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Abstract

We study the local unitary equivalence of arbitrary dimensional multipartite quantum mixed states. We present a necessary and sufficient criterion of the local unitary equivalence for general multipartite states based on matrix realignment. The criterion is shown to be operational even for particularly degenerated states by detailed examples. Besides, explicit expressions of the local unitary operators are constructed for locally equivalent states. In complement to the criterion, an alternative approach based on partial transposition of matrices is also given, which makes the criterion more effective in dealing with generally degenerated mixed states.

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Quantum entanglement is one of the most extraordinary features of quantum physics. Multipartite entanglement plays a vital role in quantum information processing [1, 2] and interferometry [3]. One fact is that the degree of entanglement of a quantum state remains invariant under local unitary transformations, while two quantum states with the same degree entanglement, e.g. entanglement of formation [4, 5] or concurrence [6, 7]), may be not equivalent under local unitary transformations. Another fact is that two entangled states are said to be equivalent in implementing quantum information tasks, if they can be mutually exchanged under local operations and classical communication (LOCC). LOCC equivalent states are interconvertible also by local unitary transformations [8]. Therefore, it is important to classify and characterize quantum states in terms of local unitary transformations.

To deal with this problem, one approach is to construct invariants of local unitary transformations. The method developed in [9, 10], in principle, allows one to compute all the invariants of local unitary transformations for bipartite states, though it is not easy to do this operationally. In [11] a complete set of 18 polynomial invariants is presented for the local unitary equivalence of two qubits mixed states. Partial results have been obtained for three qubits states [12, 13], some generic mixed states [14–16], tripartite pure and mixed states [17]. The local unitary equivalence problem for multipartite pure qubits states has been solved in [18]. By exploiting the high order singular value decomposition technique and local symmetries of the states, Ref. [19] presents a practical scheme of classification under local unitary transformations for general multipartite pure states with arbitrary dimensions, which extends results of n-qubit pure states [18] to that of n-qudit pure states. For mixed states, Ref. [20] solved the local unitary equivalence problem of arbitrary dimensional bipartite non-degenerated quantum systems by presenting a complete set of invariants, such that two density matrices are local unitary equivalent if and only if all these invariants have equal values. In [21] the case of multipartite systems is studied and a complete set of invariants is presented for a special class of mixed states. Recently, the authors in [22] have studied the local unitary equivalence problem for multi-qubit states in terms of Bloch representation.

In this paper, we study the local unitary equivalence problem in terms of matrix realignment [23, 24] and partial transposition [25, 26], the techniques used in dealing with the separability problem of quantum states and also in generating local unitary invariants [27]. We present a necessary and sufficient criterion for the local unitary equivalence of multipartite states, together with explicit forms of the local unitary operators. This generalizes the results in [20, 33] from non-degenerated states to generally degenerated states for bipartite case. The criterion is shown to be still operational for states having eigenvalues with multiplicity no more than 2. It also generalizes the results in [20, 33] from bipartite states to generally multipartite states. Alternative ways are presented to deal with generally degenerated states by using our criterion.

We first review some definitions and results about matrix realignment from matrix analysis [28]. For any $M \times N$ matrix A with entries a_{ij}, $vec(A)$ is defined by

$$vec(A) \equiv [a_{11}, \cdots, a_{M1}, a_{12} \cdots, a_{M2}, \cdots, a_{1N}, \cdots, a_{MN}]^T,$$

where T denotes transposition. Let Z be an $M \times M$ block matrix with each block of size $N \times N$, the realigned matrix $\widetilde{Z}$ is defined by

$$\widetilde{Z} \equiv [vec(Z_{11}), \cdots, vec(Z_{M1}), \cdots, vec(Z_{1M}), \cdots, vec(Z_{MM})]^T.$$

Based on these operations, the authors in [29, 30] proved that

Lemma 1: Assume that the matrix $\widetilde{Z}$ has singular value decomposition, $\widetilde{Z} = U\Sigma V^\dagger$, then $Z = \sum_{i=1}^r X_i \otimes Y_i$, where $vec(X_i) = \sqrt{\alpha_i \sigma_i}\mu_i$, $vec(Y_i) = \sqrt{\frac{1}{\alpha_i}\sigma_i}\nu_i^*$, $\alpha_i \neq 0$, $\Sigma = diag(\sigma_i)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_q \geq 0$, $\{\sigma_i\}_{i=1}^q$ are the singular values of the matrix $\widetilde{Z}$, $q = min(M^2, N^2)$, r is the number of nonzero singular values σ_i (the rank of the matrix $\widetilde{Z}$), $U = [\mu_1 \mu_2 \cdots \mu_{M^2}] \in \mathbb{C}^{M^2 \times M^2}$ and $V = [\nu_1 \nu_2 \cdots \nu_{N^2}] \in \mathbb{C}^{N^2 \times N^2}$ are unitary matrices, with μ_i and ν_i the singular vectors of σ_i.

Lemma 1 implies that [31],

Lemma 2 An $MN \times MN$ unitary matrix U can be expressed as the tensor product of an $M \times M$ unitary matrix u_1 and an $N \times N$ unitary matrix u_2 such that $U = u_1 \otimes u_2$ if and only if rank $(\widetilde{U}) = 1$.

Remark 1: Following Lemma 1, when rank$(\widetilde{U}) = 1$, $vec(X) = \sqrt{\alpha_1 \sigma_1}\mu_1$ and $vec(Y) = \sqrt{\frac{1}{\alpha_1}\sigma_1}\nu_1^*$, where μ_1 and ν_1 are the eigenvectors of $\widetilde{U}\widetilde{U}^\dagger$ and $\widetilde{U}^\dagger\widetilde{U}$ corresponding to non-zero eigenvalues. Therefore, from Lemma 2, the detailed form of u_1 and u_2 can be obtained.

Now consider the case of multipartite states. Let $H_1, H_2, \cdots, H_n$ be complex Hilbert spaces of finite dimensions $N_1, N_2, \cdots, N_n$, respectively. Let $\{|j\rangle_k\}_{j=1}^{N_k}$, $k = 1, 2, \cdots, n$, be an orthonormal basis of H_k. A mixed state $\rho \in H_1 \otimes H_2 \otimes \cdots \otimes H_n$ can be written in terms of the spectral decomposition form of ρ, $\rho = \sum_{i=1}^{N_1 N_2 \cdots N_n} \lambda_i |\phi_i\rangle\langle\phi_i|$, where $|\phi_i\rangle = \sum_{j=1}^{N_1}\sum_{k=1}^{N_2}\cdots\sum_{l=1}^{N_n} a_{jk\cdots l}^i |j\rangle_1 |k\rangle_2 \cdots |l\rangle_n$, $a_{jk\cdots l}^i \in \mathbb{C}$ satisfying $\sum_{j=1}^{N_1}\sum_{k=1}^{N_2}\cdots\sum_{l=1}^{N_n} a_{jk\cdots l}^i a_{jk\cdots l}^{i\dagger} = 1$.

Two multipartite mixed states ρ and ρ' in $H_1 \otimes H_2 \otimes \cdots \otimes H_n$ are said to be equivalent under local unitary transformations if there exist unitary operators u_i on the i-th Hilbert space H_i such that

$$\rho' = (u_1 \otimes u_2 \otimes \cdots \otimes u_n)\rho(u_1 \otimes u_2 \otimes \cdots \otimes u_n)^\dagger. \tag{1}$$

In the following, for any $N_1N_2\cdots N_n \times N_1N_2\cdots N_n$ matrix T, we denote $T_{i|\widehat{i}}$ the $N_i \times N_i$ block matrix with each block of size $N_1N_2\cdots N_{i-1}N_{i+1}\cdots N_n \times N_1N_2\cdots N_{i-1}N_{i+1}\cdots N_n$. Namely, we view T as a bipartite partitioned matrix $T_{i|\widehat{i}}$ with partitions H_i and $H_1 \otimes H_2 ... H_{i-1} \otimes H_{i+1} ... H_n$. Accordingly, we have the realigned matrix $\widetilde{T_{i|\widehat{i}}}$.

Lemma 3 Let U be an $N_1N_2\cdots N_n \times N_1N_2\cdots N_n$ unitary matrix, there exist $N_i \times N_i$ unitary matrices u_i, $i = 1, 2, \cdots, n$, such that $U = u_1 \otimes u_2 \otimes \cdots \otimes u_n$ if and only if the $\mathrm{rank}(\widetilde{U_{i|\widehat{i}}}) = 1$ for all i.

Proof First, if there exist $N_i \times N_i$ unitary matrices u_i, $i = 1, 2, \cdots, n$, such that $U = u_1 \otimes u_2 \otimes \cdots \otimes u_n$, by viewing U in bipartite partition and using Lemma 2, one has directly that $\mathrm{rank}(\widetilde{U_{i|\widehat{i}}}) = 1$ for all i.

On the other hand, if $\mathrm{rank}(\widetilde{U_{i|\widehat{i}}}) = 1$, for any given i, we prove the conclusion by induction. First, for $n = 3$, from Lemma 2, we have $U = u_1 \otimes u_{23} = u_2 \otimes u_{13}$, i.e, $(u_1^\dagger \otimes I_2 \otimes I_3)U = I_1 \otimes u_{23} = u_2 \otimes ((u_1^\dagger \otimes I_3)u_{13})$. By tracing over the first subsystem, we get $N_1 u_{23} = u_2 \otimes Tr_1((u_1^\dagger \otimes I_3)u_{13})$, i.e, $u_{23} = u_2 \otimes u_3'$ with $u_3' = Tr_1((u_1^\dagger \otimes I_3)u_{13})/N_1$. Assume that the conclusion is also true for $n-1$. Then for n, from Lemma 2, we have $U = u_1 \otimes u_{\widehat{1}} = u_2 \otimes u_{\widehat{2}} = \cdots = u_n \otimes u_{\widehat{n}}$, where u_i is an $N_i \times N_i$ unitary matrix and $u_{\widehat{i}}$ is an $N_1N_2\cdots N_{i-1}N_{i+1}\cdots N_n \times N_1N_2\cdots N_{i-1}N_{i+1}\cdots N_n$ unitary matrix, $i = 1, 2, \cdots, n$. Hence $(I_1 \otimes \cdots \otimes I_{n-1} \otimes u_n^\dagger)U = (I_1 \otimes \cdots \otimes I_{n-1} \otimes u_n^\dagger)(u_1 \otimes u_{\widehat{1}}) = \cdots = u_{\widehat{n}} \otimes I_{N_n}$. By tracing the last subsystem we get $u_1 \otimes (Tr_n(I_2 \otimes \cdots \otimes I_{N_{n-1}} \otimes u_n^\dagger)u_{\widehat{1}}) = \cdots = (Tr_n(I_1 \otimes \cdots \otimes I_{n-2} \otimes u_n^\dagger)) \otimes (u_{n-1}) = N_n u_{\widehat{n}}$. Based on the assumption, we have that $u_{\widehat{n}}$ can be written as the tensor of local unitary operators. Therefore, U also can be written as the tensor product of local unitary operators. $\square$

If two density matrices ρ_1 and ρ_2 in $H_1 \otimes H_2 \otimes \cdots \otimes H_n$ are equivalent under local unitary transformations, they must have the same set of eigenvalues λ_k, $k = 1, 2, \cdots, N_1N_2\cdots N_n$. Let $X = (x_1, x_2, \cdots, x_{N_1N_2\cdots N_n})$ and $Y = (y_1, y_2, \cdots, y_{N_1N_2\cdots N_n})$ be the unitary matrices that diagonalize the two density matrices, respectively,

$$\rho_1 = X\Lambda X^\dagger, \quad \rho_2 = Y\Lambda Y^\dagger, \tag{2}$$

where $\{x_i\}$ and $\{y_i\}$ are the normalized eigenvectors of states ρ_1 and ρ_2,

$$\Lambda = diag(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \cdots, \lambda_r I_{n_r}),$$

with $r \leq N_1 N_2 \cdots N_n$, $\sum_{k=1}^{r} n_k = N_1 N_2 \cdots N_n$, n_k is the multiplicity of the kth eigenvalue λ_k. Therefore $X^\dagger \rho_1 X = \Lambda = Y^\dagger \rho_2 Y$. Due to the degeneracy of ρ_1 and ρ_2, X and Y are not fixed in the sense that $X^\dagger \rho_1 X = Y^\dagger \rho_2 Y$ is inveriant under $X \to XU$ and $Y \to YU$, for any

$$U = diag(u_{n_1}, u_{n_2}, \cdots, u_{n_r}), \tag{3}$$

where u_{n_k} are $n_k \times n_k$ unitary matrices, $k = 1, \cdots, r$. Thus for given X and Y, $YU^\dagger X^\dagger \rho_1 XUY^\dagger = \rho_2$.

Theorem 1 Let ρ_1 and ρ_2 be two multipartite mixed quantum states given in (2), $\rho_1 = X\Lambda X^\dagger$ and $\rho_2 = Y\Lambda Y^\dagger$. ρ_1 and ρ_2 are local unitary equivalent if and only if there exists an $N_1 N_2 \cdots N_n \times N_1 N_2 \cdots N_n$ unitary matrix U of the form (3) such that rank$(\widetilde{XUY^\dagger})_{i|\hat{i}} = 1$ for $i = 1, 2, \cdots, n$.

Proof: If ρ_1 and ρ_2 are equivalent under local unitary transformations, i.e. $(u_1 \otimes u_2 \otimes \cdots \otimes u_n)\rho_1(u_1 \otimes u_2 \otimes \cdots \otimes u_n)^\dagger = \rho_2$, then there exists a unitary matrix U of the form (3) such that $Y = (u_1 \otimes u_2 \otimes \cdots \otimes u_n)XU$. From Lemma 3 the rank$(\widetilde{XUY^\dagger})_{i|\hat{i}} = 1$, where $(XUY^\dagger)_{i|\hat{i}} = u_i \otimes (u_1 \otimes \cdots \otimes u_{i-1} \otimes u_{i+1} \otimes \cdots \otimes u_n)$, $i = 1, 2, \cdots, n$.

On the other hand, if there is an $N_1 N_2 \cdots N_n \times N_1 N_2 \cdots N_n$ unitary matrix U such that rank$(\widetilde{XUY^\dagger})_{i|\hat{i}} = 1$, for any i, by Lemma 3 we have $XUY^\dagger = u_1 \otimes u_2 \otimes \cdots \otimes u_n$. Then $YU^\dagger X^\dagger \rho_1 XUY^\dagger = \rho_2$ gives rise to $(u_1 \otimes u_2 \otimes \cdots \otimes u_n)^\dagger \rho_1 (u_1 \otimes u_2 \otimes \cdots \otimes u_n) = \rho_2$, which ends the proof. $\square$

Remark 2: If there exists an $N_1 N_2 \cdots N_n \times N_1 N_2 \cdots N_n$ unitary matrix U of the form (3) such that rank$(\widetilde{XUY^\dagger})_{i|\hat{i}} = 1$, for any $i = 1, 2, \cdots, n$, then ρ_1 and ρ_2 are local unitary equivalent. From Lemma 1, we can get the explicit expressions of the local unitary matrices u_i in the following way. First, we view $XUY^\dagger$ as an $N_1 \times N_1$ block matrix with each block of size $N_2 N_3 \cdots N_n \times N_2 N_3 \cdots N_n$. Following Lemma 1, we have that $XUY^\dagger = u_1 \otimes u_{\hat{1}}$, where u_1 and $u_{\hat{1}}$ have explicit expressions from Remark 1, and $u_{\hat{1}}$ is an $N_2 N_3 \cdots N_n \times N_2 N_3 \cdots N_n$ unitary matrix. By viewing $u_{\hat{1}}$ as an $N_2 \times N_2$ block matrix with each block of size $N_3 N_4 \cdots N_n \times N_3 N_4 \cdots N_n$, we get the expression of u_2 in $u_{\hat{1}} = u_2 \otimes u_{\hat{2}}$. In this way, we can get all the detailed expressions of $u_1, u_2, \cdots, u_n$, such that $XUY^\dagger = u_1 \otimes u_2 \otimes \cdots \otimes u_n$.

Theorem 1 has many advantages compared with the previous results about local unitary equivalence. It generalizes the results for non-degenerated bipartite states in [20] to general bipartite mixed states including degenerated ones, for which the problem becomes quite difficult usually and many criteria become non-operational [20]. Our criterion can be also operational for particular degenerated bipartite states. Let us consider that $\rho_1, \rho_2 \in H_1 \otimes H_2$ have s different eigenvalues with multiplicity 2 and the rest eigenvalues with multiplicity 1. According to Theorem 1, ρ_1 and ρ_2 are local unitary equivalent if and only if there exists a unitary matrix

$$U = diag(u_1, \cdots, u_s, e^{i\theta_{s+1}}, \cdots, e^{i\theta_{N_1N_2}}), \tag{4}$$

with $u_r \in U(2)$, $r = 1, \cdots, s$, $s = 0, 1, \cdots, [\frac{N_1N_2}{2}]$, such that $\mathrm{rank}(\widetilde{XUY^\dagger}) = 1$, where $[x]$ denotes the integer part of x.

Any unitary matrix in $U(2)$ can be written as, up to a constant phase, $tI + i\sum_{j=1}^{3} z_j\sigma_j$ with $t^2 + \sum_{j=1}^{3} z_j^2 = 1$, where I is the 2×2 identity matrix and σ_j are the Pauli matrices. Therefore U has the following form:

$$U = \begin{pmatrix} t_1 + iz_3 & z_1 + iz_2 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ -z_1 + iz_2 & t_1 - iz_3 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & t_s + iz_{3s} & z_{3s-2} + iz_{3s-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -z_{3s-2} + iz_{3s-1} & t_s - iz_{3s} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & e^{i\theta_{s+1}} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & e^{i\theta_{N_1N_2}} \end{pmatrix}, \tag{5}$$

where $t_j^2 + z_{3j}^2 + z_{3j-1}^2 + z_{3j-2}^2 = 1$ for $j = 1, ..., s$. One just needs to verify the existence of the unitary matrix U such that $\mathrm{rank}(\widetilde{XUY^\dagger}) = 1$. The calculation of the rank of $\widetilde{XUY}$ only concerns the quadratic homogeneous equations and can be done simply by using the algorithm in Ref. [32] for solving systems of multivariate polynomial equations called XL (eXtended Linearizations or multiplication and linearlization) algorithm. As an example, let us consider

$$\rho_1 = \begin{pmatrix} 1/4 & 0 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 0 \\ 0 & 1/4 & 1/4 & 0 \\ 1/4 & 0 & 0 & 1/4 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 1/4 & 0 & 1/4 & 0 \\ 0 & 1/4 & 0 & -1/4 \\ 1/4 & 0 & 1/4 & 0 \\ 0 & -1/4 & 0 & 1/4 \end{pmatrix}.$$

Here ρ_1 and ρ_2 are degenerated states with the eigenvalues set $\Lambda = diag(\frac{1}{2}, \frac{1}{2}, 0, 0)$. Following (3), U has the form

$$U = \begin{pmatrix} t_1 + iz_3 & z_1 + iz_2 & 0 & 0 \\ -z_1 + iz_2 & t_1 - iz_3 & 0 & 0 \\ 0 & 0 & t_2 + iz_6 & z_4 + iz_5 \\ 0 & 0 & -z_4 + iz_5 & t_2 - iz_6 \end{pmatrix}.$$

Correspondingly,

$$X = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}.$$

It is easily verified that there are many matrices of the form (5) satisfying rank$(\widetilde{XUY^\dagger}) = 1$, for instance,

$$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Therefore ρ_1 and ρ_2 are local unitary equivalent. In fact, from singular values decomposition of $\widetilde{XUY^\dagger}$, we can get the unique nonzero singular values $\frac{1}{2}$ with multiplicity 2. Using Lemma 1, we have $\mu_1 = \frac{1}{\sqrt{2}}(-1, 0, 0, 1)$ and $\nu_1 = \frac{1}{2}(1, -1, 1, 1)$. Therefore, from Lemma 2, we can choose $vec(X_1) = \sqrt{2}u_1$ and $vec(Y_1) = \sqrt{2}v_1$, such that X_1 and Y_1 are unitary matrices, and $(X_1 \otimes Y_1)\rho_1(X_1 \otimes Y_1)^\dagger = \rho_2$.

Concerning multipartite mixed states, let us consider two density matrices in $H_1 \otimes H_2 \otimes H_3$

with $N_1 = N_2 = N_3 = 2$,

$$\rho_1 = \frac{1}{K}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho_2 = \frac{1}{K}\begin{pmatrix} \frac{1+b}{2} & 0 & \frac{b-1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{a+c}{2} & 0 & \frac{c-a}{2} & 0 & 0 & 0 & 0 \\ \frac{b-1}{2} & 0 & \frac{1+b}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{c-a}{2} & 0 & \frac{a+c}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2c}+\frac{1}{2a} & 0 & \frac{1}{2a}-\frac{1}{2c} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2b}+\frac{1}{2} & 0 & \frac{1}{2}-\frac{1}{2b} \\ 0 & 0 & 0 & 0 & \frac{1}{2a}-\frac{1}{2c} & 0 & \frac{1}{2c}+\frac{1}{2a} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2}-\frac{1}{2b} & 0 & \frac{1}{2}+\frac{1}{2b} \end{pmatrix},$$

where the normalization factor $K = 2 + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. ρ_1 and ρ_2 have the same eigenvalue set $\Lambda = \frac{1}{K} diag(2, 0, \frac{1}{a}, a, \frac{1}{b}, b, \frac{1}{c}, c)$. For the case $a \neq b \neq c \neq 0 \neq 1 \neq 2 \neq \frac{1}{2}$, ρ_1 and ρ_2 are not degenerated. In this case, one has

$$X = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}.$$

From (3), U is of the form $U = diag(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}, e^{i\theta_5}, e^{i\theta_6}, e^{i\theta_7}, e^{i\theta_8})$. Hence

$$XUY^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{e^{i\theta_1}+e^{i\theta_2}}{2} & 0 & -\frac{e^{i\theta_1}+e^{i\theta_2}}{2} & 0 & 0 & \frac{e^{i\theta_1}-e^{i\theta_2}}{2} & 0 & \frac{e^{i\theta_1}-e^{i\theta_2}}{2} \\ 0 & -e^{i\theta_4} & 0 & e^{i\theta_4} & 0 & 0 & 0 & 0 \\ e^{i\theta_6} & 0 & e^{i\theta_6} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\theta_8} & 0 & e^{i\theta_8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -e^{i\theta_7} & 0 & e^{i\theta_7} & 0 \\ 0 & 0 & 0 & 0 & 0 & -e^{i\theta_5} & 0 & e^{i\theta_5} \\ 0 & 0 & 0 & 0 & e^{i\theta_3} & 0 & e^{i\theta_3} & 0 \\ \frac{e^{i\theta_1}-e^{i\theta_2}}{2} & 0 & \frac{e^{i\theta_2}-e^{i\theta_1}}{2} & 0 & 0 & \frac{e^{i\theta_1}+e^{i\theta_2}}{2} & 0 & \frac{e^{i\theta_1}+e^{i\theta_2}}{2} \end{pmatrix}.$$

It is easily verified that $rank(\widetilde{XUY^\dagger})_{i|\hat{i}} = 1$ for $\theta_1 = \theta_2 = \theta_3 = \theta_6 = \theta_8 = 0, \theta_4 = \theta_5 = \theta_7 = \pi$, $i = 1, 2, 3$. Therefore from Theorem 1 ρ_1 and ρ_2 are local unitary equivalent.

In fact, taking $i = 1$, from the singular values decomposition of $(\widetilde{XUY^\dagger})_{1|\hat{1}}$, we can get the unique nonzero singular values $2\sqrt{2}$. From Lemma 1, we get $u_1 = \frac{1}{\sqrt{2}}(1,0,0,1)$ and $v_1 = \frac{1}{2\sqrt{2}}(1,0,1,0,0,1,0,1,-1,0,1,0,0,-1,0,1)$. Therefore, we can choose $vec(X_1) = \sqrt{2}u_1$ and $vec(X_2) = 2v_1$ such that they are unitary. Then $X_1 = I_2 \in H_1$ and

$$X_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \in H_2 \otimes H_3.$$

One can easily find that $\text{rank}(\widetilde{X_2}) = 1$. From the singular value decomposition of $\widetilde{X_2}$, using Lemma 1 again, we get $Y_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, $Y_2 = I_2$, such that $X_2 = Y_1 \otimes Y_2$ is unitary. That is, $(X_1 \otimes Y_1 \otimes Y_2)\rho_1(X_1 \otimes Y_1 \otimes Y_2)^\dagger = \rho_2$.

Our criterion is both necessary and sufficient for local equivalence of arbitrary multipartite mixed quantum systems. However, for general degenerated states, it could be less operational. In the following, complement to Theorem 1, we present an alternative way to judge the local equivalence based on partial transposition of matrices. For a density matrix $\rho \in H_1 \otimes H_2$ with entries $\rho_{m\mu,n\nu} = \langle e_m \otimes f_\mu|\rho|e_n \otimes f_\nu\rangle$, the partial transposition of ρ is defined by [26]:

$$\rho^{T_2} = (I \otimes T)\rho = \sum_{mn,\mu\nu} \rho_{m\nu,n\mu}|e_m \otimes f_\mu\rangle\langle e_n \otimes f_\nu|,$$

where ρ^{T_2} denotes the transposition of ρ with respect to the second system, $|e_n\rangle$ and $|f_\nu\rangle$ are the bases associated with H_1 and H_2 respectively.

Theorem 2 Two mixed states ρ_1 and ρ_2 in $H_1 \otimes H_2$ are local unitary equivalent if and only if $\rho_1^{T_2}$ and $\rho_2^{T_2}$ are local unitary equivalent.

Proof Without loss of generality, we assume that $\rho_1 = \sum \rho_{m\mu,n\nu}|e_m\rangle\langle e_n| \otimes |f_\mu\rangle\langle f_\nu|$. Then $\rho_1^{T_2} = \sum \rho_{m\nu,n\mu}|e_m\rangle\langle e_n| \otimes |f_\mu\rangle\langle f_\nu|$. On the one hand, if ρ_1 and ρ_2 are equivalent under local unitary transformations, one has

$$\rho_2 = (u_1 \otimes u_2)\rho_1(u_1 \otimes u_2)^\dagger = \sum \rho_{m\mu,n\nu}(u_1|e_m\rangle\langle e_n|u_1^\dagger) \otimes (u_2|f_\mu\rangle\langle f_\nu|u_2^\dagger).$$

Hence

$$\begin{aligned}\rho_2^{T_2} &= \sum \rho_{m\mu,n\nu}(u_1|e_m\rangle\langle e_n|u_1^\dagger) \otimes (u_2^*|f_\nu\rangle\langle f_\mu|u_2^T) \\ &= \sum \rho_{m\mu,n\nu}(u_1 \otimes u_2^*)(|e_m\rangle\langle e_n| \otimes |f_\nu\rangle\langle f_\mu|)(u_1 \otimes u_2^*)^\dagger \\ &= (u_1 \otimes u_2^*)\rho_1^{T_2}(u_1 \otimes u_2^*)^\dagger.\end{aligned}$$

Therefore, $\rho_2^{T_2}$ and $\rho_1^{T_2}$ are also local unitary equivalent.

On the other hand, since $(\rho^{T_2})^{T_2} = \rho$, if $\rho_1^{T_2}$ and $\rho_2^{T_2}$ are equivalent under local unitary transformations, one can derive that ρ_1 and ρ_2 are also equivalent under local unitary transformations. $\square$

The Theorem 2 is also true for ρ^{T_1}. Generally the partial transposed states are no longer semi positive. They are just Hermitian matrices. Nevertheless Theorem 2 still works for the local unitary equivalence for Hermitian matrices. Theorem 2 can be directly generalized to multipartite systems:

Theorem 3 Two mixed states ρ_1 and ρ_2 in $H_1 \otimes H_2 \otimes \cdots \otimes H_n$ are local unitary equivalent if and only if $\rho_1^{T_k}$ and $\rho_2^{T_k}$ are local unitary equivalent, where $k \in \{1, 2, ..., n\}$, ρ^{T_k} denotes the transposition of ρ with respect to the kth system.

Theorem 3 provides us an alternative way to determine the local unitary equivalence of multipartite states. If the given states are degenerated, the criterion given by Theorem 1 would be less operational. In this case one may consider the partial transposition of the states. For bipartite states, if the partially transposed states are not degenerated, we can check the local unitary equivalence by using the Theorem 1 and obtain the explicit local unitary matrices. There are many degenerated states such that their partially transposed ones are not degenerated, for example,

$$\rho = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{16} \\ 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ \frac{1}{16} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

If the partially transposed states are still degenerated, but less degenerated such that they have s different eigenvalues with multiplicity 2 and the rest eigenvalues with multiplicity 1, then the Theorem 1 can applied to determine the local unitary equivalence. For the multipartite states, the Theorem 3 could be applied to simplify problem.

In summary, based on matrix realignment we have presented a necessary and sufficient criterion of the local unitary equivalence for general multipartite mixed quantum states, and the corresponding explicit expression of the local unitary operators. The criterion proposed in [33] is a special case of Theorem 1 for bipartite case. Our criterion is even operational for a class of degenerated states. To deal with the general degenerated states, we have also presented another criterion based on state partial transpositions, which, in complement to our criterion based on matrix realignment, may transform an un-operational problem to be an operational one, so as to make our criteria more effective. Detailed examples have been presented. Our approach gives a new progress toward to the local equivalence of multipartite mixed states.

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