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Critical Probabilities and Convergence Time of Stavskaya's Probabilistic Cellular Automata

by

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Critical probabilities and convergence time of Stavskaya's Probabilistic Cellular Automata

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Abstract

We consider a class of probabilistic cellular automata undergoing a phase transition with an absorbing-state. Denoting by $\mathcal{U}(s)$ the neighbourhood of the site s, the transition probability is $T(\eta_s = 1|\eta_{\mathcal{U}(s)}) = 0$ if $\eta_{\mathcal{U}(s)} = \mathbf{0}$ or p otherwise, $\forall s \in \mathbb{Z}$. For any $\mathcal{U}$ there exists a non-trivial critical probability $p_c(\mathcal{U})$ which separates a phase with an absorbing-state from a fluctuating phase. We study how the neighbourhood affects the value of $p_c(\mathcal{U})$ and we provide lower bounds for $p_c(\mathcal{U})$. Furthermore, using techniques of dynamic renormalization, we prove that the expected convergence time of the processes on a finite space with periodic boundaries grows exponentially (resp. logarithmically) with the system size if $p > p_c$ (resp. $p < p_c$). This appears as an open problem in [4, 5, 6].

1 Introduction

Probabilistic cellular automata (PCA) are discrete-time Markov processes modelling the time evolution of a multicomponent system. Their main feature is the synchronous update of the states of the components, which take values in a finite set and interact with their neighbours according to a given probabilistic interaction rule.

PCA are favourable models to study non-equilibrium phenomena. In fact on the one hand their definition is simple, as space of configurations is discrete and interactions are local. On the other hand, despite this simplicity, they exhibit a variety of complex behaviours. In particular the interest concentrates on the study of phase transition in the context of non-equilibrium statistical physics. A phase transition occurs when, after turning a free parameter above or below a certain critical threshold, the process at infinite time preserves part of the information on its initial condition (non-ergodic behaviour). Namely the probability measure at infinite time depends on the initial state of the dynamics. On the contrary if the process is ergodic, it admits a unique, attracting invariant measure.

In the last 50 years PCA have been object of intense analytical and numerical investigations (e.g. [5, 6, 12, 13]). However, as far as we know, many questions involving e.g. the rate of convergence to equilibrium or the characterization of the invariant measures still remain open, even for the simplest models (see e.g. [4, 5] for a survey).

In the present paper we consider a well known class of PCA, called *Stavskaya's* (see e.g. [1, 2, 3, 4, 5, 6, 15, 16].) Usually this name refers to a specific model where units are located on a 1-dimensional lattice and each of them interacts only with itself and with its right-adjacent neighbour. In this article we refer to the class of models with the same interaction rule, but we keep general the choice of the neighbourhood. This class of PCA is also known as *Percolation PCA* [4, 5].

The reason to consider the Stavskaya's processes is that their simplicity, combined with the presence of a phase transition, make them an interesting test case for attempts to characterize transient behaviour and stationary measures for spatially extended stochastic dynamics. In particular Stavskaya's PCA are a prominent model for studying *absorbing-state phase transitions* ([17]), i.e. there exists a phase characterized by almost sure convergence into an "absorbing state" (a realization where the process remains for ever whenever reached) and a fluctuating phase, where the process remains active at all times.

In the present paper we discuss two distinct aspects of the Stavskaya's process. In section 3 we consider Stavskaya's processes with distinct (translation invariant) neighbourhoods $\mathcal{U}$ and we study how the neighbourhood affects the critical probability. We provide a lower bound for the critical probabilities $p_c(\mathcal{U})$. Our result is stated in Theorem 2.1. With our estimations we improve the previous lower bound [18] showing that $p_c(\mathcal{U}) > 1/2$ strictly in case of neighbourhood $\mathcal{U}(0) = \{-1, 0, 1\}$. Furthermore we provide new bounds in case of neighbourhoods not considered before (as far as we know). Estimations are obtained using the random walk method, [5, Chapter 6], which is based on the analysis of the temporal evolution of "absorbed sets" (sets of adjacent sites all in state "zero") . If these sets are in average expanding, the process is ergodic. In particular our estimations take into account a certain aspect of the dynamics, i.e. absorbed sets can dynamically merge one with the other. This ends up in an improvement of the bound. The comparison with numerical estimations from our supplementary information paper [19] shows that our bounds are sharp.

In section 4 we consider the Stavskaya's processes on a finite 1D lattice with periodic boundaries and we study the time of convergence of the process into the absorbing-state. Our second main result is stated in Theorem 2.2. We show that at p_c there exists a transition from a fast to a slow convergence regime. In particular we prove that the expected convergence time of the model grows exponentially (resp. logarithmically) with the size of the system if $p > p_c(\mathcal{U})$ (resp. $p < p_c(\mathcal{U})$). This appears as an open problem in the surveys [4, 5] and in [6]. If compared with [5], where the fast (resp. slow) convergence behaviour is proved for p small enough (resp. close enough to 1), our result provides a sharp estimation. The slow convergence regime can be interpreted as a metastable behaviour of the model, as the process spends an extraordinary long time into a non-stable state before falling into the absorbing-state. Similar studies on the metastable behaviour of PCA models have been presented recently also in [8, 9, 10], although the methods used there do not apply to our case, as the Stavskaya's processes are not reversible and do not have a naturally associated potential. Numerical estimations of $p_c(\mathcal{U})$ (e.g. [20, 21, 22]) are obtained assuming that the metastable regime (the actual regime observed in numerical simulations, as there is no way to really simulate "infinite space" in computers) is observed only for all values of p at which the infinite process is in the fluctuating phase. If maybe obvious from the point of view of the physical intuition, the Theorem 2.2 provides a justification of this assumption from a rigorous mathematical point of view.

The proof of our result relies almost entirely on the correspondence between Stavskaya's process and Oriented Percolation in dimension 2, as described in [5, 7]. In particular the probability that the Stavskaya's process on finite space of size $2n$ has not fallen into the absorbing state until time t is equal to the probability that an open oriented percolation path connects the top of a $2n \times t$ box to the bottom. In particular the proof of the fast convergence regime for $p < p_c$ follows directly from percolation estimations from [23]. The proof of the slow convergence regime for $p > p_c$ is more technical and it is based on **(1)** the generalization of the dynamic-block argument provided by [23] to the case of non symmetric neighbourhood with more than two elements and **(2)** the estimation of the probability of a certain event that involves the dual lattice construction.

We end this introductory section presenting the structure of the paper. In section 2 we define the model and we present our main results, Theorem 2.1 and Theorem 2.2. In section 3 we prove Theorem 2.1, introducing the random walk method from [5][Chapter 6]. In section 4 we prove Theorem 2.2. The section is divided into three subsections. In Subsection 4.1 we describe the correspondence between Stavskaya's process and Oriented Percolation in two dimension, following [5, 7]. In Subsection 4.2 we present several percolation estimations from [23] used in the proof of the theorem. In Subsection 4.3 we prove Theorem 2.2.

2 Definition and Results

Probabilistic Cellular Automata (PCA) are discrete-time Markov chains on a product space, $\Sigma = X^S$ whose transition probability is a product measure.

In this paper we consider both the case of infinite space, $S = \mathbb{Z}$, and of finite space, $S = \mathbb{S}_n$, $\mathbb{S}_n = \{-n, -n+1, \dots n-2, n-1\}$.

We consider the case of boolean variables, $X = \{0, 1\}$. Realizations of the system are denoted by $\eta \in \Sigma$. For any $s \in S$ and any $K \subset S$, we denote by η^t_s the s-th component of the vector η^t and by η^t_K the set components corresponding to the sites of K.

We introduce now a *neighbourhood function* on S. We first define the neighbourhood of the origin,

$$\mathcal{U}(0) = \{s_1, s_2, \dots s_u\}, \tag{1}$$

, where s_1, $s_2 \dots s_u$ are some elements of S and u is finite. We will always assume $s_1 < s_2 < \dots s_u$. Then, assuming translation invariance, we define the neighbourhood of a site $s \in S$ as $\mathcal{U}(s) = \mathcal{U}(0) + s$ and the neighbourhood of a set $K \subset S$ as $\mathcal{U}(K) = \bigcup_{s \in K} \mathcal{U}(s)$.

In case of finite space we assume *periodic boundaries*, i.e. $\eta_s = \eta_{s+2kn}$ for all $k \in \mathbb{Z}$. This means that the neighbourhood of a site s can also be rewritten as,

$$\mathcal{U}(s) = \{|s + s_1 + n|_{2n} - n,\ |s + s_2 + n|_{2n} - n,\ \dots,\ |s + s_u + n|_{2n} - n\}. \tag{2}$$

In PCA the states of the process are synchronously updated at every site according to a certain *transition probability*. For the Stavskaya's process the transition probability is $\forall s \in S$,

$$T_s(\, \eta'_s = 1 \,|\, \eta_{\mathcal{U}(s)} \,) = \begin{cases} 0 & \text{if } \eta_{\mathcal{U}(s)} = \mathbf{0} \\ p & \text{otherwise} \end{cases}, \tag{3}$$

where $p \in [0, 1]$ is a free parameter. [1] Note that by definition the transition probability is translation invariant.

The temporal evolution of the PCA can be represented by introducing a linear operator $\mathcal{P}$ which acts on the space of probability measures $\mathcal{M}(\Sigma)$. For any $\mu \in \mathcal{M}(\Sigma)$, we denote by $\mu\mathcal{P}$ the measure obtained applying $\mathcal{P}$ to μ. Denoting by $\overline{\eta'}_K$ the cylinder set $\overline{\eta'}_K = \{\eta \in \Sigma \,:\, \eta_K = \eta'_K\,\}$, with $K \subset S$, the measure $\mu\mathcal{P}$ is defined as,

$$\mu\mathcal{P}(\, \overline{\eta'}_K \,) = \sum_{\eta_{\mathcal{U}(K)} \in \{0,1\}^{\mathcal{U}(K)}} \mu(\eta_{\mathcal{U}(K)}) \prod_{s \in K} T_s(\, \eta'_s \,|\, \eta_{\mathcal{U}(s)} \,). \tag{4}$$

Observe that if μ is a Dirac measure then $\mu\mathcal{P}$ is a product measure. This is the main feature of PCA.

In order to characterize the time evolution of PCA it is useful to introduce the set of *space-time realizations*, $\tilde{\Sigma} = \{0, 1\}^V$, where $V = S \times \mathbb{N}$ is the

[1] We use a different notation from [4, 5, 6]: here p corresponds to $1 - \epsilon$ and zeroes and ones are inverted.

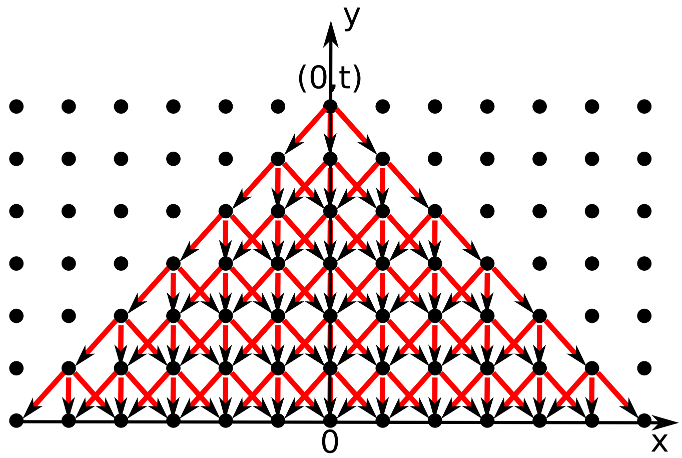


Figure 1: Representation of the graph $\mathcal{G}_{\mathcal{U}}$ with neighbourhood $\mathcal{U}(0) = \{-1, 0, 1\}$. In this figure only edges between vertices belonging to the evolution cone of $(0, t)$ have been drawn.

space-time set. The elements of $\tilde{\Sigma}$ are the realizations of the process at all times, $\tilde{\eta} = (\eta^t)_{t=0}^{\infty} \in \tilde{\Sigma}$. We introduce then a directed graph $\mathcal{G}_{\mathcal{U}} = (V, \vec{E}_{\mathcal{U}})$, whose edges connect any vertex $(s, t) \in V$ to the vertices $(k, t-1) \in V$, where $k \in \mathcal{U}(s)$, and are oriented. The vertices reachable from $(s, t) \in V$ through a path on $\mathcal{G}_{\mathcal{U}}$ constitute the *evolution cone* of $(0, t)$.

We now introduce some definitions that will be used along the whole article. First for any $t \in \mathbb{N}$ we define the set $S^t = \{(s, t) \in V,\ s \in S\}$, which is a copy of S.

Definition 1 (Evolution Measure)**.** *Consider the Stavskaya's process (4) with $S = \mathbb{Z}$ (respectively $S = \mathbb{S}_n$ and periodic boundaries). For every $\mu \in \mathcal{M}(\Omega)$, we define the evolution measure $\tilde{\mu}$ (respectively $\tilde{\mu}^{(n)}$) as the joint probability distribution of measures $\mu\mathcal{P}^t$ at all times* $0, 1, 2, \ldots$.

For example, we denote by $\tilde{\delta}_{\mathbf{1}}^{(n)}$ the evolution measure of the Stavskaya's process on finite space, starting with initial probability measure "all ones" almost surely.

Definition 2 (Expectation on the evolution space)**.** *Consider the Stavskaya's process (4) with $S = \mathbb{Z}$ (respectively $S = \mathbb{S}_n$ and periodic boundaries). We denote by $\mathbb{E}_{\tilde{\mu}}(\,\cdot\,)$ (respectively $\mathbb{E}_{\tilde{\mu}}^{(n)}(\,\cdot\,)$) the expectation with respect to the evolution measure $\tilde{\mu}$ (respectively $\tilde{\mu}^{(n)}$).*

Monotonicity It is immediate from the definition that the Dirac measure $\delta_{\mathbf{0}}$, where $\mathbf{0} = (0, 0, 0, \ldots)$, is stationary, i.e. $\delta_{\mathbf{0}} = \delta_{\mathbf{0}}\mathcal{P}$. Furthermore the operator $\mathcal{P}$ of this stochastic process is *monotone*. Monotonicity of $\mathcal{P}$ means that it preserves partial order among elements of $\mathcal{M}(\Sigma)$. We first introduce partial order " $\prec$ " in Σ defining for any two configurations $\eta, \eta' \in \Sigma$, $\eta \prec \eta' \Leftrightarrow \forall s \in S\ \eta_s \leq \eta'_s$. Then we introduce the functions $\varphi : \Sigma \longmapsto \mathbb{R}$ dependent only on a finite number of sites. We call φ *monotone iff* for any $\eta, \eta' \in \Sigma$, $\eta \prec \eta' \Rightarrow \varphi(\eta) \leq \varphi(\eta')$. Then we introduce partial order in $\mathcal{M}(\Sigma)$ defining $\mu \prec \mu' \Leftrightarrow$ for any *monotone* function φ , $\int \varphi\, d\mu \leq \int \varphi\, d\mu'$. An operator $\mathcal{P} : \mathcal{M}(\Sigma) \longmapsto \mathcal{M}(\Sigma)$ is called monotone if for any pair of measures $\mu, \mu' \in \mathcal{M}(\Sigma)$, $\mu \prec \mu' \Rightarrow \mu\mathcal{P} \prec \mu'\mathcal{P}$. For the transition operator of the Stavskaya's process this property is a consequence of the fact that the transition probability (3) preserves order locally, i.e. for any $s \in S$,

$$\eta^1_{\mathcal{U}(s)} \prec \eta^2_{\mathcal{U}(s)} \;\Rightarrow\; T_p(\eta_s = 1 \,|\, \eta^1_{\mathcal{U}(s)}) \leq T_p(\eta_s = 1 \,|\, \eta^2_{\mathcal{U}(s)}),$$

(see for example [5, page 28] for a proof). Monotonicity of $\mathcal{P}$ implies that the probability measure,

$$\nu_p := \lim_{t \to \infty} \delta_{\mathbf{1}}\, \mathcal{P}^t, \tag{5}$$

is well defined.

Definition 3 (Critical Probability)**.** *Consider the Stavskaya's process on $\mathbb{Z}$ with finite neighbourhood $\mathcal{U}(0) \subset \mathbb{Z}$. We define the critical probability as,*

$$p_c(\mathcal{U}) = \sup_p \{\nu_p = \delta_{\mathbf{0}}\}. \tag{6}$$

By definition, for any $p < p_c$ the process is ergodic, i.e. there exists a unique invariant measure on which the process converges. By monotonicity this holds for any initial probability measure. Alternatively, for any $p > p_c$, the process is not ergodic. In fact in this case $\delta_{\mathbf{0}}$ and $\nu_p \neq \delta_{\mathbf{0}}$ are two different invariant measures and any convex combination of them is still an invariant measure. Using the counting path method and the Peierls argument it is possible to show that for any neighbourhood $\mathcal{U}$ s.t. $|\mathcal{U}(0)| \geq 2$, $p_c(\mathcal{U}) \in (0, 1)$ [5, 7].

Our first result involves the estimation of p_c and it is stated in the following theorem.

Theorem 2.1. *Consider the Stavskaya's process on $\mathbb{Z}$ with neighbourhood $\mathcal{U}(0) = \{s_1, s_2, s_3, \ldots, s_u\}$, where $s_1, s_2 \ldots s_u$ are some arbitrary integers. Then, $p_c(\mathcal{U}) \geq p^2$, where,*

$$p^2 = p^1 \cdot [\ 1 \ +\ (1 - p^2)^{(s_u - s_1)}\, \frac{3\ -\ 2\,p^2}{2\ -\ p^2}\ ], \tag{7}$$

and,

$$p^1 = \frac{2}{2 + s_u - s_u}. \tag{8}$$

The proof of the theorem is presented in Section 3. From equation (7) it follows that $p^2 > p^1$. Our analytical lower bound can be compared with numerical estimation from our supplementary information article [19] in the following tables.

$\mathcal{U}(0)$	p^1	p^2	Num. Est.
$\{-1, 0\}$	2/3	0.672	0.705
$\{-1, 0, 1\}$	1/2	0.505	0.538
$\{-1, 0, 1, 2\}$	2/5	0.404	0.430
$\{-1, 0, 1, 2, 3\}$	1/3	0.338	0.339

As we can see from the following table, where neighbourhoods with 3 elements and different radius $s_u - s_1$ are considered, bounds are not so sharp in case the neighbourhood does not contain some site between the two extremal ones, s_1 and s_u.

$\mathcal{U}(0)$	p^1	p^2	Num. Est.
$\{-1, 0, 1\}$	1/2	0.505	0.538
$\{-1, 0, 2\}$	2/5	0.404	0.489
$\{-1, 0, 3\}$	1/3	0.338	0.469

Our main result involves the time of convergence into the absorbing state of the Stavskaya's process with finite space $\mathbb{S}_n$ and periodic boundaries, as defined at the beginning of this section. The result is stated in Theorem 2.2.

Naturally when S is finite there is no phase transition. This simply follows from the fact that for any configuration $\eta^t \in \Sigma$ at time t, the probability that $\eta^{t+1} = \delta_{\mathbf{0}}$ is bounded from below by the constant $(1-p)^{|S|}$. This implies that there exists almost surely a finite time $\tau \in \mathbb{N}$ such that $\eta^t =$ "all zeroes" for all $t \geq \tau$.

In order to estimate the time of convergence into the absorbing state we define the *absorbing-time*.

Definition 4. *Denote* $[[a, b]] = [a, b] \bigcap \mathbb{Z}$. *For all* $k \in \mathbb{N}$, *we call the absorbing time of the interval* $[[-k, k-1]]$ *the random variable* $\tau_k : \tilde{\Sigma} \to \mathbb{N}$,

$$\tau_k(\tilde{\eta}) = \min\{t \in \mathbb{N}_0 \ s.t. \ \tilde{\eta}^t_s = 0 \ \ \forall s \in [[-k, k-1]]\}. \tag{9}$$

It represents the first time the line $[[-k, k-1]]$ contains all zeroes in the course of the evolution of the process. In case $S = \mathbb{S}_n$, this random variable is well defined only if $k \leq n$.

Theorem 2.2. *Consider the Stavskaya's process with space* $\mathbb{S}_n$, *periodic boundaries and neighbourhood* $\mathcal{U}(0) = \{s_1, s_2, \ldots s_u\}$, *where* $s_1, s_2, \ldots s_u$ *are some integers in* $\mathbb{S}_n$. *There exist* $n_0 \in \mathbb{N}$ *and some positive constants* K_1, K_2, c_1, c_2 *(dependent on p) such that for all* $n > n_0$, *a)* $\forall p < p_c$, $\mathbb{E}^{(n)}_{\tilde{\delta}_{\mathbf{1}}}[\tau_n] \le K_1 \log(c_1\, n)$,

b) $\forall p > p_c$, $\mathbb{E}^{(n)}_{\tilde{\delta}_{\mathbf{1}}}[\tau_n] \ge K_2 \exp(c_2\, n)$.

The proof of the theorem is presented in Section 4..

3 Critical Probabilities

In this section we prove Theorem 2.1, which provides a lower bound for p_c as a function of the neighbourhood of the model. Theorem 2.1 follows from the random walk method, summarized in Theorem 3.1 and developed in [5][Chapter 6] and from our combinatorial and probabilistic arguments. Also the following statement is needed for the proof of the Theorem 2.1.

Proposition 1. *Consider two Stavskaya's processes in $\mathbb{Z}$ with neighbourhoods respectively $\mathcal{U}(0)$ and $\mathcal{U}'(0)$, both finite subsets of $\mathbb{Z}$, such that $\mathcal{U}(0) \subset \mathcal{U}'(0)$. Then $p_c(\mathcal{U}(0)) \ge p_c(\mathcal{U}'(0))$.*

The proof of this proposition is a simple consequence of Proposition 2, presented and proved in Section 4.1, and of the fact that the evolution cone (defined at the beginning of Section 2) for the model with neighbourhood $\mathcal{U}(0)$ is a subset of the evolution cone for the model with neighbourhood $\mathcal{U}'(0)$.

Now we start with the introduction of definitions and notations that allow to present Theorem 3.1. We define the Dirac measure δ_{ρ^L}, where $L \in \mathbb{N}$ and $\rho^L \in \Sigma$ is a configuration such that $\rho^L_s = 0$ if $-L \le s \le L$ and $\rho^L_s = 1$ otherwise. We say that this configuration contains a *massif of zeroes* of size $2L+1$ (see Figure 2). We observe that if the initial configuration is δ_{ρ^L}, then any space-time realization at time 1 has a massif of zeroes of size at least $2L+1-(s_u-s_1)$. This follows from (3). Analogously at time t any space-time realization has a massif of zeroes of size at least $2L+1-t(s_u-s_1)$ if $t \le (2L+1)/(s_u-s_1)$. The actual size of the massif at time t depends on the specific space-time realization. Thus we introduce the random variables ξ^t_+ and ξ^t_-,

$$\xi^t_+, \xi^t_- \, : \tilde{\Sigma} \to \mathbb{Z},$$

for any $t \in \mathbb{Z}_+$. ξ^t_+ (resp. ξ^t_-) returns the site of the rightmost (resp. leftmost) 0 belonging to the massif at time $t \in \mathbb{N}_0$. See also Figure 2 for an example. In case the massif disappears at time t^*, their value is defined to be 0 for all $t \ge t^*$. The following theorem provides a method to estimate analytically p_c from below.

Theorem 3.1. *For all $p \in [0,1]$ such that there exists a positive integer τ such that,*

$$\mathbb{E}_{\tilde{\delta}_{\rho^L}}[\xi^\tau_+ - \xi^\tau_-] > 2L, \tag{10}$$

for some $L \ge (\tau(s_u - s_1) - 1)/2$, the operator $\mathcal{P}$ is ergodic.

The proof of the theorem can be found in [5][Chapter 6]. The proof is based on the following heuristic arguments: if (10) holds, then by the law of large numbers there is a non-zero probability that the massif of zeroes of the configuration ρ^L grows with time with linear speed. Secondly, starting from a configuration $\delta_{\mathbf{1}}$ at time 0, an infinite number of massifs of zeroes of size at least L is present already at time 1 a.s. The two previous facts imply that with probability 1 at least one of the massifs of zeroes present at time 1 will be growing linearly with time. This means that the spatial realization at infinite time is "all zeroes" a.s. Thus values of p satisfying (10) for some integer τ are sub-critical. A lower bound for p_c follows.

Proof of Theorem 2.1 Referring to the statement of Theorem 3.1, we use combinatorial and probabilistic arguments to provide a $\tau = 2$ estimation that holds for all neighbourhoods.

We consider neighbourhoods of type $\mathcal{U}(0) = \{s_1, s_1+1, \ldots, s_u-1, s_u\}$, (i.e. containing all sites between s_1 and s_u) for some arbitrary finite integer u and we provide for those a lower bound that depends on the difference $s_u - s_1$. From Proposition 1, this lower bound holds also for all neighbourhoods contained into $\mathcal{U}(0)$.

We start with the proof of the $\tau = 1$ estimation. We denote with $[[a,b]]$ the set of integers in $[a,b]$. We need to compute $\mathbb{E}_{\tilde{\delta}_{\rho^L}}[\xi_+^1]$, $\mathbb{E}_{\tilde{\delta}_{\rho^L}}[\xi_-^1]$,

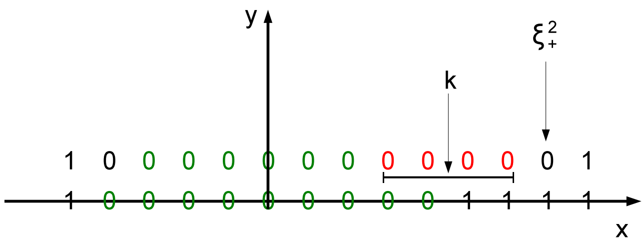
$$
\begin{aligned}
\mathbb{E}_{\tilde{\delta}_{\rho^L}}[\xi_+^1] &= \sum_{k=1}^{\infty} \tilde{\delta}_{\rho^L}(\xi_+^1 - L + s_u \geq k) - s_u + L, \\
\mathbb{E}_{\tilde{\delta}_{\rho^L}}[\xi_-^1] &= -\sum_{k=1}^{\infty} \tilde{\delta}_{\rho^L}(\xi_-^1 + L + s_1 \leq -k) - s_1 - L.
\end{aligned} \tag{11}
$$

For any $k \in \mathbb{N}_0$, $\tilde{\delta}_{\rho^L}(\{\xi_+^1 - L + s_u \geq k\})$ equals the probability that the set $[[L - s_u + 1, L - s_u + k]]$ contains all zeroes, i.e. $(1-p)^k$ (see Figure 2a). Analogously for any $k \in \mathbb{N}_0$, $\tilde{\delta}_{\rho^L}(\{\xi_-^1 + L + s_1 \leq -k\}) = (1-p)^k$. Computing the expectation, condition (10) is satisfied for all $p \geq p^1$, where p^1 is defined in (8).

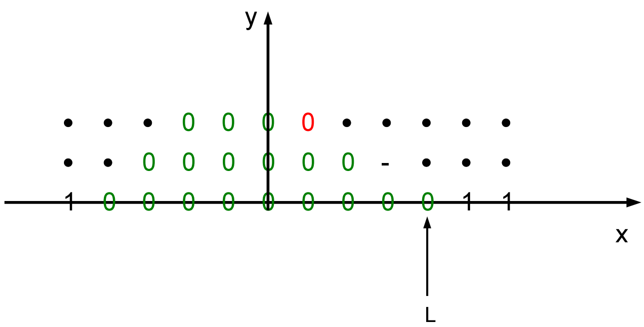
Now we try to solve (10) with $\tau = 2$. We need to estimate,

$$
\begin{aligned}
&\tilde{\delta}_{\rho^L}(\{\xi_+^2 - L + 2s_u \geq j\}), \\
&\tilde{\delta}_{\rho^L}(\{\xi_-^2 + L + 2s_1 \leq -j\}).
\end{aligned} \tag{12}
$$

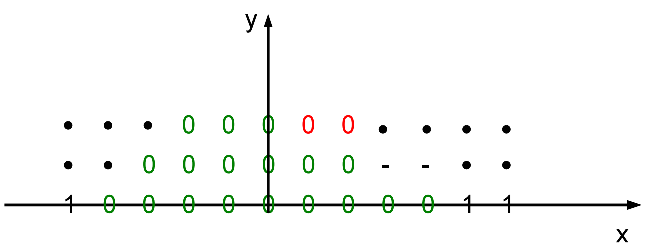
We will show how to estimate the former and then we use the same approximation for the latter. Using (4), (12) equals the sum over all realizations



(a) For any $k \in \mathbb{N}_0$, $\xi_+^1 - L + s_u \geq k$ if the k sites over the horizontal line are in state 0. Because of (3), one should see that this occurs with probability $(1-p)^k$.



(b) The outcome of the site $(L - 2s_u + 1, 2)$ (red zero in the figure) is zero with probability 1 if the site signed with "$-$" has outcome zero, otherwise it is one with probability p and zero with probability $1-p$.



(c) The distribution of the outcomes of the sites $(L-2s_u+1, 2)$ and $(L - 2s_u + 2, 2)$ (red zero in the figure) depends on the outcomes of the sites signed with "$-$", as specified in the text.

Figure 2: In all figures we assume $\mathcal{U}(0) = \{-1, 0, 1, 2\}$ and initial configuration ρ^L, $L = 5$. Observe that all space-time realizations must have zeroes in the regions coloured by green on the lines $y = 1, 2$.

of S^1 of the products of the transition probabilities (3). Namely,

$$\tilde{\delta}_{\rho^L}(\{\xi_+^2 - L + 2s_u \geq j\}) = \sum_{\eta^1 \in \{0,1\}^{S^1}} \tilde{\delta}_{\rho^L} \mathcal{P}_p(\{\eta^1\}) \prod_{s \in [[L-2s_u, L-2s_u+j]]} T(\eta_s^2 = 0 | \eta^1_{\mathcal{U}(s)}). \quad (13)$$

As the outcomes of the variables η_s^2, $s \in [[L - 2s_u + 1, L - 2s_u + j]]$ depend only on the outcomes of the variables η_s^1, $s \in [[L - 2s_u + s_1 + 1, L + s_1 + j]]$, and as $\eta_s^1 = 1$ for any $s \in [[-L - s_1, L - s_u]]$, the previous sum reduces to a sum over the realizations of the set $[[L - 2s_u + 1, j]]$. See Figure 3.

Computing the exact value of (13) for any j is a difficult combinatorial problem, as for each of the 2^j possible configurations one should determine the corresponding product of transition probabilities. The cases $j = 1, 2$ are easily solvable exactly. Looking at Figure 2b and using (3),

$$\begin{aligned} &\tilde{\delta}_{\rho^L}(\{\xi_+^2 - L + 2s_u \geq 1\}) = \tilde{\delta}_{\rho^L}(\{\eta^2_{L-2s_u+1} = 0\}) = \\ &\tilde{\delta}_{\rho^L}(\{\eta^1_{L-s_u+1} = 0\})\, 1 + \tilde{\delta}_{\rho^L}(\{\eta^1_{L-s_u+1} = 1\})\,(1-p) = 1 - p^2. \end{aligned} \quad (14)$$

Looking at Figure 2c and using (3),

$$\begin{aligned} &\tilde{\delta}_{\rho^L}(\{\xi_+^2 - L + 2s_u \geq 2\}) = \tilde{\delta}_{\rho^L}(\{\eta^2_{L-2s_u+1} = \eta^2_{L-2s_u+2} = 0\}) = \\ &\tilde{\delta}_{\rho^L}(\{\eta^1_{L-s_u+1} = 0,\, \eta^1_{L-s_u+2} = 0\})\ 1\ + \\ &\tilde{\delta}_{\rho^L}(\{\eta^1_{L-s_u+1} = 0,\, \eta^1_{L-s_u+2} = 1\})\ (1-p)\ + \\ &\tilde{\delta}_{\rho^L}(\{\eta^1_{L-s_u+1} = 1,\, \eta^1_{L-s_u+2} = 0\})\ (1-p)^2\ + \\ &\tilde{\delta}_{\rho^L}(\{\eta^1_{L-s_u+1} = 1,\, \eta^1_{L-s_u+2} = 1\})\ (1-p)^2\ = \\ &(1-p)^2\ +\ p(1-p)^2\ +\ p(1-p)^3\ +\ p^2(1-p)^2. \end{aligned} \quad (15)$$

In case of $j > 2$ we provide the lower bound (20). In order to consider the contribution from several different realizations of the sum (13), we first fix a value of the index $k \in [[0, j-1]]$, later we sum over k. We start considering the configurations $\eta^{a,k} \in \{0,1\}^{S^1}$, which are those such that $\eta_s^{a,k} = 0$ for all $s \in [[L - s_u + 1, L - s_u + k]]$, $\eta^{a,k}_{L-s_u+k+1} = 1$, $\eta_s^{a,k} = 0$ for all $s \in [[L - s_u + k + 2, L - s_u + j]]$. See also Figure 3a. All these configurations give a contribution,

$$p(1-p)^{j-1}(1-p)^{s_u - s_1}, \quad (16)$$

if $0 \leq k \leq j - 1 - r$ or a contribution,

$$p(1-p)^{j-1}(1-p)^{j-k}, \quad (17)$$

if $j - r \leq k \leq j - 1$. In both expressions, the first two factors (i.e. $p(1-p)^j$) represent the probability to observe one of such configurations at time 1 and the last factor equals the transition probability to a state $\{\xi_+^2 + L - 2s_u \geq j\}$.

Then we consider configurations of type $\eta^{b,k} \in \{0,1\}^{S^1}$, which are those such that $\eta^{b,k}_s = 0$ for all $s \in [[L - s_u + 1, L - s_u + k]]$, $\eta^{b,k}_{L-s_u+k+1} = 1$ and such that set $[[L - s_u + k + 2, L - s_u + j]]$ does not contain all zeroes (on the contrary of $\eta^{a,k}$). Those are represented in Figure 3b. They give a contribution

$$(1-p)^k p(1-(1-p)^{j-k-1})(1-p)^{j-k}, \tag{18}$$

to the sum (13). The first three factors equal the probability that one of these configurations appears at time 1, the last factor (i. e. $(1-p)^{j-k}$) is obtained considering $T(\eta^2_s = 0 \,|\, \eta_{\mathcal{U}(s)}) = 1$ for all $s \in [[L - 2s_u + 1, L - 2s_u + k]]$ and minimizing $T(\eta^2_s = 0 \,|\, \eta_{\mathcal{U}(s)}) \geq (1-p)$ for all $s \in [[L-2s_u+1+k, L-2s_u+j]]$. We note also that for all $k \in [[0, j-u-1]]$, the set of configurations $\eta^{b,k}$, $\eta^{a,k}$ are disjoint. This means that, in order to estimate (13), we can sum over k without counting more times the same configurations.

Third, we have a contribution $(1-p)^j$ from the configuration η^c, represented in Figure 3c, not considered yet, as k doesn't take the value $k = j$. This is the realization $\eta^c_s = 0$ for all $s \in [[L - s_u + 1, L - s_u + j]]$. In this case all transition probabilities $T(\eta^2_s = 0 \,|\, \eta_{\mathcal{U}(s)})$ take value 1.

At last we sum over all the remaining configurations, which give a contribution,

$$[1 - \sum_{k=0}^{j-1} p(1-p)^k - (1-p)^j](1-p)^j, \tag{19}$$

for all k. The factor inside the square bracket represents the probability that one of these configurations appears at time 1. This equals one minus the probability that one of the configurations η^a, η^b or η^c occurs. The factor $(1-p)^j$ that multiplies the square bracket is obtained using the bound, $T(\eta^2_s = 0 \,|\, \eta_{\mathcal{U}(s)}) \geq (1-p)$ for all $s \in [[L - 2s_u + 1, L - 2s_u + j]]$.

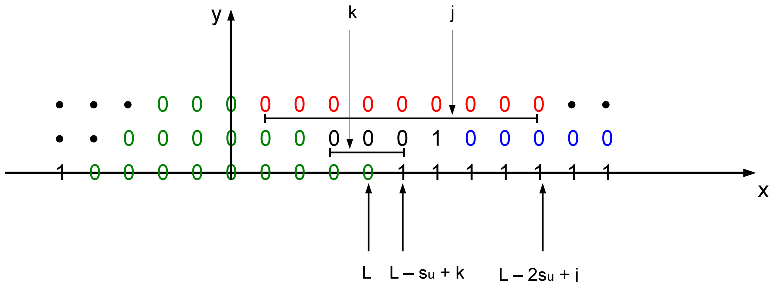
All previous terms together give the lower bound,

$$\begin{aligned}
&\tilde{\delta}_{\rho^L}(\{\xi^2_+ - L + 2s_u \geq j\}) \geq \\
&p(1-p)^j + \\
&(1-p)^j \mathbb{1}_{j \geq s_u - s_1}[(j-u)p(1-p)^{j-1+s_u-s_1} - (1-p)^{2j}(1-(1-p)^{s_u-s_1-j})].
\end{aligned} \tag{20}$$

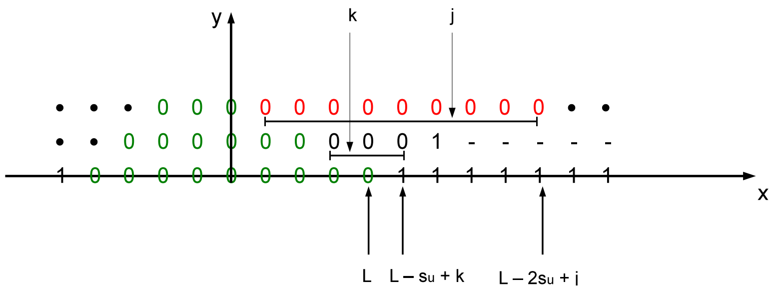
Thus with the previous approximation a lower bound for the expectation follows,

$$\mathbb{E}_{\tilde{\delta}_{\rho^L}}[\xi^2_+] - L \geq \frac{2(1-p)}{p} - 2s_u + \frac{(1-p)^{2(s_1-s_1)}}{p}\frac{3-2p}{p}, \tag{21}$$

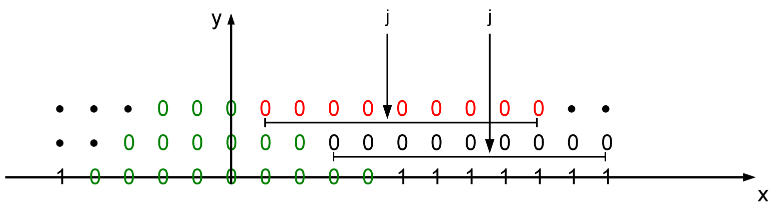
where the last term is a correction of the estimation of order $\tau = 1$. Without the contribution from the "blue" massif of zeroes of Figure 3a, that term would not be present and there wouldn't be an improvement of the estimation of order $\tau = 1$. In this sense our $\tau = 2$ estimation takes into account



(a) Representation of a realization $\eta^{a,k}$, defined in the text, at time 1. The conditions $\xi_+^1 - L + 2s_u \geq j$ is satisfied.



(b) Representation of a realizations $\eta^{b,k}$, defined in the text, at time 1. In order the realization to be of type $\eta^{b,k}$, the set of sites signed with the symbols "$-$" can be everything except "all zeroes". The condition $\xi_+^1 - L + 2s_u \geq j$.



(c) Representation of the realization $\eta_s = 0$ for all $s \in [[L - 2s_u + 1, L - 2s_u + j]]$, as considered in the text.

Figure 3: In all figures the neighbourhood is assumed to be $\mathcal{U}(0) = \{-1, 0, 1, 2\}$. This means that sites coloured by green on the lines $y = 1$ and $y = 2$ have outcome 0 with probability one.

the fact that massifs of zeroes can dynamically merge one with the other one. Using the same approximation for ξ^2_-, (7) follows.

4 Time of convergence of the finite process

In this section we prove Theorem 2.2. In Section 4.1 we describe the connection between Stavskaya's process and Oriented Percolation, following [5, 7]. In Section 4.2 we list some percolation estimates from [23]. In Section 4.3 we finally prove the Theorem.

4.1 Connections between Stavskaya's process and oriented percolation

Consider Stavskaya's process in $\mathbb{Z}$ and define $\Theta(p) = \nu_p(\eta_0 = 1)$, omitting the dependence on the neighbourhood. By definition of p_c, for any $p > p_c$ the probability $\Theta(p)$ is positive and for any $p < p_c$ it is zero. As pointed out in [7], we can interpret $\Theta(p)$ as a percolation probability for the infinite graph $\mathcal{G}_\mathcal{U} = (V, \vec{E}_\mathcal{U})$, where percolation is Bernoulli, site and oriented. We follow [5, 7].

We observe that the evolution measure $\tilde{\mu}$ on the evolution space $\tilde{\Sigma}$ is representable as induced by the Bernoulli product measure $\mathbb{P}_p$ on the auxiliary space $\Omega = \{0,1\}^{\mathbb{Z}\times\mathbb{N}}$ by the mappings $\eta^t_s : \Omega \mapsto \{0,1\}$,

$$\eta^t_s = \min\{\, \omega_{s,t},\ \max_{k\in\mathcal{U}(s)} \{\eta^{t-1}_k\,\}\},\ s \in \mathbb{Z},\, t \in \mathbb{Z}_+, \tag{22}$$

where $(\omega_{s,t})_{s\in S, t\in\mathbb{N}}$ are elements of Ω. This mapping defines any η^T_K, $K \subset V$, $T \in \mathbb{Z}_+$ as a function of a finite set of variables $\omega_{s,t}$, η^0_s. The following proposition points out a connection between the values of the variables η^t_s and certain percolation events on the graph $\mathcal{G}_\mathcal{U}$.

Proposition 2. *The function $\eta^t_s : \Omega \mapsto \{0,1\}$ is such that $\eta^t_s = 1$ iff there exists a sequence $\{s_0, s_1, s_2, \dots s_t\} \subset \mathbb{Z}$ satisfying the three following properties,*

1. *$s_t = s$ and $s_{i-1} \in \mathcal{U}(s_i)$ for any $i \in \{1, 2, \dots t\}$,*
2. *$\omega_{i,s_i} = 1$ for any $i \in \{1, 2, \dots t\}$,*
3. *$\eta^0_{s_0} = 1$.*

Proof. The proof of Proposition 2 is by induction. Assume $\eta^t_s = 1$ and assume that properties 1, 2, 3 hold for a sequence of sites $s_{t-k}, s_{t-k+1}, \dots s_t$. From (22) it follows that $\eta^{t-k}_{s_{t-k}} = 1 \Leftrightarrow \omega_{s_{t-k},t-k} = 1$ and $\exists\, s_{t-k-1} \in \mathcal{U}(s_{t-k})$ s.t. $\eta^{t-k-1}_{s_{t-k-1}} = 1$. This implies that there exists an element $s_{t-k-1} \in S$ such that properties 1, 2, 3 hold for the sequence s_{t-k-1}, s_{t-k}, $\dots$ s_t. By induction the proposition is proved. □

We end this section introducing some more definitions that will be used in the following sections.

Definition 5 (Open sites and paths)**.** *If $\omega_{s,t} = 1$ we call the site $(s,t) \in V$ open. Analogously if all the variables $\omega_{s,t}$ along a path of the graph $\mathcal{G}_{\mathcal{U}}$ are equal to one, we call the path "open". We denote the event {"the site (s,t) is connected to S^0 by an open path in $\mathcal{G}_{\mathcal{U}}$"} by,*

$$\{(s,t) \xrightarrow{\mathcal{G}_{\mathcal{U}}} S^0\},$$

reminding that S^0 is the set of vertices on the line $y = 0$.

See Figure 1 for an example. Note that, choosing as initial probability measure $\delta_{\mathbf{1}}$ and using Proposition 2, it follows that $\delta_{\mathbf{1}}\mathcal{P}^t(\eta_s = 1) = \mathbb{P}_p(\{(s,t) \xrightarrow{\mathcal{G}_{\mathcal{U}}} S^0\},)$ for any $s \in S$. Calling $\mathcal{C}$ the limit event $\lim_{t\to\infty}\{(s,t) \xrightarrow{\mathcal{G}_{\mathcal{U}}} S^0\}$, it follows that $\Theta(p) = \mathbb{P}_p(\mathcal{C})$. Thus the stochastic process is ergodic *iff* the probability that $(0,\infty)$ belongs to an infinite open path in $\mathcal{G}_{\mathcal{U}}$ is zero (subcritical percolation), otherwise it is not ergodic (supercritical percolation).

Let $s \in [[-n, n+1]]$, $t \in \mathbb{N}$ and define the following event.

Definition 6. *We say that the event,*

$$\{(s,t) \xrightarrow{\mathcal{G}_{\mathcal{U}(n)}} S^0\},$$

occurs if the two following properties hold,

1. *an open path in $\mathcal{G}_{\mathcal{U}}$ connects the vertex (s,t) to the bottom of the box $2n \times t$,*
2. *the path never crosses the boundary of such box (e.g. see the path b in Figure 4) or it reaches one side of the box and re-appears at the same heigh on the other side (e.g. see the path $a \circ c$ in Figure 4).*

Remind that τ_k can be intended also as a function $\tau_k : \Omega \to \mathbb{N}$, as, from (22), $(\eta_s^t)_{s\in S, t\in\mathbb{N}}$ is a mapping from Ω to $\tilde{\Sigma}$. The following proposition follows directly from Proposition 2 and it connects the occurrence of $\{\tau_n > t\}$ for the Stavskaya's process in finite space and periodic boundaries starting from $\delta_{\mathbf{1}}$ with the occurrence of the percolation event defined above.

Proposition 3. *Consider Stavskaya's process with space $S = \mathbb{S}_n$, periodic boundaries and neighbourhood $\mathcal{U}(0) = \{s_1, s_2, \dots, s_u\}$, where $s_1, s_2, \dots, s_u$ are some integers in $\mathbb{S}_n$. Then,*

$$\tilde{\delta}_{\mathbf{1}}^{(n)}(\tau_n > t) = \mathbb{P}_p(\ \exists s \in [[-n, n-1]] \ s.t.\ (s,t) \xrightarrow{\mathcal{G}_{\mathcal{U}(n)}} S^0\ ). \tag{23}$$

Observe that in the previous statement the left-side probability measure is defined in the probability space of $\tilde{\Sigma} = \{0,1\}^{\mathbb{S}_n}$, while the right-side probability measure is defined in the probability space of $\Omega = \{0,1\}^{\mathbb{Z}}$, although the event depends only on the variables contained in the box $2n \times t$.

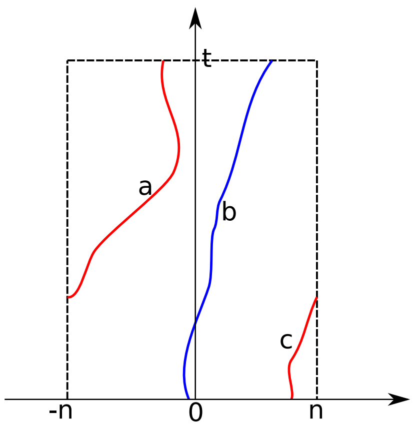


Figure 4: If $S = \mathbb{S}_n$ and boundaries are periodic, the event $\{\tau_n > t\}$ occurs if at least one open path joins the top to the bottom of the box without ever going out from the box or going out from one of its vertical sides and re-appearing at the same heigh on the other side (e.g. see the path $a \circ c$).

4.2 Percolation estimates

We list now some properties involving the cluster of vertices connected to $(0,t)$ by an open path in $\mathcal{G}_{\mathcal{U}}$. These percolation estimates come from [23], where the model with neighbourhood $\mathcal{U}(0) = \{-1, 1\}$ has been considered. We first present the estimations and then we show that they hold also for the more general case considered in the present paper.

We begin with some definitions. For every $t, m \in \mathbb{N}$ we define the sets,

$$\begin{aligned} \xi^{m,t} &= \{x \in S^{t-m} : (0,t) \xrightarrow{\mathcal{G}_{\mathcal{U}}} (x, t-m)\}, \\ \overline{\xi}^{m,t} &= \{x \in S^{t-m} : \exists\, z \le 0 \text{ s.t. } (z,t) \xrightarrow{\mathcal{G}_{\mathcal{U}}} (x, t-m)\}, \\ \overline{\chi}^{m,t} &= \{x \in S^{t-m} : \exists\, z \ge 0 \text{ s.t. } (z,t) \xrightarrow{\mathcal{G}_{\mathcal{U}}} (x, t-m)\}. \end{aligned} \tag{24}$$

Note that $\xi^{m,t} \subset \{s_1\, m,\ s_1\, m+1,\ s_1\, m+2, \ldots,\ s_u\, m\}$. We define then the variables,

$$\begin{aligned} r^{m,t} &= \sup\{\xi^t_m\}, \\ \ell^{m,t} &= \inf\{\xi^t_m\}, \\ \overline{r}^{m,t} &= \sup\{\overline{\xi}^t_m\}, \\ \overline{\ell}^{m,t} &= \inf\{\overline{\chi}^t_m\}, \end{aligned} \tag{25}$$

and we set $r^{m,t} = -\infty$, $\ell^{m,t} = \infty$ if $\xi^{m,t} = \varnothing$. We observe that for every t, m, the probability that $\overline{\xi}^{m,t} = \varnothing$ is zero, as there are infinite vertices on the line $y = t$, $x \ge 0$ and each of them has a non-zero probability to be connected to S^0 in $\mathcal{G}_{\mathcal{U}}$. The same holds for the event $\overline{\chi}^{m,t} = \varnothing$. By definition,

$$\begin{aligned} r^m &\leq \overline{r}^m, \\ \ell^m &\geq \overline{\ell}^m. \end{aligned} \tag{26}$$

As the distributions of $r^{m,t}$, $\ell^{m,t}$, $\overline{r}^{m,t}$, $\overline{\ell}^{m,t}$, $\xi^{m,t}$, $\overline{\xi}^{m,t}$ and $\overline{\chi}^{m,t}$ depend only on the difference $t-m$, in the following we omit the dependence on t, that will be some positive integer. Furthermore we consider the space $\mathcal{G}_{\mathcal{U}}$ as before, but with vertices $V = \mathbb{Z}^2$ instead of $\mathbb{Z} \times \mathbb{N}$, allowing in this way $(0,t)$ to be connected to an infinite open cluster. In this way we recover the notation of [23], with the difference that in this article paths are oriented from up to down. Under the same assumption and omitting the dependence on t in the notation we introduce the following quantity,

$$\begin{aligned} \overline{r}^{m,n} = \sup\{x - r^m \,:\, x \in S^{t-n} \text{ and } \exists z \in S^{t-m} \text{ s.t.} \\ z \leq r^m \text{ and } (z, t-m) \xrightarrow{\mathcal{G}_{\mathcal{U}}} (x, t-n)\}, \end{aligned} \tag{27}$$

where $n \geq m$.

It is easy to see that,

$$\overline{r}^m + \overline{r}^{m,n} \geq \overline{r}^n. \tag{28}$$

To prove (28) one should observe that $\overline{r}^m + \overline{r}^{m,n}$ is the rightmost point on the line $y = t-n$ which can be reached from any of the points $(x, t-m)$ with $x \leq \overline{r}^m$, while $\overline{r}^n$ is the rightmost point on the line $y = t-n$ which can be reached from any of the points $(x, t-m)$ with $x \leq \overline{r}^m$ and with the restriction that there exists an open path in $\mathcal{G}_{\mathcal{U}}$ from (z,t) to $(x, t-m)$ for some $z \leq 0$. As explained in [23][pag. 1004], where the case of symmetric neighbourhood with two elements and edges percolation has been considered, (28) implies that there exists a constant $\alpha \in [-\infty, 1]$ such that,

$$\overline{r}^m / m \to \alpha \text{ a.s.} \tag{29}$$

From this result the following proposition follows (see [23][pag. 1004]).

Proposition 4. *For all $p > p_c$ there exist constants α, β (dependent on p) such that, conditioning on the existence of an infinite open path which includes the vertex $(0,t)$,*

$$\lim_{m\to\infty} r^m/m = \alpha \quad \textit{almost surely}, \tag{30}$$

$$\lim_{m\to\infty} \ell^m/m = \beta \quad \textit{almost surely}, \tag{31}$$

Referring to the symmetric case, as considered in [23], one concludes that $\beta = -\alpha$. Furthermore, conditioning on the existence of an infinite open path which includes $(0,t)$, one has that $r^m \geq \ell^m$ for all m. This means that if $\alpha < 0$, then necessarily $p < p_c$. In the other direction, as proved in [23, pages 1005 and 1006] the following proposition holds.

Proposition 5. *For all $p > p_c$, $\alpha > 0$.*

The following property has also been proved in [23, page 1017] and later generalized in [27] to both oriented and non-oriented models, edges and sites percolation and dimensions higher than 2.

Proposition 6. *There exists a constant $h > 0$ (dependent on p) such that for any $p < p_c$,*

$$\mathbb{P}_p((0,t) \xrightarrow{\mathcal{G}_{\mathcal{U}}} S^0) \leq \exp(-ht). \tag{32}$$

The three main differences between the case considered in this article and the model studied in [23] are that **(1)** our model corresponds to site percolation, while the model considered in [23] corresponds to edges percolation, **(2)** we consider also the case of neighbourhood with more than 2 elements, **(3)** in our case the neighbourhood is not necessarily symmetric. These differences do not play any role in the proof of (28). Thus Proposition 4 holds also for the case considered in the present article, with the difference that without symmetry not necessarily $\beta = -\alpha$. For what concerns Proposition 5, observe that if $(0,t)$ belongs to an infinite open cluster, then $r^m \geq \ell^m$. This means that in the $\gamma = \alpha - \beta < 0$ implies that $p < p_c$. Following the same steps of the proof of proposition 5 and replacing α with $\gamma/2$, one should see that in the case of neighbourhood considered in the present article, for all $p > p_c$, $\gamma > 0$. We end this section recalling a property proved in [28].

Proposition 7. *Consider Stavskaya's process with any neighbourhood.*

$$\mathbb{P}_p((0,t) \xrightarrow{\mathcal{G}_{\mathcal{U}(n)}} S^0) \leq \mathbb{P}_p((0,t) \xrightarrow{\mathcal{G}_{\mathcal{U}}} S^0). \tag{33}$$

4.3 Proof of Theorem 2.2

Recall the definitions given just before the statement of the theorem. The main goal is to estimate from below and from above the probability $\tilde{\delta}_{\mathbf{1}}^{(n)}\{\tau_n > t\}$, which gives the expectation,

$$\mathbb{E}_{\tilde{\delta}_1}^{(n)}[\tau_n] = \sum_{t=0}^{\infty} \tilde{\delta}_{\mathbf{1}}^{(n)}(\tau_n > t), \tag{34}$$

Proof of part (a). The proof of the statement (a) follows directly from the propositions considered above. Considering (34) it follows that,

$$\begin{aligned}\mathbb{E}^{(n)}_{\tilde{\delta}_1}[\,\tau_n\,] &= \sum_{t=1}^{\infty}\mathbb{P}_p(\bigcup_{s=-n}^{n-1}\{(s,t)\overset{\mathcal{G}_{\mathcal{U}}(n)}{\longrightarrow} S^0\})\\ &\leq \sum_{t=1}^{\infty}\min\{1, 2n\,\mathbb{P}((0,t)\overset{\mathcal{G}_{\mathcal{U}}(n)}{\longrightarrow} S^0)\}\\ &\leq \sum_{t=1}^{\infty}\min\{1, 2n\exp(-ht)\}\\ &\leq \frac{\log(2\,n)}{h}+K,\end{aligned} \tag{35}$$

where K is some positive constant. In first equality we used Proposition 3, in the second-last inequality we used (32) and (33).

Proof of part (b). The statement (b) of the theorem follows from Proposition 8 and from some further estimations based on path constructions. For every $a\in\mathbb{R}$ we define the event, $\mathcal{D}_{n,t,a} := \{\exists\, x\in[[-n,n-1]],$ such that (x,t) is connected to S^0 by an open path in $\mathcal{G}_{\mathcal{U}}$ which never goes out from the lines $y=\pm n - a(x-t)$, $x\in\mathbb{R}$ $\}$. See also Figure 5 - up. Observe that,

$$\mathbb{P}(\mathcal{D}_{n,t,a}) \leq \tilde{\delta}_{\mathbf{1}}(\tau_n > t). \tag{36}$$

This is because for the latter event the restriction that the path must be contained inside the region delimited by the two lines is not required, having assumed periodic boundaries condition. Consider now the following change of coordinates,

$$\begin{cases} x' = x - b(t-y)\\ y' = y\end{cases}, \tag{37}$$

under which the graph $\mathcal{G}_{\mathcal{U}}$ is transformed into the new graph $T^t_b\,\mathcal{G}_{\mathcal{U}}$. We denote by $T^t_b\,\mathcal{D}_{n,t,a}$ the event $\mathcal{D}_{n,t,a}$, defined for the graph $T^t_b\,\mathcal{G}_{\mathcal{U}}$, (i. e. replace $\mathcal{G}_{\mathcal{U}}$ with $T^t_b\,\mathcal{G}_{\mathcal{U}}$ in the definition of the event above). From now on we will use the same notation for all events. This means that if a certain event $\mathcal{E}$ is defined for the graph $\mathcal{G}_{\mathcal{U}}$, then the event $T^t_b\,\mathcal{E}$ is defined for the transformed graph $T^t_b\mathcal{G}_{\mathcal{U}}$. The following equation holds,

$$\mathbb{P}_p(T^t_b\,\mathcal{D}_{n,t,a}) = \mathbb{P}_p(\,\mathcal{D}_{n,t,a-b}), \tag{38}$$

as the change of coordinates preserves connection between vertices. See also Figure 5. Now we introduce the event $\mathcal{H}_n$,

$$\begin{aligned}\mathcal{H}_n = \{\exists\, y\,,\, y' \;:\; y\in[[4n,6n]],\, y'\in[[0,2n]]:\\ (-n,y)\overset{\mathcal{G}_{\mathcal{U}}}{\longrightarrow}(n,y')\},\end{aligned} \tag{39}$$

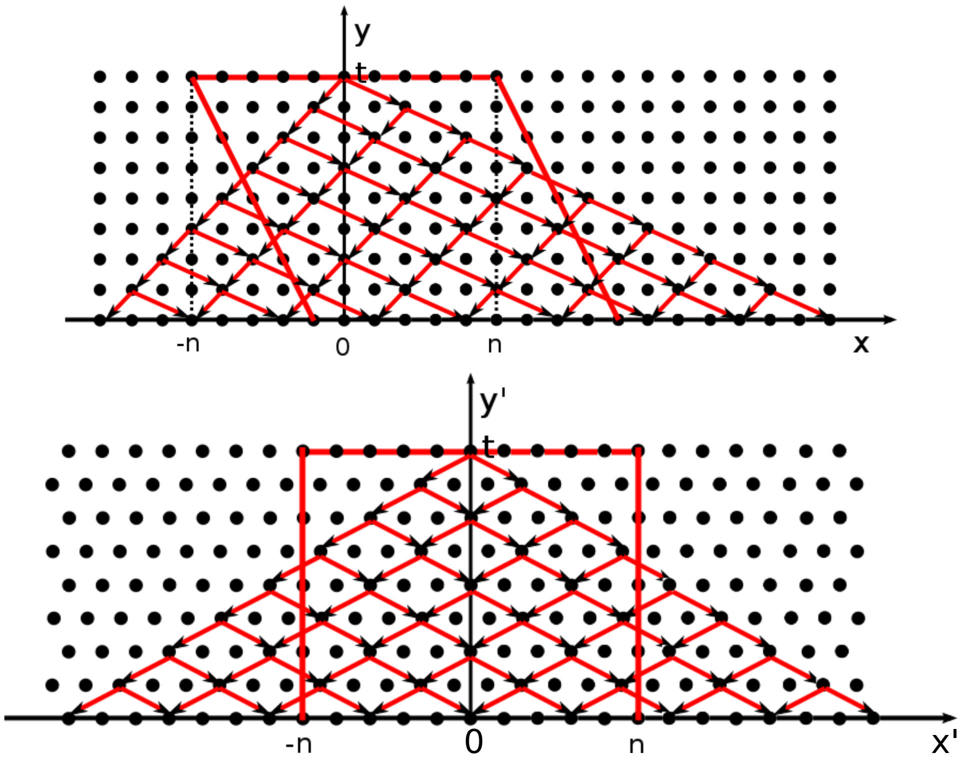


Figure 5: *Up*: Representation of $\mathcal{G}_{\mathcal{U}}$ in case of neighbourhood $\mathcal{U}(0) = \{-1, 2\}$. For graphical reasons only edges belonging to the evolution cone of $(0, t)$ have been drawn. The event $\mathcal{D}_{n,t,a}$, defined in the text, occurs if an open path joins the top of the red parallelogram to the bottom without crossing its sides. In the figure $a = \frac{s_1+s_u}{2}$. *Down*: the same graph of the figure above, transformed via (37) with parameter $b = a$.

which is represented in Figure 6-right. The following proposition is about this event.

Proposition 8. *There exist positive constants A, b (dependent on p) such that for any $p > p_c$, for any $t \in \mathbb{N}$ and for n large enough,*

$$\mathbb{P}_p(T^t_{\frac{\alpha+\beta}{2}} \mathcal{H}_n) \geq 1 - A \exp(-b\, n). \tag{40}$$

We show first that it implies statement (b) of the theorem and later we prove the proposition. Define then the new event $\mathcal{F}_{n,t}$, which is represented in Figure 6. $\mathcal{F}_{n,t}$ occurs *iff* (a) and (b) hold:

(a) for every odd $j \in [[0, \frac{t}{2n}]]$ there is a vertex $(-n, y)$, with $y \in [[2nj, 2n(j+1)]]$, connected to (n, y') by an open path in $\mathcal{G}_{\mathcal{U}}$, with $y' \in [[2n(j-2), 2n(j-1)]]$,

(b) for any even $j \in [[0, \frac{t}{2n}]]$ there is a vertex (n, y), with $y \in [[2nj, 2n(j+1)]]$, connected by an open path in $\mathcal{G}_{\mathcal{U}}$ to $(-n, y')$, with $y' \in [[2n(j-2), 2n(j-1)]]$.

Note first that,

$$\mathbb{P}_p(T^t_{\frac{\alpha+\beta}{2}}\, \mathcal{F}_{n,t}) \leq \mathbb{P}_p(T^t_{\frac{\alpha+\beta}{2}}\, \mathcal{D}_{n,t,0}), \tag{41}$$

because if $\mathcal{F}_{n,t}$ occurs, then the top of the box $2n \times t$ is connected to the bottom by a path that never goes out from the box (compare figures 6-left and 6-middle). Secondly, we observe that the event $T^t_{\frac{\alpha+\beta}{2}} \mathcal{F}_{n,t}$ equals the intersection of $\lfloor \frac{t}{n} \rfloor$ events of type $T^t_{\frac{\alpha+\beta}{2}}\, \mathcal{H}_n$, represented in Figure 6-right. Being the event $\mathcal{H}_n$ increasing, the FKG inequality is applicable, i.e.

$$\mathbb{P}_p(T^t_{\frac{\alpha+\beta}{2}}\, \mathcal{H}_n)^{\lfloor \frac{t}{n} \rfloor} \leq \mathbb{P}_p(T^t_{\frac{\alpha+\beta}{2}}\, \mathcal{F}_{n,t}) \tag{42}$$

Then using (40) finally we get,

$$\mathbb{P}_p(T^t_{\frac{\alpha+\beta}{2}}\, \mathcal{F}_{n,t}) \geq \ (1 - An \exp(-nb))^{\frac{t}{n}} \tag{43}$$

Then, from (34) and for n large enough,

$$\begin{aligned}
\mathbb{E}^{(n)}[\tau_n] &\geq \sum_{t=1}^{\infty} \mathbb{P}_p(T^t_{\frac{\alpha+\beta}{2}}\, \mathcal{D}_{n,t,0}) \\
&\geq \sum_{t=1}^{\infty} \mathbb{P}_p(T^t_{\frac{\alpha+\beta}{2}}\, \mathcal{F}_{n,t})) \\
&\geq \sum_{t=1}^{\infty} (1 - A \exp(-bn))^{\frac{t}{n}} \\
&\geq j(1 - \frac{Ae^{-bn} j}{n}),
\end{aligned} \tag{44}$$

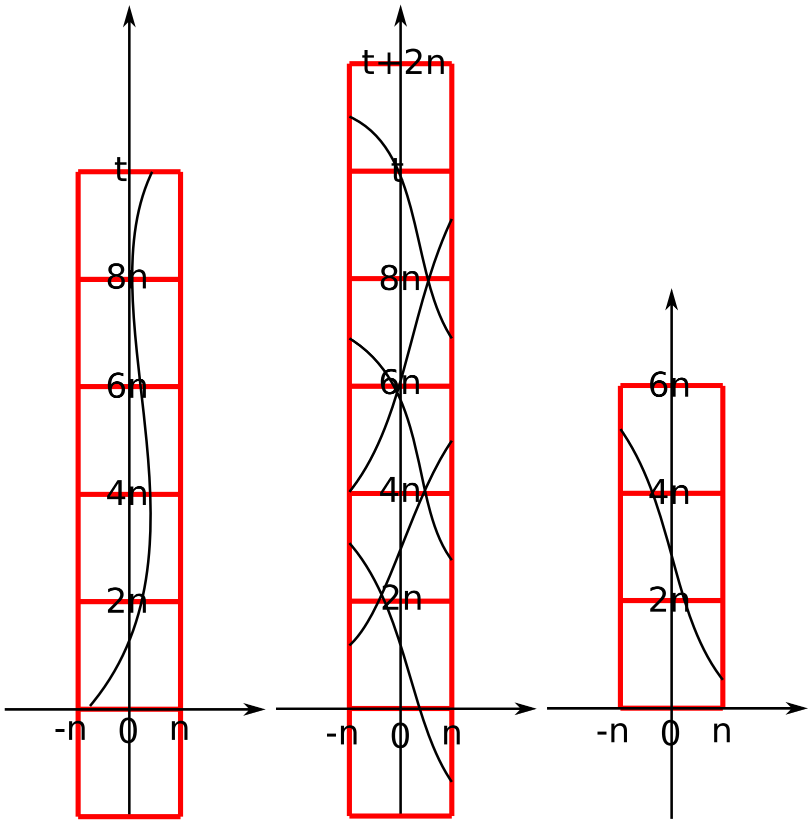


Figure 6: *Left*: representation of the event $\mathcal{D}_{n,t,0}$: the top of the rectangle $2n \times t$ is connected to the bottom by a path that never crosses the vertical sides. *Center*: representation of the event $\mathcal{F}_{n,t}$, defined in the text. *Right*: representation of the event $\mathcal{H}_n$, defined in the text. In all figures the details of the structure of the graph have been omitted.

where j is an arbitrary integer. In the previous expression we have used Proposition 3, (36) and (38) for the first inequality, (41) for the second inequality and (43) for the third one. Choosing finally $j = \lfloor \frac{ne^{bn}}{2A} \rfloor$, the part (b) of the theorem follows.

Proof of Proposition 8 We consider two graphs, $T^t_{\frac{\alpha+\beta}{2}}\,\mathcal{G}_{\mathcal{U}} = (V^1, \vec{E}^1_{\mathcal{U}})$ and $T^t_{\frac{s_1+s_u}{2}}\,\mathcal{G}_{\mathcal{U}} = (V^2, \vec{E}^2_{\mathcal{U}})$, reminding that s_1 and s_u are respectively the min and the max of $\mathcal{U}(0)$, recalling that the definitions of α and β are given in Section 4.2 and that the definitions of the transformation T^t is given in (37). Observe that vertices of both graphs could take non integer positions. The proof is divided into two parts. In the first part we generalize the dynamic block argument presented in [23] to the more general case of non-symmetric neighbourhood and more than 2 neighbours. This is the most technical part of the article. The method consists in introducing a coupling between realizations in $T^t_{\frac{s_1+s_u}{2}}\,\mathcal{G}_{\mathcal{U}}$ and those in $T^t_{\frac{\alpha+\beta}{2}}\,\mathcal{G}_{\mathcal{U}}$. We show that if the probability of the event $\mathcal{H}_n$ in $T^t_{\frac{s_1+s_u}{2}}\,\mathcal{G}_{\mathcal{U}}$ satisfies (40) for p large enough in a certain dependent oriented percolation model, then for all $p > p_c$, the probability of $\mathcal{H}_{Ln}$ in $T^t_{\frac{s_1+s_u}{2}}\,\mathcal{G}_{\mathcal{U}}$ cannot be smaller for a rescaling parameter L large enough. In the second part we define a sub-graph of $T^t_{\frac{s_1+s_u}{2}}\,\mathcal{G}_{\mathcal{U}}$, that we call $\mathcal{L}$, and we use the Peierls argument for dependent oriented percolation model in this graph to show that $\mathcal{H}_n$ occurs with probability satisfying (40) for p large enough. As $\mathcal{L}$ is a sub-graph of $T^t_{\frac{s_1+s_u}{2}}\,\mathcal{G}_{\mathcal{U}}$, the same event must occur with non smaller probability also in $T^t_{\frac{s_1+s_u}{2}}\,\mathcal{G}_{\mathcal{U}}$.

Part 1: Dynamic blocks construction We divide $T^t_{\frac{\alpha+\beta}{2}}\,\mathcal{G}_{\mathcal{U}}$ into *macro-regions* $R_{x,y}$ centred around the point $C_{x,y}$, where $(x, y) \in V^2$ and

$$C_{x,y} = (x\frac{\gamma}{s_u - s_1}(1-\delta), yL),$$

$$R_{x,y} = C_{x,y} + [(-1-\delta)\frac{\gamma}{2}L, (1+\delta)\frac{\gamma}{2}L] \times [0, -(1+\delta)L].$$

We recall that $\gamma = \alpha - \beta > 0$ for all $p > p_c$, as pointed out in the previous section. δ and L are parameters to be properly chosen. In order the argument to work rigorously, $(1-\delta)\gamma L$ and L should be even integers. To not complicate things, as in [23] here we ignore these details. Each vertex $(x, y) \in V^2$ is associated to a random variable $\varphi_{x,y}$ which takes value 1 if a certain event $\mathcal{B}_{x,y}$ occurs in the region $R_{x,y}$ of $(V^1, \vec{E}^1_{\mathcal{U}})$ or 0 otherwise. In order to define such event we introduce the following points in space (see also Figure 7),

$$\begin{aligned}
u &= (\frac{\delta\gamma L}{2}\ ,\ 0),\\
v &= (\frac{3\delta\gamma L}{4}\ ,\ 0),\\
-u &= (-\frac{\delta\gamma L}{2}\ ,\ 0),\\
-v &= (-\frac{3\delta\gamma L}{4}\ ,\ 0),\\
u_s^R &= (\frac{\delta\gamma L}{2} + (s - \frac{s_1+s_u}{2})\cdot\frac{(1-\delta)\gamma L}{s_u - s_1}\ ,\ -L(1+\delta)),\\
v_s^R &= (\frac{3\delta\gamma L}{4} + (s - \frac{s_1+s_u}{2})\cdot\frac{(1-\delta)\gamma L}{s_u - s_1}\ ,\ -L(1+\delta)),\\
u_s^L &= (-\frac{\delta\gamma L}{2} + (s - \frac{s_1+s_u}{2})\cdot\frac{(1-\delta)\gamma L}{s_u - s_1}\ ,\ -L(1+\delta)),\\
v_s^L &= (-\frac{3\delta\gamma L}{4} + (s - \frac{s_1+s_u}{2})\cdot\frac{(1-\delta)\gamma L}{s_u - s_1}\ ,\ -L(1+\delta)),\\
u_s^U &= (-\frac{\delta\gamma L}{2} + (s - \frac{s_1+s_u}{2})\cdot\frac{(1-\delta)\gamma L}{s_u - s_1} + \frac{\gamma}{2}(1+\delta)L\ ,\ 0),\\
v_s^U &= (-\frac{3\delta\gamma L}{4} + (s - \frac{s_1+s_u}{2})\cdot\frac{(1-\delta)\gamma L}{s_u - s_1} + \frac{\gamma}{2}(1+\delta)L\ ,\ 0),
\end{aligned} \tag{45}$$

for any $s \in \mathcal{U}(0)$. As one can see in the example from Figure 7, where the box $R_{0,t}$ is represented, these points identify some *target zones* (red horizontal segments in the figure) on the right and on the left side of $C_{0,t}$ and of the points $C_{s,t-1}$ for $s \in T^t_{\frac{s_1+s_u}{2}}\,\mathcal{U}(0)$. Consider now the parallelograms obtained connecting the following quadruplets of points, (see also Figure 7),

$$\begin{aligned}
P_R &= (-v,\ -u,\ u^R_{s_u},\ v^R_{s_u}),\\
P_L &= (u,\ v,\ u^L_{s_1},\ v^L_{s_1})\\
P_s &= (u^L_s, v^L_s, u^U_s, v^U_s),
\end{aligned} \tag{46}$$

for all s in $\mathcal{U}(0)$ different from s_1 and s_u. Define the translated parallelograms $P_R(x,y) = P_R + C_{x,y}$, $P_L(x,y) = P_R + C_{x,y}$, $P_s(x,y) = P_s + C_{x,y}$ for all s in $\mathcal{U}(0)$ different from s_1 and s_u. The event $\mathcal{B}_{x,y}$ occurs if the top of the parallelograms $P_R(x,y)$, $P_L(x,y)$ and of $P_s(x,y)$, for all $s \in \mathcal{U}(0)$ which is not s_1 or s_u, is connected to the bottom by an open path in $T^t_{\frac{\alpha+\beta}{2}}\,\mathcal{G}_{\mathcal{U}}$ which remains always inside the parallelogram. This construction is such that the following properties are satisfied,

1. the random variables $\varphi_{x,y}$ are $s_u - s_1$-dependent. With this we mean that $\varphi_{x,y}$ and $\varphi_{x',y'}$, with $(x,y), (x',y') \in V^2$, are independent if $|x - x'| > s_u - s_1$ or $|y - y'| > 1$.

2. Denote by $z_1 \dots z_m$ the vertices of a path in $T^t_{\frac{s_1+s_u}{2}}\,\mathcal{G}_{\mathcal{U}}$ and assume that the path is open, i.e. $\varphi_{z_i} = 1$ for all $i \in \{1, 2, \dots m\}$. Then there exists an open path in $T^t_{\frac{\alpha+\beta}{2}}\,\mathcal{G}_{\mathcal{U}}$ that connects a vertex in $C_{z_1} + [-v, v]$ to a vertex in $C_{z_m} + [-v, v]$ and which remains always inside the parallelograms that connect $C_{z_i} + [-v, v]$ to $C_{z_{i+1}} + [-v, v]$, for all $i \in \{1, 2, \dots m\}$.
3. if $\delta, \epsilon > 0$ and $p > p_c$, we can pick L large enough so that for any $(x, y) \in V^2$, $\mathbb{P}_p(\varphi_{x,y} = 1) > 1 - \epsilon$.

Property 1 follows by the fact that if $R_{x,y}$ and $R_{x',y'}$ have empty intersection, then the variables $\varphi_{x,y}$ and $\varphi_{x',y'}$ are independent. Property 2 follows by construction. Looking also at Figure 7-down, one can see that if the event $\mathcal{B}_{0,t}$ and the event $\mathcal{B}_{s_1,t-1}$ occur, then there is a path connecting at least one vertex of the interval $C_{0,t} + [-v, v]$ to at least one vertex of the intervals $C_{s'_1+s'_1,t} + [-v, v]$, $C_{s'_1+s'_2,t} + [-v, v]$, $C_{s'_1+s'_3,t} + [-v, v]$ $C_{s'_1+s'_4,t} + [-v, v]$, where s'_i are the elements of $T^t_{\frac{s_u+s_1}{2}}\,\mathcal{U}(0)$. By induction, property 2 follows. Property 3 follows by Proposition 4, which is a sort of law of large numbers for $\overline{r}_n$, $\overline{\ell}_n$. In fact for the transformed graph $T^t_{\frac{\alpha+\beta}{2}}\,\mathcal{G}_{\mathcal{U}}$, $\overline{r}_n/n \overset{n\longrightarrow\infty}{\longrightarrow} \gamma/2$ a.s. and $\overline{\ell}_n/n \overset{n\longrightarrow\infty}{\longrightarrow} -\gamma/2$ a. s. These parallelograms are constructed in such a way that the slopes of the vertical sides is $2/\gamma$ and the length of their sides (both horizontal and diagonal) is proportional to L. Thus, heuristically, as in order to go out from the parallelogram a path should keep a slope different from $2/\gamma$, by the law of large number the probability of this to happen will go to 0 as L goes to infinity. Thus the probability that there is an infinite open path which starts from the top of the parallelogram and which never goes out from it goes to 1 as L goes to infinity. For the proof we refer to [23, pages 1025-1026], observing that one should replace α there with our $\gamma/2$.

Part 2: Peierls argument Now we use the Peierls argument for the $(s_u - s_1)$-dependent oriented percolation model to prove that there exists $p_1 > p_c$ and positive constants A', b' (dependent on p) such that for all $p > p_1$,

$$\mathbb{P}_p(T^t_{\frac{s_1+s_u}{2}}\,\mathcal{H}_n) \geq 1 - A'ne^{-b'n}. \tag{47}$$

Then the third property of the dynamic-blocks construction implies that for all $p > p_c$,

$$\mathbb{P}_p(T^t_{\frac{\alpha+\beta}{2}}\,\mathcal{H}_{Ln}) \geq 1 - A'ne^{-b'n}. \tag{48}$$

Defining new constants $A = A'/L$ and $b = b'L$, the statement of Proposition 8 follows. Thus it just remains to prove (47). Then we define a new graph $\mathcal{L}$, which is a sub-graph of $T_{\frac{s_1+s_u}{2}}\,\mathcal{G}_{\mathcal{U}}$, whose vertices (x, y) are,

$$V' = \{(x, y) : \; x = (s_u - s_1)z - (y - t)\frac{s_u - s_1}{2}, \; z \in \mathbb{Z}, y \in \mathbb{Z}\}, \tag{49}$$

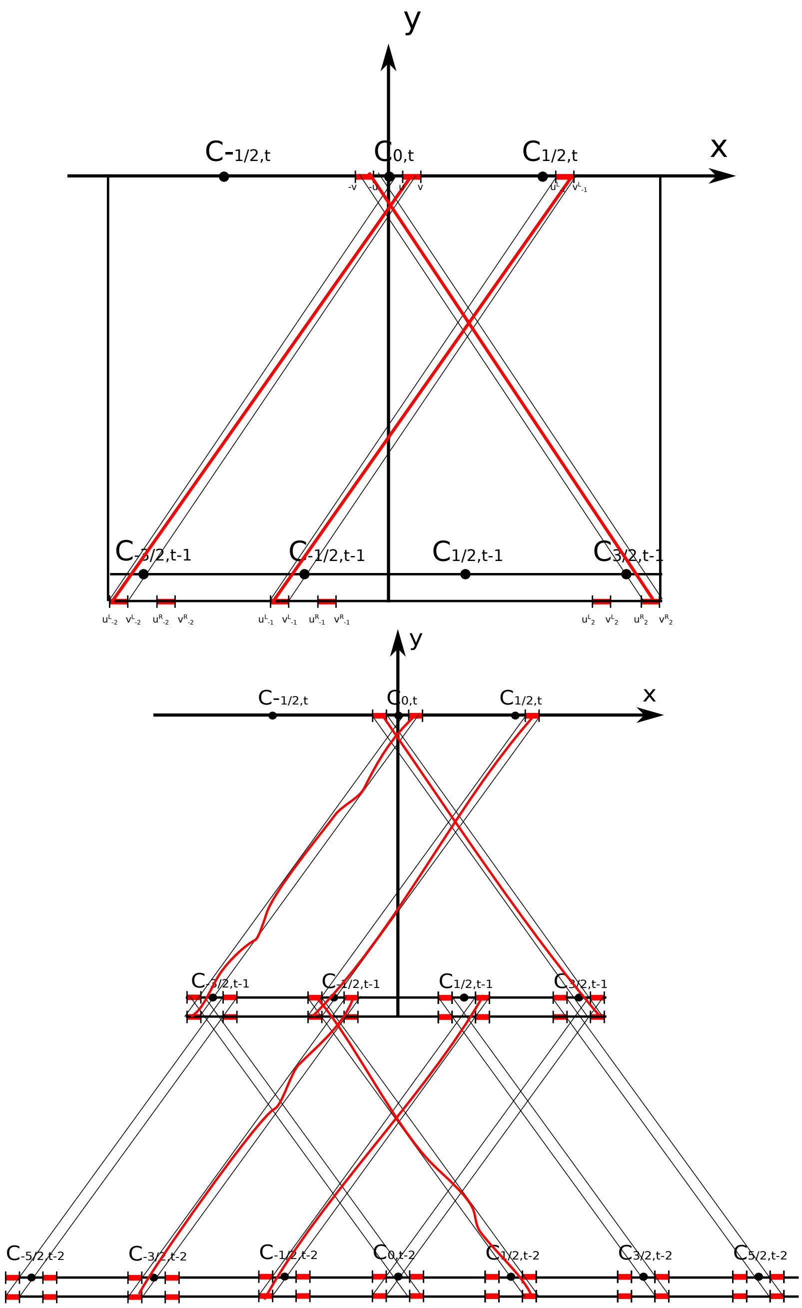


Figure 7: *Up*: The rectangle in the figure represents the region $R_{0,t}$ of the graph $T^{t}_{\frac{s_u+s_1}{2}}\,\mathcal{G}_{\mathcal{U}}$. Target zones and paths are represented by red. The event $\mathcal{B}_{0,t}$ occurs if an open path joins the top to the bottom of the parallelograms in the figure, as explained in the text. *Down*: In the figure, the events $\mathcal{B}_{0,t}$ and $\mathcal{B}_{-1/2,t-1}$ occur.

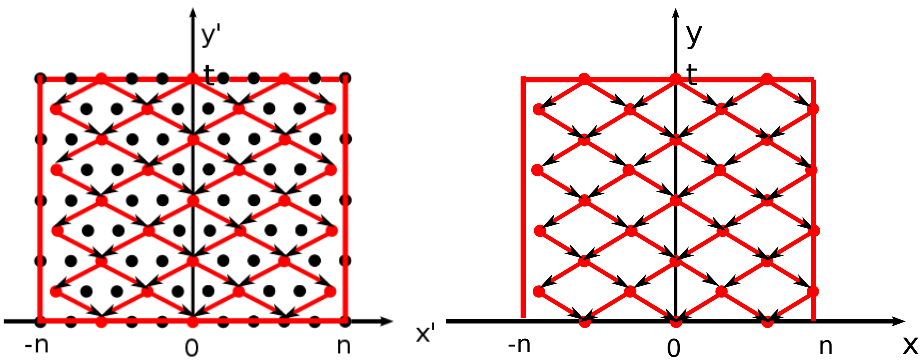


Figure 8: On the left we represented the graph $T^t_{\frac{s_1+s_u}{2}}\mathcal{G}_{\mathcal{U}}$, with $\mathcal{U}(0) = \{-2, 1, 0\}$. Note that only edges connecting vertices belonging to V', defined in the text, have been represented. *Right*: representation of $\mathcal{L}$, defined in the text.

and whose edges connect vertices (x, y) to $(x \pm \frac{s_1+s_u}{2}, y-1)$. The new graph is represented in Figure 8. As $\mathcal{L}$ is a sub-graph of $T_{\frac{s_1+s_u}{2}}\mathcal{G}_{\mathcal{U}}$, the following inequality holds,

$$\mathbb{P}_p(\mathcal{H}_n^{\mathcal{L}}) \leq \mathbb{P}_p(T^t_{\frac{s_1+s_u}{2}}\mathcal{H}_n). \tag{50}$$

In the previous expression, the superscript $^{\mathcal{L}}$ denotes that the event $\mathcal{H}_n$, defined in (39), occurs on the graph $\mathcal{L}$. Call then $\mathcal{L}_D$ the dual graph of $\mathcal{L}$. Its vertices are denoted by (x^*, y^*) and they are located at $(x, y)+(\frac{s_u-s_1}{2}, +1/2)$, where (x, y) are the vertices of $\mathcal{L}$. Our construction is similar to the one presented in [5][Chapter 8] and in [7], where the case of neighbourhood $\mathcal{U}(0) = \{0, 1\}$ is considered. In Figure 9-right, vertices (x^*, y^*) correspond to the intersection of the blue lines. Edges of $\mathcal{L}_D$ are oriented and they are of three different types:

1. they connect points (x^*, y^*) to points $(x^* + s_u - s_1\,,\, y^*)$
2. they connect points (x^*, y^*) to points $(x^* - \frac{s_u-s_1}{2}\,,\, y^* - 1)$
3. they connect points (x^*, y^*) to points $(x^* - \frac{s_u-s_1}{2}\,,\, y^* + 1)$

As in [5][Chapter 8], we declare every horizontal edge of the dual graph "open" if the site above in the original lattice is "closed" and vice versa. A path in the dual lattice is composed of horizontal, up-left or down-left edges and it is considered "open" if all its horizontal edges are open. The state "open" or "closed" for diagonal connections is not defined. The following is true: referring to Figure 9, there is an open path in $\mathcal{L}$ connecting the top to the bottom of the box and never crossing its vertical sides *iff* there is no "open" path in the dual lattice which connects the segment $\overline{AD}$ to $\overline{AB}$ or to $\overline{BC}$.

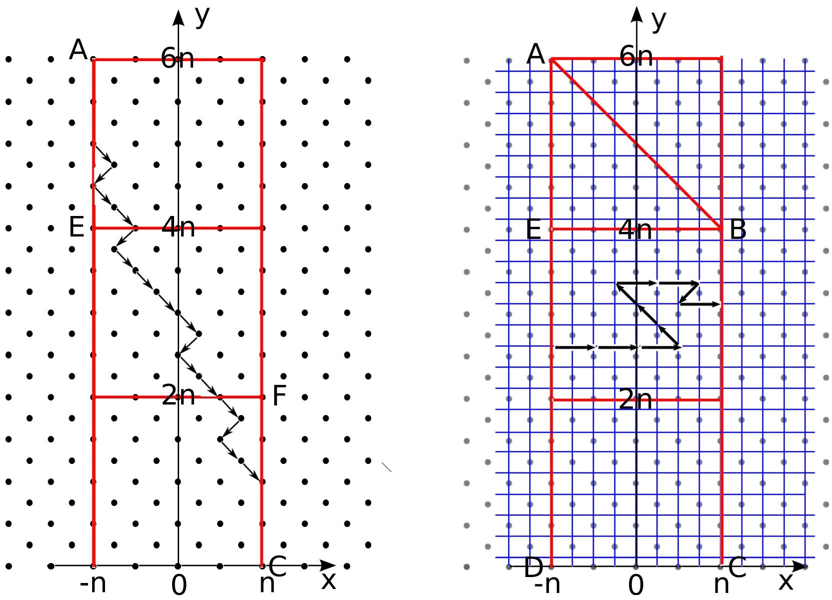


Figure 9: *Left*: representation of the event $\mathcal{H}_n$ in $\mathcal{L}$. In both figures the x-axes has been rescaled by $s_u - s_1$. The event occurs if the side $\overline{AE}$ is connected to the side $\overline{FC}$ by an open path in $\mathcal{L}$. Points represent sites and arrows the edges of a path joining $\overline{AE}$ to $\overline{FC}$. *Right*: representation of $\overline{\mathcal{H}}_n$ (the complementary of $\mathcal{H}_n$) in the dual graph. $\mathcal{H}_n$ does not occur *iff* the side $\overline{AD}$ is connected to the segments $\overline{AB}$ or $\overline{BC}$ by an open path in the dual lattice (as for instance the one represented with the black arrows).

Consider then a vertex $(-n, y)$, $y \in [[0, 4n]]$. Call $N_{y,h}$ the number of paths connecting this point to one of the sides $\overline{AB}$, $\overline{BC}$ and having h horizontal steps. Call dl the number of down-left steps, ul the number of up-left steps of one of these paths. As the last edge of the path cannot be on the left of the first edge, $2h - ul - dl \geq 0$. This implies that the sum $h + ul + dl$ is bounded from above by $3h$. Furthermore the total number of possible paths having h horizontal steps cannot be larger than 3^{3h}, as there are only 3 different types of steps. Thus $N_{y,h} \leq 3^{3h}$ for every y. A rough upper bound for $\mathbb{P}_p(\overline{\mathcal{H}_n^{\mathcal{L}}})$, where $\overline{\mathcal{H}_n^{\mathcal{L}}}$ is the complementary of $\mathcal{H}_n^{\mathcal{L}}$, follows,

$$\mathbb{P}_p(\overline{\mathcal{H}_n^{\mathcal{L}}}) \leq \sum_{y=0}^{6n} \sum_{h=2n}^{\infty} N_{y,h} \, (1-p)^{h/2} \leq A' n \exp(-b' n), \tag{51}$$

where in the first one we used the fact that the probability of the union of events "the path is open" is less of the sum of the probabilities of the same events and the second inequality is true if $p > 1 - \frac{1}{3}^6$, with A', b' positive constants, as the sum converges. In the previous expression we have multiplied the factor $1 - p$ only over $h/2$ horizontal edges, as percolation is $(s_u - s_1)$-dependent. This means that, following a path and starting from the first edge, we ignore the state of every second edge. This completes the proof of the proposition.

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