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Metric Curvature Revisited - A Brief Overview

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by

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METRIC CURVATURES REVISITED – A BRIEF OVERVIEW

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Abstract. We survey metric curvatures, special accent being placed upon the Wald curvature, its relationship with Alexandrov curvature, as well as its application in defining a metric Ricci curvature for PL cell complexes and a metric Ricci flow for PL surfaces. In addition, a simple, metric way of defining curvature for metric measure spaces is proposed.

1. Introduction

We begin with a brief motivation for our interest in the material of this chapter: The reason we study *Metric* Geometry is due to the fact that it provides us with a minimalistic framework that requires no additional smoothness, nor does it impose any supplementary, ad hoc structure upon a given geometric object. It is therefore our firm belief that this property renders the metric method as an approach ideally suited for the study of the various structures and problems encountered in computer Science in general, and in Graphics, Imaging and Vision in particular and that, moreover, this Newtonian stance of "hypotheses non fingo" represents not only the philosophically correct attitude, it also is ideally suited for the truthful intelligence of Digital Spaces. This terse and somewhat vague argument was

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augmented in [94] with examples drawn from Wavelet Theory (and practice), DNA Microarray Analysis, Imaging and Graphics, and we briefly mention them here, too, as well as a possible application to Networking.

To be sure, the term “Metric Geometry” is extensively used elsewhere and there exists a quite extensive literature on Metric Geometry and its various applications to Computer Science. (See, e.g. [16], [56], [51], [1], to mention only very small number of titles from an impressive literature – without any chance whatsoever of exhausting it). Unfortunately, the “linear”, or “first order Geometry” considered in the articles mentioned above does not fulfill all its expectations, and particularly so in Manifold Learning (as well as in Imaging and Graphics). To explain this, we could emphasize the essential importance of “second order Geometry”, or succinctly, the Geometry of curvature, in Graphics, Imaging and Manifold Learning. We did this in some detail in our previous work already alluded to, so we refrain repeating ourselves, except to quote again the phrase so aptly coined by the regretted Robert Brooks: “The fundamental notion of differential geometry is the concept of curvature” ([18]) – And our goal here is to sketch the very basis of a metric (“discrete”) Differential Geometry.

Before we conclude the introduction, we wish to outline the structure of the main part of the paper: In Section 2 we briefly discuss some metric notions of curvature for 1-dimensional geometric objects, i.e. curves. The accent is placed on the so called *Haantjes* (or *Finsler-Haantjes*) *curvature*, partly because we shall make appeal to it in the last section, where we suggest, amongst others, a simple approach to the introduction of a curvature for metric measure spaces. Section 3 represents the very nucleus, of this chapter. In its beginning we introduce the “main character” of this paper, namely the *Wald* (or *Wald-Berestovskii*) *curvature*, we discuss its properties and its relationship to the much better known notion of *Alexandrov* curvature and we apply it to develop a metric Ricci flow for PL surfaces as well as notion of metric Ricci curvature for cell complexes. In the forth – and last – we recall notion of *snowflaking* and we apply it to the introduction of a simple approach to curvature on metric measure spaces, with applications to Sampling Theory.[1]

[1]As the reader will become aware while progressing with this text, we have written previously two book chapters on metric curvatures. However the present paper does not

2. Metric Curvature for Curves

We do not detail here all the basic, simple geometric ideas that reside behind the various notions of metric curvature; the reader who might feel a slower, more graduate and detailed exposition is needed is referred to our previous expositions [90] and [94]. (He might also consult with much benefit [11], and we refer the reader to this book, as well as to [13], [14] for any missing notions in our exposition.) Neither do we (as we already warned the reader) go into details over all the metric curvatures for curves. However, we find impossible to write an overview on the subject of metric curvatures without mentioning them, even if only very briefly.

2.1. **Menger Curvature.** The simplest and most direct version of metric curvature for curves is the so called *Menger curvature* κ_M. It's idea is to mimic the definition of the *osculatory circle*, by first defining the (metric) curvature of triangles (or triples of points), by defining the curvature $K(T)$ of a triangle T to be just $1/R(t)$, where $R(T)$ is the radius of the circle circumscribed to the triangle, then passing to the limit (exactly like in the standard, classical osculatory circle definition). To define $K(T)$, one makes appeal to some very elementary and quite well know formulas of high-school geometry. Unfortunately, this impose on the space under scrutiny an intrinsically Euclidean notion of curvature. Nevertheless, Menger curvature has been employed with considerable success to the study of such problems as finding estimates (obtained via the Cauchy integral) for the regularity of fractals and the flatness of sets in the plane (see [75]). As far as practical implementations are concerned, Menger curvature has been used for curve reconstruction ([28]). Also, one might consider its use in for the obvious task of approximating the *principal curvatures*[2], hence the computation of

represent a calque of any of these previous ones. For one, the present one is addressed to a much more mathematically literate (not to say "very well educated") audience than the previous expositions. To be sure, certain repetitions are, unfortunately, unavoidable: After all, the same subject represents the common theme of all these three papers. However, we have strived to keep these at an inevitable minimum. Moreover, we did our best to emphasize different aspects (in general, more modern ones) as well as introducing some novel applications.

[2]We know that in mentioning this here we anticipate, somewhat, the reminder of the paper.

Gauss and mean curvature, of triangulated (polyhedral) surfaces, and their manifold uses in Graphics, Imaging and other, related fields. Experiments with the Menger curvature (see Figure 1) and with the Haantjes curvature (see below and [98]) have shown, unfortunately, only moderate success.

FIGURE 1. A standard test image (left) and its Menger curvature (right).

The reader may have already recognized Menger curvature to be nothing else than Gromov's $\mathbf{K}_3(\{p,q,r\})$, where p,q,r represent the vertices of a triangle, introduced in Memoli's exposition in the present volume. However, while the general setting of the modern discourse are certainly important, there still is use for the classical, "parochial" Menger curvature, as demonstrated by its many contemporary uses, thus interest and mathematical activity around the "old fashioned" Menger curvature still exists, even though the much more general and modern setting of $\mathbf{K}_n(X)$ is much more alluring. Moreover, it should be noted that, beyond this generality there rests a lot of uncertainty, as Gromov himself notes [32].

Remark 2.1. Discussing Menger curvature, and mainly the idea behind it, one can hardly not mention the very recent development [7][3]. Here a kind of "comparison Menger" curvature is introduced. Very loosely formulated, whereas in the classical Menger curvature a specific, Euclidean in nature, formula is developed for the circumradius, here it is only compared to that

[3]Although, when these notes were started, the mentioned work was still not published.

of a triangle of the same sides, in different model (or gauge) surfaces. This is similar (and presumably inspired by) the Alexandrov curvature (which we shall discuss later on).

Remark 2.2. While again anticipating higher dimensional curvatures (and mainly 2-dimensional ones), we can not leave this short section on Menger curvature without mentioning the *Menger curvature measure*:

$$\mu(\mathcal{T}) = \mu_p(\mathcal{T}) = \sum_{T \in \mathcal{T}} \kappa_M^p(T)(\operatorname{diam} T)^2\,, \tag{2.1}$$

for some $p \geq 1$, where $\kappa_M(T)$ denotes the Menger curvature (of the triangle T).

(See, e.g. [53] and, for a somewhat different approach and another range of problems altogether, [107].)

In the applications range, one possible use of the Menger curvature measure (in its alternative variant) is in the field of Pattern Recognition, for texture segmentation and classification – see [29]. This represents an approach to non local "operators" for Imaging, radically different (being "purely" geometric) from those of Osher et al. [50], [30] and Jost et al. [45], [46].

2.2. **Haantjes Curvature.** Having only very briefly presented the Menger curvature, we shall expound in somewhat more detail upon another, less commonly known notion of metric curvature, namely the so called *Haantjes curvature* or *Finsler-Haantjes curvature*.[4] We have chosen to do so not only because this curvature does not mimic $\mathbb{R}^2$ (we already have emphasized this point in [94] and elsewhere), nor because of its adaptivity to applications (again, see [94] and the bibliography within, but see also Remark 2.7 below). The reasons behind our choice are that we both want to present some of Haantjes connections with other notions (not just of curvature), and because we wish to suggest it represents a simple and direct alternative – at least in certain applications – of more involved and fashionable concepts.

Definition 2.3 (*Haantjes curvature)*. Let (M, d) be a metric space and let $c : I = [0, 1] \stackrel{\sim}{\rightarrow} c(I) \subset M$ be a homeomorphism, and let $p, q, r \in c(I)$, $q, r \neq$

[4]Named after Haantjes [42], who extended to metric spaces an idea introduced by Finsler in his PhD Thesis.

p. Denote by $\widehat{qr}$ the arc of $c(I)$ between q and r, and by qr line segment from q to r.

We say that c has *Haantjes curvature* $\kappa_H(p)$ at the point p iff:

$$\kappa_H^2(p) = 24 \lim_{q,r \to p} \frac{l(\widehat{qr}) - d(q,r)}{\big(l(\widehat{qr})\big)^3}\ ; \tag{2.2}$$

where "$l(\widehat{qr})$" denotes the length – in intrinsic metric induced by d – of $\widehat{qr}$.

Remark 2.4. Since for points/arcs where Haantjes curvature exists, $\frac{l(\widehat{qr})}{d(q,r)} \to 1$, as $d(q,r) \to 0$ (see [42]), κ_H can alternatively be defined (see, e.g. [49]) as

$$\kappa_H^2(p) = 24 \lim_{q,r \to p} \frac{l(\widehat{qr}) - d(q,r)}{\big(d(q,r)\big)^3}\ ; \tag{2.3}$$

As it turns out, in applications it is this alternative form of the definition of Haantjes curvature proves itself to be more malleable (see [94] for some details).

The definition of Haantjes curvature (in both its versions) is quite intuitive and even the $\big(d(q,r)\big)^3$ factor is clearly inserted for scaling reasons. Far less intuitive (and somewhat puzzling) is the "24" factor. However, it arises quite naturally in the proof of following basic (and reassuring, so to say[5]) theorem:

Theorem 2.5. *Let $\gamma \in \mathcal{C}^3$ be smooth curve in $\mathbb{R}^3$ and let $p \in \gamma$ be a regular point. Then the metric curvature $\kappa_H(p)$ exists and equals the classical curvature of γ at p.*

(For a proof, see [42] or – probably somewhat more easily accessible – [13].)

Moreover, the same result holds for Menger curvature (see [13]). In fact, the two curvatures (when both applicable) coincide, as shown by the next result:

Theorem 2.6 (*Haantjes*)**.** *Let γ be a rectifiable arc in a metric space (M,d), and let $p \in \gamma$. If κ_M and κ_H exist, then they are equal.*

[5]since it proves us that, indeed, for smooth curves, Haantjes curvature coincides with the classical notion of curvature.

Remark 2.7. With the risk of being somewhat redundant, but to mirror our presentation of Menger curvature, we mention here, in brief, that Haantjes has proved its versatility in such diverse fields as Imaging and Graphics, Wavelets (with applications for texture segmentation) and DNA Microarray Analysis. (For details see again [94] and the bibliography within. See also [65] for an application to the quasi-conformal and quasi-isometric planar representation of (medical) images.)

FIGURE 2. A natural image (left) and its average Haantjes curvature (right).

2.2.1. *Haanjes curvature and excess.* A metric geometry concept that has been proven itself as both flexible and powerful, in many mathematical settings, and in particular in studying the Global Geometry of Manifolds (see, e.g., [34], [35] and the bibliography therein), is the *excess*:

Definition 2.8 (*Excess*)**.** Given a triangle[6] $T = \triangle(pxq)$ in a metric space (X, d), the *excess* of T is defined as

$$e = e(T) = d(p, x) + d(x, q) - d(p, q). \tag{2.4}$$

A local version of this notion was introduced (seemingly by Otsu [74]), namely the *local excess* (or, more precisely, the *local d-excess*):

$$e_d(X) = \sup_p \sup_{x \in B(p,\rho)} \inf_{q \in S(p,\rho)} \left(e(\triangle(pxq)) \right), \tag{2.5}$$

[6]not necessarily geodesic

where $\rho \leq \mathrm{rad}(X) = \inf_p \sup_q d(p,q)$, (and where $B(p,\rho), S(p,\rho)$ stand – as they usually do – for the ball and respectively sphere of center p and radius ρ).

In addition, global variations of this quantity have also been defined:

$$e(X) = \inf_{(p,q)} \sup_x \left(e(\triangle(pxq))\right), \tag{2.6}$$

and, the so called *global big excess* (see [74]):

$$E(X) = \sup_q \inf_p \sup_x \left(e(\triangle(pxq))\right). \tag{2.7}$$

Intuitively, it is clear that (local) excess and curvature are closely related concepts, since the geometric "content" of the notion of local excess resides in the fact that, for any $x \in B(p,\rho)$, there exists a (minimal) geodesic γ from p to $S(p,\rho)$ such that γ is close to x. More precisely, we have the following relation between the two notions:

$$\kappa_H^2(T) = \frac{e}{\rho^3}, \tag{2.8}$$

where $\rho = \rho(p,q)$, and where by the curvature of a triangle $T = T(pxq)$ we mean the curvature of the path $\widehat{pxq}$. Here and below we have used a simplified notation and discarded (for sake of simplicity and clarity) the normalizing constant "24".[7] Thus Haantjes curvature can be viewed as a *scaled* version of excess. Keeping this in mind, one can define also a global version of this type of metric curvature, namely by defining, for instance:

$$\kappa_H^2(X) = \frac{E(X)}{\mathrm{diam}^3(X)}, \tag{2.9}$$

or

$$\kappa_H^2(X) = \frac{e(X)}{\mathrm{diam}^3(X)}, \tag{2.10}$$

as preferred.

To be sure, one can proceed in the opposite direction and express the proper (i.e. point-wise) Haantjes curvature by means of the definition (2.5) of local excess, as

$$\kappa_H^2(x) = \lim_{\rho \to 0} e(x). \tag{2.11}$$

[7] In any case, it is not truly required and, in fact, even cumbersome in practical applications (see [97], [3] for two such examples).

Remark 2.9. We should mention that both Menger and Haantjes curvatures have their more modern ("updated" and "sophisticated") respective versions – see [2]. However, let us add here that, at least as far as applications are concerned, we favour the older notions over their more modern "avatars", not solely for their simplicity, but also for a number of reasons, mainly appertaining to their potential:

(1) First and foremost, while the Alexander-Bishop variants are more "tight", so to say, they coincide with their classical counterparts on all but the most esoteric spaces;
(2) They are applicable to more general settings, fact that represents a further incentive in their application in discrete (i.e. Computer Science driven) settings;
(3) In addition, no *a priori* knowledge of the global geometry of the ambient space (i.e. Alexandrov curvature) is presumed, nor is it necessary to first determine the curves of constant curvature (see [2]) in order to compute these curvatures; furthermore
(4) They are easy to compute in a direct fashion in the discrete setting (at least amongst those discrete versions we encountered), thus they are simpler and far more intuitive;
(5) Last – but certainly not least – they are more ready to lend themselves to discretization, hence admit easy and direct "semi-discrete" (or "semi-continuous") versions, as the one mentioned in Remark 2.7 above. In view of this and their simplicity noted above, they prove to be more conducive towards practical applications.

3. Metric Curvature for Surfaces: Wald Curvature

3.1. **Wald Curvature.** We introduce here the main type of metric curvature that we overview in this chapter, namely the so called *Wald curvature*. Wald's seminal idea was to go back to Gauss' original method of defining surface curvature by comparison to a standard, model surface (i.e. the unit sphere in $\mathbb{R}^3$), while extending it to general gauge surfaces, rather than restrict himself to the unit sphere. Moreover, instead of comparing infinitesimal areas (which would be an impossible task in general metric space not endowed with a measure), he compared quadrangles. More precisely, his starting point was the following definition:

Definition 3.1. Let (M, d) be a metric space, and let $Q = \{p_1, ..., p_4\} \subset M$, together with the mutual distances: $d_{ij} = d_{ji} = d(p_i, p_j)$; $1 \leq i, j \leq 4$. The set Q together with the set of distances $\{d_{ij}\}_{1 \leq i,j \leq 4}$ is called a *metric quadruple.*

Remark 3.2. The reader has undoubtedly already recognized that the definition above conducts toward $\mathbf{K}_4(X)$. Indeed, we can view, in a sense, this chapter as representing an extended overview of and discussion on $\mathbf{K}_4(X)$.

Remark 3.3. The following slightly more abstract definition can be also considered, one that does not make appeal to the ambient space: a metric quadruple being a 4 point metric space, i.e. $Q = \big(\{p_1, ..., p_4\}, \{d_{ij}\}\big)$, where the distances d_{ij} verify the axioms for a metric. However, this comes at a price, as we shall shortly see in Remark 3.5.

Before being able to pass to the next definition we need to introduce some additional notation: S_κ denotes *the complete, simply connected surface of constant Gauss curvature* κ (or *space form*), i.e. $S_\kappa \equiv \mathbb{R}^2$, if $\kappa = 0$; $S_\kappa \equiv \mathbb{S}^2_{\sqrt{\kappa}}$, if $\kappa > 0$; and $S_\kappa \equiv \mathbb{H}^2_{\sqrt{-\kappa}}$, if $\kappa < 0$. Here $S_\kappa \equiv \mathbb{S}^2_{\sqrt{\kappa}}$ denotes the sphere of radius $R = 1/\sqrt{\kappa}$, and $S_\kappa \equiv \mathbb{H}^2_{\sqrt{-\kappa}}$ stands for the hyperbolic plane of curvature $\sqrt{-\kappa}$, as represented by the Poincaré model of the plane disk of radius $R = 1/\sqrt{-\kappa}$.

Definition 3.4. The *embedding curvature* $\kappa(Q)$ of the metric quadruple Q is defined to be the curvature κ of the gauge surface S_κ into which Q can be isometrically embedded – if such a surface exists.

Remark 3.5. Even though the basic idea of embedding curvature is, in truth, quite intuitive, care is needed if trying to employ it directly, since there are a number of issues that arise (as we have anticipated in Remark 3.3 above):

(1) If one uses the second (abstract) definition of the metric curvature of quadruples, then the very existence of $\kappa(Q)$ is not assured, as it is shown by the following

Counterexample 3.6. The metric quadruple of lengths

$$d_{12} = d_{13} = d_{14} = 1;\ d_{23} = d_{24} = d_{34} = 2$$

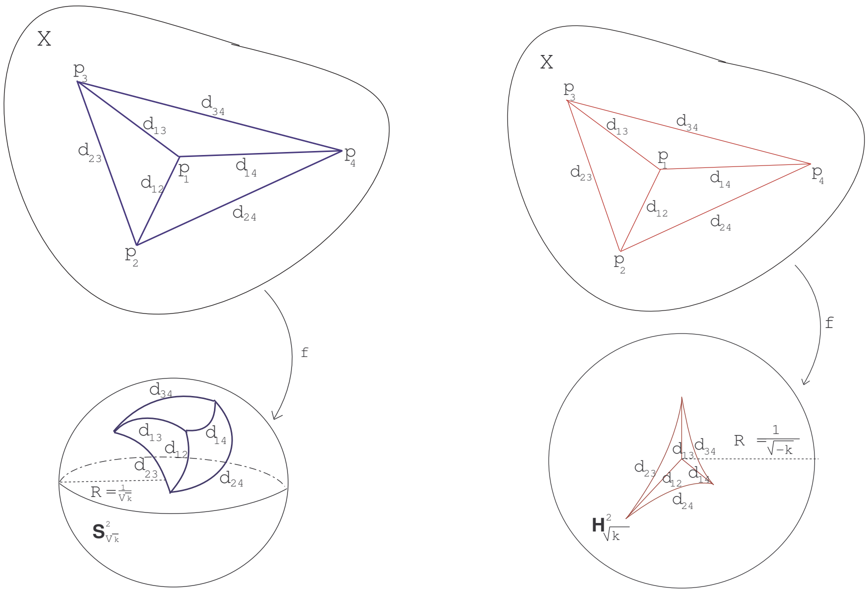

FIGURE 3. Isometric embedding of a metric quadruple in a gauge surface: $\mathbb{S}^2_{\sqrt{\kappa}}$ (left) and $\mathbb{H}^2_{\sqrt{\kappa}}$ (right) .

admits no embedding curvature.

(2) Any linear quadruple is embeddable, apart from the Euclidean plane, in all hyperbolic planes (i.e. of any strictly negative curvature), as well as in infinitely many spheres (whose radii are sufficiently large for the quadruple to be realized upon them).

(3) Moreover, even if a quadruple has an embedding curvature, it still may be not unique (even if Q is not linear); as it is illustrated by the following examples:

Example 3.7. (a) For each $\kappa > 0$, each neighbourhood of any point $p \in S_\kappa$ contains a non-degenerate quadruple that is also isometrically embeddable in $\mathbb{R}^2$. (For the proof see [13], pp. 372-373).

(b) The quadruple Q of distances $d_{13} = d_{14} = d_{23} = d_{24} = \pi$, $d_{12} = d_{34} = 3\pi/2$ admits exactly two embedding curvatures: $\kappa_1 = \frac{1}{2}$ and $\kappa_2 \in \left(\frac{1}{4}, \frac{4}{9}\right)$. (See [14].)

We are now able to define the *Wald curvature* [115],[116] (or, more precisely, its modification due to Berestovskii [10]):

Definition 3.8. Let (X, d) be a metric space. An open set $U \subset X$ is called a *region of curvature* $\geq \kappa$ iff any metric quadruple can be isometrically embedded in S_m, for some $m \geq k$. A metric space (X, d) is said to have *Wald-Berestovskii curvature* $\geq \kappa$ iff any $x \in X$ is contained in a region $U = U(x)$ of curvature $\geq \kappa$.

The embedding curvature at a point is now definable most naturally as a limit. However, we need first yet another preparatory definition:

Definition 3.9. (M, d) be a metric space, let $p \in M$ and let N be a neighbourhood of p. Then N is called *linear* iff N is contained in a geodesic curve.

Definition 3.10. Let (M, d) be a metric space, and let $p \in M$ be an accumulation point. Then M has (*embedding*) *Wald curvature* $\kappa_W(p)$ at the point p iff

(1) Every neighbourhood of p is non-linear;
(2) For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $Q = \{p_1, ..., p_4\} \subset M$ and if $d(p, p_i) < \delta\,, i = 1, ..., 4$; then $|\kappa(Q) - \kappa_W(p)| < \varepsilon$.

Fortunately, for "nice" metric spaces – i.e. spaces that are locally sufficiently "plane like" – the embedding curvature exists and it is unique (see, e.g, [13] and, for a briefer but more easily accessible presentation, [90]). Moreover – and this represents a fact that is very important for some of our own goals, as detailed further on (see Section 6.1) – this embedding curvature coincides with the classical Gaussian curvature. Indeed, one has the following result due to Wald:

Theorem 3.11 (Wald [116])**.** *Let $S \subset \mathbb{R}^3$, $S \in \mathcal{C}^m$, $m \geq 2$ be a smooth surface. Then, given $p \in S$, $\kappa_W(p)$ exists and $\kappa_W(p) = K(p)$, where $K(p)$ denotes the Gaussian curvature at p.*

Remark 3.12. In the theorem above the metric considered in the computation of Wald curvature is the intrinsic one of the surface. (Indeed, the reciprocal Theorem 3.13 below is formulated, at least *prima facie*, for a much more general class of metric spaces than mere smooth surfaces embedded in Euclidean 3-space.) However, in applications Euclidean (extrinsic) distances are used instead. However, this does not represent a theoretical obstruction (only, perhaps, a practical one) – see, for instance, [94], Section 4.3 and the references therein.[8]

Moreover, Wald also has shown that the following partial reciprocal theorem also holds:

Theorem 3.13. Let M be a compact and convex metric space. If $\kappa_W(p)$ exists, for all $p \in M$, then M is a smooth surface and $\kappa_W(p) = K(p)$, for all $p \in M$.

Remark 3.14. Obviously, here the metric considered is the abstract one of the given metric space, that is proven to coincide with the intrinsic one of a smooth surface.

The results above, in conjunction, show that Wald curvature represents, indeed, a proper metrization of the classical (smooth) notion, and not just a mathematical "divertissement", lacking any significant geometric content.

We continue with a definition whose full significance will become more clear in the sequel, where it will be viewed in the correct perspective.

Definition 3.15. A metric quadruple $Q = Q(p_1, p_2, p_3, p_4)$, of distances $d_{ij} = dist(p_i, p_j)$, $i = 1, ..., 4$, is called *semi-dependent* (or a *sd-quad*, for brevity), there exist 3 indices, e.g. 1,2,3, such that: $d_{12} + d_{23} = d_{13}$.

Remark 3.16. The condition in the definition above implies, in fact, that the three points in question lie on a common *metric segment* i.e. a subset of a given metric space that is isometric to a segment in $\mathbb{R}$ (see [13], p. 246).

Perhaps the main advantages of sd-quads stems from in the following fact:

Proposition 3.17. *An sd-quad admits at most one embedding curvature.*

In fact, there also exists a classification criterion – due to Berestovskii [10], see also [82], Theorem 18 – for embedding curvature possibilities in the general case:

Theorem 3.18. *Let M, Q be as above. Then one and only one of the following assertion holds:*

(1) *Q is linear.*

[8] The literature on the subject being too vast to even begin and enumerate it here.

(2) *Q has exactly one embedding curvature.*

(3) *Q can be isometrically embedded in some $\mathcal{S}^m_\kappa$, $m \geq 2$; where $\kappa \in [\kappa_1, \kappa_2]$ or $(-\infty, \kappa_0]$, where $\mathcal{S}^m_\kappa \equiv \mathbb{R}^m$, if $\kappa = 0$; $\mathcal{S}^m_\kappa \equiv \mathbb{S}^m_{\sqrt{\kappa}}$, if $\kappa > 0$; and $\mathcal{S}^m_\kappa \equiv \mathbb{H}^m_{\sqrt{-\kappa}}$, if $\kappa < 0$. Moreover, $\kappa \in \{\kappa_0, \kappa_1, \kappa_2\}$. represent the only possible values of* planar *embedding curvatures, i.e. such that $m = 2$. (Here $\mathbb{S}^m_{\sqrt{\kappa}}$ denotes the m-dimensional sphere of radius $R = 1/\sqrt{\kappa}$, and $\mathbb{H}^m_{\sqrt{-\kappa}}$ stands for the m-dimensional hyperbolic space of curvature $\sqrt{-\kappa}$, as represented by the Poincaré model of the ball of radius $R = 1/\sqrt{-\kappa}$).*

(4) *There exist no m and k such that Q can be isometrically embedded in $\mathcal{S}^m_\kappa$.*

3.1.1. *A Local-to-Global Result.* Before passing to the actual computation of Wald curvature, we include here a result who's full importance and relevance will become much clearer later on. More precisely, we bring the fitting version of the Toponogov (or Alexandrov-Toponogov) Comparison Theorem:

Theorem 3.19 (Toponogov's Comparison Theorem for Wald Curvature)**.** *Let (X, d) be an inner metric space of curvature $\geq k$. Then the entire X is a region of Wald curvature $\geq k$.*

Since the proof is somewhat lengthy and technical we do not bring it here – see [78] (see also [82]).

3.1.2. *Computation of Wald Curvature I: The Exact Formula.* A non-negligible part of the attractiveness of Wald curvature does not reside in its simplicity and intuitiveness, but also that it comes endowed, so to say, with a simple formula for its actual computation. (This is in stark contrast with the *Alexandrov* (*comparison*) *curvature* at least in its usual presentation – but we shall elaborate later on on this subject.) More precisely, we have the following formula:

(3.1)
$$\kappa(Q) = \begin{cases} 0 & \text{if } D(Q) = 0\,; \\ \kappa,\ \kappa < 0 & \text{if } det(\cosh\sqrt{-\kappa}\cdot d_{ij}) = 0\,; \\ \kappa,\ \kappa > 0 & \text{if } det(\cos\sqrt{\kappa}\cdot d_{ij}) \text{ and } \sqrt{\kappa}\cdot d_{ij} \leq \pi \\ & \quad \text{and all the principal minors of order 3 are} \geq 0; \end{cases}$$

where $d_{ij} = d(x_i, x_j), 1 \leq i, j \leq 4$, $(\cosh\sqrt{-\kappa} \cdot d_{ij})$ is a shorthand for $(\cosh\sqrt{-\kappa} \cdot d_{ij})_{1 \leq i,j \leq n}$, etc., and $D(Q)$ denotes the so called *Cayley-Menger determinant*:

$$D(x_1, x_2, x_3, x_4) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{12}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{13}^2 & d_{23}^2 & 0 & d_{34}^2 \\ 1 & d_{14}^2 & d_{24}^2 & d_{34}^2 & 0 \end{vmatrix} . \tag{3.2}$$

There is, in fact, nothing mysterious about the formula above. Indeed, it has a very simple geometric meaning ensuing from the following fact:

$$D(p_1, p_2, p_3, p_4) = 8\big(Vol(p_1, p_2, p_3, p_4)\big)^2 , \tag{3.3}$$

where $Vol(p_1, p_2, p_3, p_4)$ denotes the (un-oriented) volume of the parallelepiped determined by the vertices $p_1, ..., p_4$ (and with edges $\overrightarrow{p_1p_2}, \overrightarrow{p_1p_3}, \overrightarrow{p_1p_4}$).[9] From here immediately follows that

Proposition 3.20. *The points $p_1, ..., p_4$ are the vertices of a non-degenerate simplex in $\mathbb{R}^3$ iff $D(p_1, p_2, p_3, p_4) \neq 0$:*

Clearly, this also implies the opposite assertion, namely that a simplex of vertices $p_1, ..., p_4$ is degenerate, i.e. isometrically embeddable in the plane $\mathbb{R}^2 \equiv S_0$.

It is now relatively easy to guess that the expressions appearing in Formula (3.2) for the cases where $\kappa \neq 0$ represent the equivalents of $D(Q)$ in the hyperbolic, respective spherical cases, using the well known fact, that, in the spherical (resp. hyperbolic) metric, the distances d_{ij} are replaced by $\cos d_{ij}$ (resp. $\cosh d_{ij}$). However, the proof of this fact, as well for the analogous formulas and results in higher dimension diverge from the boundaries of this restricted exposition, therefore we refer the reader to [13].

Remark 3.21. A stronger result along these lines also exists. Moreover, it is readily generalized to any dimension. For proofs and further details, see [13].

[9]As a historical note, it is perhaps worthwhile to recall that Formula 3.3 above was proved by Cayley in his very first mathematical paper [22] (published while he was still begrudgingly making his living as a lawyer!...)

3.1.3. *Computation of Wald Curvature II: An Approximation.* Unfortunately, using Formula (3.1) for the actual computation of $\kappa(Q)$ is anything but simple, since the equations involved are – apart from the Euclidean case – transcendental, therefore not solvable, in general, using elementary methods. Moreover, they tend to display a numerical instability when solved with computer assisted methods. (See [89], [98] for a more detailed comments and some numerical results.)

Note that Formula (3.1) implies that, in practice, a renormalization might be necessary for some of the vertices of positive Wald-Besetkovskii curvature, which represents yet another impediment in it use.

Fortunate enough, there exists a good approximation result, due to Robinson. Not only does his result give a rational formula for approximating $\kappa(Q)$ and provide good error estimates, it also solves one other problem inherent in the use of the Wald curvature, namely the possible lack of uniqueness of the computed curvature. The way to circumvent this difficulty and the other pitfalls of Formula (3.1) is to make appeal to the simpler geometric configuration of sd-quads:

Theorem 3.22 ([85])**.** Given the metric semi-dependent quadruple $Q = Q(p_1, p_2, p_3, p_4)$, of distances $d_{ij} = d(p_i, p_j),\ i, j = 1, ..., 4$; the embedding curvature $\kappa(Q)$ admits a rational approximation given by:

$$K(Q) = \frac{6(\cos\measuredangle_0 2 + \cos\measuredangle_0 2')}{d_{24}\big(d_{12}\sin^2(\measuredangle_0 2) + d_{23}\sin^2(\measuredangle_0 2')\big)} \tag{3.4}$$

where: $\measuredangle_0 2 = \measuredangle(p_1p_2p_4)\,,\ \measuredangle_0 2' = \measuredangle(p_3p_2p_4)$ represent the angles of the Euclidian triangles of sides d_{12}, d_{14}, d_{24} and d_{23}, d_{24}, d_{34} , respectively (see also Figure 4). Moreover the *absolute error* R satisfies the following inequality:

$$|R| = |R(Q)| = |\kappa(Q) - K(Q)| < 4\kappa^2(Q)\mathrm{diam}^2(Q)/\lambda(Q)\,, \tag{3.5}$$

where $\lambda(Q) = d_{24}(d_{12}\sin\measuredangle_0 2 + d_{23}\sin\measuredangle_0 2')/S^2$, and where $S = Max\{p, p'\}$; $2p = d_{12} + d_{14} + d_{24}\ ,\ 2p' = d_{32} + d_{34} + d_{24}$.

We do not bring here the proof of Robinson' result – the interested reader can see [89] and in [98] (and, of course, to Robinson's original paper [85]). However, we would like to underline that basic idea of the proof is to basically calque, in a general metric setting, the original way of defining Gaussian curvature – in this case, rather than accounting for the area distortion, one

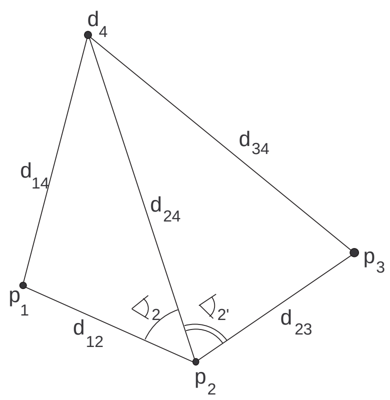

FIGURE 4. The Euclidean triangles corresponding to an sd-quad.

measures the curvature by the amount of "bending" one has to apply to a general planar quadruple so that it may be "straightened" to a triangle $\triangle(p_1p_3p_4)$, with p_2 lying on the edge p_1p_3 – i.e. isometrically embedded as a *sd-quad* – in some S_κ.

Remark 3.23. In some special cases (e.g. when $d_{12} = d_{32}$, etc.) simpler formulas are obtained instead of (3.4) – see, again, [85], or [89], [98]

Naturally, there raises the question whether Formula (3.4) (or any of its variations mentioned above) is truly efficient in applications. The following example, due also to Robinson, indicates that, at least in some cases, the actual computed error is far smaller then the theoretical one provided by Formula (3.5).

Example 3.24 ([85]). Let Q_0 be the quadruple of distances $d_{12} = d_{23} = d_{24} = 0.15, d_{14} = d_{34}$ and of embedding curvature $\kappa = \kappa(Q_0) = 1$. Then $\kappa S^2 < 1/16$ and $K(Q_0) \approx 1.0030280$, whereas the error computed using formula (3.5) is $|R| < 0.45$.

For some experimental results and comparison to other metric curvatures for images, see [98]. However, we should emphasize that the results therein do comply to the expectations arising from the following (quite expected, but nevertheless necessary) theorem:

Theorem 3.25. *Let S be a smooth (differentiable) surface. Then, for any point $p \in S$:*

$$K_G(p) = \lim_{n\to 0} K(Q_n)\,;$$

for any sequence $\{Q_n\}$ of sd-quads that satisfy the following conditions:

$$Q_n \to Q = \Box p_1pp_3p_4\,;\ \mathrm{diam}(Q_n) \to 0\,.$$

Sketch of Proof Recall that the Gaussian curvature $K_G(p)$ at a point p is given by:

$$K_G(p) = \lim_{n\to 0} \kappa(Q_n)\,;$$

where $Q_n \to Q = \Box p_1pp_3p_4\,;\ \mathrm{diam}(Q_n) \to 0$. But, if Q is any sd-quad, then $\kappa^2(Q)\mathrm{diam}^2(Q)/\lambda(Q) < \infty$. Moreover, $|R|$ is small if Q is not close to linearity. In this case $|R(Q)| \sim \mathrm{diam}^2(Q)$, for any given Q (see [85]). The theorem now follows easily.

$\Box$

Remark 3.26. The convergence result provided in Theorem 3.25 is not just in the sense of measures and errors of different signs do not simply cancel each other. Indeed, $\mathrm{sign}(\kappa(Q)) = \mathrm{sign}(K(Q))$, for any metric quadruple Q.

Wald Curvature and Isometric Embeddings. Proposition 3.20 and Remark 3.21 rise the general problem of the existence of isometric embeddings of generic metric metric spaces into gauge spaces. While in its full generality this is, of course, an unattainable goal, one would still be interested in the much more restricted, but important in the applied setting (Graphics, Imaging, Mathematical Modeling, Networking etc.), problem of isometric embedding of PL surfaces in $\mathbb{R}^3$.

A partial result in this direction is a criterion for the local isometric embedding of polyhedral surfaces in $\mathbb{R}^3$, resemblant to the classical Gauss fundamental (compatibility) equation in the classical differential geometry of surfaces, that we proved in [93]. However, to be able to formulate it we need first some additional notations and results:

First, let us note that, in the context of polyhedral surfaces, the natural choice for the set U required in Definition 3.8 is the *star* of a given vertex v, that is, the set $\{e_{vj}\}_j$ of edges incident to v. Therefore, for such surfaces, the set of metric quadruples containing the vertex v is finite.

Equipped with this quite simple and intuitive choice for U (and in in analogy with Alexandrov spaces – see also Section 5.1 below) it is quite natural to consider, for PL surfaces, the following definition of the *discrete* (PL, or "finite scale") Wald curvature $K_W(v)$ at the vertex v:

$$K_W(v) = \min_{v_i,v_j,v_k \in \mathrm{Lk}(v)} K_W^{ijk}(v)\,, \tag{3.6}$$

where $K_W^{ijk}(v) = \kappa(v; v_i, v_j, v_k)$, and where $\mathrm{Lk}(v)$ denotes the *link* of the vertex v.[10] Note that here we consider the (intrinsic) PL distance between vertices.

Let $Q = \{x_1, x_2, x_3, x_4\}$ be a *metric quadruple* and let $V_\kappa(x_i)$ be defined as follows:

$$V_\kappa(x_i) = \alpha_\kappa(x_i; x_j, x_l) + \alpha_\kappa(x_i; x_j, x_m) + \alpha_\kappa(x_i; x_l, x_m) \tag{3.7}$$

where $x_i, x_j, x_l, x_m \in Q$ are distinct, and κ is any number, and where the angles $\alpha_i\,, i = 1, 2, 3$ are as in Figure 5.

Proposition 3.27 ([82], Theorem 23)**.** *Let (X, d) be a metric space and let $U \in X$ be an open set. U is a region of curvature $\geq \kappa$ iff $V_\kappa(x) \leq 2\pi$, for any metric quadruple $\{x, y, z, t\} \subset U$.*

We can now state the desired result for local isometric embedding of polyhedral surfaces in $\mathbb{R}^3$: Given a vertex v, with metric curvature $K_W(v)$, the following system of inequalities should hold:

$$\begin{cases} \max A_0(v) \leq 2\pi; \\ \alpha_0(v; v_j, v_l) \leq \alpha_0(v; v_j, v_p) + \alpha_0(v; v_l, v_p), \quad \text{for all } v_j, v_l, v_p \sim v; \\ V_\kappa(v) \leq 2\pi; \end{cases} \tag{3.8}$$

Here

$$A_0 = \max_i V_0\,; \tag{3.9}$$

[10]Recall that the *link* $\mathrm{lk}(v)$ of a vertex v is the set of all the faces of $\overline{\mathrm{St}}(v)$ that are not incident to v. Here $\overline{\mathrm{St}}(v)$ denotes the *closed star* of v, i.e. the smallest subcomplex (of the given simplicial complex K) that contains $\mathrm{St}(v)$, namely $\overline{\mathrm{St}}(v) = \{\sigma \in \mathrm{St}(v)\} \cup \{\theta \mid \theta \leqslant \sigma\}$, where $\mathrm{St}(v)$ denotes the *star* of v, that is the set of all simplices that have v as a face, i.e $\mathrm{St}(v) = \{\sigma \in K \mid v \leqslant \sigma\}$.

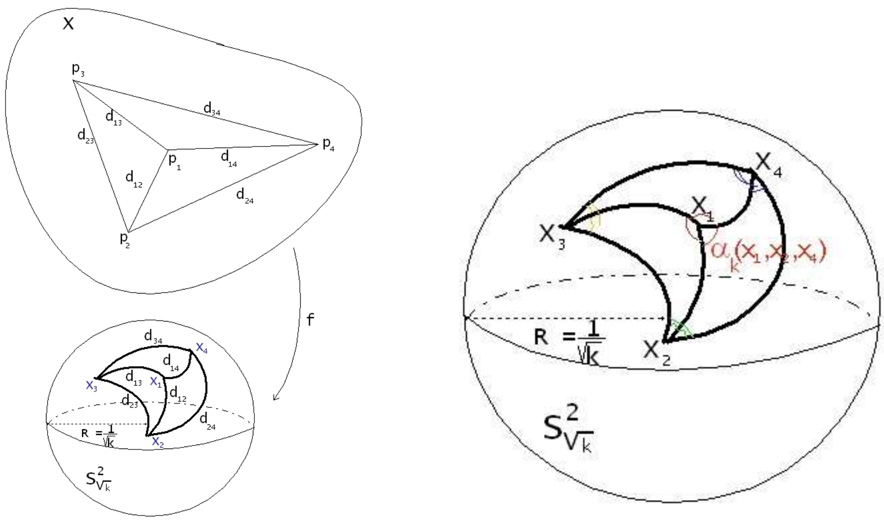

FIGURE 5. The angles $\alpha_\kappa(x_i, x_j, x_l)$ (right), induced by the isometric embedding of a metric quadruple in $\mathbb{S}^2_{\sqrt{\kappa}}$ (left).

"$\sim$" denotes incidence, i.e. the existence of a connecting edge $e_i = vv_j$ and, of course, $V_\kappa(v) = \alpha_\kappa(v; v_j, v_l) + \alpha_\kappa(v; v_j, v_p) + \alpha_\kappa(v; v_l, v_p)$, where $v_j, v_l, v_p \sim v$, etc.

Returning to the analogy with the Gauss compatibility equation, the first two inequalities represent the (extrinsic) embedding condition, while the third one represents the intrinsic curvature (of the PL manifold) at the vertex v.

Also, for details and a corresponding global embedding criterion see [93].

Remark 3.28. Before passing to more general issues, let us mention here that, precisely as a Menger measure was introduced, one can also consider (and, in fact, much more naturally) a *Wald measure* (for surfaces);

$$\mu_W(v) = K_W(v) \cdot \mathrm{Area}(St(v)), \tag{3.10}$$

where $St(v)$ denotes the star of the vertex v.

We defer the investigation for future study of its usefulness in practice.

4. WALD CURVATURE UNDER GROMOV-HAUSDORFF CONVERGENCE

It is practically impossible, both from a purely mathematical viewpoint, as well as considering the background of this volume as a whole (see mainly

Memoli's contribution) to introduce any notion of curvature without discussing its behaviour under the Gromov-Hausdorff convergence. We begin with a much more general discussion and continue with some results regarding Wald curvature and Gromov-Hausdorff convergence.

4.1. **Intrinsic Properties and Gromov-Hausdorff Convergence.** Given that many of Graphics and Imaging tasks (and Sampling Theory, as well), reduce, in the end, to better and better approximation by certain nets (or graphs), be they triangular meshes in the first case, or square grids, in the second, it is most natural (and, indeed, necessary) to have a comprehensive, and sound approach to investigating the behaviour and convergence, under limits, of the relevant properties.[11] We overview here some significant results regarding convergence of nets in metric spaces (basically due, seemingly, to Gromov).

We begin by reminding the reader the following basic definition:

Definition 4.1. Let (X, d) be a metric space. A set $\{p_1, \ldots, p_m\} \subset X$ is called an ε-*net* on(in) X iff the balls $B(p_k, \varepsilon)$, $k = 1, \ldots, m$ cover X.

It turns out that ε-nets in compact metric spaces have the following important property:

Proposition 4.2. *Let* $X, \{X_n\}_{n=1}^{\infty}$ *be compact metric spaces. Then* $X_n \underset{GH}{\longrightarrow} X$ *iff for all* $\varepsilon > 0$*, there exist* finite ε-*nets* $S \subset X$ *and* $S_n \subset X_n$*, such that* $S_n \underset{GH}{\longrightarrow} S$ *and, moreover,* $|S_n| = |S|$*, for large enough* n*.*

The importance of the result above does not reside only in the fact that compact metric spaces can be approximated by finite ε-nets – after all, just the existence of *some* approximation by such sets is hardly surprising – but rather in the fact that, as we shall shortly see, it assures the convergence of geometric properties of S_n to those of S, as $X_n \underset{GH}{\longrightarrow} X$. This would be, in a nutshell, the real significance of the proposition above.

One can also reformulate Proposition 4.2 in an equivalent, in a less concise and elegant manner yet, on the other hand, far more useful in concrete instances (to say nothing of the fact that it is far more familiar in the Applied Mathematics community):

[11] It was, it would appear, Gromov's observation that, in the geometric setting, the relevant convergence is the Gromov-Hausdorff one.

Proposition 4.3. *Let X, Y be compact metric spaces. Then:*
(a) If Y is a (ε, δ)-approximation of X, then $d_{GH}(X, Y) < 2\varepsilon + \delta$.
(b) If $d_{GH}(X, Y) < \varepsilon$, then Y is a 5ε-approximation of X.

Recall that ε-δ-*approximations* are defined as follows:

Definition 4.4. Let X, Y be compact metric spaces, and let $\varepsilon, \delta > 0$. X, Y are called ε-δ-*approximations* (of each-other) iff: there exist sequences $\{x_i\}_{i=1}^N \subset X$ and $\{y_i\}_{i=1}^N \subset Y$ such that
(a) $\{x_i\}_{i=1}^N$ is an ε-net in X and $\{y_i\}_{i=1}^N$ is an ε-net in Y;
(b) $|\,d_X(x_i, x_j) - d_(y_i, y_j)\,| < \delta$ for all $i, j \in \{1, ..., N\}$.
An $(\varepsilon, \varepsilon)$-approximation is called, for short an ε-*approximation*.

Recall that a metric spaces whose metric d is *intrinsic*, i.e.induced by a *length structure* (i.e. path length) by the ambient metric on a subset of a given metric space is called a *length space*. Such spaces are, for obvious reasons, of special interest in Geometry. (As a basic motivation both theoretical and practical, for considering such spaces, would be that of surfaces in $\mathbb{R}^3$.) The following theorem shows that length spaces are closed in the *Gromov-Hausdorff topology*:

Theorem 4.5. *Let $\{X_n\}_{n=1}^{\infty}$ be length spaces and let X be a complete metric space such that $X_n \underset{GH}{\longrightarrow} X$. Then X is a length space.*

Using ε-*approximations* one can prove the following theorem and corollary, that are quite important, not only for the specific purpose of this overview, but in a far more general and powerful context (see e.g. [32] and [19]):

Theorem 4.6 (Gromov)**.** *Any compact length space is the GH-limit of a sequence of finite graphs.*

The proof of the theorem above is constructive, therefore potentially adaptable in practical applications (such as those arrising in Graphics, Imaging and related fields). For this very reason, and for essential simplicity we bring it below:

Proof. Let ε, δ ($\delta \ll \varepsilon$) small enough, and let S be a δ-net in X. Also, let $G = (V, E)$ be the graph with $V = S$ and $E = \{(x, y)\,|\,d(x, y) < \varepsilon\}$. We shall prove that G is an ε-approximation of X, for δ small enough, more precisely, for $\delta < \frac{\varepsilon^2}{4\mathrm{diam}(X)}$.

But, since S is an ε-net both in X and in G, and since $d_G(x,y) \geq d_X(x,y)$, it is sufficient to prove that:

$$d_G(x,y) \leq d_X(x,y) + \varepsilon\,.$$

Let γ be the shortest path between x and y, and let $x_1, ..., x_n \in \gamma$, such that $n \leq \mathrm{length}(\gamma)/\varepsilon$ (and $d_X(x_i, x_{i+1}) \leq \varepsilon/2$). Since for any x_i there exists $y_i \in S$, such that $d_X(x_i, y_i) \leq \delta$, it follows that $d_X(y_i, y_{i+1}) \leq d_X(x_i, x_{i+1}) + 2\delta < \varepsilon$.

Therefore, (for $\delta < \varepsilon/4$), there exists an edge $e \in G, e = y_i y_{i+1}$. From this we get the following upper bound for $d_G(x,y)$:

$$d_G(x,y) \leq \sum_{n=0}^{n} d_X(y_i, y_{i+1}) \leq \sum_{n=0}^{n} d_X(x_i, x_{i+1}) + 2\delta n$$

But $n < 2\,\mathrm{length}(\gamma)/\varepsilon \leq 2\mathrm{diam}(X)/\varepsilon$. Moreover: $\delta < \varepsilon^2/4\,\mathrm{diam}(X)$. It follows that:

$$d_G(x,y) \leq d_X(x,y) + \delta\frac{4\mathrm{diam}(X)}{\varepsilon} < d_X(x,y) + \varepsilon\,.$$

Thus, for any $\varepsilon > 0$, there exists a graph an ε-approximation of X by a graph G, $G = G_\varepsilon$. Hence $G_\varepsilon \underset{\varepsilon}{\longrightarrow} X$. $\square$

In fact, one can strengthen the theorem above as follows:

Corollary 4.7. *Let X be a compact length space. Then X is the Gromov-Hausdorff limit of a sequence $\{G_n\}_{n\geq 1}$ of finite graphs, isometrically embedded in X.*

Remark 4.8. A certain amount of care is needed when applying the theorem above, as the following facts show:

(1) If $G_n \underset{\varepsilon}{\to} X$, $G_n = (V_n, E_n)$. If there exists $N_0 \in \mathbb{N}$ such that

$$(*) \quad |E_n| \leq N_0, \text{ for all } n \in \mathbb{N}\,,$$

then X is a finite graph.

(2) If condition $(*)$ is replaced by:

$$(**) \quad |V_n| \leq N_0, \text{ for all } n \in \mathbb{N}\,,$$

then X will still be always a graph, but not necessarily finite.

Remark 4.9. Theorem 4.6 can be strengthened as follows: Compact inner metric spaces can be, in fact, Gromov-Hausdorff approximated by smooth surfaces that, moreover, are embedded in $\mathbb{R}^3$, as shown by Cassorla [21] (see also [32], p. 99 and the reference therein). In other words, one can "visualize" in $\mathbb{R}^3$ (up to some predetermined but arbitrarily small error) *any* compact inner metric space. Unfortunately, the genus of the approximating surfaces can not remain bounded. (In consequence, in order that a good approximation even of a simple space be obtained, using the method given in Cassorla's proof, one has to increase the topological complexity of the approximating surface.)

Note also that there is no geometric (curvature) restriction on the approximating surfaces. In fact, it is also stated in [21] that one can approximate the given spaces with a series of smooth surfaces having Gaussian curvature bounded from above by -1 (this being, however a seemingly unpublish result). Unfortunately, to obtain this, one has to abandon the embeddability in $\mathbb{R}^3$ of the approximating surfaces.

We conclude this remark by adding a few words regarding Cassorla's proof: He begins by constructing an approximation by graphs, following Gromov, then he considers the (smooth) boundaries of canonical tube neighborhoods or, in other words, he builds the smooth surfaces having as axes (or nerve) the graph constructed previously.

4.2. **Wald Curvature and Gromov-Hausdorff Convergence.** It turns out (not very surprisingly, in fact, in view of the facts that we shall present in the next section) that it is somewhat naive to hope for a generic result for Wald curvature as such. It turns out that the upper and lower bound for K_W display quite different behaviours. There are very few results we can state, therefore, in the generic case. The basic one is

Lemma 4.10. *Let (X_i, d_i) be compact metric spaces, such that $X_i \underset{GH}{\longrightarrow} X$. If $B_i \subset X_i$, $B_i = B(p_i, r)$ is a region of curvature $\geq k$, for all $i \geq 1$, and if $p_i \underset{GH}{\longrightarrow} p \in X$, then $B = B(p, r)$ is a region of curvature $\geq k$ in X.*

(Recall that by Theorem 4.5 X is also an inner metric.)

In view of Toponogov's Theorem we can now formulate:

Theorem 4.11. *Let (X_i, d_i) be (compact) metric spaces, such that $X_i \underset{GH}{\longrightarrow} X$. If X_i has curvature $\geq k$, for all $i \geq 1$, then X has curvature $\geq k$.*

Remark 4.12. We should note in his context that the diameter function is continuous under the the Gromov-Hausdorff convergence.

4.2.1. *The case $K_W \geq K_0$.* This is the case where a plethora of powerful results exist, mainly due to Plaut [78], [79], [81].

The first such result represents a generalization of the classical by now compactness theorem of Gromov (see, e.g. [32]). For its formulation we need an additional notation: We denote by $\mathcal{M}(k, n, D)$ the class of all finitely dimensional[12] spaces of curvature $\geq k$, dimension $\leq n$ and diameter $\leq D$. We can now formulate the theorem in question:

Theorem 4.13 (Plaut [82])**.** *$\mathcal{M}(k, n, D)$ is compact in the Gromov-Hausdorff metric.*

Remark 4.14. Let us denote by $\mathfrak{M}(k, n, D)$ the set of all Riemannian manifolds satisfying the same conditions as the spaces in $\mathcal{M}(k, n, D)$ (k denoting, in this case, sectional curvature). Then (clearly) $\mathfrak{M}(k, n, D) \subsetneq \mathcal{M}(k, n, D)$.

The results above have quite important consequences for a variety of practical fields (or, at least, for their more theoretical, basic aspects): Since by the now classical Gromov Precompactness Theorem [32], any element in a compact (hence a fortiori precompact) collection of compact metric spaces admits, for any $\varepsilon > 0$, an *ε-net* with at most $N(\varepsilon)$ number of elements, they represent quite general sampling theorems, giving, moreover, a strong upper bound on the number of sampling points – a number that depends, apart from the class on the manifold, only on the quality of the sampling (as given by ε). (The price to be paid, so to say, for the strengths above, is represented by the non-algorithmic nature of these results.)

[12]The dimension can be taken as the topological dimension or the Hausdorff dimension – see, e.g. [82].

In fact, even for a larger class of metric spaces, namely $\mathcal{M}(k,n,\varepsilon)$, where $\varepsilon > 0$ denotes a lower bound for the *injectivity radius*[13] some important facts can be asserted, such as the following theorem:

Theorem 4.15 (Plaut [79])**.** *The elements of $\mathcal{M}(k,n,\varepsilon)$ are smoothable, for any n.*

Again, the theorem above is also highly relevant to Sampling Theory, since it shows that the traditional approach in the field, that is of considering smoothings ("filtrations") of the given signals/images and sampling them according to a more traditional, Gauss curvature based scheme is theoretically valid.

Moreover, the following compactness result also holds:

Theorem 4.16 (Plaut [79])**.** *$\mathcal{M}^*(k,n,D,\varepsilon)$ is compact in the Gromov-Hausdorff metric, where $\mathcal{M}^*(k,n,D,\varepsilon)$ denotes the class of spaces of dimension equal to n, curvature $\geq k$, diameter $\leq D$ and injectivity radius $\geq \varepsilon$.*

In addition to these compactness results, the following finiteness theorems, representing generalizations of classical results of Cheeger [23], respectively Grove-Petersen [36] and Grove-Petersen-Wu [37] also hold :

Theorem 4.17 (Perelman [76])**.** *The class $\mathcal{M}^*(k,n,D)$ has finitely many homeomorphism types.*

(We believe that by now the notation must be clear to the reader.)

Theorem 4.18 (Perelman [76])**.** *$\mathfrak{M}(k,n,D,v)$, where $v > 0$ denotes a lower bound on volume, has finitely many types of homeomorphism for all n, and diffeomorphism, for all $n \neq 4$.*

Remark 4.19. At first glance Theorems 4.17 and 4.18 seemingly are contradicted by the existence of infinitely many homotopy types of lens spaces. However, this is not the case, since they fail to satisfy the conditions of the theorem even as Riemannian manifolds, thus, *a fortiori* as Alexandrov

[13]Without getting into the technical subtleties of the definition of the space of directions S_p at a point p in a space of bounded curvature, the injectivity radius at p is defined as $\inf_{\gamma \in S_p} \sup_t \{\gamma|_{[0,t]} \text{is minimal}\}$.

spaces (see [36], p. 196). For instance, the lens spaces $\mathbb{S}^{2n+1}/\mathbb{Z}\neg$ have constant curvature equal to 1 and diameter $\pi/2$, but have no lower bound on the volume (and, indeed, they belong to infinitely many homotopy classes. (Up to a scaling of volume, the same spaces show that the upper bound on diameter is also necessary).

These last two theorems, as well as the previous, related results of Grove et al. mentioned above have quite practical importance in the Recognition Problem (in Manifold Learning and related fields), in particular in determining the complete so called "Shape DNA".

More geometrically interesting, powerful results exists, but for lack of space and for the sake of cohesiveness of the text, we do not bring here – see [82] and the bibliography cited therein.

4.2.2. *The case $K_W \leq K_0$.* As already mentioned in the introduction of this section, spaces of Wald curvature bounded from above display a behaviour widely divergent from those satisfying the opposite inequality. Most notable they do not satisfy a fitting Toponogov type theorem, the example of the flat torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$ being the basic one (see [82], p. 886 for details). However, such spaces still have some very interesting geometric properties, see again [82] as well as [2].

However, the most powerful results, due to Berestovskii [9] and Nikolaev [69], [70], [71] (see also [80]) are obtained when combining the lower and upper curvature bounds, the main such theorem being:

Theorem 4.20. *A topological space admits a smooth manifold structure (with or without boundary) iff (i) it is finite dimensional; and (ii) has a metric curvature bounded both from above and from below.*

5. Wald and Alexandrov Curvatures Comparison

We have alluded many times to the notion of Alexander curvature. Moreover, it is quite probably that many (if not most) of the readers are quite familiar with this concept permeating modern Mathematics, certainly much more than with the rather esoteric (for some) Wald curvature. It is therefore only fitting that we finally discuss the relationship between these two types of comparison curvature.

5.1. **Alexandrov Curvature.** We begin by reminding the reader the definition of Alexandrov comparison curvature. Before bringing the formal definition, let us just specify the main difference between this approach and Wald curvature: In defining Alexandrov curvature, one makes appeal to *comparison triangles* in the model space (i.e. gauge surface $\mathcal{S}_k$), rather than quadrangles, as in the definition of Wald curvature.

A space is said to be of *Alexandrov curvature* $\geq k$ iff any of the following equivalent conditions holds *locally*, and to be of *Alexandrov curvature* $\leq k$ iff any of the conditions below holds with the opposite inequality:

Definition 5.1. Let X be an inner metric space, let $T = \triangle(p,q,r)$ be a geodesic triangle, of sides $\overline{pq}, \overline{pr}, \overline{qr}$, and let $\widetilde{T}$ denote its(a) representative triangle in $\mathcal{S}_k$.

A0 Given the triangle $T = \triangle(p,q,r)$ and points $x \in \overline{pq}$ and $y \in \overline{pr}$, there exists a representative triangle $\widetilde{T}$ in $\mathcal{S}_k$. Let $\tilde{x}, \tilde{y}$ represent the corresponding points on the sides of $\widetilde{T}$. Then $d(x,y) \geq d(\tilde{x},\tilde{y})$.

A01 Given the triangles $T = \triangle(p,q,r)$ and $\widetilde{T}$ as above, and a point $x \in \overline{pq}$, $d(x,c) \geq d(\tilde{x},\tilde{c})$.

A1 Given the triangle $T = \triangle(p,q,r)$, there exists a representative triangle $\widetilde{T}$ in $\mathcal{S}_k$, and $\measuredangle(\overline{pq}, \overline{pr}) \geq \measuredangle(\overline{\tilde{p}\tilde{q}}, \overline{\tilde{p}\tilde{r}})$, where $\measuredangle(\overline{pq}, \overline{pr})$ denotes the angle between $\overline{pq}$ and $\overline{pr}$, etc.

A2 For any hinge $H = (\overline{pq}, \overline{pr})$, there exists a representative hinge $\widetilde{H} = (\overline{\tilde{p}\tilde{q}}, \overline{\tilde{p}\tilde{r}})$ in $\mathcal{S}_k$ and, moreover, $d(p,q) \leq d(\tilde{p},\tilde{q})$, where a *hinge* is a pair of minimal geodesics with a common end point.

Remark 5.2. (1) Axiom $A0$ represents the basic one in defining Alexandrov comparison. It is also the one used by Rinow [84], in his seemingly (semi-)independent development of comparison geometry. (For a shorter presentation of his approach, but somewhat more accessible and in English, see [13], [14] and, for an even briefer one, but easy to reach, [90].) This condition represents nothing more the transformation into an axiom of the following essential geometric fact: Thales Theorem does not hold in Spherical and Hyperbolic Geometry. (In particular – and most spectacular – the line connecting the midpoints of two of the edges of a triangle does not equate half of the third one.[14])

(2) Condition $A01$ clearly represents just a particular case of $A0$, by fixing y to be one of the end points of $\overline{pr}$, say $y = r$. However, it is, actually, equivalent to $A0$ and it is, in fact, usually easier to check. It is sometimes used as the basic definition of Alexandrov comparison, saying, for instance that "negatively curved spaces have short ties" (the figure of speech being, we believe, self explanatory) – see, e.g. [114]. The role of *sd*-quads in such fundamental results as Theorem 3.14 becomes now less strange and, in fact, it will become quite clear once the result in the next section is introduced. We anticipate somewhat by adding that now Robinson's method in Theorem 3.18 shows itself as it truly is: A method[15] of approximating the relevant k appearing in Axiom $A01$.

(3) We have used here (for the most part) Plaut's notation in [82]. For an excellent, detailed, clear (and by now already classical) presentation of the various comparison conditions, see [19].

(4) Conditions $A1$ and $A2$ show that one can introduce comparison Geometry via angle comparison. However, we prefer a "purely" metric approach, even if it is somewhat illusory. (See [94] for the application of this approach to a "purely" metric Regge calculus.)

5.2. **Alexandrov Curvature vs. Wald Curvature.** Loosely formulated, the important fact regarding the connection between Wald and Alexandrov curvatures is that *(in the presence of sufficiently many minimal geodesics) Wald curvature is (essentially) equivalent to Alexandrov curvature* or, slightly more precisely, that inner metric spaces with Wald curvature $\geq k$ satisfy the condition of having sufficiently many geodesics. (This fact may be viewed as an extended, weak Hopf-Rinow type theorem.) The formal enouncement of this result requires yet more technical definitions, which we present below for the sake of completeness:

[14]This well known "paradox" of the foundations of Geometry is, unfortunately, generally overlooked in certain applications in Imaging and Graphics, which results in a penalty on the quality of the numerical results.

[15]developed *avant la lettre*

Definition 5.3. Let X be an inner metric space and let $\gamma_{pq} \subset X$ be a minimal geodesic connecting the points p and q. γ is called

(1) *extendable* beyond q if there exists a geodesic $\widetilde{\gamma}$, such that $\gamma = \widetilde{\gamma}|_{(p,q)}$ and $q \in \mathrm{int}\widetilde{\gamma}$.

(2) *almost extendable* beyond q if for any $\varepsilon > 0$, there exists an $r \in X \setminus \{p, q\}$, such that $\sigma(q; p, r) < \varepsilon$, where $\sigma(q; p, r)$ denotes the *strong excess*

$$\sigma(q; p, r) = \frac{e(T)}{\min d(p, r), d(r, q)}, \tag{5.1}$$

where $T = \Delta(q, p, r)$ (and where $e(T)$, stands, as above, for its excess).

We shall also need the following

Definition 5.4. Let X be as above and let $p \in X$. We denote

$$J_p = \{q \in X \mid \exists! \text{ minimal geodesic } \gamma_{pq} \text{ almost extendable beyond } q.\}\,. \tag{5.2}$$

We have the following

Theorem 5.5 (Plaut [82],[81])**.** *Let X be an inner metric space of Wald-Berestovskii curvature $\geq \kappa$. Then, for any $p \in X$, J_p contains a dense G_δ set.*

Since, by the Baire Category Theorem, the intersection of countably many dense G_δ sets is a dense G_δ set, we obtain the following corollary:

Corollary 5.6. *Let X be an inner metric space of Wald-Berestovskii curvature $\geq \kappa$, and let $p_1, p_2, \ldots \in X$. Then there exist points $p'_1, p'_2, \ldots \in X$ such that*

(1) *p_i is arbitrarily close to p_i, for all i;*

(2) *There exists a unique minimal geodesic connecting p_i and p'_i.*

Moreover, one can take $p'_1 = p_1$.

In other words, given any three points p, q, r in X, there exist points p_1, q_1, r_1 arbitrarily close to them (respectively) such that p_1, q_1, r_1 represent the vertices of a triangle whose sides are minimal geodesics or, simply put, one can construct (minimal geodesic) triangles "almost everywhere".

From the corollary above and from the Toponogov Comparison Theorem we obtain the announced theorem that establishes the essential equivalence between Alexandrov and Wald curvature, once the existence of "enough geodesics" is assured:

Theorem 5.7 (Plaut [81],)**.** *Let X be an inner metric space. X is a space of Alexandrov curvature bonded from below iff for every $x \in X$, there exists an open set U, $x \in U$, such that for every $y \in U$ the set J_y contains a dense G_δ set of U.*

(For a different formulation of the results above see [82], Corollary 40.)

In view of the result above, it is easy to recognize finitely dimensional spaces of $Wald$ curvature $\geq k$ in the garb that they are widely known in the modern terminology, namely *Alexandrov spaces.* The reader can, therefore, substitute, in the results in the previous section "Alexandrov space" whenever this is possible – this is the form in which many of the theorems in question are better known.

However, Wald curvature allows us to discard conditions as are usually used when employing the Alexandrov triangle comparison, e.g. local compactness, while still being able to obtain many important theorems, such as the Toponogov Theorem and the Hopf-Rinow Theorem that we have discussed above, as well as fitting variants of the Maximal Radius Theorem and of the Sphere Theorems, that we have only alluded too, unfortunately. For details the reader is invited to consult [81].

6. A Metric Approach to Ricci Curvature

In this section we concentrate on the application of the metric approach, and more precisely of Wald curvature, to the defining Ricci flow for cell complexes and, in dimension 2, to the development of a fitting metric Ricci flow. Since these problems were studied in detail in our papers [38] and [95], we present only the main ideas, and hope that the interested reader will consult the original papers (especially [95], were a more detailed discussion is contained.[16]) In consequence, we follow here the exposition in the much shorter and restricted proceedings paper. However, we emphasize whenever

[16]Also, we warn the eventual reader of an unfortunate previously unnoted typo towards the end of [38].

possible those correlations with the theoretical material included in the previous section. Also, a new approach to the problem of the Ricci flow for PL surfaces, also concordant to the theory in the previous section is also included.

6.1. **Metric Ricci Flow for PL Surfaces.** Motivated mainly by Perelmans work, the Ricci flow has become lately[17] an object of active interest and research in Graphics and Imaging. Previously, various approaches have been suggested, encompassing such methods as classical approximations of smooth differential operators, as well as discrete, combinatorial methods.

Among these diverse approaches, the most successful so far proved to be the one based on the discrete Ricci flow of Chow and Luo [25], due to Gu (see, e.g. [47] and, for more details, [39]). In truth, the paper [95] was largely motivated by our desire to get a better understanding of the discrete, circle-packing based Ricci flow of Chow and Luo, and its relation with the Ricci flow for smooth surfaces introduced by Hamilton [42] and Chow [24].

6.1.1. *A Smoothing based Approach.* Our approach, as developed in [95] to this problem is to pass from the discrete context to the smooth one and explore the already classical results known in this setting, by applying Theorem 3.11. To this end we have first to make a few observations: One can pass from the PL surfaces to smooth ones by employing *smoothings*, defined in the precise sense of PL differential Topology (see [64]). Since, by [64], Theorem 4.8, such smoothings are δ-*approximations*, and therefore, for δ small enough, also α-approximations of the given piecewise-linear surface S^2_{Pol}, they approximate arbitrarily well both distances and angles on S^2_{Pol}. (Due to space restrictions, we do not bring here these technical definitions, but rather refer the reader to [64].) It should be noted that, while Munkres' results concern PL manifolds, they can be applied to polyhedral ones as well, because, by definition, polyhedral manifolds have simplicial subdivisions (and furthermore, such that all vertex links[18] are combinatorial

[17] in what would have been probably consider to be a strange – not to say bizarre – development even only a few years ago

[18] Recall that the *link* $\mathrm{Lk}(v)$ of a vertex v is the set of all the faces of $\overline{\mathrm{St}}(v)$ that are not incident to v. Here $\overline{\mathrm{St}}(v)$ denotes the *closed star* of v, i.e. the smallest subcomplex (of the given simplicial complex K) that contains $\mathrm{St}(v)$, namely $\overline{\mathrm{St}}(v) = \{\sigma \in \mathrm{St}(v)\} \cup \{\theta \mid \theta \leqslant \sigma\}$,

manifolds). Of course, for different subdivisions, one may obtain different polyhedral metrics. However, by the *Hauptvermutung* Theorem in dimension 2 (and, indeed, for smooth triangulations of diffeomorphic manifolds in any dimension) (see e.g. [64] and the references therein), any two subdivisions of the same space will be combinatorially equivalent, hence they will give rise to the same polyhedral metric. It follows from the observations above that *metric quadruples* on S_{Pol} are also arbitrarily well approximated (including their angles) by the corresponding metric quadruples) on the smooth approximating surfaces S_m. But, by Theorem 3.11, $K_{W,m}(p)$ – the *Wald metric curvature* of S_m, at a point p – equals the classical (Gauss) curvature $K(p)$. Hence the Gauss curvature of the smooth surfaces S_m approximates arbitrarily well the metric one of S_{PL} (and, as in [17], the smooth surfaces differ from polyhedral one only on (say) the $\frac{1}{m}$-neighbourhood of the 1-skeleton of S_{Pol} – see also the discussion below). Moreover, this statement can be made even more precise, by assuring that the convergence is in the Hausdorff metric. This follows from results of Gromov (see e.g. [98] for details).

We can now introduce the *metric* Ricci flow: By analogy with the classical flow

$$\frac{dg_{ij}(t)}{dt} = -2K(t)g_{ij}(t)\,. \tag{6.1}$$

we define the *metric* Ricci flow by

$$\frac{dl_{ij}}{dt} = -2K_i l_{ij}\,, \tag{6.2}$$

where $l_{ij} = l_{ij}(t)$ denote the edges (1-simplices) of the triangulation (PL or piecewise flat surface) incident to the vertex $v_i = v_i(t)$, and $K_i = K_i(t)$ denotes the Wald curvature at the same vertex, where, as above, we employ the discrete version of Wald's curvature defined by Formula (3.6).

Remark 6.1. Before continuing further on, it is important to remark the asymmetry in equation 6.2, that is caused by the fact that the curvature on two different vertices acts, so to say, on the same edge. However, passing to the smooth case, is that the asymmetry in the metric flow that we observed above disappears automatically via the smoothing process. To this end it is

where $\mathrm{St}(v)$ denotes the *star* of v, that is the set of all simplices that have v as a face, i.e $\mathrm{St}(v) = \{\sigma \in K \mid v \leqslant \sigma\}$.

important to note that while the formula of $K_W(v)$ involves the edges incident to v, it is – precisely by this incidence criterion – a curvature attached to the vertex v. (For further details see [94] as well as Section 6.1.2. below.)

We also consider the close relative of (6.1), the normalized flow

$$\frac{dg_{ij}(t)}{dt} = (K - K(t))g_{ij}(t)\,, \tag{6.3}$$

and its metric counterpart

$$\frac{dl_{ij}}{dt} = (\bar{K} - K_i)l_{ij}\,, \tag{6.4}$$

where $K, \bar{K}$ denote the average classical, respectively Wald, sectional (Gauss) curvature of the initial surface S_0: $K = \int_{S_0} K(t)dA / \int_{S_0} dA$, and $\bar{K} = \frac{1}{|V|}\sum_{i=1}^{|V|} K_i$, respectively. (Here $|V|$ denotes, as usually, the cardinality of the vertex set of S_{Pol}.)

An Approximation Result. The first result that we can bring is a metric curvature version of classical result of Brehm and Kühnel [17] (where the combinatorial/defect definition of curvature for polyhedral surfaces is used.

Proposition 6.2. *Let S^2_{Pol} be a compact polyhedral surface without boundary. Then there exists a sequence $\{S^2_m\}_{m\in\mathbb{N}}$ of smooth surfaces, (homeomorphic to S^2_{Pol}), such that*

1. (a) *$S^2_m = S^2_{Pol}$ outside the $\frac{1}{m}$-neighbourhood of the 1-skeleton of S^2_{Pol},*
   (b) *The sequence $\{S^2_m\}_{m\in\mathbb{N}}$ converges to S^2_{Pol} in the Hausdorff metric;*
2. *$K(S^2_m) \to K_W(S^2_{Pol})$, where the convergence is in the weak sense.*

Remark 6.3. As we have already noted above, the converse implication – namely that Gaussian curvature $K(\Sigma)$ of a smooth surface Σ may be approximated arbitrarily well by the Wald curvatures $K_W(\Sigma_{Pol,m})$ of a sequence of approximating polyhedral surfaces $\Sigma_{Pol,m}$ – is quite classical.

For a more in-depth discussion and analysis of the convergence rate in the proposition above, see [94].

Remark 6.4. In view of the equivalence of the Alexandrov and Wald curvatures), one can view the result above as a elementary, restricted to dimension

2, but on other hand a more specific and constructive version of Theorem 4.11.

As already stressed, the "good", i.e. metric and curvature, approximations results mentioned above, imply that one can study the properties of the metric Ricci flow via those of its classical counterpart, by passing to a smoothing of the polyhedral surface. The use of the machinery of metric curvature considered has the benefit that, by using it, the "duality" between the combinatorics of the packings (and angles) and the metric disappears: The flow becomes purely metric and, moreover, the curvature at each stage (i.e. for every "t") is given – as in the smooth setting – in an intrinsic manner, that is in terms of the metric alone.

We bring here the most important properties that follow immediately using this approach (for further results and additional details, see [95]).

Existence and Uniqueness. The first result that we should bring here is the following

Proposition 6.5. *Let (S^2_{Pol}, g_{Pol}) be a compact polyhedral 2-manifold without boundary, having bounded discrete Wald curvature. Then there exists $T > 0$ and a smooth family of polyhedral metrics $g(t), t \in [0,T]$, such that*

$$\begin{cases} \frac{\partial g}{\partial t} = -2K_W(t)g(t) \quad t \in [0,T]\,; \\ g(0) = g_{Pol}\,. \end{cases} \tag{6.5}$$

(Here $K_W(t)$ denotes the Wald curvature induced by the metric $g(t)$.)

Moreover, both the forwards and the backwards (when existing) Ricci flows have the uniqueness of solutions property, that is, if $g_1(t), g_2(t)$ are two Ricci flows on S^2_{Pol}, such that there exists $t_0 \in [0,T]$ such that $g_1(t_0) = g_2(t_0)$, then $g_1(t) = g_2(t)$, for all $t \in [0,T]$.

Beyond the theoretical importance, the existence and uniqueness of the backward flow would allow us to find surfaces in the conformal class of a given circle packing (Euclidean or Hyperbolic). More importantly, the use of purely metric, Wald curvature based, approach adopted, rather than the combinatorial (and metric) approach of [25], allows us to give a preliminary and purely theoretical at this point, answer to Question 2, p. 123, of [25], namely whether there exists a Ricci flow defined on the space of all piecewise constant curvature metrics (obtained via the assignment of lengths to a given triangulation of 2-manifold). Since, by the results of Hamilton's [42] and Chow [24], the Ricci flow exists for all compact surfaces, it follows that the fitting metric flow exists for surfaces of piecewise constant curvature. In consequence, given a surface of piecewise constant curvature (e.g. a mesh with edge lengths satisfying the triangle inequality for each triangle), one can evolve it by the Ricci flow, either forward, as in the works discussed above, to obtain, after the suitable area normalization, the polyhedral surface of constant curvature conformally equivalent to it; or backwards (if possible) to find the "primitive" family of surfaces – including the "original" surface obtained via the backwards Ricci flow, at time T – conformally equivalent to the given one.

Convergence Rate. A further type of result, quite important both from the theoretical viewpoint and for computer-driven applications, is that of the *convergence rate* (see [39], [38] for the precise definition).

Since we already know that the solution exists and it is unique (see also the subsection below for the nonformation of singularities), by appealing to the classical results of [42] and [24], we can control the convergence rate of the curvature, as follows:

Theorem 6.6. *Let (S^2_{Pol}, g_{Pol}) be a compact polyhedral 2-manifold without boundary. Then the normalized metric Ricci flow converges to a surface of constant metric curvature. Moreover, the convergence rate is*

(1) *exponential, if $\bar{K} = \bar{K}_W < 0$ (i.e. $\chi(S^2_{Pol}) < 0$) ;*
(2) *uniform; if $\bar{K} = 0$ (i.e. $\chi(S^2_{Pol}) = 0$);*
(3) *exponential, if $\bar{K} > 0$ (i.e. $\chi(S^2_{Pol}) > 0$).*

Singularities Formation. Another extremely important aspect of the Ricci flow, both smooth or discrete, is that of singularities formation. Again, a certain (theoretical, at least) advantage of the proposed method presents itself. Indeed, by [25], Theorem 5.1, for compact surfaces of genus ≥ 2, the combinatorial Ricci flow evolves without singularities. However, for surfaces of low genus no such result exists. Indeed, in the case of the Euclidean background metric, that is the one of greatest interest in graphics, singularities do appear. Moreover, such singularities are always combinatorial in nature and amount to the fact that, at some t, the edges of at least one triangle do not satisfy the triangle inequality. These singularities are removed in heuristic manner. However, by [42], Theorem 1.1, the smooth Ricci flow exists at all times, i.e. no singularities form. From the considerations above, it follows that the metric Ricci flow also exists at all times without the formation of singularities.

Embeddability in $\mathbb{R}^3$. The importance of the embeddability of the flow is not solely theoretical (e.g. if one considers the problem of the Ricci flow for surfaces of piecewise constant curvature), as it is essential in Imaging (see [4], [108]), and of high importance in Graphics. Indeed, even our very capability of seeing (grayscale) images is nothing but a translation, in the field of vision, of the embeddability of the associated height-surface into $\mathbb{R}^3$. (Or, perhaps one should view the mathematical aspect as a formalization of a physical/biological phenomenon...) We should note here that in this respect there exists a certain (mainly theoretical, at this point in time) advantage of our proposed metric flow over the combinatorial Ricci flow [39], [47]. Indeed, in the combinatorial flow, the goal is to produce, via the circle packing metric, a conformal mapping from the given surface to a surface of constant (Gauss) curvature. Since in the relevant cases (see, e.g. [39]) the surface in question is a planar region (usually a subset of the unit disk), its embeddability (not necessarily isometric) is trivial. Moreover, in the above mentioned works, there is no interest (and indeed, no need) to consider the (isometric) embeddability of the surfaces S^2_t (see below) for an intermediate time $t \neq 0, T$.

The tool that allows us to obtain this type of results is making appeal (again) to δ-approximations, in combination with classical results in embedding theory. Indeed, by [64], Theorem 8.8 a δ-approximation of an embedding is also an embedding, for small enough δ. Since, as we have already mentioned, smoothing represent δ-approximations, the possibility of using results regarding smooth surfaces to infer results regarding polyhedral embeddings is proven. (The other direction – namely from smooth to PL and polyhedral manifolds – follows from the fact that the *secant approximation* is a δ-approximation if the simplices of the PL approximation satisfy a certain nondegeneracy condition – see [64], Lemma 9.3.) We state here the relevant facts:

Let S_0^2 be a smooth surface of positive Gauss curvature, and let S_t^2 denote the surface obtained at time t from S_0^2 via the Ricci flow. (For all omitted background material (proofs, further results, etc.) we refer to [43].)

Proposition 6.7. *Let S_0^2 be the unit sphere $\mathbb{S}^2$, equipped with a smooth metric g, such that $\chi(S_0^2) > 0$. Then the surfaces S_t^2 are (uniquely, up to a congruence) isometrically embeddable in $\mathbb{R}^3$, for any $t \geq 0$.*

In fact, this results can be slightly strengthened as follows:

Corollary 6.8. *Let S_0^2 be a compact smooth surface. If $\chi(S_0^2) > 0$, then there exists some $t_0 \geq 0$, such that the surfaces S_t^2 are isometrically embeddable in $\mathbb{R}^3$, for any $t \geq t_0$.*

In stark contrast with this positive result regarding surfaces uniformized by the sphere, for (complete) surfaces uniformized by the hyperbolic plane we only have the following negative result:

Proposition 6.9. *Let (S_0^2, g_0) be a complete smooth surface, and consider the normalized Ricci flow on it. If $\chi(S_0^2) < 0$, then there exists some $t_0 \geq 0$, such that the surfaces S_t^2 are not isometrically embeddable in $\mathbb{R}^3$, for any $t \geq t_0$.*

6.1.2. *An Alexandrov Surfaces Based Approach.* After a completed version of our paper [95] was essentially finished, we noted that there are other works regarding the Ricci flow on surfaces with conical singularities, and especially Richards paper [84] on the smoothing of (compact) Alexandrov surfaces via the Ricci flow. We would like to stress that our approach as developed is different from Richard's work, being much more direct and, in a sense, more elementary. Moreover, we should also accentuate the fact that our method facilitates concrete, computational treatment of the flow. On the other hand, Richard's method uses the very Ricci flow for smoothing, and makes no appeal to approximations, making it much more alluring for theoretical ends. However, its proof is far from trivial and we don't even sketch it here, since it would take us to far afield, and the interested reader is invited too study Richard's paper.[19]

More important to our purpose here, Richard's method also provides us with a smoothing of the given PL surface, hence all the theoretical results in the previous section also follow via this route.

However, Richard's method of proof seems to be adaptable in order to solve the following

Problem 1. *Devise a purely metric flow.*

Surely, such a flow, independent both from smoothing and to the advanced (and somewhat abstract) mathematical apparatus of [84] would provide a powerful and flexible tool for many Imaging and graphics tasks, akin to the one based on Chow and Luo's paper (see the relevant bibliography mentioned above).

One basic observation that needs to be made in this context is that the lack of symmetry that we mentioned when we first introduced the metric flow, will not disappear by passing to the limit, and has to be dealt with in a different and direct manner. From symmetry reasons, a natural way of defining the flow is (using the same notation as before):

$$\frac{dl_{ij}}{dt} = -\frac{K_i + K_j}{2} l_{ij} \,, \tag{6.6}$$

where in this case, K_i, K_j denote, of course, the Wald curvature at the vertices v_i and v_j, respectively. It is also important to notice that, in fact, this expression appears also in the practical method of computing the combinatorial curvature, where it is derived via the use of a conformal factor (see [39]).

6.1.3. *An Application: Smoothable Metrics on Cube Complexes.* We illustrate our belief in the many possible applications of the metric Ricci flow with only one such example (due to space and time restrictions), appertaining to the corpus of "Pure" Mathematics. The following seemingly well

[19]Note that to apply Richard's result we have only to consider our surfaces as an Alexandrov surface having curvature bounded from below, condition that is, evidently, satisfied. (In this regard and for a discussion on the definition of Wald/Alexandrov curvature for PL surfaces, see [95], pp. 26-27.

known problem in the theory of cube complexes[20] was posed to the author by Joel Haas [41], together with the basic idea of the first part of the proof, for which the author is deeply grateful.

Let $\mathcal{C}$ be a cube complex, satisfying the following conditions:

(1) $\mathcal{C}$ is negatively curved (i.e. such that $\#_v Q \geq 4$, for all vertices v,where $\#_v Q$ denotes the number of cubes incident to the vertex v;
(2) The link $\mathrm{lk}(v)$ of any vertex is a *flag complex*, i.e. a simplicial complex such that any 3-arcs closed curve bounds a triangle (2-simplex), i.e. no such curve separates without being a boundary.[21]

Question 1. *Does there exist a Riemannian metric g (on $\mathcal{C}$) such that $K_g \equiv K$, where K denotes the comparison (Alexandrov) curvature of $\mathcal{C}$?*

In other words: Does there exist a smoothing of (M, g) (i.e. Riemannian manifold) of a given cube complex $\mathcal{C}$ (that has a manifold structure), such that $K \equiv K_g$? Evidently, an important particular case would be that "cubical version of PL approximations"), i.e. that of "cubulations" of a (given) Riemannian manifold.

Remark 6.10. The similar problem can be also posed, of course, for positively curved complexes (i.e. such that $\#_v Q \leq 4$). However, we address here only the negatively curved case. The similar results for polyhedral manifolds of non-negative curvature was also proved recently – see [52].[22]

Evidently, the answer to Question 1 above is "No", even if $\mathcal{C}$ is actually a manifold, since it is not always possible to recover the Riemannian metric from the discrete ("cubical") one. (Recall that each edge is supposed to be of length 1.) However, in the special case of 3-dimensional cube complexes the question has a positive answer.

We sketch below the proof:

(1) *Away from the vertices*, i.e. around the edges,[23] one can use a method developed by Gromov and Thurston [33] to produce a generalized

[20] For a formal definition and more details see, e.g. [86].

[21] Alternatively, this condition may be expressed either as $\mathrm{lk}(v)$ "has no missing simplices (M. Sageev, [86]), or as "a nonsimplex contains a non edge" (W. Dicks, see [12]).

[22] The author would like to thank the anonymous reviewer for bringing to his attention this paper.

[23] obviously, in the interiors of the faces the metric is already smooth.

type of branched cover (in any dimension). More precisely, (a) construct negatively curved conical surfaces of revolution, with vertex at a vertex v and with apex angle $\alpha = 2\pi/n$, where $n = \#_v Q$. Each such cone can be canonically mapped upon a Euclidean cone of apex angles $\pi/2$; then (b) glue the outcome of this process to the result of Step (2) below.

(2) *Around the vertices* excise an ε-ball neighbourhood B_ε of v. On the boundary of B_ε, i.e. on the sphere S_ε one has the natural triangulation by the intersections of S_ε with the cubes of $\mathcal{C}$ incident with v. Moreover, the curvature of the vertices of this triangulation will be $K_\varepsilon \equiv c/\varepsilon^2$, where c is some constant.

   However, while the gluing itself is trivial, one still has to ensure that the result is indeed endowed with a Riemannian metric. For this one has to go through Step 3 of the construction:

(3) *Smoothen the ball B_ε.* In general dimension this represents a daunting problem. Indeed, even in dimension 3, Ricci flow – who represents a natural candidate for smoothing with control of curvature –is yet not attainable, since we can offer, at this point in time, no PL (metric) Ricci flow. However, due to Perelman's resolution of the Poincarè conjecture, in dimension 3 suffices to smoothen the boundary S_ε. It is at this point where the method described in Section 3 is applied, producing the required smooth ball $\tilde{S}_\varepsilon$, that has the same curvature as the PL[24] one S_ε.

6.2. PL **Ricci for Cell Complexes.** Following [38], we briefly review here a definition of a metric Ricci curvature for PL manifolds in dimension higher than 2, as well as its immediate consequence, a method that does not make appeal to smoothings, as we did in the previous section.

6.2.1. *The Definition.* While the results in the preceding sections might be encouraging, one would still like to recover in the metric setting a "full" Ricci curvature, namely one that holds for 3− and higher dimensional manifolds, and not just in the degenerate case of surfaces. Our approach (as developed in [38]) is to start from the following classical formula:

[24]but not piecewise Euclidean.

$$\text{Ric}(e_1) = \text{Ric}(e_1, e_1) = \sum_{i=2}^{n} K(e_1, e_i)\,. \tag{6.7}$$

for any orthonormal basis $\{e_1, \cdots, e_n\}$, and where $K(e_1, e_j)$ denotes the sectional curvature of the 2-sections containing the directions e_1.

To adapt this expression for the Ricci curvature to the PL case, we first have to be able to define (*variational*) *Jacobi fields*. In this we heavily rely upon Stones's work [110], [111]. Note, however, that we do not need the full strength of Stone's technical apparatus, only the capability determine the relevant two sections and, of course, to decide what a direction at a vertex of a PL manifold is.

We start from noting that, in Stone's work, combinatorial Ricci curvature is defined both for the given simplicial complex $\mathcal{T}$, and for its *dual complex* $\mathcal{T}^*$ (see, e.g. [72], pp. 55-56). For the dual complex, cells – playing here the role of the planes in the classical setting of which sectional curvatures are to be averaged – are considered. Unfortunately, Stone's approach for the given complex, where one computes the Ricci curvature $\text{Ric}(\sigma, \tau_1 - \tau_2)$ of an n-simplex σ in the direction of two adjacent $(n-1)$-faces, τ_1, τ_2, is not natural in a geometric context (even if useful in his purely combinatorial one), except for the 2-dimensional case, where it coincides with the notion of Ricci curvature in a direction. However, passing to the dual complex will not restrict us, since $(\mathcal{T}^*)^* = \mathcal{T}$ and, moreover – and more importantly – considering *thick* triangulations enables us to compute the more natural metric curvature for the dual complex and use the fact that the dual of a thick triangulation is thick (for details, see [38]). Recall that thick (also called *fat*) triangulations are defined as follows:

Definition 6.11. Let $\tau \subset \mathbb{R}^n$; $0 \leq k \leq n$ be a k-dimensional simplex. The *thickness* (or *fatness*) φ of τ is defined as being:

$$\varphi(\tau) = \frac{\text{dist}(b, \partial\sigma)}{\text{diam}\,\sigma}\,, \tag{6.8}$$

where b denotes the *barycenter* of σ and $\partial\sigma$ represents the standard notation for the *boundary* of σ (i.e the union of the $(n-1)$-dimensional faces of σ). A simplex τ is φ_0-*thick*, for some $\varphi_0 > 0$, if $\varphi(\tau) \geq \varphi_0$. A triangulation (of a submanifold of $\mathbb{R}^n$) $\mathcal{T} = \{\sigma_i\}_{i\in\mathbf{I}}$ is φ_0-*thick* if all its simplices are φ_0-thick.

A triangulation $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ is *thick* if there exists $\varphi_0 \geq 0$ such that all its simplices are φ_0*-thick*.

Keeping in mind the notions and facts above, we can now return to the definition of Ricci curvature for simplicial complexes: Given a vertex v_0 in the dual complex, corresponding to a n-dimensional simplicial complex, a *direction* at v_0 is just an oriented edge $e_1 = v_0 v_1$. Since there exist precisely n 2-cells, $\mathfrak{c}_1, \ldots, \mathfrak{c}_n$, having e_1 as an edge and, moreover, these cells form part of n relevant variational (Jacobi) fields (see [110]), the Ricci curvature at the vertex v, in the direction e_1 is simply

$$\mathrm{Ric}(v) = \sum_{i=1}^{n} K(\mathfrak{c}_i)\,, \tag{6.9}$$

where we define the sectional curvature of a cell $\mathfrak{c}$ in the following manner:

Definition 6.12. Let $\mathfrak{c}$ be a cell with vertex set $V_{\mathfrak{c}} = \{v_1, \ldots, v_p\}$. The *embedding curvature* $K(\mathfrak{c})$ of $\mathfrak{c}$ is defined as:

$$K(\mathfrak{c}) = \min_{\{i,j,k,l\} \subseteq \{1,\ldots,p\}} \kappa(v_i, v_j, v_k, v_l)\,. \tag{6.10}$$

Remark 6.13. Note that by choosing to work with the dual complex we have restricted ourselves largely to considering solely submanifolds of $\mathbb{R}^N$, for some N sufficiently large. However, in the case of 2-dimensional PL manifolds this does nor represent restriction, since, by a result of Burago and Zalgaller [20] (see also [93]) such manifolds admit isometric embeddings in $\mathbb{R}^3$.

Remark 6.14. Evidently, the definition above presumes that cells in the dual complex have at least 4 vertices. However, except for some utterly degenerate (planar) cases, this condition always holds. Still, even in this case Ricci curvature can be computed using a slightly different approach – see the following remark.

Remark 6.15. It is still possible (by dualization) to compute Ricci curvature according, more or less, to Stone's ideas, at least for the 2-dimensional case. Indeed, according to [111],

$$\mathrm{Ric}(\sigma, \tau_1 - \tau_2) = 8n - \sum_{j=1}^{2n-1} \{|\beta_j| \mid \beta_j < \tau_1 \text{ or } \beta_j < \tau_2; \dim \beta_j = n-2\}\,. \tag{6.11}$$

For details and implications of this alternative approach, see [38].

6.2.2. *Main Results.* The first results one wants to ascertain are those ensuring the convergence of the newly defined Ricci curvature. These are quite straightforward, so here we content ourselves with simply stating them (for further details, see [38]).

Theorem 6.16. *Let $\mathcal{T}$ be a thick simplicial complex, and let $\mathcal{T}^*$ denote his dual. Then*

$$\lim_{\mathrm{mesh}(\mathcal{T})\to 0} \mathrm{Ric}(\sigma) = \lim_{\mathrm{mesh}(\mathcal{T}^*)\to 0} C\cdot \mathrm{Ric}^*(\sigma^*)\,, \tag{6.12}$$

where $\sigma \in \mathcal{T}$ and where $\sigma^* \in \mathcal{T}^*$ is (as suggested by the notation) the dual of σ.

Theorem 6.17. *Let M^n be a (smooth) Riemannian manifold and let $\mathcal{T}$ be a thick triangulation of M^n. Then*

$$\mathrm{Ric}_{\mathcal{T}} \to C_1\cdot \mathrm{Ric}_{M^n},\ \text{as}\ \mathrm{mesh}(\mathcal{T}) \to 0\,, \tag{6.13}$$

where the convergence is the weak convergence (of measures).

Beyond these convergence and approximations results, one would like to address deeper issues. Indeed, having introduced a metric Ricci curvature for PL manifolds, one naturally wishes to verify that this represents a proper notion of Ricci curvature, and not just an approximation of the classical notion. According to the synthetic approach to Differential Geometry, a proper notion of Ricci curvature should satisfy adapted versions of the main, essential theorems that hold for the classical notions. The first and foremost among such theorems is the Bonnet-Myers Theorem and, as expected, fitting versions for combinatorial cell complexes and weighted cell complexes were proven by Stone [110], [111], [112]. and Forman [27]. Moreover, the Bonnet part of the Bonnet-Myers theorem, that is the one appertaining to the sectional curvature, was also proven for PL manifolds, again by Stone – see [112], [109].

In [38] we proved a series of increasingly more general variants of the Bonnet-Myers Theorem, with proofs adapted to the various settings and/or notions of curvature (metric, combinatorial, Alexandrov comparison). Here we bring only two more representative ones.

Theorem 6.18 (PL Bonnet-Myers – metric). *Let M^n_{PL} be a complete, n-dimensional PL, smoothable manifold without boundary, such that*

(i) There exists $d_0 > 0$, such that $\mathrm{mesh}(M^n_{PL}) \leq d_0$*;*

(ii) $K_W(M^n_{PL}) \geq K_0 > 0\,,$

where $K_W(M^n_{PL})$ denotes the sectional curvature of the "combinatorial 2-sections".

Then M^n_{PL} is compact and, moreover

$$\mathrm{diam}(M^n_{PL}) \leq \frac{\pi}{\sqrt{K_0}}\,. \tag{6.14}$$

Unfortunately, determining whether a general PL complex has Wald curvature bounded from below can be, in practice, a daunting task. However, in the special case of thick complexes in $\mathbb{R}^N$, for some N one can determine a simple criterion as follows:

Theorem 6.19 (PL Bonnet-Myers – Thick Complexes). *Let $M = M^n_{PL}$ be a complete, connected PL manifold thickly embedded in some $\mathbb{R}^N$, such that $K_W(M^2) \geq K_0 > 0$, where M^2 denotes the 2-skeleton of M. Then M^n_{PL} is compact and, moreover*

$$\mathrm{diam}(M^2_{PL}) \leq \frac{\pi}{\sqrt{K_0}}\,. \tag{6.15}$$

Remark 6.20. The embedding condition in the theorem above necessitates, perhaps, further elaboration. One can, for instance, start with a (PL-)submanifold of $\mathbb{R}^N$, endowed with a thick triangulation (as it is the case in Graphics and Imaging, for instance). Alternatively, one can begin with an abstract metric PL manifold (recall that thickness is a purely metric concept — see Definition 6.11 above and embed it isometrically, or even just quasi-isometrically in $\mathbb{R}^N$. Moreover, one can be given a combinatorial PL manifold, i.e. such that the lengths of all the edges equals 1, and consider a quasi-conformal embedding of this object.

6.2.3. *Scalar Curvature and a Comparison Theorem.* Up to this point of we were concerned, in this section, solely with Ricci curvature. However, since Ricci curvature is the mean of sectional curvatures we had to consider them too (and, in fact, even more so in view of our definition of Ricci curvature for PL complexes). We did not discuss, however, scalar curvature. It is only fitting, therefore, for us to add a number of observation regarding this invariant, in particular since a immediate, but significant result presents itself.

Of course, we first have to define the scalar curvature $K_W(M)$ of a PL manifold M. In light of our preceding discussion and results, the following definition is quite natural:

Definition 6.21. Let $M = M^n_{PL}$ be an n-dimensional PL manifold (without boundary). The *scalar metric curvature* scal_W of M is defined as

$$\mathrm{scal}_W(v) = \sum_{\mathfrak{c}} K_W(\mathfrak{c}), \tag{6.16}$$

the sum being taken over all the cells of M^* incident to the vertex v of M^*.

Remark 6.22. Observe that the definition of scalar curvature of M is defined, somewhat counterintuitively, by passing to its dual M^*. However, this is consistent with our approach to Ricci curvature (and also similar to Stone's original approach – see the discussion in 4.1 above).

From this definition and our previous results (see [38]), we immediately[25] obtain, the following generalization of the classical curvature bounds comparison in Riemannian geometry:

Theorem 6.23 (Comparison theorem)**.** *Let $M = M^n_{PL}$ be an n-dimensional PL manifold (without boundary), such that $K_W(M) \geq K_0 > 0$, i.e. $K(\mathfrak{c}) \geq K_0$, for any 2-cell of the dual manifold (cell complex) M^*. Then*

$$K_W \lesseqqgtr K_0 \Rightarrow \mathrm{Ric}_W \lesseqqgtr nK_0\,. \tag{6.17}$$

Moreover

$$K_W \lesseqqgtr K_0 \Rightarrow \mathrm{scal}_W \lesseqqgtr n(n+1)K_0\,. \tag{6.18}$$

Remark 6.24. (1) Inequality (6.18) can be formulated in the seemingly weaker form:

$$\mathrm{Ric}_W \lesseqqgtr nK_0 \Rightarrow \mathrm{scal}_W \lesseqqgtr n(n+1)K_0\,. \tag{6.19}$$

[25] and, in truth rather trivially, since the result holds, regardless of the specific definition for the curvature of a cell.

(2) Note that in all the inequalities above, the dimension n appears, rather then $n-1$ as in the smooth, Riemannian case (hence, for instance one has in (6.18), $n(n+1)K_0$, instead of $n(n-1)K_0$[26] as in the classical case). This is due to our definition (6.9) of Ricci (and scalar) curvature, via the dual complex of the given triangulation, hence imposing standard and simple combinatorics, at the price of allowing only for such weaker bounds.[27]

7. Metric Curvatures for Metric Measure Spaces

While divagating somewhat from our professed goal, namely that of studying metric curvatures, it is impossible, especially in the context of this volume, not to mention the Ricci curvature for metric measure spaces, either as it was developed by Lott, Sturm and Villani [58], [113][28] (and its further elaborations [15]), or in the form pioneered by Ollivier [73] (and its further developments, mostly for graphs – due to Yau, Jost and their collaborators [54], [48], but also for polyhedral surfaces [57][29].).

Let us begin with the following observation: However simple and alluring the probabilistic approach may appear, to the geometer it seems somewhat unnatural, and even more so to those whose interest is drove mainly by possible implementations, e.g. people working in information geometry, image processing, manifold learning, etc.

Therefore, without diminishing whatsoever, the extensive theoretical merits of the Lott-Sturm-Villani approach, it still is a natural desire to find a new metric that encapsulates the behaviors of both the original metric and of the given measure. Here the accent should be understood as being placed on "simple", and by simple, we mean a "(geo-)metric", that is a metric that, while incorporating the measure, can still be investigated with very direct geometric methods, such as the ones discussed in detail above, or like the more analytic ones that we shall describe below. This is in some contrast to other means of "coalescing" metric-and-measure into a unique metric, such

[26] but, on the other hand, this holds even if $n = 3$!...

[27] without affecting the analogue of the Bonnet-Myers Theorem – see Section 2 above.

[28] See also [114]

[29] When mentioning generalized curvatures for surfaces, one can not fail to mention Morgan's [63] and his students' [26] work on "weighted" surfaces and curves.

as the Gromov-Prokhorov and Gromov-Hausdorff-Prokhorov distances (see, e.g. [114] for these and also for some variations, as well as their "practical" versions (such as those in [60], [61]).

7.1. **The Basic Idea: The Snowflaking Operator.** We begin by introducing a number of definitions and facts required in the sequel. (As general bibliographical references for the material in this subsection, including missing proofs, we have used [44], [105], [106].)

7.1.1. *Quasimetrics.* We begin with the following basic definition:

Definition 7.1. Let X be a nonempty set. $q : X \times X \to \mathbb{R}_+$ is called a *K-quasimetric* iff

(1) $q(x, y) = 0$ iff $x = y$;
(2) $q(x, y) = q(y, x)$, for any $x, y \in X$;
(3) $q(x, y) \leq K(q(x, z) + q(z, y))$, for any $x, y, z \in X$.

Remark 7.2. Some authors replace condition (2) above by the following weaker one: There exists $C_0 \geq 1$ such that $q(x, y) \leq C_0 q(y, x)$, for any $x, y \in X$.

Remark 7.3. A number of brief comments:

- A quasimetric is not necessarily a metric (while obviously, any metric is a quasimetric with $K = 1$).

  Counterexample 7.4. The following counterexample is not only the basic one, it is – as we shall shortly see – very important to us in the sequel:

$$q_s(x, y) = (d(x, y))^s \ , \tag{7.1}$$

  where d is a metric, is a quasimetric for any $s > 0$, but not, in general, a metric, for $s > 1$ (but it still is for $0 < s < 1$).

- Quasimetric balls can be defined precisely like metric balls, and the constitute the basis for a topology on X.
- For the next remark we need a definition that may appear a bit superfluous at this point, but it will prove to be highly relevant later on:

Definition 7.5. Let (X, q) and (Y, ρ) be quasimetric spaces, and let $f : X \to Y$ be an injection. f is called η-*quasisymmetric*, where $\eta : [0, \infty) \to [0, \infty)$ is a homeomorphism iff

$$\frac{\rho(f(x), f(a))}{\rho(f(x), f(b))} \leq \eta\left(\frac{q(x,a)}{q(x,b)}\right), \tag{7.2}$$

for any distinct points $x, a, b \in X$.

Intuitively, while quasisymmetric mappings may change the size of balls quite dramatically, they do not change very much their shape. This fact is important in the next proposition (see, e.g. [105] for its proof), that shows that whereas, as we noted above, q_s is not a metric, the canonical injection $(X, d) \hookrightarrow (X, q_s)$ is quasisymmetric.

Proposition 7.6. *Let q be a K-quasimetric on X. Then, there exists $s_0 = s_0(K)$ such that, for any $0 < s \leq s_0$ there exists a metric d_s on X, and a constant $C = C(s, K) \geq 1$, such that*

$$\frac{1}{C} q_s(x,y) \leq d_s(x,y) \leq C q_s(x,y)\,, \tag{7.3}$$

where q_s is as in (7.1), i.e. $q_s(x,y) = (q(x,y))^s$.

Remark 7.7. If q is a K-quasimetric $(K \geq 1)$, then q_s is bilipschitz equivalent to d_s, for any $s > 0$, such that $(2K)^{2s} \leq 2$, that is for any $s > 0$ such that

$$s \leq \frac{1}{2}(\log_2 K + 1)\,.$$

Moreover, the bilipschitz constant can be chosen to be

$$C = (2K)^{2s}\,.$$

The importance of the proposition above (augmented by the precise estimates in its succeeding remark) is quite evident, but we would still like to emphasize its relevance for our goal, namely that of combining the metric and measure into a new metric, that is simple yet unifying of the metric and measure properties. What we have succeeded to show so far is that given a quasimetric q_d obtained by snowflaking from a metric d, one can find a metric quantifiable close to it. Therefore, such metric curvatures as, say, Haantjes curvature, can be defined for (curves in) quasimetric spaces via those of the new metric d_s. The properties of the new metric curvature κ_{H,d_s} are clearly close to those of $\kappa_{H,d}$ (where the notation is, we hope, self-explanatory). However, we postpone a more detailed comparative analysis for future work. Also, instead of choosing to incorporate the new metric in the Haantjes curvature (that functions as *geodesic* curvature), one can as well use it to compute a fitting Wald curvature (as a metric analogue of *sectional* curvature).

We still have, however, to be able to produce enough expressive quasimetrics. Here, by "expressive", we mean quasimetrics that not only approximate the original metric, but also incorporate, according the our goal detailed above, as faithful (or significantly) as possible the given measure as well. It turns out that, again, this is quite standard and easy, as we shall see in the next subsection.

7.1.2. *From doubling measures to quasimetrics.* We first remind the reader the following basic definition:

Definition 7.8. Let (X, d, μ) be a metric measure space X is called *doubling* iff the measure μ itself is doubling, i.e. iff there exists a constant D such that, for any $x \in X$ and any $r > 0$,

$$\mu\left(B_d[x, 2r]\right) \leq D\mu\left(B_d[x, r]\right) . \tag{7.4}$$

(Here $B_d[x, r]$ denotes – as it standardly does – the closed ball of radius r, in the metric d.) A metric measure space (X, d, μ), where μ is doubling is sometimes called *of homogeneous type*.

For the record, a *metric measure space* is a triple $\mathcal{X} = (X, d, \mu)$ where (X, d) is a metric *Polish* space[30] (i.e. complete and having a countable base), and μ is a *Borel measure* on X.

Remark 7.9. If (X, d, μ) is doubling, then it admits *atoms*, i.e. points of positive mass, only at isolated points.

Remark 7.10. The notion of doubling measures is, in fact, intrinsically related to that of curvature, more precisely with that of Ricci curvature: Any Riemannian manifold of nonnegative Ricci curvature is doubling (with respect to the volume measure) – see, e.g. [77]. (Indeed, it may be that this

[30] at least, this is the usual convention

case represents one of the original motivations for studying doubling spaces.) Moreover, this implication is also preserved for the generalized Ricci curvature of Lott-Villani and Sturm (see, [114]).

We have now the necessary ingredient that allows us to construct the desired quasimetric, starting from a metric and a doubling measure: For any $s > 0$, we define the quasimetric $q_{\mu,s}$ as

$$q_{\mu,s}(x,y) = \big(\mu\left(B[x,d(x,y)]\right) + \mu\left(B[y,d(x,y)]\right)\big)^s \,. \tag{7.5}$$

(This can be written in compact form as $q_{\mu,s}(x,y) = (\mu(B_{x,y}))^s$, where $B_{x,y} = B[x,d(x,y)] \cup B[y,d(x,y)]$.)

Example 7.11. If $X = \mathbb{R}^n$, with $\mu \equiv Vol_n$, and if $s = 1/n$, then $q_{\mu,s} \equiv$ const $\cdot\, d_{Eucl}$. (In particular, for $n = 2$, $q_{\mu,s} = \frac{\sqrt{\pi}}{2} d_{Eucl}$.)

Remark 7.12. For $X = \mathbb{R}^n$, one can define $q_{\mu,s}(x,y)$ simply by $q_{\mu,s}(x,y) = \left(\mu\left(B[m, \frac{x+y}{2}]\right)\right)^s$, where m denotes the midpoint of the segment $\overline{xy}$. However, in the general case, and in particular for graphs, one has to use the more general expression (7.5).

Note that, if K is the quasimetric constant of $q_{\mu,s}$, then $K = K(\mu,s)$.

Also, by Proposition 7.6, there exists $s_0 = s_0(\mu) > 0$, such that $q_{\mu,s}$ is bilipschitz equivalent to a metric $d_{\mu,s}$, for any $0 < s \leq s_0$. This fact will play a crucial role in the sequel, as already hinted.

Remark 7.13. Obviously, the geometry induced by the quasimetric $q_{\mu,s}$, and a fortiori by the metric $d_{\mu,s}$, will diverge widely from the geometry given by the original metric d. This is most evident in the properties of the "new" geodesics, in comparison with the "old" ones (e.g. when $X = \mathbb{R}^n$ equipped with the standard Euclidean metric and with μ being the volume element.) However, the deformation of the geometry produced by (7.5) is controlled, and many essential properties are preserved. (For further details, see [105], [106].)

Remark 7.14. To be certain, one would like to explore the relevance of the "snowflaking" above to Imaging, etc. While further applications will be discussed below (see Theorems 7.23 and 7.27), even the very definition might prove to be useful, for instance in texture analysis and segmentation.

In the case of images the substrate distance d can be chosen to be the preferred discrete distance (Euclidean, L_1, etc.) For the measure of the balls $B[x, d(x,y)]$ one can settle, of course, just for the d-area. However, more interesting and relevant for textures measures present themselves, such as the Hausdorff measure (texture are viewed as fractals, sometimes) or the energy (of a *texton*).

As far as the choice of the points x and y is concerned, one possible (typical) choice would be the centers of adjacent neighbourhoods or textons.

Clearly, in this case, one take $q_{\mu,s}(x,y)$ to be $\left(\mu\left(B[m, \frac{x+y}{2}]\right)\right)^s$, where m denotes the midpoint of the segment $\overline{xy}$, even if d is not the Euclidean distance. However, this will not be true when working with (communication) networks.

Also, the relevant parameters s have to be chosen such that $s \leq s_0$, where s_0 should be determined from the proof of Proposition 7.6, as restricted to the given specific, concrete context.

Unfortunately, the existence of the doubling measure required in producing the snowflaked quasimetrics may prove to be a quite daunting task (to say nothing about the lesser degree of geometric intuitiveness we are endowed with, in comparison with our grasp of the distance). Luckily enough, a simpler, purely metric condition exists that is, essentially, equivalent to that of doubling measure, at least as far as compete spaces are concerned. More precisely, we have the following

Definition 7.15. A metric space (X,d) is called *doubling* iff there exists $D_1 \geq 1$, such that any ball in X, of radius r, can be covered by at most D_1 balls of radius $r/2$.

(Obviously, there is nothing special about $r/2$, and the metric doubling condition can be formulated in terms of general sets of bounded diameter.)

Remark 7.16. Clearly, the metric and measures arising in Imaging and Vision are (quite trivially) doubling.

As expected (and alluded to above), there exists a connection between the notions of doubling metric and doubling measure. More precisely, we have the following

Lemma 7.17. *Let (X, d) be a metric space such that there exists a doubling measure μ on X. Then (X, d) is doubling (as a metric space).*

(A proof of this fact can be found in [44] or [105].)

The converse statement does not hold in general, a counterexample being $(\mathbb{Q}, d_{eucl})$ (see [44], p. 103). However, it does hold for the important case of complete spaces:

Theorem 7.18 (Luukkainen-Saksman [59])**.** *Let (X, d) be a doubling, complete metric space. Then X carries a doubling measure.*

The following consequence since it obviously includes the important particular case of finite graphs:

Corollary 7.19. *Any compact, doubling metric spaces carries a doubling measure μ.*

The corollary above obviously holds for finite graphs.

Before we formulate the important theorem below, we give, for convenience, the following definition:

Definition 7.20. If (X, d) is a metric space, then the metric space $(X, d^{\varepsilon}), 0 < \varepsilon < 1$, is called a *snowflaked version* of (X, d).

Theorem 7.21 (Assouad [5], [6])**.** *Let (X, d) be a doubling metric space. Then, for each $0 < \varepsilon < 1$, there exists N, such that its ε-snowflaked version is bilipschitz equivalent to a subset of $\mathbb{R}^N$, quantitatively.*

Here, *quantitatively* means that the embedding dimension N and the bilipschitz constant L depend solely on the doubling constant D of X and on the "snowflaking" factor ε, i.e.

$$N = N(D, \varepsilon),\ L = L(D, \varepsilon)\,.$$

Remark 7.22. Assouad's result does not hold, in general, for $\varepsilon = 1$. (For a counterexample, see [44], p. 99).

From a practical, applicative point of view Assouad's theorem above allows us to "translate" the highly nonintuitive geometry of metric measure spaces to that of the familiar setting of subsets in (some) Euclidean space. In particular, in combination with our geometric, curvature based approach to sampling of images and higher dimensional signals (see [99], [100]), enables us to enunciate the following sampling "meta-theorem":

Theorem 7.23. *Sampling of Ahlfors regular metric measure spaces is quasisymmetrically equivalent, quantitatively, to the sampling of sets in* $\mathbb{R}^N$*, for some* N*.*

Before passing further on, let us mention briefly here that the sampling density is, roughly formulated, proportional to $1/K$ (or $1/\mathrm{Ric}$. (For the full details see, for instance, [99], [100], [36] and the references therein.)

Unfortunately, while the theorem above grants the desired framework for geometric sampling of metric measure spaces and, even more, it provides with numerical control over the distortion/error during the embedding in $\mathbb{R}^N$, the constants involved are far from being ideal - see the remark below for a more detailed discussion.

Remark 7.24. The beauty of Assouad's Theorem – and even more so its applicability in the sampling of real data – is marred by the "course of dimensionality": Given that $N = N(D, \varepsilon)$, the fear exist that, as in the case of Nash's Embedding Theorem [67], [68], the embedding dimension is prohibitively high for general manifolds (i.e. data). Obviously, this is even more important if low distortion – i.e. (bi-)lipschitz constant – is an imperative (as it usually is), that is for ε close to 0. And, indeed, Assouad's original construction provides $\lim_{\varepsilon \to 0} N(D, \varepsilon) = \infty$. So it would seem that, the price to pay for low distortion is a high embedding dimension. It is a quite recent result of Naor and Neiman [66] (itself based on ideas of Abraham, Bartal and Neiman [1]), that, in fact, given a (separable) D-doubling metric space, there exist $N = N(D) \in \mathbb{N}$ and $L = L(D, \varepsilon)$, such that for any $\varepsilon \in (0, 1/2)$, the $(1-\varepsilon)$-snowflaked version of X admits a bilipschitz embedding in $\mathbb{R}^N$, with distortion L. Moreover, specific upper bounds for N and L are given: $N \leq a \log D, L \leq b \left(\frac{\log K}{\varepsilon}\right)^2$, where a and b are constants. So it appears that, at least as far as Assouad's Theorem is concerned, the snowflaking-based embedding is feasible.

At this point, one has to ask oneself whether this result can be improved. The belief in the possibility of such an improvement rests upon the following two facts: One one hand, Assouad's Theorem assures the existence of a bilipschitz embedding, which represents a much stronger condition then mere quasisymmetry[31]. On the other hand, as we have seen, Ahlfors rigidity is not the most easy property to check directly on a metric measure space, therefore one naturally would wish to find a sampling result similar to Theorem 7.23, that would hold for general doubling spaces. Such a result does exist, and it makes appeal again to the quasimetric $q_{\mu,s}$ as defined by (3.3). However, we have to make an additional assumption, that ensures that $q_{\mu,s}$-lengths of curves in $\mathbb{R}^N$ do not "shrink" too much, due to the presence of the measure μ in the definition of $q_{\mu,s}$ (see [105]). We encode this restriction via

Definition 7.25. A doubling measure μ on $\mathbb{R}^N$ is called a *metric doubling measure* iff there exist a constant C_6, and a metric δ, such that

$$\frac{1}{C_6}\delta(x,y) \leq q_{\mu,\frac{1}{n}} \leq C_6\delta(x,y)\,,$$

for any $x, y \in X$, where $q_{\mu,\frac{1}{n}}$ is associated to μ as in (3.3), with $s = 1/n$.

We can now formulate the desired result, in terms of metric doubling measures:

Theorem 7.26 (Semmes [101] Theorem 1.15, [105], Proposition B. 20.2)**.** *Let (X, d) be a doubling metric space. Then there exists a natural number N and a metric doubling measure μ, such that (X, d) is bilipschitz equivalent to a subset of $(\mathbb{R}^N, q_{\mu,\frac{1}{N}})$, where $q_{\mu,\frac{1}{N}}$ is as above.*

Clearly PL surfaces (a.k.a. in Graphics as triangular meshes), endowed with a specific additional measure (e.g. luminosity), as well as images satisfy the metric doubling condition. (Even if modelled as fractals, certain textures are not truly such objects, due to the inherent discreteness, hence finiteness, therefore they also can be viewed as metric doubling measure spaces.) Again, this implies, in view of the theorem above, that, at least theoretically, images and meshes can be sampled as "weighted" manifolds, but using classical by now, geometric means of sampling subsets (hypersurfaces) of Euclidean space. The theorem is also relevant for sampling weighted

[31]However, quasisymmetry represents a much more flexible analytic tool, than the rigid bilipschitz condition – see [44], [105], [106] for a deeper and far more detailed discussion.

networks, such as communication networks, for instance for choosing positions for routers/sensors [31], load balancing [87], or in-network sensor data storage [88].

Theorem 7.26 above represents a most encouraging result, and the idea of the proof is quite simple: By Assouad's Theorem, $(M, d^{\frac{1}{2}})$ is bilipschitz equivalent to a subset Y of some $\mathbb{R}^N$. The sought for measure on $\mathbb{R}^N$ will be define as $\mu = \mathrm{dist}(x, Y^n)dx$ – for details of the proof see [101].

One would naturally would hope that $(\mathbb{R}^n, q_{\mu,\frac{1}{n}})$ can be bilipschitzly embedded in some $\mathbb{R}^N$, for any doubling measure μ. This is a quite ambitious wish and, unfortunately, it is not true in general (see [101]). However, such an embedding exists for "most" metric doubling measures – for a precis formulation and the proof see [101]. Still, in view of the discussion proceeding Theorem 7.23 above, we can formulate the fitting sampling result (recall that given the quasimetric $q_{\mu,s}$, there exists a metric d_s bilipschitz equivalent to it):

Theorem 7.27. *Sampling of doubling metric spaces is bilipschitz equivalent quantitatively to the sampling of sets in $(\mathbb{R}^N, d_{\frac{1}{N}})$, for some N, where $d_{\frac{1}{N}}$ represents the snowflaked version of d, associated to a certain metric doubling measure μ.*

How relevant this approach to metric curvature for metric measure spaces will turn prove itself to be, besides providing a sound, intuitive and convenient theoretical setting for a wide range of signals is, unfortunately, to early to ascertain. However, in view of the success of the basic snowflaking approach (and related ideas) in solving such problems as the existence of ("enough") Lipschitz functions and Poincaré and Sobolev inequalities (i.e. that of "novel types" of "decent calculus" – see [105], [106]), as well as the existence of fitting versions on metric measure spaces satisfying the $CD(K,N)$ condition (see, for instance, [114]), one can display at least a moderate amount of optimism. For a different approach to sampling spaces satisfying a $CD(K,N)$ condition, as well as an application of the approach exposed in this section to the sampling of weighted graphs/networks, see [92]. Moreover, it turns out that the particular case of $N = +\infty$, when the $CD(K,N)$ condition reduces to the generalized Ricci curvature of Bakry, Emery and Ledoux it is natural and easy to implement of grayscale images,

FIGURE 6. A standard rest image (above, left) as a grayscale surface (above, right). One can compare the geometric sampling methods: The Gaussian curvature based one (bottom, left) and one using the generalized Ricci curvature (bottom, right).

where the density function appearing in the formula of generalized Ricci curvature is nothing the grayscale level – see [55] for further details. We illustrate this new approach to sampling of grayscale images (natural images, but also range images, as well as cartoons) in Figure 6. This is approach is even more natural and having far higher potential benefits in the context of medical images, such as CT and MRI images since, for instance, the density of many types of MRI images is equal to the proton density. A further application of the geenarlized Ricci curvature is in Graphics, where the density can be taken as, e.g. luminosity or shading of a mesh model. (This represents work in progress.) We should also note that while the generalized Ricci curvature (of metric measure spaces) might appear, prima facie, as an unnecessary complication, it allows not only for the sampling of wider range of images and signals, but it is also an *intrinsic* curvature, i.e. independent of the specific embedding considered. This is not a purely theoretical advantage, for it allows us to dispense with the need to compute the embedding curvature (e.g. tubular radius, reach, etc.), a task that is, in general, both non-trivial and cumbersome.

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