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A Note on Resistance of NPT to Mixture of Separable States

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Abstract. We study the stability of NPT property of an arbitrary pure entangled state under the mixture of arbitrary pure separable states. For bipartite pure states with Schmidt number n ($n > 1$) which is NPT, we show that this state is still NPT when it is mixed with no more than $\frac{n(n-1)}{2} - 1$ arbitrary pure separable states. This result is generalized to multipartite case.

Keywords: partial transposition; separable state; entanglement

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Quantum entangled states are used as key resources in quantum information processing such as quantum teleportation, cryptography, dense coding, error correction and parallel computation. Due to the decoherence maximally entangled pure states may evolve into non-maximally entangled ones. Distillation is an important protocol in improving the quantum entanglement against the decoherence due to noisy channels in information processing. However, there are two kinds of quantum entangled states, distillable and non distillable.

Let H be a d -dimensional complex Hilbert spaces, with $\{|i\rangle\}_{i=1}^d$ the orthonormal basis of the space H . Any bipartite quantum state $\rho \in H \otimes H$ can be written as $\rho = \sum_{i,j,k,l} \rho_{ij,kl} |ij\rangle\langle kl|$, $\rho_{ij,kl} \in \mathbb{C}$. The partial transposition of ρ with respect to the first (resp. second) system is $\rho^{T_1} = \sum_{i,j,k,l} \rho_{ij,kl} |kj\rangle\langle il|$ (resp. $\rho^{T_2} = \sum_{i,j,k,l} \rho_{ij,kl} |il\rangle\langle kj|$). The transpositions with respect to the two systems are related by $\rho^{T_1} = (\rho^{T_2})^T$, with T denoting the transposition of the whole matrix. Hence the positivity of ρ^{T_1} is equivalent to the positivity of ρ^{T_2} . A quantum state that its partially transposed matrices ρ^{T_1} and ρ^{T_2} are positive is called a PPT (positive partial transposition) state. It has been shown [1] that any entangled PPT states are not distillable. These states are called bound entangled [2–7].

If ρ^{T_1} and ρ^{T_2} have negative eigenvalues, the state ρ is called NPT (non positive partial transposition). For example, all entangled pure states, entangled isotropic states [8] and entangled Werner states [9] are all NPT. Moreover, an NPT state is necessarily entangled

[10] and believed to be free entangled (distillable). NPT states are significant resources for quantum information and quantum computation [11].

In this paper, we study the stability of NPT property of an arbitrary pure entangled state, namely the resistance of an entangled pure state to the mixture with pure separable perturbation. Suppose $|\chi_0\rangle$ is any bipartite entangled pure state. Let $|\chi_i\rangle$, $i = 1, \dots, K$, be arbitrary pure separable states. Consider the mixed quantum state,

$$\rho = \lambda_0|\chi_0\rangle\langle\chi_0| + \sum_{i=1}^K \lambda_i|\chi_i\rangle\langle\chi_i|, \quad (1)$$

where $0 < \lambda_i < 1$, $i = 0, 1, \dots, K$, $\sum_{i=0}^K \lambda_i = 1$. It is interesting to ask how large the number K can be such that ρ is still NPT. For bipartite pure state $|\chi_0\rangle$ with Schmidt number n ($n > 1$), we show that ρ is still NPT for $K \leq \frac{n(n-1)}{2} - 1$. This result is then generalized to multipartite case.

Denote n ($n > 1$) the Schmidt number of the state $|\chi_0\rangle$. Under some local unitary transformations $|\chi_0\rangle$ can be expressed in Schmidt form, $|\chi_0\rangle = \sum_{i=1}^n \mu_i |ii\rangle$, $\mu_i > 0$, $\sum_{i=1}^n \mu_i^2 = 1$. The state $|\chi_0\rangle\langle\chi_0|$ is NPT, because the eigenvalues of $(|\chi_0\rangle\langle\chi_0|)^{T_1}$ are μ_i^2 , $\pm\mu_i\mu_j$, $i, j = 1, 2, \dots, n$ and $i \neq j$. Hence $(|\chi_0\rangle\langle\chi_0|)^{T_1}$ has $\frac{n(n-1)}{2}$ negative eigenvalues. We first present a result for a simple case.

Theorem 1 *If $n = 2$ and $K = 1$, then the state (1) is NPT.*

Proof. In this case the state (1) has the form, $\rho = \lambda_0|\chi_0\rangle\langle\chi_0| + \lambda_1|\chi_1\rangle\langle\chi_1|$. Since $|\chi_0\rangle$'s Schmidt number is 2, there are exist unitary operators U and V such that $|\tilde{\chi}_0\rangle \equiv U \otimes V |\chi_0\rangle = \mu_1|11\rangle + \mu_2|22\rangle$. Instead of ρ , we consider the state $\tilde{\rho} \equiv (P \otimes P)(U \otimes V)\rho(U^\dagger \otimes V^\dagger)(P \otimes P)$, where $P = |1\rangle\langle 1| + |2\rangle\langle 2|$ is a project operator. Then $\tilde{\rho}$ has the following form,

$$\tilde{\rho} = \lambda_0|\tilde{\chi}_0\rangle\langle\tilde{\chi}_0| + \lambda_1|\tilde{\chi}_1\rangle\langle\tilde{\chi}_1|, \quad (2)$$

where $|\tilde{\chi}_1\rangle = (P \otimes P)(U \otimes V)|\chi_1\rangle$ is still a separable state, as $|\chi_1\rangle$ is separable. Therefore $|\tilde{\chi}_1\rangle$ is generally of the form, $|\tilde{\chi}_1\rangle = (a|1\rangle + b|2\rangle) \otimes (c|1\rangle + d|2\rangle)$ with $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$. And the determinant of $\tilde{\rho}^{T_1}$ is given by

$$-\lambda_0^4 \mu_1^4 \mu_2^4 - \lambda_1 \lambda_0^3 \mu_1^2 \mu_2^2 |\mu_1 b d + \mu_2 a c|^2 < 0.$$

Hence $\tilde{\rho}$ is NPT, which implies that the state ρ is NPT too. \square

Theorem 2 *If $n > 2$ and $K \leq \frac{n(n-1)}{2} - 1$, the quantum state (1) is still NPT.*

Proof. Since $(|\chi_0\rangle\langle\chi_0|)^{T_1}$ has $\frac{n(n-1)}{2}$ negative eigenvalues, the linear subspace V_- spanned by all the eigenvectors associated with the negative eigenvalues of $(|\chi_0\rangle\langle\chi_0|)^{T_1}$ is $\frac{n(n-1)}{2}$ dimensional, $\dim V_- = \frac{n(n-1)}{2}$. Note that the dimension of the subspace V_s spanned by K arbitrary separable pure states $\{|\chi_i\rangle\}_{i=1}^K$ is at most $\frac{n(n-1)}{2} - 1$, $\dim V_s \leq \frac{n(n-1)}{2} - 1$. As the dimension of the whole vector space $H \otimes H$ is d , $d \geq n^2$, the dimension of the complementary space V_s^\perp of V_s satisfies $\dim V_s^\perp \geq d + 1 - \frac{n(n-1)}{2}$. Subsequently, $\dim V_- + \dim V_s^\perp \geq d + 1$, which implies that there exists a vector $|\xi\rangle$ belonging to the intersection of the subspaces V_- and V_s^\perp , such that

$$\begin{aligned}\langle\xi|(|\chi_i\rangle\langle\chi_i|)^{T_1}|\xi\rangle &= \langle\xi|\chi_i\rangle\langle\chi_i|\xi\rangle = 0, \quad \forall i, \\ \langle\xi|(|\chi_0\rangle\langle\chi_0|)^{T_1}|\xi\rangle &< 0,\end{aligned}$$

and further

$$\langle\xi|\rho^{T_1}|\xi\rangle = \lambda_0\langle\xi|(|\chi_0\rangle\langle\chi_0|)^{T_1}|\xi\rangle + \sum_{i=1}^K \lambda_i\langle\xi|\chi_i\rangle\langle\chi_i|\xi\rangle < 0.$$

Therefore ρ is NPT. □

Comment. If $K > \frac{n(n-1)}{2}$, then the quantum state ρ in (1) can be either NPT, or PPT entangled or PPT separable. This can be seen from the following examples.

Example 1. Consider the $3 \otimes 3$ pure state $|\chi_0\rangle = 0.5|11\rangle + 0.8|22\rangle + \sqrt{0.11}|33\rangle$, and the following four separable states,

$$\begin{aligned}|\chi_1\rangle &= (0.4|1\rangle - 0.6|2\rangle + \sqrt{0.48}|3\rangle) \otimes (0.3|1\rangle + 0.95|2\rangle + \sqrt{0.0075}|3\rangle), \\ |\chi_2\rangle &= (0.27|1\rangle + 0.5|2\rangle + \sqrt{0.6771}|3\rangle) \otimes (-0.75|1\rangle - 0.1|2\rangle + \sqrt{0.4275}|3\rangle), \\ |\chi_3\rangle &= (-0.2|1\rangle + 0.4|2\rangle + \sqrt{0.8}|3\rangle) \otimes (-0.05|1\rangle + 0.01|2\rangle - \sqrt{0.9974}|3\rangle), \\ |\chi_4\rangle &= (0.2|1\rangle + 0.6|2\rangle - \sqrt{0.6}|3\rangle) \otimes (0.8|1\rangle - 0.55|2\rangle - \sqrt{0.0575}|3\rangle).\end{aligned}$$

Take $\lambda_0 = 0.01, \lambda_1 = 0.6, \lambda_2 = 0.09, \lambda_3 = \lambda_4 = 0.15$. In this case $n = 3, K = 4$. The quantum state ρ in Eq. (1) is PPT because the minimal eigenvalue of ρ^{T_1} is 0.00006.

Example 2. The Horodecki's $3 \otimes 3$ state [3],

$$\sigma_\alpha = \frac{2}{7}|\psi^+\rangle\langle\psi^+| + \frac{\alpha}{21}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|) + \frac{5-\alpha}{21}(|10\rangle\langle 10| + |21\rangle\langle 21| + |02\rangle\langle 02|),$$

where $|\psi^+\rangle = (|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$ is a maximally entangled state. σ_α is just of the form (1), a maximally entangled state $|\psi^+\rangle$ mixed with six pure separable states. It is (PPT) separable for $2 \leq \alpha \leq 3$, PPT entangled for $3 < \alpha \leq 4$, and NPT entangled for $4 \leq \alpha \leq 5$.

Utilizing the proof of Theorem 2, one can get a similar result for mixed states.

Corollary 1 *For arbitrary mixed state ρ_0 , if $\rho_0^{T_1}$ has $\frac{n(n-1)}{2}$ negative eigenvalues, then*

$$\rho = \lambda_0 \rho_0 + \sum_{i=1}^K \lambda_i |\chi_i\rangle\langle\chi_i| \quad (3)$$

is still NPT for $K \leq \frac{n(n-1)}{2}$, where $|\chi_i\rangle$ is an arbitrary pure separable state, $0 < \lambda_i < 1$, $i = 0, 1, \dots, K$, $\sum_{i=0}^K \lambda_i = 1$.

Our conclusions can be generalized to multipartite case. For a multipartite quantum state ρ , we view it as a bipartite quantum state with partition \mathcal{Y} and $\bar{\mathcal{Y}}$, where the subsystems \mathcal{Y} and subsystems $\bar{\mathcal{Y}}$ span the whole quantum system, $\mathcal{Y} \cap \bar{\mathcal{Y}} = \emptyset$. Let $\rho^{T_{\mathcal{Y}}}$ denote the partial transposition with respect to the subsystems \mathcal{Y} . For a multipartite pure state $|\chi_0\rangle$, assume that $(|\chi_0\rangle\langle\chi_0|)^{T_{\mathcal{Y}}}$ have $p_{\mathcal{Y}}$ negative eigenvalues. We set $p_{\mathcal{Y}_0} \equiv \max_{\mathcal{Y}} p_{\mathcal{Y}}$, where the maximum goes over all possible partitions \mathcal{Y} . Similar to the Theorem 2, we have the following result,

Theorem 3 *If $K \leq p_{\mathcal{Y}_0} - 1$, then the quantum state $\rho = \lambda_0 |\chi_0\rangle\langle\chi_0| + \sum_{i=1}^K \lambda_i |\chi_i\rangle\langle\chi_i|$ is still NPT, where $\{|\chi_i\rangle\}_{i=1}^K$ are arbitrary biseparable states under the partition between \mathcal{Y} and $\bar{\mathcal{Y}}$, $0 < \lambda_i < 1$, $i = 0, 1, \dots, K$, $\sum_{i=0}^K \lambda_i = 1$. Especially, ρ is NPT if $\{|\chi_i\rangle\}_{i=1}^K$ are K fully separable states.*

We have studied the stability of the NPT property of an entangled pure state under the mixture of arbitrary pure separable states. For bipartite pure state with Schmidt number n ($n > 1$), we have shown that it is still NPT under mixing with no more than $\frac{n(n-1)}{2} - 1$ arbitrary pure separable states, with a generalization to multipartite cases.

For $n \geq 2$ and $K = \frac{n(n-1)}{2}$, we have the evidence that the quantum state (1) is still NPT. This result holds true at least for $n = 2$, as shown in Theorem 1. However it still remains open whether the Theorem 2 is still valid for $K \leq \frac{n(n-1)}{2}$.

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