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THE EINSTEIN-HILBERT ACTION WITH COSMOLOGICAL CONSTANT AS A FUNCTIONAL OF GENERIC FORM

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Abstract. The geometrical underpinnings of a certain class of Dirac operators is discussed. It is demonstrated how these Dirac operators allow to relate various geometric functionals like, for example, the Yang-Mills and the functional of non-linear $\sigma-$models (i.e. (Dirac) harmonic maps). In fact, all these functionals are shown to be intimately related to the Einstein-Hilbert action with cosmological constant (EHC). Therefore, the latter may be regarded as a kind of "generic functional". In addition, the geometrical setup presented also allows to avoid the issue of the "fermion doubling".

Contents

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1. Introduction

The Einstein-Hilbert action with a cosmological constant (EHC), see (22) below, is known to be the most general functional whose Euler-Lagrange equation yields a covariantly constant tensor that can be build from the metric and its first and second derivatives, only (see Section 3). In contrast to the pure Einstein-Hilbert action:

$$\mathcal{I}_{\rm EH}(g_{\rm M}) := \int_M *scal(g_{\rm M})\,, \tag{1}$$

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which can be expressed in terms of "quantized Clifford connections" (for the notation and terminology used, please see the next section), the EHC can be expressed in terms of "Dirac operators of simple type". This class of Dirac operators provide a natural generalization of quantized Clifford connections in the sense that the Bochner-Laplacian associated with a Dirac operator is still defined in terms of a Clifford connection. When expressed in terms of Dirac operators of simple type, we demonstrate that the EHC has a "generic form". We discuss the functional of non-linear $\sigma-$models and the Yang-Mills action from this point of view. Though these functionals yield rather different Euler-Lagrange equations, both functionals may nonetheless be recast into a form similar to the EHC. For this a certain class of Clifford module bundles is introduced. This class of bundles also allows to avoid the "doubling of fermions" needed to geometrically describe the Standard Model action in terms of Dirac operators (see [10] and the References sited therein).

Dirac operators of simple type describe the dynamics of the Standard Model fermions. In loc site it has been shown how the bosonic functional of the Standard Model can be derived from Dirac operators of simple type. As a premise for this, however, it is necessary to assume a specific bi-module structure of the underlying Clifford module bundle. Moreover, one has to double the Clifford module bundle to introduce curvature terms into Dirac operators. In the Standard Model fermions are described in terms of spinors. But twisted spinor bundles do not permit a bi-module structure necessary to describe the Standard Model action in terms of Dirac operators of simple type. Without a bi-module structure, however, the pure "kinetic term" of the Higgs sector of the Standard Model cannot be described in terms of the setup introduced in loc. site. For the same reason the geometrical setup discussed in [10] cannot be used to describe functionals like the functional of Dirac harmonic maps (non-linear $\sigma-$models). Also, in order to describe Yang-Mills gauge theory within the geometrical scheme presented in loc site one has to use a class of Dirac operators (of "Pauli type"), which do not form a distinguished subset of all Dirac operators. This is different to the case of simple type Dirac operators.

The main motivation of the present paper is to remedy these flaws and to demonstrate that Dirac operators of simple type actually provide a "generic root" of a variety of various seemingly different functionals, including non-linear $\sigma-$models and the Yang-Mills action. All of these functionals can therefore be re-cast into the form of Einstein's "biggest blunder"[1] [7].

2. The geometrical setup and basic definitions

In this section we introduce the basic geometrical setup and fix the notation used. For sake of self-consistency we recapitulate some basic facts about Dirac operators acting on sections of general Clifford module bundles.

In the sequel, $(M, g_{\rm M})$ always denotes a smooth orientable (semi-)Riemannian manifold of finite dimension $n \equiv p + q$. The index of the (semi-)Riemannian metric $g_{\rm M}$ is $s \equiv p - q \not\equiv 1 \bmod 4$. The *bundle of exterior forms* of degree $k \geq 0$ is denoted by $\Lambda^k T^*M \to M$ with its canonical projection. Accordingly, the *Grassmann bundle* is given by $\Lambda T^*M \equiv \bigoplus_{k\geq 0} \Lambda^k T^*M \to M$. It naturally inherits a metric denoted by $g_{\Lambda \rm M}$,

[1] Actually, there seems no written text in which Einstein himself called his introduction of the cosmological constant "the biggest blunder of my life". See, however, Ref. [3].

such that the direct sum is orthogonal and the restriction of $g_{\Lambda\mathrm{M}}$ to degree one equals to the fiber metric g^*_{M} of the cotangent bundle $T^*M \to M$.

The bundle of (complexified) *Clifford algebras* is denoted by $Cl_{\mathrm{M}} \to M$, where, again, we do not explicitly mention its canonical projection. As a vector bundle the Clifford bundle is canonically isomorphic to the Grassmann bundle (see below). Accordingly, the Clifford bundle also inherits a natural metric structure, such that its restriction to the generating sub-space $T^*M \subset Cl_{\mathrm{M}}$ again reduces to g^*_{M}. In what follows, the Grassmann and the Clifford bundle are mainly regarded as complex bundles, though we do not explicitly indicate their complexification. Accordingly, all (linear) maps are understood as complex linear extensions of the underlaying real linear maps.

The mutually inverse "musical isomorphisms" in terms of g_{M} (resp. g^*_{M}) are denoted by ${}^{\flat/\sharp} : TM \simeq T^*M$, such that, for instance, $g_{\mathrm{M}}(u,v) = g^*_{\mathrm{M}}(u^\flat, v^\flat)$ for all $u, v \in TM$.

A smooth complex vector bundle $\pi_{\mathcal{E}} : \mathcal{E} \to M$ is called a *Clifford module bundle*, provided there is a *Clifford map*. That is, there is a smooth linear (bundle) map (over the identity on M)

$$\begin{aligned} \gamma_{\mathcal{E}} : T^*M &\longrightarrow \mathrm{End}(\mathcal{E}) \\ \alpha &\mapsto \gamma_{\mathcal{E}}(\alpha)\,, \end{aligned} \tag{2}$$

satisfying $\gamma_{\mathcal{E}}(\alpha)^2 = \epsilon g^*_{\mathrm{M}}(\alpha,\alpha)\mathrm{Id}_{\mathcal{E}}$. Here, $\epsilon \in \{\pm 1\}$ depends on how the *Clifford product* is defined. That is, $\alpha^2 := \pm g^*_{\mathrm{M}}(\alpha,\alpha)1_{\mathrm{Cl}} \in Cl_{\mathrm{M}}$, for all $\alpha \in T^*M \subset Cl_{\mathrm{M}}$ and $1_{\mathrm{Cl}} \in Cl_{\mathrm{M}}$ denotes the unit element.

A Clifford map (2) is known to induce a unique homomorphism $\Gamma_{\mathcal{E}} : Cl_{\mathrm{M}} \to \mathrm{End}(\mathcal{E})$ of associative algebras with unit, such that $\Gamma_{\mathcal{E}}(\alpha) = \gamma_{\mathcal{E}}(\alpha)$, for all $\alpha \in T^*M \subset Cl_{\mathrm{M}}$. To explicitly mention the underlying structure we denote a Clifford module bundle also by

$$\begin{aligned} \pi_{\mathcal{E}} : (\mathcal{E}, \gamma_{\mathcal{E}}) &\longrightarrow (M, g_{\mathrm{M}}) \\ z &\mapsto x = \pi_{\mathcal{E}}(z)\,. \end{aligned} \tag{3}$$

If the Clifford module bundle is $\mathbb{Z}_2-$graded, with grading involution being given by $\tau_{\mathcal{E}} \in \mathrm{End}(\mathcal{E})$, then the Clifford map $\gamma_{\mathcal{E}}$ is assumed to be odd: $\gamma_{\mathcal{E}}(\alpha)\tau_{\mathcal{E}} = -\tau_{\mathcal{E}}\gamma_{\mathcal{E}}(\alpha)$, for all $\alpha \in T^*M$. Furthermore, if the vector bundle is supposed to be hermitian, with the hermitian product being denoted by $\langle \cdot,\cdot \rangle_{\mathcal{E}}$, then the Clifford map and the grading involution are supposed to bei either hermitian or anti-hermitian. In this case, we call (3) an *odd hermitian Clifford module bundle*.

The sheaf of sections of any bundle $\pi_{\mathcal{W}} : \mathcal{W} \to M$ is denoted by $\mathfrak{S}ec(M, W)$. In the particular case of the cotangent bundle, however, we follow the common notation and denote the corresponding sheaf of sections by $\Omega(M) \equiv \mathfrak{S}ec(M, \Lambda T^*M)$. Accordingly, $\Omega^k(M, \mathrm{End}(\mathcal{E})) \equiv \mathfrak{S}ec(M, \Lambda^k T^*M \otimes_{\mathrm{M}} \mathrm{End}(\mathcal{E}))$ are the "$\mathrm{End}(\mathcal{E})-$valued forms" of degree $k \geq 0$.

From the Wedderburn structure theorems about invariant algebras one infers that (see [1], [2] and [4])

$$\mathrm{End}(\mathcal{E}) \simeq Cl_{\mathrm{M}} \otimes_{\mathrm{M}} \mathrm{End}_\gamma(\mathcal{E})\,, \tag{4}$$

where $\mathrm{End}_\gamma(\mathcal{E}) \subset \mathrm{End}(\mathcal{E})$ denotes the sub-algebra of endomorphisms on (3) which commute with the Clifford action provided by $\gamma_{\mathcal{E}}$.

As a consequence,

$$\Omega^0(M, \mathrm{End}(\mathcal{E})) \simeq \Omega(M, \mathrm{End}_\gamma(\mathcal{E}) \equiv \bigoplus_{k\geq 0} \Omega^k(M, \mathrm{End}_\gamma(\mathcal{E})\,. \tag{5}$$

The linear map

$$\begin{aligned} \delta_\gamma : \Omega(M, \mathrm{End}(\mathcal{E})) &\longrightarrow \Omega^0(M, \mathrm{End}(\mathcal{E})) \\ \alpha \otimes \mathfrak{B} &\mapsto \gamma_{\mathcal{E}}\big(\sigma_{\mathrm{Ch}}^{-1}(\alpha)\big)\mathfrak{B} \end{aligned} \tag{6}$$

is called the *"quantization map"*. It is determined by the linear isomorphism called *symbol map*:

$$\begin{aligned} \sigma_{\mathrm{Ch}} : Cl_{\mathrm{M}} &\xrightarrow{\simeq} \Lambda T^*M \\ \mathfrak{a} &\mapsto \Gamma_{\mathrm{Ch}}(\mathfrak{a})1_\Lambda \, . \end{aligned} \tag{7}$$

Here, $1_\Lambda \in \Lambda T^*M$ is the unit element. The homomorphism $\Gamma_{\mathrm{Ch}} : Cl_{\mathrm{M}} \to \mathrm{End}(\Lambda T^*M)$ is given by the canonical Clifford map:

$$\begin{aligned} \gamma_{\mathrm{Cl}} : T^*M &\longrightarrow \mathrm{End}(\Lambda T^*M) \\ v &\mapsto \left\{ \begin{array}{ccc} \Lambda T^*M & \longrightarrow & \Lambda T^*M \\ \omega & \mapsto & \epsilon int(v)\omega + ext(v^\flat)\omega \, , \end{array} \right. \end{aligned} \tag{8}$$

where, respectively, "*int*" and "*ext*" indicate "interior" and "exterior" multiplication.

Definition 2.1. *A Clifford module bundle $\pi_{\mathcal{E}'} : (\mathcal{E}', \gamma_{\mathcal{E}'}) \longrightarrow (M, g_M)$ is called an "extension" of the Clifford module bundle* (3)*, provided there is a bundle embedding $\iota : \mathcal{E} \hookrightarrow \mathcal{E}'$ (over the identity on M), such that for all $\alpha \in T^*M$ and $z \in \mathcal{E}$:*

$$\gamma_{\mathcal{E}'}(\alpha)\iota(z) = \iota(\gamma_{\mathcal{E}}(\alpha)z) \, . \tag{9}$$

Furthermore, in the case of odd hermitian Clifford module bundles one assumes that

$$\begin{aligned} \tau_{\mathcal{E}'}\iota(z) &= \iota(\tau_{\mathcal{E}}z) \, , \\ \big\langle \iota(z_1), \iota(z_2) \big\rangle_{\mathcal{E}'} &= \big\langle z_1, z_2 \big\rangle_{\mathcal{E}} \, , \end{aligned} \tag{10}$$

for all $z, z_1, z_2 \in \mathcal{E}$.

Definition 2.2. *A smooth vector bundle $\pi_{\mathcal{E}} : \mathcal{E} \longrightarrow (M, g_M)$ is called a "Clifford bi-module bundle", if there are Clifford maps: $\gamma_{\mathcal{E}}, \gamma'_{\mathcal{E}} : T^*M \longrightarrow \mathrm{End}(\mathcal{E})$, such that for all $\alpha, \beta \in T^*M$:*

$$\gamma_{\mathcal{E}}(\alpha)\gamma'_{\mathcal{E}}(\beta) = \gamma'_{\mathcal{E}}(\beta)\gamma_{\mathcal{E}}(\alpha) \, . \tag{11}$$

Every Clifford module bundle (3) possesses a natural extension to a Clifford bi-module bundle, which is given by

$$\begin{aligned} \iota : \mathcal{E} &\hookrightarrow \mathcal{E}' := \mathcal{E} \otimes_{\mathrm{M}} Cl_{\mathrm{M}} \\ z &\mapsto z \equiv z \otimes 1_{\mathrm{Cl}} \, . \end{aligned} \tag{12}$$

The extension (12) also provides an extension of odd hermitian Clifford module bundles with respect to the grading involution and hermitian structure, respectively

$$\begin{aligned} \tau_{\mathcal{E}'} &:= \tau_{\mathcal{E}} \otimes \mathrm{Id}_{\mathrm{Cl}} \, , \\ \big\langle \cdot, \cdot \big\rangle_{\mathcal{E}'} &:= \big\langle \cdot, \cdot \big\rangle_{\mathcal{E}} \big\langle \cdot, \cdot \big\rangle_{\mathrm{Cl}} \, . \end{aligned} \tag{13}$$

Here, $\big\langle \cdot, \cdot \big\rangle_{\mathrm{Cl}}$ denotes the hermitian structure that is defined in terms of the symbol map (7) and the canonical extension of g^*_{M} to the Grassmann bundle.

Notice that the extension (12) completely fixes the Clifford action to be given by $\gamma_{\mathcal{E}'} = \gamma_{\mathcal{E}} \otimes \mathrm{Id}_{\mathrm{Cl}}$. We call the canonical bi-module extension (12) the *Clifford twist* of the Clifford module bundle (3).

Similar to (4), one has

$$\mathrm{End}(\mathcal{E}') \simeq Cl_{\mathrm{M}} \otimes_{\mathrm{M}} \mathrm{End}_\gamma(\mathcal{E}) \otimes_{\mathrm{M}} \big(Cl_{\mathrm{M}} \otimes_{\mathrm{M}} Cl^{\mathrm{op}}_{\mathrm{M}}\big) \, , \tag{14}$$

where $Cl^{\rm op}_{\rm M} \to M$ is the bundle of *opposite Clifford algebras*.

We call in mind that a *Dirac operator* $\not{D}$ on a Clifford module bundle is a first order differential operator acting on sections $\psi \in \mathfrak{S}ec(M, \mathcal{E})$, such that $[\not{D}, df]\psi = \gamma_{\mathcal{E}}(df)\psi$ for all smooth functions $f \in \mathcal{C}^\infty(M)$. The set of all Dirac operators on (3) is denoted by $\mathfrak{Dir}(\mathcal{E}, \gamma_{\mathcal{E}})$. It provides an affine set over the vector space $\Omega^0(M, {\rm End}(\mathcal{E}))$. Moreover, on odd Clifford module bundles Dirac operators are *odd* operators, i.e. $\not{D}\tau_{\mathcal{E}} = -\tau_{\mathcal{E}}\not{D}$.

We call the Dirac operator $\not{\nabla}^{\mathcal{E}} \equiv \delta_\gamma(\nabla^{\mathcal{E}})$ the "quantization" of a connection $\nabla^{\mathcal{E}}$ on (3). Let $e_1, \ldots, e_n \in \mathfrak{S}ec(U, TM)$ be a local frame and $e^1, \ldots, e^n \in \mathfrak{S}ec(U, T^*M)$ its dual frame. For $\psi \in \mathfrak{S}ec(M, \mathcal{E})$ one obtains

$$\not{\nabla}^{\mathcal{E}}\psi := \sum_{k=1}^n \delta_\gamma(e^k)\nabla^{\mathcal{E}}_{e_k}\psi = \sum_{k=1}^n \gamma_{\mathcal{E}}(e^k)\nabla^{\mathcal{E}}_{e_k}\psi\,, \tag{15}$$

where the canonical embedding $\Omega(M) \hookrightarrow \Omega(M, {\rm End}(\mathcal{E}))$, $\omega \mapsto \omega \equiv \omega \otimes {\rm Id}_{\mathcal{E}}$ is taken into account.

Every Dirac operator has a canonical first-order decomposition:

$$\not{D} = \not{\partial}_{\rm B} + \Phi_{\rm D}\,. \tag{16}$$

Here, $\partial_{\rm B}$ denotes the *Bochner connection* on (3), that is defined by $\not{D}$ as

$$2ev_g\big(df, \partial_{\rm B}\psi\big) := \epsilon\big([\not{D}^2, f] - \delta_g df\big)\psi \qquad \big(\psi \in \mathfrak{S}ec(M, \mathcal{E})\big)\,, \tag{17}$$

with ev_g" being the evaluation map with respect to $g_{\rm M}$ and δ_g the dual of the exterior derivative (see [2]).

The zero-order section $\Phi_{\rm D} := \not{D} - \not{\partial}_{\rm B} \in \mathfrak{S}ec(M, {\rm End}(\mathcal{E}))$ is thus also uniquely determined by $\not{D}$. We call the Dirac operator $\not{\partial}_{\rm B}$ the *"quantized Bochner connection"*.

A (linear) connection on (3) is called a *Clifford connection* if the corresponding covariant derivative $\nabla^{\mathcal{E}}$ "commutes" with the Clifford map $\gamma_{\mathcal{E}}$ in the following sense:

$$[\nabla^{\mathcal{E}}_X, \gamma_{\mathcal{E}}(\alpha)] = \gamma_{\mathcal{E}}\big(\nabla^{T^*M}_X\alpha\big) \qquad \big(X \in \mathfrak{S}ec(M, TM),\ \alpha \in \mathfrak{S}ec(M, T^*M)\big)\,. \tag{18}$$

We denote Clifford connections as $\partial_{\rm A}$ since a Clifford connection is seen to be parametrized by a family of locally define forms $A \in \Omega^1(U, {\rm End}_\gamma(\mathcal{E}))$. This basically follows from (4). Therefore, Clifford connections certainly provide a distinguished class of connections of a Clifford module bundle.

Since $\mathfrak{Dir}(\mathcal{E}, \gamma_{\mathcal{E}})$ is an affine space, every Dirac operator can be written as

$$\not{D} = \not{\partial}_{\rm A} + \Phi\,. \tag{19}$$

However, this decomposition is far from being unique, as opposed to the first-order decomposition (16). In particular, the section $\Phi \in \mathfrak{S}ec(M, {\rm End}(\mathcal{E}))$ depends on the chosen Clifford connection $\partial_{\rm A}$. In general, a Dirac operator does not uniquely determine a Clifford connection.

Definition 2.3. *A Dirac operator is said to be of "simple type" provided that Φ_D anti-commutes with the Clifford action:*

$$\Phi_D\gamma_{\mathcal{E}}(\alpha) = -\gamma_{\mathcal{E}}(\alpha)\Phi_D \qquad \big(\alpha \in T^*M\big)\,. \tag{20}$$

It follows that a Dirac operator of simple type uniquely determines a Clifford connection $\partial_{\rm A}$ together with a zero-order operator $\phi_{\rm D} \in \mathfrak{S}ec(M, {\rm End}_\gamma(\mathcal{E}))$, such that (c.f. [10])

$$\not{D} = \not{\partial}_{\rm A} + \tau_{\mathcal{E}}\phi_{\rm D}\,. \tag{21}$$

These Dirac operators play a basic role in the geometrical description of the Standard Model (c.f. [10]). They are also used in the context of the family index theorem (see, for instance, [2]).

Apparently, Dirac operators of simple type provide a natural generalization of quantized Clifford connections. They are distinguished (and fully characterized!) by the fact that they build the most general class of Dirac operators with the property that their Bochner connections (17) are also Clifford connections (see the next section).

Notice that $\phi_{\rm D} \in \mathfrak{Sec}(M, {\rm End}^-_\gamma(\mathcal{E}))$ in the case of odd Clifford module bundles, with ${\rm End}^\pm(\mathcal{E}) \subset {\rm End}(\mathcal{E})$ denoting the sub-algebras of even and odd endomorphisms.

3. The EHC action as a generic functional

In appropriate physical units, the *Einstein-Hilbert functional* (action of gravity) with a cosmological constant added is given by

$$\mathcal{I}_{\rm EHC} := \int_M *\big(scal(g_{\rm M}) + \Lambda\big)\,. \tag{22}$$

Here, "$*$" denotes the Hodge map with respect to $g_{\rm M}$ and a chosen orientation of M. The smooth function $scal(g_{\rm M}) \in \mathcal{C}^\infty(M)$ is the scalar curvature and $\Lambda \in \mathbb{R}$ denotes the *cosmological constant*.

Due to *Lovelock's Theorem*, the structure of the functional $\mathcal{I}_{\rm EHC}$ is uniquely determined by the requirement that the Euler-Lagrange equation is tensorial and only depends on the metric and its first and second derivative (c.f. [6] and the editors remark H2 on page 285 in [9]; for a refinement of this statement in terms of "natural geometry" we refer to [8]).

We demonstrate that the functional $\mathcal{I}_{\rm EHC}$ has a generic form in the sense that several other geometrical functionals may be recast into the form (22). As an example, we present in this vein the functional of non-linear $\sigma-$models (Dirac harmonic maps) and the Yang-Mills action.

On a given Clifford module bundle (3) every Dirac operator naturally defines two connections: The Bochner connection (17) and the *"Dirac connection"*

$$\partial_{\rm D} := \partial_{\rm B} + \omega_{\rm D}\,. \tag{23}$$

Here, $\omega_{\rm D} \equiv \Theta\Phi_{\rm D} \in \Omega^1(M, {\rm End}(\mathcal{E}))$ is the *"Dirac form"*, with $\Theta(v) := \frac{\epsilon}{n}\gamma_{\mathcal{E}}(v^\flat)$ being the *canonical one-form* for all $v \in TM$. It is the right-inverse of the quantization map (6) restricted to $\Omega^1(M, {\rm End}(\mathcal{E}))$. The canonical one-form also plays a basic role in the construction of twister operators. In terms of the canonical one-form Clifford connections may be characterized as follows: A connection on a Clifford module bundle (3) is a Clifford connection if and only if it leaves the canonical one-form covariantly constant:

$$\nabla_X^{T^*M\otimes{\rm End}(\mathcal{E})}\Theta \equiv 0 \qquad \big(X \in \mathfrak{Sec}(M, TM)\big)\,. \tag{24}$$

On a Clifford module bundle with a chosen Dirac operator:

$$\pi_{\mathcal{E}} : (\mathcal{E}, \gamma_{\mathcal{E}}, \not{D}) \longrightarrow (M, g_{\rm M})\,, \tag{25}$$

the Dirac connection is distinguished for it is uniquely determined by $\not{D}$. Also, the Dirac connection has the property that $\not{\partial}_{\rm D} \equiv \delta_\gamma(\partial_{\rm D}) = \not{D}$. Notice that neither $\not{\partial}_{\rm B} = \not{D}$, nor are $\partial_{\rm B}$ and $\partial_{\rm D}$ are Clifford connections, in general.

Every Dirac operator is known to have a unique *second order decomposition*

$$\not{D}^2 = \triangle_{\rm B} + V_{\rm D}\,, \tag{26}$$

where the *Bochner-Laplacian* (or "trace Laplacian") is given in terms of the Bochner connection as $\triangle_{\rm B} := \epsilon ev_g\big(\partial_{\rm B}^{T^*M\otimes\mathcal{E}} \circ \partial_{\rm B}\big)$. The trace of the zero-order operator $V_{\rm D} \in \mathfrak{Sec}(M, {\rm End}(\mathcal{E}))$ explicitly reads (c.f. [10]):

$$tr_{\mathcal{E}} V_{\rm D} = tr_\gamma\big(curv(\not{D}) - \varepsilon\, {\rm ev}_g(\omega_{\rm D}^2)\big) - \varepsilon\delta_g\big(tr_{\mathcal{E}}\omega_{\rm D}\big)\,, \tag{27}$$

where $curv(\not{D}) \in \Omega^2(M, {\rm End}(\mathcal{E}))$ denotes the curvature of the Dirac connection of $\not{D} \in \mathfrak{Dir}(\mathcal{E}, \gamma_{\mathcal{E}})$ and $tr_\gamma := tr_{\mathcal{E}} \circ \delta_\gamma$ the "quantized trace".

Let M be closed compact. We call the functional

$$\begin{aligned}\mathcal{I}_{\rm D} :\, \mathfrak{Dir}(\mathcal{E}, \gamma_{\mathcal{E}}) &\to \mathbb{C}\\ \not{D} &\mapsto \int_M *tr_{\mathcal{E}} V_{\rm D}\end{aligned} \tag{28}$$

the *"universal Dirac action"* and

$$\begin{aligned}\mathcal{I}_{\rm D,tot} :\, \mathfrak{Dir}(\mathcal{E}, \gamma_{\mathcal{E}}) \times \mathfrak{Sec}(M, \mathcal{E}) &\to \mathbb{C}\\ (\not{D}, \psi) &\mapsto \int_M *\big(\langle\psi, \not{D}\psi\rangle_{\mathcal{E}} + tr_{\mathcal{E}} V_{\rm D}\big)\end{aligned} \tag{29}$$

the *"total Dirac action"*. Here, "$*$" is the Hodge map with respect to $g_{\rm M}$ and a chosen orientation of M.

If the Dirac connection of $\not{D}$ is a Clifford connection, then $\partial_{\rm D} = \partial_{\rm B}$. In this case, the Dirac action (28) reduces to the Einstein-Hilbert functional (1).

In contrast, for Dirac operators of simple type the Dirac action becomes

$$\mathcal{I}_{\rm D}\big(\not{\partial}_{\rm A} + \tau_{\mathcal{E}}\phi_{\rm D}\big) = \int_M *\big(-\epsilon\tfrac{rk(\mathcal{E})}{4} scal(g_{\rm M}) + tr_{\mathcal{E}}\phi_{\rm D}^2\big)\,, \tag{30}$$

with $rk(\mathcal{E}) \geq 1$ being the rank of (3). This is a direct consequence of Lemma 4.1 and the Corollary 4.1 of Ref. [10] (see also Sec. 6 in loc site).

Apparently, the restriction of the Dirac action (28) to Dirac operators of simple type (21) formally coincides with the Einstein-Hilbert action (22) with a cosmological constant, where (up to numerical factors)

$$\Lambda = tr_{\mathcal{E}}\phi_{\rm D}^2 \equiv \pm\|\phi_{\rm D}\|^2\,. \tag{31}$$

Similar to (22), the Einstein equation of (30) yields $\|\phi_{\rm D}\| = const.$ as long as the section $\phi_{\rm D} \in \mathfrak{Sec}(M, {\rm End}_\gamma^-(\mathcal{E}))$ does not depend on $g_{\rm M}$. In the case of a transitive action, this reduces the gauge group $\mathcal{G} \equiv \mathfrak{Sec}(M, {\rm Aut}_\gamma(\mathcal{E}))$ to the stabilizer group of a chosen point on the (hyper) sphere $\|\phi_{\rm D}\| = const.$ and therefore spontaneously breaks the (gauge) symmetry that is provided by the structure of the underlying Clifford module bundle (3). This reduction of the gauge group is in complete analogy to the symmetry breaking induced by the Higgs potential of the Standard Model (we refer to [10], for a more thorough discussion of this point).

To proceed let $\chi \in \Omega(M, {\rm End}_{\gamma'}^-(\mathcal{E}'))$. With respect to a local (oriented) orthonormal basis $e^1, \ldots, e^n \in \mathfrak{Sec}(U, T^*M)$ we may write ($U \subset M$, open):

$$\begin{aligned}\chi &\overset{loc.}{=} \sum_{k=0}^n \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} e^{i_1}\wedge e^{i_2}\wedge\cdots\wedge e^{i_k}\otimes\chi_{i_1 i_2\cdots i_k}\\ &\equiv \sum_I e^I \otimes \chi_I\,.\end{aligned} \tag{32}$$

By abuse of notation we put

$$\not{\chi} := \sum_I \chi_I e^I \in \mathfrak{Sec}(M, \mathrm{End}^-_{\gamma'}(\mathcal{E}'))\,, \tag{33}$$

where for $\chi_I = \sum_{J,K} \varphi_{IJK} \otimes \mathfrak{a}_J \otimes \mathfrak{b}_K \in \mathrm{End}_{\gamma'}(\mathcal{E}') = \mathrm{End}^-_\gamma(\mathcal{E}) \otimes Cl_{\mathrm{M}} \otimes Cl^{\mathrm{op}}_{\mathrm{M}}$:

$$\chi_I e^I \equiv \sum_{J,K} \varphi_{IJK} \otimes \mathfrak{a}_J \sigma^{-1}_{\mathrm{Ch}}(e^I) \otimes \mathfrak{b}_K\,. \tag{34}$$

The explicit form of the coefficients $\varphi_{IJK} \in \mathrm{End}_\gamma(\mathcal{E})$ is related to the structure of the Clifford module bundle (3). This structure may yield a metric dependent "cosmological constant"

$$\Lambda = \pm\|\chi\|^2\,. \tag{35}$$

The EHC then gives rise to first order constraints on the fields φ_{IJK}.

As an application we discuss the functional of Dirac harmonic maps and the Yang-Mills action as special cases of (30).

4. The functional of non-linear $\sigma-$models

In this section we specify the Clifford module bundle (3) and Dirac operators of simple type.

For $k = 1, 2$ let $\pi_k : (\mathcal{E}_k, \gamma_k) \to (M_k, g_k)$ be odd hermitian Clifford module bundles over smooth orientable (semi-)Riemannian manifolds of dimensions $n_k = p_k + q_k \equiv dim(M_k)$ and signatures $s_k = p_k - q_k \in \mathbb{Z}$. The corresponding Clifford bundles are denoted by $Cl_k \to (M_k, g_k)$. The grading involution and hermitian products read τ_k and $\langle\cdot,\cdot\rangle_k$, respectively. In the sequel we assume M_1 to be closed compact.

Let $\varphi : M_1 \to M_2$ be a smooth map. We set

$$\pi_{\mathcal{E}} : \mathcal{E} := \mathcal{E}_1 \otimes_{M_1} \varphi^*\mathcal{E}_2 \longrightarrow (M_1, g_1)\,, \tag{36}$$

as well as

$$\begin{aligned} \gamma_{\mathcal{E}} &:= \gamma_1 \otimes \mathrm{Id}_{\varphi^*\mathcal{E}_2}\,,\\ \tau_{\mathcal{E}} &:= \tau_1 \otimes \tau_2|_\varphi\,,\\ \langle\cdot,\cdot\rangle_{\mathcal{E}} &:= \langle\cdot,\cdot\rangle_1 \langle\cdot,\cdot\rangle_2|_\varphi\,. \end{aligned} \tag{37}$$

That is, in the case considered the Clifford module bundle (3) is a *twisted Clifford module bundle* with the twisting provided by the pull-back of the odd hermitian vector bundle $\pi_2 : \mathcal{E}_2 \to M_2$ with respect to the smooth map φ.

Finally, we set

$$\pi_{\mathcal{E}'} : \mathcal{E}' := \mathcal{E} \otimes_{M_1} Cl_1 \longrightarrow (M_1, g_1) \tag{38}$$

and

$$\begin{aligned} \gamma_{\mathcal{E}'} &:= \gamma_{\mathcal{E}} \otimes \mathrm{Id}_{\mathrm{Cl}_1}\,,\\ \tau_{\mathcal{E}'} &:= \tau_{\mathcal{E}} \otimes \mathrm{Id}_{\mathrm{Cl}_1}\,,\\ \langle\cdot,\cdot\rangle_{\mathcal{E}'} &:= \langle\cdot,\cdot\rangle_{\mathcal{E}} \langle\cdot,\cdot\rangle_{\mathrm{Cl}_1}\,. \end{aligned} \tag{39}$$

That is, the odd hermitian Clifford module bundle $\pi_{\mathcal{E}'} : (\mathcal{E}', \gamma_{\mathcal{E}'}) \to (M_1, g_1)$ is the Clifford twist of the odd hermitian Clifford module bundle $\pi_{\mathcal{E}} : (\mathcal{E}, \gamma_{\mathcal{E}}) \to (M_1, g_1)$. Likewise, the bundle (38) may be regarded as a twisted Clifford module bundle with the twisting given by the twisted Clifford module bundle $\varphi^*\mathcal{E}_2 \otimes_{M_1} Cl_1 \to (M_1, g_1)$.

A Clifford connection on (38) reads

$$\partial_{\mathrm{A}'} = \partial_{\mathrm{A}} \otimes \mathrm{Id}_{\mathrm{Cl}_1} + \mathrm{Id}_{\mathcal{E}} \otimes \nabla^{\mathrm{Cl}_1}\,. \tag{40}$$

Here,

$$\partial_{\mathrm{A}} = \partial_{\mathrm{A}_1} \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes \nabla^{\varphi^*\mathcal{E}_2} \tag{41}$$

denotes a general Clifford connection on (36), with, respectively, ∇^{Cl_1} and $\nabla^{\mathcal{E}_2}$ being arbitrary connections on the Clifford module bundle $Cl_1 \to (M_1, g_1)$ and $\pi_2 : (\mathcal{E}_2, g_2) \to (M_2, g_2)$. Clearly, on $Cl_1 \to (M_1, g_1)$ there is a canonical choice provided by the Levi-Civita connection with respect to g_1.

Note that for all sections $\psi \in \mathfrak{Sec}(M_1, \mathcal{E})$:

$$\partial_{\mathrm{A}'}(\psi \otimes 1) = \partial_{\mathrm{A}}\psi \otimes 1\,. \tag{42}$$

Let $e_1, e_2, \ldots, e_{n_1} \in \mathfrak{Sec}(U_1, TM_1)$ be a local (oriented orthonormal) frame on the open subset $U_1 \subset M_1$ with the dual frame denoted as $e^1, e^2, \ldots, e^{n_1} \in \mathfrak{Sec}(U_1, T^*M_1)$.

The quantization of (40) then reads

$$\partial\!\!\!/_{\mathrm{A}'} = \partial\!\!\!/_{\mathrm{A}} \otimes \mathrm{Id}_{\mathrm{Cl}_1} + \sum_{a=1}^{n_1} \gamma_{\mathcal{E}}(e^a) \otimes \nabla^{\mathrm{Cl}_1}_{e_a}\,. \tag{43}$$

It follows that $\partial\!\!\!/_{\mathrm{A}'}(\psi \otimes 1) = \partial\!\!\!/_{\mathrm{A}}\psi \otimes 1$.

The Jacobi map of φ can be identified with the section $d\varphi \in \Omega^1(M_1, \varphi^*TM_2)$. We set for all $t \in U_1$: $\varphi_a(t) \equiv d\varphi(t)e_a(t) \in T_{\varphi(t)}M_2$, such that $d\varphi \overset{loc.}{=} e^a \otimes \varphi_a$ and consider

$$\begin{aligned} \chi &\overset{loc.}{=} e^a \otimes \chi_a \in \Omega^1(M_1, \mathrm{End}^-_\gamma(\mathcal{E}))\,, \\ \chi_a &:= \big(\mathrm{Id}_{\mathcal{E}_1} \otimes \gamma_2\big)\big(\mathrm{Id}_{\mathcal{E}_1} \otimes \varphi^\flat_a\big) \\ &= \mathrm{Id}_{\mathcal{E}_1} \otimes \gamma_2(\varphi^\flat_a) \in \mathcal{C}^\infty\big(U_1, \mathrm{End}^-_\gamma(\mathcal{E})\big)\,. \end{aligned} \tag{44}$$

Accordingly, we set

$$\chi\!\!\!/ \overset{loc.}{:=} \sum_{a=1}^{n_1} \mathrm{Id}_{\mathcal{E}_1} \otimes \gamma_2(\varphi^\flat_a) \otimes e^a \in \Omega^0(M_1, \mathrm{End}^-_{\gamma'}(\mathcal{E}')) \tag{45}$$

and consider the following Dirac operator of simple type, which acts on sections of (38):

$$D\!\!\!\!/ := \partial\!\!\!/_{\mathrm{A}'} + \tau_{\mathcal{E}'}\chi\!\!\!/\,. \tag{46}$$

For sections $\psi \equiv \psi \otimes 1 \in \mathfrak{Sec}(M_1, \mathcal{E}')$ one gets

$$\big\langle \psi, D\!\!\!\!/\psi \big\rangle_{\mathcal{E}'} = \big\langle \psi, \partial\!\!\!/_{\mathrm{A}}\psi \big\rangle_{\mathcal{E}}\,. \tag{47}$$

Furthermore,

$$\begin{aligned} \|\chi\!\!\!/\|^2 &\equiv \epsilon_1 g^*_1(e^a, e^b)\, tr_{\mathcal{E}}\chi^\dagger_a\chi_b \\ &= \pm\epsilon_1\epsilon_2 rk\mathcal{E}\|d\varphi\|^2\,, \end{aligned} \tag{48}$$

where $\|d\varphi\|^2 \equiv g^*_1(e^a, e^b)\, g_2|_\varphi(\varphi_a, \varphi_b) = g^*_1(e^a, e^b)\, \varphi^*g_2(e_a, e_b)\,.$

We thus proved the following

Proposition 4.1. *The total Dirac action* (29) *with respect to the simple type Dirac operators* (46) *reads*

$$\begin{aligned} \mathcal{I}_{D,tot}(D\!\!\!\!/, \psi) &= \int_{M_1} \Big(\big\langle \psi, D\!\!\!\!/\psi \big\rangle_{\mathcal{E}'} + tr_{\gamma'}curv(\partial\!\!\!/_{A'}) \pm \|\chi\!\!\!/\|^2\Big) dvol(g_1) \\ &= \int_{M_1} \Big(-\epsilon_1 \tfrac{rk(\mathcal{E}')}{4} scal(g_1) + \big\langle \psi, \partial\!\!\!/_A\psi \big\rangle_{\mathcal{E}} \pm \epsilon_1\epsilon_2 rk(\mathcal{E})\|d\varphi\|^2\Big) dvol(g_1)\,. \end{aligned} \tag{49}$$

Therefore, up to the Einstein-Hilbert action (e.g. for fixed metric on M_1), the Dirac action basically coincides with the functional of non-linear $\sigma-$models (Dirac harmonic maps). This holds true, especially, in the case $dim(M_1) = 2$.

4.1. **Geodesics as a specific example.** To present a prominent geometrical example, let $(M_2, g_2) \equiv (M, g_{\rm M})$ be an arbitrary $n-$dimensional smooth Riemannian manifold and $(M_1, g_1) := [0,1] \subset \mathbb{R}^{1,0}$. For $(\mathcal{E}_1, \gamma_1)$ we set $\mathcal{E}_1 := [0,1] \times {}^2\mathbb{R}$ and consider the canonical Clifford map

$$\begin{aligned} \gamma_1 : \mathbb{R} &\longrightarrow {}^2\mathbb{R} \\ dt &\mapsto e \equiv (1,-1). \end{aligned} \tag{50}$$

Here, ${}^2\mathbb{R} \simeq Cl_{1,0}$ denotes the two-dimensional real algebra of *Study numbers* and $Cl_{1,0}$ is the Clifford algebra of the one-dimensional Euclidean space $\mathbb{R}^{1,0}$.

We call in mind that the real algebra of Study numbers equals the two-dimensional real vector space $\mathbb{R}^2$ with component wise multiplication. The unit is given by $1 \equiv (1,1)$, such that $\mathbb{R} \hookrightarrow {}^2\mathbb{R}$ is contained as a canonical sub-algebra. Furthermore, $e \equiv (1,-1) \in {}^2\mathbb{R}$ is analogues to $i \equiv (1,-1) \in \mathbb{C}$. It follows that ${}^2\mathbb{R} \simeq_{\mathbb{R}} {\rm End}(S \oplus \check{S})$, where $S := \{(u,0)\,|\,u \in \mathbb{R}\} \subset {}^2\mathbb{R} \simeq Cl_{1,0}$ and $\check{S} := \{(0,v)\,|\,v \in \mathbb{R}\} \subset {}^2\mathbb{R} \simeq Cl_{1,0}$ are the spinor modules with respect to the primitive idempotents $(1 \pm e)/2 \in {}^2\mathbb{R}$.

We put $(\mathcal{E}_2, \gamma_2) := (\Lambda T^*M, \gamma_{\rm Ch})$, such that

$$\mathcal{E} = [0,1] \times \big({}^2\mathbb{R} \otimes \varphi^*\Lambda T^*M\big) \simeq \varphi^*\Lambda T^*M \oplus \varphi^*\Lambda T^*M\,. \tag{51}$$

Any section $\psi \in \mathfrak{S}ec([0,1], \mathcal{E})$ thus corresponds to a pair of sections of the (trivial) algebra bundle $\varphi^*\Lambda T^*M \simeq [0,1] \times \Lambda\mathbb{R}^n \to [0,1]$.

Let again $I_k = (i_1, \ldots, i_k)$ be a multi-index for all $1 \leq k \leq n = dim(M)$ and $i_l = 1, \ldots, n$ $(l = 1, \ldots, k)$. We make again usage of the shorthand $\sum_I \sigma_I \equiv \sum_{k=1}^n \sum_{I_k} \sigma_{I_k}$. Then, $\psi(t) = \sum_I \psi_I(t) \otimes e_I$, whereby $e_{I_k} : [0,1] \to \varphi^*\Lambda T^*M,\ t \mapsto (t, \mathbf{e}_{I_k})$ are the canonical sections with $\mathbf{e}_{I_1}, \ldots, \mathbf{e}_{I_n} \in \Lambda\mathbb{R}^n$ being the standard basis. Furthermore, $\psi_{I_k}(t) = (\alpha_{I_k}(t), \beta_{I_k}(t)) \in {}^2\mathbb{R}$ for all $k = 1, \ldots, n$.

We choose the trivial connection to define $\nabla^{\mathcal{E}_1}$, such that for all smooth sections $\chi = (\chi_1, \chi_2) \in \mathfrak{S}ec([0,1], \mathcal{E}_1) \simeq \mathcal{C}^\infty([0,1], {}^2\mathbb{R})$ the action of the corresponding Dirac operator reads: $\not{\partial}_1\chi(t) = (\dot\chi_1(t), -\dot\chi_2(t)) \in {}^2\mathbb{R}$. Here, $\dot\chi_k(t) := d\chi_k(t)\partial_t|_t$, whereby $\partial_t : [0,1] \to [0,1] \times \mathbb{R},\ t \mapsto (t,1)$ is the canonical tangent vector field. On the Clifford module bundle $\Lambda T^*M \to M$ we take the induced Levi-Civita connection of $(M, g_{\rm M})$.

With respect to our notation the action of the Dirac operator $\not{\partial}$ reads:

$$\begin{aligned} &\not{\partial}\psi(t) = \\ &\sum_I \Big(\dot\alpha_I(t) + \sum_{I'} \Gamma_{II'}|_{\varphi(t)}(\dot\varphi(t))\alpha_{I'}(t),\ -\dot\beta_I(t) - \sum_{I'} \Gamma_{II'}|_{\varphi(t)}(\dot\varphi(t))\beta_{I'}(t)\Big) \otimes \mathbf{e}_I\,, \end{aligned} \tag{52}$$

where for all $k = 1, \ldots, n$:

$$\sum_{l=1}^n \sum_{I_l} \Gamma_{I_k I_l}|_{\varphi(t)}(\dot\varphi(t)) e_{I_l}(t) := \nabla^{\varphi^*\Lambda T^*M}_{\partial_t} e_{I_k}(t)\,, \tag{53}$$

defines the induced Levi-Civita connection coefficients of the pull-back connection and $\dot\varphi(t) := d\varphi(t)\partial_t|_t \in T_{\varphi(t)}M$ is the velocity vector of the smooth curve $\varphi : [0,1] \to M$.

After appropriate normalization, the total Dirac action (29) simplifies to

$$\mathcal{I}_{\rm D,tot}(\not{D}, \psi) = \int_0^1 \Big(\big\langle \psi, \not{\partial}\psi \big\rangle_{\mathcal{E}} + \|d\varphi\|^2\Big) dt\,. \tag{54}$$

In particular, the Dirac action (30) reduces to what is referred to as the *energy functional* of the curve φ:

$$\mathcal{I}_{\mathrm{D}}\big(\not{\partial} + \tau_{\mathcal{E}'}\not{\chi}\big) = \int_0^1 g_{\mathrm{M}}|_{\varphi(t)}\big(\dot\varphi(t), \dot\varphi(t)\big)\, dt\,, \tag{55}$$

where $d\varphi = dt \otimes \dot\varphi \in \Omega^1([0,1], \varphi^*TM)$.

For fixed metric, the minima of (55) are known to be given by the geodesics on (M, g_{M}).

5. The Yang-Mills action

A Dirac operator of simple type on an arbitrary Clifford module bundle (3) is uniquely defined in terms of a Clifford connection $\partial_{\mathrm{A}} = \partial_{\mathrm{B}}$, together with a section $\phi_{\mathrm{D}} \in \mathfrak{S}ec(M, \mathrm{End}^-_\gamma(\mathcal{E}))$. It is thus natural to consider Dirac operators of simple type $\not{D}$, which are fully determined by the Clifford connection ∂_{A} of $\not{D}$. Of course, the Dirac operators $\not{\partial}_{\mathrm{A}}$ are special cases thereof. A straightforward construction of Dirac operators of simple type which are fully determined by Clifford connections ∂_{A} but $\not{D} \neq \not{\partial}_{\mathrm{A}}$ goes as follows.

Let $M_1 = M_2 \equiv M$ be a smooth orientable manifold of dimension $n \geq 1$. We assume M to be closed compact. Also, let $g_1 = g_2 \equiv g_{\mathrm{M}}$ be a (semi-)Riemannian metric of arbitrary signature. For $\varphi = \mathrm{Id}_{\mathrm{M}}$ we denote by

$$F_{\mathrm{A}} = F_{\mathrm{A}_1} \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes F_{\mathrm{A}_2} \in \Omega^2(M, \mathrm{End}^+_\gamma(\mathcal{E})) \tag{56}$$

the *twisting (relative) curvature* on (36) of the hermitian Clifford connection

$$\partial_{\mathrm{A}} := \partial_{\mathrm{A}_1} \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes \partial_{\mathrm{A}_2}\,. \tag{57}$$

Let again $e_1, \ldots, e_n \in \mathfrak{S}ec(U, TM)$ be a local (oriented orthonormal) frame with dual frame $e^1, \ldots, e^n \in \mathfrak{S}ec(U, T^*M)$. We set

$$\begin{aligned} \chi &\overset{loc.}{=} \sum_{a=1}^n e^a \otimes \chi_a \in \Omega^1(M, \mathrm{End}^-_\gamma(\mathcal{E}))\,, \\ \chi_a &:= \big(\mathrm{Id}_{\mathcal{E}_1} \otimes \gamma_2\big)\big(int(e_a)F_{\mathrm{A}}\big) \\ &= F_{\mathrm{A}_1}(e_a, e_b) \otimes \gamma_2(e^b) + \mathrm{Id}_{\mathcal{E}_1} \otimes F_{\mathrm{A}_2}(e_a, e_b)\gamma_2(e^b) \in \mathcal{C}^\infty(U, \mathrm{End}^-_\gamma(\mathcal{E}))\,. \end{aligned} \tag{58}$$

The reader may compare this with the case (44) of non-linear $\sigma-$models, whereby the correspondence is given by $\varphi^\flat_a(v) \leftrightarrow F_{\mathrm{A}}(e_a, v)$, for all $v \in TU$. In contrast to the case (44), however, the choice (58) is most natural, as already mentioned, for it allows in a canonical way to also define the zero-order part ϕ_{D} of a simple type Dirac operator $\not{D}$ in terms of the Clifford connection of $\not{D}$. In this case one has

$$\|\chi\|^2 = \epsilon_1\epsilon_2 \|F_{\mathrm{A}}\|^2\,, \tag{59}$$

where $\|F_{\mathrm{A}}\|^2 \equiv -g^*_{\mathrm{M}}(e^a, e^c)\, g^*_{\mathrm{M}}(e^b, e^d)\, tr_{\mathcal{E}}\big(F_{\mathrm{A}}(e_a, e_b)F_{\mathrm{A}}(e_c, e_d)\big) \equiv -tr_{\mathcal{E}} F_{ab}F^{ab} \in \mathcal{C}^\infty(M)$.

We thus proved the following

Proposition 5.1. *When restricted to the class of simple type Dirac operators considered, the total Dirac action decomposes as*

$$\mathcal{I}_{D,tot} = \int_M *\Big(-\epsilon_1 \tfrac{rk(\mathcal{E}')}{4} scal(g_M) + \big\langle \psi,\, \not{\partial}_A \psi \big\rangle_{\mathcal{E}} + \epsilon_1\epsilon_2\, \|F_A\|^2 \Big)\,. \tag{60}$$

We finally discuss the case of *twisted spinor bundles*, usually encountered in the literature dealing with Dirac operators. To this end let M be a *spin manifold* and $\pi_{\rm S} : S \to M$ be a *spinor bundle*. Moreover, let $\pi_{\rm E} : E \to M$ be a smooth (odd) hermitian vector bundle. In this particular case, we set $\mathcal{E}_1 := S \otimes_{\rm M} E$ and $\mathcal{E}_2 := Cl_{\rm M}$. The canonical embedding $\mathcal{E}_1 \hookrightarrow \mathcal{E},\ z \mapsto z \otimes 1$ then yields $\partial_{\rm A}(\psi \otimes 1) = \partial_{{\rm A}_1}\psi \otimes 1$. The Clifford connection $\partial_{{\rm A}_1} \equiv \nabla^{S\otimes E}$ is but the *twisted spin connection* with the twisting curvature being given by $F_{{\rm A}_1} = {\rm Id}_{\rm S} \otimes F^{\rm E}$, where $F^{\rm E} \in \Omega^2(M, {\rm End}^+(E))$ denotes the curvature of some (even) hermitian connection $\nabla^{\rm E}$ on $\pi_{\rm E} : E \to M$.

When restricted to the sub-bundle $\pi_{\mathcal{E}}|_{\mathcal{E}_1} : \mathcal{E}_1 \subset \mathcal{E} \to M$ the twisting curvature (56) reduces to

$$F_{\rm A} = {\rm Id}_{\rm S} \otimes F^{\rm E} \otimes {\rm Id}_{\rm Cl} = F_{{\rm A}_1} \otimes {\rm Id}_{\rm Cl}\,. \tag{61}$$

Therefore, in the case of twisted spinor bundles the total Dirac action decomposes into the sum of the Dirac, the Einstein-Hilbert and the Yang-Mills action:

$$\begin{aligned} \mathcal{I}_{\rm D,tot} &= \int_M * \Big(\big\langle \psi,\, \not{\partial}_{{\rm A}_1}\psi \big\rangle_{\mathcal{E}_1} - \epsilon_1 \tfrac{rk(\mathcal{E}')}{4} scal(g_{\rm M}) + 2^n rk(S) \epsilon_1 \epsilon_2 \, \|F^{\rm E}\|^2 \Big) \\ &\equiv \int_M * \big\langle \psi,\, \not{\partial}_{{\rm A}_1}\psi \big\rangle_{\mathcal{E}_1} - \epsilon_1 \tfrac{rk(\mathcal{E}')}{4} \int_M * scal(g_{\rm M}) + 2^{n+1} \epsilon_1 \epsilon_2 \int_M tr_{\mathcal{E}_1}(F_{{\rm A}_1} \wedge * F_{{\rm A}_1})\,. \end{aligned} \tag{62}$$

6. Conclusion

The Yang-Mills action and the functional of non-linear $\sigma-$models can both be described by Dirac operators of simple type. As such they have the same "root" and the underlying generic form of these actions is provided by the Dirac action (30) generalizing the Einstein-Hilbert action with a cosmological constant (22).

The decomposition of the Dirac action in terms of the fields defining a Dirac operator is rather similar to the decomposition of manifest supersymmetric actions in terms of the fields defining the underlying super-field. From this point of view one may argue that certain classes of Dirac operators will give rise to supersymmetric actions. Indeed, the functional of Proposition (4.1) is known to have a supersymmetric interpretation. This holds true especially for $dim(M_1) = 2$, where the supersymmetric interpretation of the functional of Dirac harmonic maps plays a basic role in the discussion of (super) Riemann surfaces (for an appreciable survey of this issue we refer to Sec. 2.4 in [5]).

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