M}(Q_h)$, where $Q_h = \sum_{i,j} Q_h^{i,j}$. In other words we can write $Q_h = \int_I T d\lambda_h$, as an integral of currents, with respect to a discrete measure λ_h supported on the finite set I of unit multiplicity oriented segments with the first extreme among the points

x_1, \dots, x_{N_-} and second extreme among the points y_1, \dots, y_{N_+} . It is also easy to see that the total variation of λ_h has eventually the following bound from above

$$\|\lambda_h\| \leq \frac{\mathbf{M}(T_h)}{\min_{i \neq j} d(x_i, y_j)} \leq \frac{\mathbf{M}(T_0) + 1}{\min_{i \neq j} d(x_i, y_j)} .$$

Hence, up to subsequences, λ_h converges to some positive measure λ on I and so the normal 1-current

$$Q = \int_{T \in I} T \, d\lambda$$

satisfies

$$(4.3) \quad \partial Q = B_0$$

and

$$\mathbf{M}(Q) \leq \mathbf{M}(T_0) = \mathcal{M}_N(B_0) .$$

In order to conclude the proof of the theorem, we need to show that Q can be replaced by an integral current R with same boundary and mass $\mathbf{M}(R) = \mathbf{M}(Q) \leq \mathcal{M}_N(B_0)$. Since I is the set of unit multiplicity oriented segments Σ^{ij} from x_i to y_j , we can obviously represent

$$Q = \sum_{i,j} k^{ij} \Sigma^{ij} \quad \text{with } k^{ij} \in \mathbb{R} ,$$

and, again, thanks to (4.3),

$$\sum_{i=1}^{N_-} k^{ij} = b_j \quad \text{and} \quad \sum_{j=1}^{N_+} k^{ij} = a_i .$$

If $k^{ij} \in \mathbb{Z}$ for any i, j , then Q itself is integral and then we are done; if not, let us consider the finite set of non-integer multiplicities

$$K_{\mathbb{R} \setminus \mathbb{Z}} := \{k^{ij} : i = 1, \dots, N_-, j = 1, \dots, N_+\} \setminus \mathbb{Z} \neq \emptyset .$$

We fix $k \in K_{\mathbb{R} \setminus \mathbb{Z}}$ and we choose an index (i_0, j_0) , such that k is the multiplicity of the oriented segment $\Sigma^{i_0 j_0}$ in Q . It is possible to track down a non-trivial cycle \overline{Q} in Q with the following algorithm: after $\Sigma^{i_0 j_0}$, choose a segment from $x_{i_1} \neq x_{i_0}$ to y_{j_0} with non-integer multiplicity, it must exist because $B_0 = \partial Q$ is integral. Then choose a segment from x_{i_1} to $y_{j_1} \neq y_{j_0}$ with non-integer multiplicity and so on. Since $K_{\mathbb{R} \setminus \mathbb{Z}}$ is finite, at some moment we will get a cycle. Up to reordering the indices i and j we can write

$$\overline{Q} = \sum_{l=1}^n (\Sigma^{i_l j_l} - \Sigma^{i_{l+1} j_l}) .$$

We will denote by

$$\begin{aligned} \alpha &:= \min_l (k^{i_l j_l} - \lfloor k^{i_l j_l} \rfloor) > 0 \\ \beta &:= \min_l (k^{i_{l+1} j_l} - \lfloor k^{i_{l+1} j_l} \rfloor) > 0 . \end{aligned}$$

Finally notice that both $Q - \alpha \overline{Q}$ and $Q + \beta \overline{Q}$ have lost at least one non-integer coefficient; in addition, we claim that either

$$(4.4) \quad \mathbf{M}(Q - \alpha \overline{Q}) \leq \mathbf{M}(Q) \quad \text{or} \quad \mathbf{M}(Q + \beta \overline{Q}) \leq \mathbf{M}(Q) .$$

In fact we can define the linear auxiliary function

$$F(t) := \mathbb{M}(Q) - \mathbb{M}(Q - t\bar{Q}) = \sum_l (k^{i_l j_l} - t) d(x_{i_l}, y_{j_l}) + (k^{i_{l+1} j_l} + t) d(x_{i_{l+1}}, y_{j_l})$$

for which $F(0) = 0$, so either

$$F(\alpha) \geq 0 \quad \text{or} \quad F(-\beta) \geq 0 .$$

Iterating this procedure finitely many times, we obtain an integral current without increasing the mass. \square

Now, we want to know whether the analog of this result holds also in the framework of 1-dimensional E -currents. Fix a 0-dimensional rectifiable G -current R in $U \subset \mathbb{R}^d$. Do the minima for the mass among 1-dimensional normal E -currents and rectifiable G -currents with boundary R coincide?

Remark 4.5. The answer to the previous question is positive if and only if the following is true: given $R = \sum_{i=1}^n g_i \delta_{x_i}$ with $\|g_i\|_G = 1$ and T a rectifiable G -current which is mass-minimizer with $\partial T = R$, then for every $k \in \mathbb{N}$ we have that

$$(4.5) \quad \min \{ \mathbb{M}(S) : S \text{ rectifiable } G\text{-current}, \partial S = kR \} = k\mathbb{M}(T) .$$

Notice that, using the notation introduced in Theorem 4.4, (4.5) can be meaningfully written as

$$(4.6) \quad \mathcal{M}_I(kR) = k\mathcal{M}_I(R) .$$

The condition 4.6 is clearly necessary to have the equality of the two minima. It is also sufficient, in fact one can approximate a normal E -current with polyhedral currents with coefficients in $\mathbb{Q}G$.

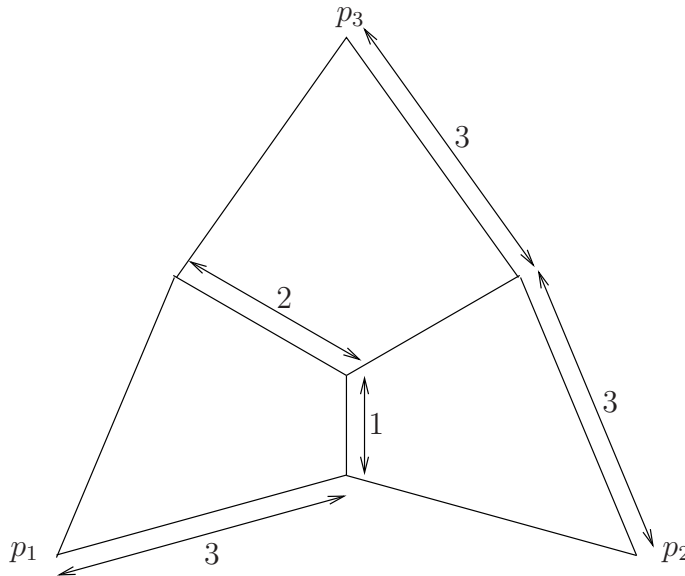


FIGURE 11. Metric space in the Example 4.6

Example 4.6. Consider the metric space⁷ $M \subset \mathbb{R}^2$ given⁸ in Figure 11. Consider the group G , with $n = 3$, introduced in Section 2 and let $R := g_1\delta_{p_1} + g_2\delta_{p_2} + g_3\delta_{p_3}$. We will show that (4.6) does not hold even when $k = 2$. In fact it is trivial to prove that

$$\mathcal{M}_I(R) = 12 .$$

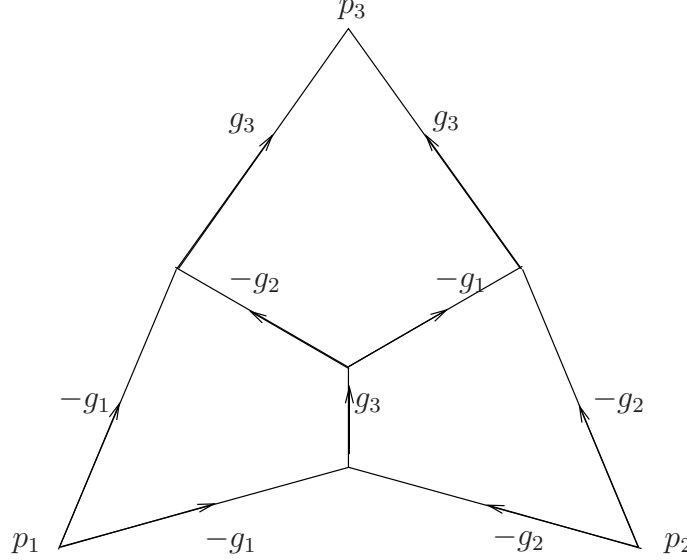


FIGURE 12. Counterexample to (4.6)

Nevertheless, concerning $\mathcal{M}_I(2R)$, it is shown in Figure 12 that

$$\mathcal{M}_I(2R) \leq 23 < 24 = 2\mathcal{M}_I(R) .$$

Remark 4.7. One can expect a behaviour like that in Example 4.6 in the metric space \mathbb{R}^2 endowed with a density which is very high outside of the subset $M \subset \mathbb{R}^2$. To be precise, let us consider a bounded continuous function $W : \mathbb{R}^2 \rightarrow \mathbb{R}$, with $W \equiv 1$ on M and $W \gg 1$ out of a small neighbourhood of M . For any couple $(x_0, x_1) \in \mathbb{R}^2$, the distance on (\mathbb{R}^2, W) is given by

$$d(x_0, x_1) = \inf \left\{ \int_0^1 |\gamma'(t)| W(\gamma(t)) dt : \gamma(0) = x_0 \text{ and } \gamma(1) = x_1 \right\} .$$

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⁷For currents in metric spaces, see [AKi].

⁸The length of each segment is explicitly declared in Figure 11, note that the set is symmetric with respect to the vertical axis.

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