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Quantitative properties of infinite and finite pseudotrajectories

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Contents

Introduction

This thesis is devoted to the study of relation between shadowing properties of dynamical systems generated by diffeomorphisms, vector field and actions of more complicated groups with such forms of hyperbolicity as structural stability, Ω-stability and partial hyperbolicity.

The shadowing problem in the most general setting is related to the following question: under which conditions for any pseudotrajectory of a dynamical system there exists a close exact trajectory? The problem of shadowing was initiated in works of Anosov [3] and Bowen [13]. Current state of shadowing theory is reflected in monographs [65, 72] and recent review [76].

In the most simple setting shadowing property is formulated as follows. Let (X, dist) be a metric space. Let $f : X \to X$ be a homeomorphism. For $d > 0$ we say that sequence of points $\{y_k\}_{k\in\mathbb{Z}}$ is a d-pseudotrajectory if

$$\mathrm{dist}(y_{k+1}, f(y_k)) < d, \quad k \in \mathbb{Z}.$$

We say that dynamical system generated by f has the shadowing property if for any $\varepsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory $\{y_k\}_{k\in\mathbb{Z}}$ there exists an exact trajectory $\{x_k\}_{k\in\mathbb{Z}}$ satisfying the following

$$\mathrm{dist}(x_k, y_k) < \varepsilon, \quad k \in \mathbb{Z}.$$

Pseudotrajectories naturally appears in numerical simulations of dynamical systems. Indeed if a diffeomorphism f (or vector field X) has the shadowing property, then approximate trajectories, attained as a result of numerical simulation of a corresponding dynamical system, reflects the behaviour of the system on infinite time interval.

Shadowing property plays important role in the smooth dynamical system theory. Indeed, if diffeomorphisms f_1, f_2 (vector fields X_1, X_2) are close in the C^1 topology then exact trajectories of f_2 (X_2) are pseudotrajectories for f_1 (X_1), hence the shadowing property is a weak analogue of structural stability.

Even that the most natural motivation of the shadowing property is justification of results numerical simulation, initially it was introduced in the notion of chain-recurrent sets and in structural stability theory.

It is well-known that dynamical systems have shadowing property in a neighborhood of a hyperbolic set [3, 13]. This statement is often called the shadowing lemma. Structurally stable systems have the shadowing property on the whole space [71, 90, 96]. Let us note that for the theory of structural stability it is important only the fact that pseudotrajectory and exact trajectory are close, for numerical simulations it is important to know qualitative characteristics of shadowing property and consider shadowing of pseudotrajectories of finite length.

It is not difficult to construct examples of nonhyperbolic systems with the shadowing property (see for instance [75, 79]), however in the modern theory of dynamical systems it is believed that shadowing and hyperbolicity are almost equivalent. At the same time shadowing property shows good results for much broader class of dynamical systems.

Hammel, Grebogi and York [32, 33] considered question about the length of shadowable pseudotrajectories. In those works based on results of numerical simulations for logistics map and Henon map authors formulate a conjecture on the length of shadowable pseudotrajectories.

Before the current work the structure of the C^1-interior of the set of diffeomorphisms, satisfying various shadowing properties was studied. In works [78,95] it was shown that such an interior coincide with the set of structurally stable diffeomorphisms for the case of orbital and standard shadowing properties. Let us also mention that Abdenur and Diaz [1] conjectured that for a C^1-generic diffeomorphism shadowing property is equivalent to structural stability; they proved this conjecture for the case of tame diffeomorphisms.

Description of the set of diffeomorphisms with shadowing property (without passage to the C^1-interior) was available only variational shadowing property [75]; it is equivalent to structural stability.

One of the important problem is the description of the set of periodic orbits of a dynamical system. In this context it is natural to consider periodic shadowing property, in which we consider shadowing of periodic pseudotrajectories by periodic exact trajectories. This notion was introduced earlier, at the same time it is still not known if shadowing property implies periodic shadowing property [43].

In order to support paradigm of equivalence of shadowing and hyperbolicity Bonatti, Diaz and Turcat [11] constructed an example of partially hyperbolic diffeomorphism without shadowing property. Hirsh, Pugh, Shub [38] proved that under some additional assumptions (plaque expansivity and dynamical coherence) central foliation of a partially hyperbolic diffeomorphism is leaf stable. At the same time it was not known which shadowing property is satisfied for partially hyperbolic diffeomorphisms.

The main difference of the shadowing problem for vector fields from discrete dynamical systems is necessity of reparametrisation of shadowing trajectories in the former case. One more difference comes from the possibility of accumulation of closed trajectories to a fixed point. As in the case of diffeomorphisms, vector fields have shadowing property in a neighborhood of a hyperbolic set [3] and structurally stable vector fields have the shadowing property on the whole manifold [71].

Described differences are essential in studying the shadowing property. For instance, in the context of C^1-interiors, it was known only that C^1-interior of the set of vector fields with shadowing property without fixed points consists only from structurally stable vector fields [47], which is much weaker then corresponding results for diffeomorphisms. It is not known if the assumption of absence of fixed points is essential or is a drawback of the proof.

We consider shadowing property for actions of finitely generated groups. Note that this notion was first time introduced in the work of the author in 2003 [82]. Since that it is widely used in the literature, see for example [10, 44, 46, 51, 61, 62].

In this thesis we for the first time systematically study quantitative aspects of shadowing property. We study the following problems in details.

- Quantitative properties of dependence between ε and d in the definition of the shadowing property for diffeomorphisms and vector fields.
- Properties of pseudotrajectories of finite length.
- The structure of the C^1-interior of vector fields with shadowing property.
- Dependence from the type of reparametrisation of the shadowing property for vector fields.
- Shadowing property for partially hyperbolic diffeomorphisms.
- Shadowing property in actions of finitely generated groups.

In Chapter 1 we study quantitative aspects of shadowing properties for diffeomorphisms.

We systematically study Lipschitz and Lipschitz periodic shadowing properties. We proved that Lipschitz shadowing property is equivalent to structural stability and Lipschitz periodic shadowing property is equivalent to Ω-stability. This result allows us to give a complete description of the sets of diffeomorphisms with Lipschitz and Lipschitz periodic shadowing properties. To prove those statements we developed new technique for studying shadowing property using inhomogeneous linear equation.

We consider pseudotrajectories of finite length with the polynomial dependence between size of the jump of a pseudotrajectory and precision of shadowing. We introduced the notion of the Finite Holder Shadowing property and gave an upper bound for length of shadowable pseudotrajectories, which agrees with the mentioned above conjecture by Hammel, Grebogi and Yorke. In the proof we introduced notion of slow growth solution for inhomogeneous linear equation and characterise it in terms of exponential dichotomy.

In Chapter 2 we study shadowing property for partially hyperbolic diffeomorphisms.

We introduce notion of the central shadowing property and prove that any dynamically coherent partially hyperbolic diffeomorphism has the central showing property. To be more precise we proved that any pseudotrajectory can be shadowed by a pseudotrajectory with the jumps along the central foliation. This statement might be considered as the shadowing lemma for partially hyperbolic diffeomorphism. Note that we do not assume that central foliation is Lipschitz.

We consider special type of partially hyperbolic system: linear skew products. For the case of nonzero Lyapunov exponent we gave sharp bounds on the length of shadowable pseudotrajectories. This result allows us to suggest that multidimensional analog of the mentioned above conjecture by Hammel, Grebogi and Yorke is not correct. We reduce the shadowing problem in this case to gambler's ruin problem for random walk.

In Chapter 3 we study shadowing property for vector fields.

We constructed an example of not structurally stable vector field on 4-dimensional manifold $S^2 \times S^2$, satisfying the oriented shadowing property together with all its small perturbations. This example shows that there is an essential difference between shadowing problem for diffeomorphisms and vector fields.

Note that constructed example is in a certain sense unique. In my Ph. D. thesis the following statements was proved: (1) on manifolds of dimension not greater than 3 vector fields satisfying the oriented shadowing property together with C^1-small perturbations are structurally stable; (2) vector fields without special semilocal construction (B-sisters) satisfying the oriented shadowing property together with C^1-small perturbations are structurally stable. In this thesis we proved that vector fields satisfying the oriented shadowing property together with C^1-small perturbations are Ω-stable.

We constructed an example of a vector field satisfying the oriented shadowing property and not satisfying the standard shadowing property. The only difference between those shadowing properties is in the restrictions on reparametrisations of shadowing trajectory. The question of existence of such a vector field was posed by Komuro in 1984 [41].

In Chapter 4 we study Lipschitz and Lipschitz periodic shadowing properties for vector fields. We proved that the Lipschitz shadowing property is equivalent to structural stability and Lipschitz periodic shadowing property is equivalent to Ω-stability. The statements are similar for the case of diffeomorphisms. At the same time the proof is quite different due to the following facts: shadowing problem has different nature in neighborhoods of fixed points and closed trajectories; we need to exclude accumulation of closed trajectories to fixed points.

In Chapter 5 we introduce and study shadowing property for actions of finitely generated groups. We show that the shadowing problem depends not only of hyperbolicity of an action but from the structure of the group as well. In particular the following results were obtained. We prove that for nilpotent groups if action of one element has the shadowing property and expansivity then the action of the whole group have the shadowing property. We consider an example of action of Baumslag-Solitar group, where shadowing property depends on quantitative characteristics of hyperbolicity of actions of particular elements We proved that any linear action of a non abelian free group does not have shadowing property.

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Chapter 1

Lipschitz shadowing for diffeomorphisms

1.1 Basic Definitions

Let (M, dist) be a metric space. Let $f : M \to M$ be a homeomorphism.

Consider an interval $I = (a, b)$, where $a \in \mathbb{Z} \cup \{-\infty\}$, $b \in \mathbb{Z} \cup \{+\infty\}$. We say that sequence of points $\{x_k\}_{k \in I}$ is an *exact trajectory* (or simply a *trajectory*) if for some $x \in M$ the following equalities hold:

$$x_k = f^k(x), \quad k \in I.$$

Definition 1.1. For $d > 0$ we call a sequence of points $\{y_k\}_{k \in I}$ a *d-pseudotrajectory* if the following inequalities hold

$$\mathrm{dist}(y_{k+1}, f(y_k)) < d, \quad k \in \mathbb{Z}, \quad k, k+1 \in I.$$

We are interested under which conditions for a pseudotrajectory there exists a close exact trajectory.

Definition 1.2. For $\varepsilon > 0$ and a d-pseudotrajectory $\{y_k\}_{k \in I}$. We say that exact trajectory $\{x_k\}_{k \in I}$ ε-shadows $\{y_k\}$ if the following inequalities hold

$$\mathrm{dist}(x_k, y_k) < \varepsilon. \quad k \in I.$$

Definition 1.3. We say that f has the *standard shadowing property* on a set $V \subset M$ if for any $\varepsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory $\{y_k\}_{k \in \mathbb{Z}} \subset V$ there exists a trajectory $\{x_k\}_{k \in \mathbb{Z}} \subset M$, which ε-shadows it:

$$\mathrm{dist}(x_k, y_k) < \varepsilon, \quad k \in \mathbb{Z}. \tag{1.1}$$

If $V = M$ we simply say that f has the standard shadowing property. Denote set of all diffeomorphisms satisfying the standard shadowing property by StSh.

It will be of a special interest for us the case when dependence between ε and d in the Definition 1.3 is Lipschitz.

Definition 1.4. We say that f has the *Lipschitz shadowing property* on a set $V \subset M$ if there exist constants $\mathcal{L}, d_0 > 0$ with the following property: For any d-pseudotrajectory $\{y_k\}_{k\in\mathbb{Z}} \subset V$ with $d \le d_0$ there exists an exact trajectory $\{x_k\}_{k\in\mathbb{Z}} \subset M$, which $\mathcal{L}d$-shadows it:

$$\operatorname{dist}(y_k, x_k) \le \mathcal{L}d, \quad k \in \mathbb{Z}. \tag{1.2}$$

If $U = M$ we simply say that f has the Lipschitz shadowing property. Denote set of all diffeomorphisms satisfying the Lipschitz shadowing property by LipSh.

Together with the shadowing property the following notion plays important role for us.

Definition 1.5. We say that a diffeomorphism f is *expansive* on a set $V \subset M$ if there exists a positive number a (expansivity constant) such that if two trajectories x_k, y_k belonging to V satisfy the inequalities

$$\operatorname{dist}(f^k(x), f^k(y)) \le a. \quad k \in \mathbb{Z}$$

then $x_k = y_k$ for all $k \in \mathbb{Z}$. If $V = M$ we simply say that f is expansive.

In most part of the text (Chapters 1–4) we are concentrated on the case, when M is a smooth compact manifold of class C^∞ without boundary with Riemannian metric dist. Endow $\operatorname{Diff}^1(M)$ by the C^1-topology. For a set $P \subset \operatorname{Diff}^1(M)$ we denote by $\operatorname{Int}^1(P)$ its C^1-interior. Denote by T_xM the tangent space of M at a point x; let $|v|$, $v \in T_xM$, be the norm of v generated by the metric dist. For any $x \in M$, $\varepsilon > 0$ we denote $B_\varepsilon(x) = \{y \in M : \operatorname{dist}(x, y) \le \varepsilon\}$. Consider a diffeomorphism $f \in \operatorname{Diff}^1(M)$.

In the shadowing theory one of the central role is played by the notion of hyperbolicity.

Definition 1.6. We say that a compact invariant set $\Lambda \subset M$ is *hyperbolic* if there exist numbers $C > 0$, $\lambda \in (0, 1)$ and a decomposition of a tangent bundle $T_xM = E^s_x \oplus E^u_x$ for $x \in \Lambda$ such that

1. $Df(x)E^{s,u}_x = E^{s,u}_{f(x)}$ for $x \in \Lambda$;
2. $|Df^k(x)v^s| \le C\lambda^k|v^s|$ for $x \in \Lambda$, $v^s \in E^s_x$, $k \ge 0$.
3. $|Df^{-k}(x)v^u| \le C\lambda^k|v^u|$ for $x \in \Lambda$, $v^u \in E^u_x$, $k \ge 0$.

The so-called *shadowing lemma* [3, 13] tells that in a neighborhood of a hyperbolic set diffeomorphism has the shadowing property, moreover shadowing property is Lipschitz [72].

Theorem 1.1. *If Λ is a hyperbolic set for a diffeomorphism f, then there exists a neighborhood V of Λ such that f has the Lipschitz shadowing property on V and is expansive on V.*

For us will be important the notion of structural stability.

Definition 1.7. We say that a diffeomorphism $f \in \mathrm{Diff}^1(M)$ is *struclurally stable* if there exists a neighborhood $U \subset \mathrm{Diff}^1(M)$ of f such that for any $g \in U$ there exists a homeomorphism $h : M \to M$ such that $h \circ f = g \circ h$.

Notion of structural stability is strongly related to the notion of hyperbolicity. It is known that diffeomorphism f is structurally stable iif it satisfies Axiom A (hyperbolicity of nonwondering set and density of periodic orbits in nonwondering set) and the strong transversality condition [56, 89].

Moreover structurally stable diffeomorphsims satisfy the shadowing property on the whole manifold [90, 96].

Theorem 1.2. *Structurally stable diffeomorphisms satisfy the Lipschitz shadowing property.*

At the same time, it is easy to give an example of a diffeomorphism that is not structurally stable but has the standard shadowing property (see [75], for instance). Thus, structural stability is not equivalent to shadowing.

One of possible approaches in the study of relations between shadowing and structural stability is the passage to C^1-interiors. At present, it is known that the C^1-interior of the set of diffeomorphisms having shadowing property coincides with the set of structurally stable diffeomorphisms [95]. Later, a similar result was obtained for orbital shadowing property (see [78] for details). Abdenur and Diaz conjectured that a C^1-generic diffeomorphism with the shadowing property is structurally stable; they have proved this conjecture for so-called tame diffeomorphisms [1].

In the present chapter we are interested in relation between shadowing and structural stability without perturbations in the C^1-topology.

Let us mention that recently it was proved that the so-called variational shadowing is equivalent to structural stability [75].

1.2 Inhomogeneous linear equation

In this paragraph we are going to introduce the main technical tool of this chapter – inhomogenious linear equation, and discuss its relation to structural stability.

Consider Euclidian spaces $E_{n\in\mathbb{Z}}$ of dimension m and a sequence $\mathcal{A} = \{A_{n\in\mathbb{Z}} : E_n \to E_{n+1}\}$ of linear isomorphisms satisfying for some $R > 0$ the following inequalities

$$\|A_n\|, \|A_n^{-1}\| < R, \quad n \in \mathbb{Z}. \tag{1.3}$$

Definition 1.8. We say that a sequence $\mathcal{A}$ has *slow growth* property with exponent $\gamma > 0$ ($\mathcal{A} \in \mathrm{SG}(\gamma)$) if there exists a constant $L > 0$ such that for any $i \in \mathbb{Z}$, $N > 0$ and a sequence $\{w_k \in E_k\}_{k\in[i+1,i+N]}$, $|w_k| \le 1$ there exists a sequence $\{v_k \in E_k\}_{k\in[i,i+N]}$ satisfying

$$v_{k+1} = A_k v_k + w_{k+1}, \quad k \in [i, i + N - 1], \tag{1.4}$$

$$|v_k| \le LN^\gamma, \quad k \in [i, i + N]. \tag{1.5}$$

If $\mathcal{A} \in \mathrm{SG}(\gamma)$ with $\gamma \in [0, 1)$ we say that it has *sublinear growth* property. If $\mathcal{A} \in \mathrm{SG}(0)$ we say that it has *bounded solution* property.

We have not found analogues of the notion of slow growth property in the literature. At the same time the notion of bounded solution property was widely investigated, for example see [8, 16, 18, 52, 66–68, 84].

To characterize sequences satisfying sublinear growth property we need notion of exponential dichotomy (see [18], for some generalisations see [8]).

Definition 1.9. We say that a sequence $\mathcal{A}$ has *exponential dichotomy* on $\mathbb{Z}^+$ if there exist numbers $C > 0$, $\lambda \in (0, 1)$ and a decomposition $E_k = E_k^{s,+} \oplus E_k^{u,+}$, $k \geq 0$ such that

$$E_{k+1}^{\sigma,+} = A_k E_k^{\sigma,+}, \quad k \geq 0, \sigma \in \{s, u\},$$

$$|A_{k+l-1} \cdot \ldots A_k v_k^s| \leq C\lambda^l |v_k^s|, \quad k \geq 0, l > 0, v_k^s \in E_k^{s,+}, \tag{1.6}$$

$$|A_{k+l-1} \cdot \ldots A_k v_k^u| \geq \frac{1}{C}\lambda^{-l} |v_k^u|, \quad k \geq 0, l > 0, v_k^u \in E_k^{u,+}. \tag{1.7}$$

Similarly we say that $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^-$ if there exist numbers $C > 0$, $\lambda \in (0, 1)$ and a decomposition $E_k = E_k^{s,-} \oplus E_k^{u,-}$, $k \leq 0$ such that

$$E_{k+1}^{\sigma,-} = A_k E_k^{\sigma,-}, \quad k < 0, \sigma \in \{s, u\},$$

$$|A_{k+l-1} \cdot \ldots A_k v_k^s| \leq C\lambda^l |v_k^s|, \quad l > 0, l + k < 0, v_k^s \in E_k^{s,-},$$

$$|A_{k+l-1} \cdot \ldots A_k v_k^u| \geq \frac{1}{C}\lambda^{-l} |v_k^u|, \quad l > 0, l + k < 0, v_k^u \in E_k^{u,-}.$$

Denote by $P_k^{s,+}$ the projection with the range $E_k^{s,+}$ and kernel $E_k^{u,+}$. Similarly we define $P_k^{u,+}$, $P_k^{s,-}$, $P_k^{u,-}$.

Remark 1.3. It is easy to show that there exists $H > 0$ such that (see for instance [40, Lemma 3.1], [106, Remark 2.3])

$$|P_k^{\sigma,a} v_k| \leq H|v_k|, \quad v_k \in E_k, \sigma \in \{s, u\}, a \in \{+, -\}, k \in \mathbb{Z}^a.$$

Remark 1.4. In Definition 1.9 we do not require the uniqueness of $E_k^{s,+}$, $E_k^{s,-}$, $E_k^{u,+}$, $E_k^{u,-}$. At the same time if $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^+$ then $E_k^{s,+}$ is uniquely defined and if $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^-$ then $E_k^{u,-}$ is uniquely defined [68, Proposition 2.3].

Recently the following were shown [106, Theorem 1, 2]:

Theorem 1.5. *A sequence $\mathcal{A}$ has bounded solution property if and only if the following two conditions hold:*

(ED) $\mathcal{A}$ has exponential dichotomy both on $\mathbb{Z}^+$ and $\mathbb{Z}^-$.

(TC) The corresponding spaces $E_0^{s,+}$, $E_0^{u,-}$ satisfy the following transversality condition

$$E_0^{s,+} + E_0^{u,-} = E_0.$$

Theorem 1.6. *The following statements are equivalent.*

(i) $\mathcal{A}$ *has exponential dichotomy on* $\mathbb{Z}^+$ *(*$\mathbb{Z}^-$*).*

(ii) There exists $L > 0$ *such that for any sequence* $\{w_k \in E_k\}$*,* $k \geq 0$ *(*$k \leq 0$*), satisfying* $|w_k| \leq 1$ *there exists sequence* $\{v_k \in E_k\}_{k\in\mathbb{Z}}$ *such that* $|v_k| \leq L$ *and*

$$v_{k+1} = A_k v_k + w_{k+1} \tag{1.8}$$

for $k \geq 0$ *(*$k \leq 0$*).*

Remark 1.7. Such type of results were also considered in [17,40,68,99], however we were not able to find in earlier literature statements which imply Theorems 1.5, 1.6. Similar results not for sequences of isomorphisms but for inhomogeneous linear systems of differential equations were obtained in [18, 52, 68, 84]. The relation between discrete and continuous settings is discussed in [73].

For us will be important connection between Exponential dichotomy and Structural stability.

First we introduce some notation. For a point $x \in M$, define the following two subspaces of T_xM:

$$B^+(x) = \{v \in T_xM \,:\, |Df^k(x)v| \to 0, \quad k \to +\infty\}$$

and

$$B^-(x) = \{v \in T_xM \,:\, |Df^k(x)v| \to 0, \quad k \to -\infty\}.$$

Proposition 1.8. *[Mañé, [54]]. The diffeomorphism* f *is structurally stable if and only if*

$$B^+(x) + B^-(x) = T_pM$$

for any $p \in M$*.*

1.3 Sublinear growth property

In this paragraph we prove the following theorem, which is interesting by itself without relation to shadowing property [104].

Theorem 1.9. *If a sequence* $\mathcal{A}$ *has sublinear growth property then it satisfies properties (ED) and (TC).*

As a consequence of this theorem we conclude that sublinear growth property and bounded solution property are in fact equivalent.

Remark 1.10. Note that sequences $\mathcal{A} \in \mathrm{SG}(1)$ do not necessarily satisfy condition (ED). A trivial example in arbitrary dimension is $\mathcal{A} = \{A_k = \mathrm{Id}\}$.

Proof of Theorem 1.9. Let us first prove the following.

Lemma 1.11. *If a sequence $\mathcal{A}$ satisfies slow growth property and (ED) then it satisfies (TC).*

Proof. Let $L, \gamma > 0$ be the constants from the definition of slow growth property and let $C > 0$, $\lambda \in (0,1)$ be the constants from the definition of exponential dichotomy on $\mathbb{Z}^{\pm}$. Let H be the constant from Remark 1.3 for exponential dichotomies on $\mathbb{Z}^{\pm}$. Assume that $E_0^{s,+} + E_0^{u,-} \neq E_0$. Let us choose a vector $\eta \in E_0 \setminus (E_0^{s,+} + E_0^{u,-})$ satisfying $|\eta| = 1$. Denote $a = \operatorname{dist}(\eta, E_0^{s,+} + E_0^{u,-})$. Consider the sequence $\{w_k \in E_k\}_{k\in\mathbb{Z}}$ defined by the formula

$$w_k = \begin{cases} 0, & k \neq 0, \\ \eta, & k = 0. \end{cases}$$

Take $N > 0$ and an arbitrary solution $\{v_k\}_{k\in[-N,N]}$ of

$$v_{k+1} = A_k v_k + w_k, \quad k \in [-N, N-1]. \tag{1.9}$$

Denote $v_k^{s,+} = P_k^{s,+} v_k$, $v_k^{u,+} = P_k^{u,+} v_k$ for $k \geq 0$. Since $w_k = 0$ for $k > 0$ we conclude

$$|v_N^{u,+}| \geq \frac{1}{C} \lambda^{-(N-1)} |v_0^{u,+}|$$

and hence

$$|v_N| \geq \frac{1}{H}\frac{1}{C} \lambda^{-(N-1)} |v_0^{u,+}|. \tag{1.10}$$

Similarly we denote $v_k^{s,-} = P_k^{s,-} v_k$, $v_k^{u,-} = P_k^{u,-} v_k$, for $k \leq 0$ and conclude

$$|v_{-N}| \geq \frac{1}{H}\frac{1}{C} \lambda^{-(N-1)} |v_{-1}^{s,-}|. \tag{1.11}$$

Equality (1.9) implies that

$$v_0 = A_{-1} v_{-1} + \eta$$

and hence

$$\max(\operatorname{dist}(v_0, E_0^{s,+} + E_0^{u,-}), \operatorname{dist}(A_{-1} v_{-1}, E_0^{s,+} + E_0^{u,-})) \geq a/2.$$

From this inequality it is easy to conclude that

$$v_0^{u,+} \geq \frac{1}{H}\frac{a}{2} \quad \text{or} \quad v_{-1}^{s,-} \geq \frac{1}{R}\frac{1}{H}\frac{a}{2}. \tag{1.12}$$

Inequalities (1.10)-(1.12) imply that

$$\max(|v_N|, |v_{-N}|) \geq \frac{1}{H}\frac{1}{C} \lambda^{-(N-1)} \frac{1}{R}\frac{1}{H}\frac{a}{2}.$$

Note that for large enough N the right hand side of this inequality is greater than $L(2N+1)^{\gamma}$ which contradicts to the sublinear growth property. $\square$

Now let us pass to the proof of Theorem 1.9. We prove this statement by induction over m (dimension of the Euclidian spaces). First we prove the following.

Lemma 1.12. *Theorem 1.9 holds for $m = 1$.*

Proof. Choose a vector $e_0 \in E_0$, $|e_0| = 1$ and consider the sequence $\{e_k \in E_k\}_{k\in\mathbb{Z}}$ defined by the relations

$$e_{k+1} = \frac{A_k e_k}{|A_k e_k|}, \; e_{-k-1} = \frac{A^{-1}_{-k-1} e_{-k}}{|A^{-1}_{-k-1} e_{-k}|} \quad k \geq 0. \tag{1.13}$$

Let $\lambda_k = |A_k e_k|$. Inequalities (1.3) imply that

$$\lambda_k \in (1/R, R), \quad k \in \mathbb{Z}. \tag{1.14}$$

Denote

$$\Pi(k,l) = \lambda_k \cdot \dots \cdot \lambda_{k+l-1}, \quad k \in \mathbb{Z}, l \geq 1. \tag{1.15}$$

Let us prove the following lemma, which is the heart of the proof of Theorem 1.9.

Lemma 1.13. *If $m = 1$ and $\mathcal{A}$ satisfies sublinear growth property then there exists $N > 0$ such that for any $i \in \mathbb{Z}$*

$$\Pi(i, N) > 2 \quad \text{or} \quad \Pi(i+N, N) < 1/2.$$

Proof. Let us fix $i \in \mathbb{Z}$, $N > 0$ and consider the sequence

$$w_k = -e_k, \quad k \in [i+1, i+2N+1].$$

By sublinear growth property there exists a sequence $\{v_k\}_{k\in[i,i+2N+1]}$ satisfying

$$v_{k+1} = A_k v_k + w_k, \quad |v_k| \leq L(2N+1)^\gamma, \quad k \in [i, i+2N].$$

Let $v_k = a_k e_k$, where $a_k \in \mathbb{R}$, then

$$a_{k+1} = \lambda_k a_k - 1, \quad |a_k| \leq L(2N+1)^\gamma, \quad k \in [i, i+2N]. \tag{1.16}$$

Those relations easily imply the following

Proposition 1.14. *If $a_k \leq 0$ for some $k \in [i, i+2N-1]$ then $a_{k+1} < 0$.*

Below we prove the following: There exists a large $N > 0$ (depending only on R, L, γ) such that

Case 1. if $a_{i+N-1} \geq 0$ then $\Pi(i, N) > 2$,

Case 2. if $a_{i+N-1} < 0$ then $\Pi(i+N, N) < 1/2$.

We give the proof of the case 1 in details, the second case is similar. Proposition 1.14 implies that $a_i, \dots, a_{i+N-2} > 0$, $a_{i+N-1} \geq 0$. Relation (1.16) implies that

$$\lambda_k = \frac{a_{k+1} + 1}{a_k}, \quad k \in [i, i+N-1].$$

The following relations hold (compare with (1.84))

$$\begin{aligned}\Pi(i,N) = \frac{a_{i+1}+1}{a_i}\frac{a_{i+2}+1}{a_{i+1}}\dots\frac{a_{i+N-1}+1}{a_{i+N-2}} = \\ = \frac{1}{a_i}\frac{a_{i+1}+1}{a_{i+1}}\frac{a_{i+2}+1}{a_{i+2}}\dots\frac{a_{i+N-2}+1}{a_{i+N-2}}(a_{i+N-1}+1) = \\ = \frac{a_{i+N-1}+1}{a_i}\prod_{k=i+1}^{i+N-2}\frac{a_k+1}{a_k} \geq \frac{1}{L(2N+1)^\gamma}\left(1+\frac{1}{L(2N+1)^\gamma}\right)^{N-2}.\end{aligned}$$

Denote the latter expression by $G_\gamma(N)$. The inclusion $\gamma \in (0,1)$ implies that

$$\lim_{N\to+\infty} G_\gamma(N) = +\infty \tag{1.17}$$

and for large enough N the inequality $G_\gamma(N) > 2$ holds, which completes the proof of Case 1.

Remark 1.15. In relation (1.17) we essentially use that $\gamma \in (0,1)$; for $\gamma \geq 1$ it does not hold.

□

Lemma 1.16. *Let N be the number from Lemma 1.13.*

(i) If $\Pi(i,N) > 2$ then $\Pi(i-N,N) > 2$.

(ii) If $\Pi(i,N) < 1/2$ then $\Pi(i+N,N) < 1/2$.

Proof. We prove statement (i); the second one is similar. Lemma 1.13 implies that either $\Pi(i-N,N) > 2$ or $\Pi(i,N) < 1/2$. By the assumptions of Lemma 1.16 the second case is not possible and hence $\Pi(i-N,N) > 2$. □

Now let us complete the proof of Lemma 1.12. It is easy to conclude from Lemmas 1.13, 1.16 that one of the following cases holds.

Case 1. For all $i \in \mathbb{Z}$ the inequality $\Pi(i,N) > 2$ holds. Then

$$\Pi(i,l) \geq R^{N-1}(2^{1/N})^l, \quad i \in \mathbb{Z}, l > 0$$

and hence $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^\pm$ with the splitting

$$E_k^{s,\pm} = \{0\}, \quad E_k^{u,\pm} = \langle e_k \rangle, \quad k \in \mathbb{Z}.$$

Case 2. For all $i \in \mathbb{Z}$ the inequality $\Pi(i,N) < 1/2$ holds. Similarly to the previous case $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^\pm$ with the splitting

$$E_k^{s,\pm} = \langle e_k \rangle, \quad E_k^{u,\pm} = \{0\}.$$

Case 3. There exist $i_1, i_2 \in \mathbb{Z}$ such that

$$\Pi(i_1, N) > 2, \quad \Pi(i_2, N) < 1/2.$$

Similarly to Case 1 the following inequality holds

$$\Pi(k,l) \geq R^{N-1}(2^{1/N})^l, \quad k + l < i_1, l > 0$$

and hence

$$\Pi(k,l) \geq R^{|i_1|+N-1}(2^{1/N})^l, \quad k + l < 0, l > 0.$$

The last inequality implies that $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^-$ with the splitting

$$E_k^{s,-} = \{0\}, \quad E_k^{u,-} = \langle e_k \rangle, \quad k \leq 0.$$

Similarly $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^+$ with the splitting

$$E_k^{s,+} = \langle e_k \rangle, \quad E_k^{u,+} = \{0\}, \quad k \geq 0.$$

In all of those cases Lemma 1.12 is proved. □

Now let us continue the proof of Theorem 1.9. Assume that Theorem 1.9 is proved for $\dim E_k \leq m$. Below we prove it for $\dim E_k = m + 1$.

Let us choose a unit vector $e_0 \in E_0$ and consider the vectors $\{e_k\}_{k \in \mathbb{Z}}$ defined by relations (1.13). Denote $\lambda_k = |A_k e_k|$. Similarly to Lemma 1.12 inclusions (1.14) hold. For $k \in \mathbb{Z}$ let S_k be the orthogonal complement of e_k in E_k and let Q_k be the orthogonal projection onto S_k. Note that $\dim S_k = m$. Consider the linear operators $B_k : S_k \to S_{k+1}$, $D_k : S_k \to \langle e_{k+1} \rangle$ defined by the following

$$B_k = Q_{k+1} A_k, \quad D_k = (\mathrm{Id} - Q_{k+1}) A_k, \quad k \in \mathbb{Z}.$$

Note that $B_k^{-1} = Q_{k-1} A_k^{-1}$ and

$$\|B_k\|, \|B_k^{-1}\|, \|D_k\| < R. \tag{1.18}$$

For any vector $b \in E_k$ denote by $b^\perp = P_k b, \quad b^1 = b - b^\perp$. We also write $b = (b^\perp, b^1)$. In such notation equations (1.4) are equivalent to

$$v_{k+1}^\perp = B_k v_k^\perp + w_{k+1}^\perp, \tag{1.19}$$

$$v_{k+1}^1 = \lambda_k v_k^1 + D_k v_k^\perp + w_{k+1}^1. \tag{1.20}$$

Let us prove that the sequence $\{B_k\}$ satisfies property $\mathrm{SG}(\gamma)$. Indeed, fix $i \in \mathbb{Z}$, $N > 0$ and consider an arbitrary sequence $\{w_k^\perp \in S_k\}_{k \in [i+1, i+N+1]}$ with $|w_k^\perp| \leq 1$. Consider the sequence $\{w_k \in E_k\}_{k \in [i+1, i+N+1]}$ defined by $w_k = w_k^\perp$. By the sublinear growth property there exists a sequence $\{v_k \in E_k\}_{k \in [i, i+N+1]}$ satisfying (1.4), (1.5) and hence (1.19). Recalling that $|v_k^\perp| \le |v_k|$ we conclude that the sequence $\{B_k\}$ satisfies sublinear growth property and hence by the induction assumption if satisfies conditions (ED) and (TC) from Theorem 1.5.

Below we prove that $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^+$. Let $\{B_i\}$ satisfy exponential dichotomy on $\mathbb{Z}^+$ with constants $C > 0$, $\lambda \in (0, 1)$ and splitting $S_k = S_k^{s,+} \oplus S_k^{u,+}$. Let H_1 be the constant from Remark 1.3 for this splitting.

First we prove that there exists a big $N > 0$ such that for any $i \ge 2N$ the following inequality hold

$$\Pi(i, N) > 2 \quad \text{or} \quad \Pi(i - N, N) < 1/2, \tag{1.21}$$

where $\Pi(k, l)$ is defined by (1.15).

Let us choose $N > 0$ satisfying

$$C\lambda^N H_1 L(4N)^\gamma < 1/(4R). \tag{1.22}$$

and consider some $i \ge 2N$. Define a sequence $\{w_k = -e_k\}_{k\in[i-2N,i+2N]}$. By slow growth property there exists a sequence $\{v_k = (v_k^\perp, v_k^1)\}_{k\in[i-2N,i+2N+1]}$ satisfying the following for $k \in [i - 2N, i + 2N]$:

$$v_{k+1}^\perp = B_k v_k^\perp \tag{1.23}$$

$$v_{k+1}^1 = \lambda_k v_k^1 + D_k v_k^\perp - 1, \tag{1.24}$$

$$|v_k| < L(4N)^\gamma. \tag{1.25}$$

Represent $v_k^\perp = v_k^{\perp,s} + v_k^{\perp,u}$, where $v_k^{\perp,s} \in S_k^{s,+}$, $v_k^{\perp,u} \in S_k^{u,+}$. Applying relations (1.23), (1.25) and Remark 1.3 we conclude that

$$|v_k^{\perp,s}|, |v_k^{\perp,u}| < H_1 L(4N)^\gamma, k \in [i - 2N, i + 2N].$$

Exponential dichotomy of $\{B_i\}$ implies that

$$|v_k^{\perp,s}|, |v_k^{\perp,u}| < C\lambda^N H_1 L(4N)^\gamma, \quad k \in [i - N, i + N].$$

By inequality (1.22) we conclude that

$$|v_k^{\perp,s}|, |v_k^{\perp,u}| < 1/(4R), \quad k \in [i - N, i + N]$$

and hence

$$|v_k^\perp| < 1/(2R), \quad k \in [i - N, i + N]. \tag{1.26}$$

Denote $b_k = D_k v_k^\perp - 1$. Inequalities (1.18) and (1.26) imply that

$$b_k \in (-3/2, -1/2), \quad k \in [i - N, i + N].$$

Using those inclusions, relations (1.24), (1.25) and arguing similarly to Lemma 1.13 (increasing N if necessarily) we conclude relation (1.21).

Arguing similarly to the proof of Lemma 1.12 we conclude that the linear operators generated by λ_i have exponential dichotomy on $\mathbb{Z}^+$.

Let us show that $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^+$. Consider an arbitrary sequence

$$\{w_k = (w_k^\perp, w_k^1) \in E_k\}_{k\geq 0}, \quad |w_k| \leq 1.$$

Since $\{B_k\}$ has exponential dichotomy on $\mathbb{Z}^+$, by Theorem 1.6 there exists a sequence $\{v_k^\perp \in S_k\}_{k\geq 0}$, satisfying (1.19) and $|v_k| \leq L_1$, where $L_1 > 0$ does not depend on $\{w_k\}$. Inequality (1.18) implies that

$$|D_k v_k^\perp + w_{k+1}^1| \leq L_1 R + 1, \quad k \geq 0.$$

Since linear operators generated by λ_k have exponential dichotomy on $\mathbb{Z}^+$, by Theorem 1.6 there exists $\{v_k^1 \in \mathbb{R}\}$ such that for $k \geq 0$ equalities (1.20) hold and $|v_k^1| \leq L_2(L_1 R + 1)$, where L_2 does not depend on $\{w_k\}$.

Hence for $k \geq 0$ the sequence $v_k = (v_k^\perp, v_k^1)$ satisfies (1.8) and

$$|v_k| \leq |v_k^\perp| + |v_k^1| \leq L_2(L_1 R + 1) + L_1.$$

Theorem 1.6 implies that $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^+$.

Similarly $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^-$ and hence satisfies property (ED). By Lemma 1.11 the sequence $\mathcal{A}$ also satisfies property (TC). This completes the induction step and the proof of Theorem 1.9.

□

It follows from Proposition 1.8 and Theorem 1.9

Theorem 1.17. *If for any trajectory x_k of a diffeomorphism f the sequence $\mathcal{A} = \{A_k = \mathrm{D}\, f(x_k)\}$ satisfies sublinear growth property then f is structurally stable.*

1.4 Lipschitz shadowing

In this paragraph, we study the relation between Lipschitz shadowing property and structural stability without the passage to C^1 topology. We show that Lipschitz shadowing property is equivalent to structural stability [81].

Theorem 1.18. *The following two statements are equivalent:*

(1) f has Lipschitz shadowing property;

(2) f is structurally stable.

As a corollary, we show that an expansive diffeomorphism having Lipschitz shadowing property is Anosov.

Corollary 1.19. *The following two statements are equivalent:*

(1) f is expansive and has Lipschitz shadowing property;

(2) f is Anosov.

Let us mention that Ombach [60] and Walters [108] showed that a diffeomorphism f is Anosov if and only if f has shadowing property and is strongly expansive (which means that all the diffeomorphisms in a C^1-small neighborhood of f are expansive with the same expansivity constant).

Proof of the Corollary 1.19. The implication (2) $\Rightarrow$ (1) is well known (see, for example, [72]). By our theorem, condition (1) of the corollary implies that f is structurally stable, and it was shown by Mañé that an expansive structurally stable diffeomorphism is Anosov (see [53]). $\square$

Now we pass to the proof of the main theorem.

The implication (2) $\Rightarrow$ (1) is well known (see, for example, [72]).

The proof of the implication (1) $\Rightarrow$ (2) is trivial consequence of Theorem 1.17 and the following lemma.

Lemma 1.20. *If f has the Lipschitz shadowing property with constants $\mathcal{L}, d_0$, then for any sequence $\{w_k \in T_{p_k}M, k \in \mathbb{Z}\}$ such that $|w_k| < 1, k \in \mathbb{Z}$, there exists a sequence $\{v_k \in T_{p_k}M, k \in \mathbb{Z}\}$ such that*

$$|v_k| \le 8\mathcal{L} + 1, \quad v_{k+1} = A_k v_k + w_k, \quad k \in \mathbb{Z}. \tag{1.27}$$

To prove Lemma 1.20, we first prove the following statement.

Lemma 1.21. *Assume that f has the Lipschitz shadowing property with constants $\mathcal{L}, d_0$. Fix a trajectory $\{x_k\}$ and a natural number n. For any sequence $\{w_k \in T_{p_k}M, k \in [-n, n]\}$ such that $|w_k| < 1$ for $k \in [-n, n]$ and $w_k = 0$ for $k \notin [-n, n]$ there exists a sequence $\{z_k \in T_{p_k}M, k \in \mathbb{Z}\}$ such that*

$$|z_k| \le 8\mathcal{L} + 1, \quad k \in \mathbb{Z}, \tag{1.28}$$

and

$$z_{k+1} = A_k z_k + w_k, \quad k \in [-n, n]. \tag{1.29}$$

Proof. First we locally "linearize" the diffeomorphism f in a neighborhood of the trajectory $\{x_k\}$.

Let exp be the standard exponential mapping on the tangent bundle of M and let $\exp_x : T_xM \to M$ be the corresponding exponential mapping at a point x.

We introduce the mappings

$$F_k = \exp_{p_{k+1}}^{-1} \circ f \circ \exp_{p_k} : T_{x_k}M \to T_{x_{k+1}}M. \tag{1.30}$$

It follows from the standard properties of the exponential mapping that $D\exp_x(0) = \mathrm{Id}$; hence, $DF_k(0) = A_k$. Since M is compact, for any $\mu > 0$ we can find $\delta > 0$ such that if $|v| \le \delta$, then

$$|F_k(v) - A_k v| \le \mu |v|. \tag{1.31}$$

Denote by $B(r, x)$ the ball in M of radius r centered at a point x and by $B_T(r, x)$ the ball in T_xM of radius r centered at the origin.

There exists $r > 0$ such that, for any $x \in M$, $\exp_x$ is a diffeomorphism of $B_T(r, x)$ onto its image, and $\exp_x^{-1}$ is a diffeomorphism of $B(r, x)$ onto its image. In addition, we may assume that r has the following property.

If $v, w \in B_T(r, x)$, then

$$\frac{\operatorname{dist}(\exp_x(v), \exp_x(w))}{|v - w|} \leq 2; \tag{1.32}$$

if $y, z \in B(r, x)$, then

$$\frac{|\exp_x^{-1}(y) - \exp_x^{-1}(z)|}{\operatorname{dist}(y, z)} \leq 2. \tag{1.33}$$

Now we pass to construction of pseudotrajectories; every time, we take d so small that the considered points of our pseudotrajectories, points of shadowing trajectories, their "lifts" to tangent spaces etc belong to the corresponding balls $B(r, x_k)$ and $B_T(r, x_k)$ (and we do not repeat this condition on the smallness of d).

Fix a sequence w_k having the properties stated in Lemma 1.21. Consider the sequence $\{\Delta_k \in T_{p_k}M, k \in [-n, n+1]\}$ defined as follows:

$$\begin{cases} \Delta_{-n} = 0, \\ \Delta_{k+1} = A_k \Delta_k + w_k, \quad k \in [-n, n]. \end{cases} \tag{1.34}$$

Let $Q = \max_{k \in [-n, n+1]} |\Delta_k|$.

Fix a small $d > 0$ and construct a pseudotrajectory $\{\xi_k\}$ as follows:

$$\begin{cases} \xi_k = \exp_{p_k}(d\Delta_k), k \in [-n, n+1], \\ \xi_l = f^{l+n}(\xi_{-n}), l \leq -n-1, \\ \xi_l = f^{l-n-1}(\xi_{n+1}), l > n+1. \end{cases}$$

Note that definition (1.34) of the vectors Δ_k and condition (1.32) imply that if d is small enough, then the following inequality holds:

$$\operatorname{dist}(\xi_{k+1}, \exp_{x_{k+1}}(dA_k\Delta_k)) < 2d.$$

Since

$$f(\xi_k) = \exp_{x_{k+1}}(F_k(d\Delta_k)),$$

condition (1.31) with $\mu < 1$ implies that if d is small enough, then

$$\operatorname{dist}(\exp_{x_{k+1}}(dA_k\Delta_k), f(\xi_k)) < 2d.$$

Hence,

$$\operatorname{dist}(f(\xi_k), \xi_{k+1}) \leq 4d.$$

Let us note that the required smallness of d is determined by the chosen trajectory $\{x_k\}$, the sequence w_k, and the number n.

The Lipschitz shadowing property of f implies that if d is small enough, then there exists an exact trajectory $\{y_k\}$ such that

$$\operatorname{dist}(\xi_k, y_k) \le 4\mathcal{L}d, \quad k \in [-n, n+1]. \tag{1.35}$$

Consider the finite sequence

$$\{t_k = \frac{1}{d}\exp_{x_k}^{-1}(y_k),\ k \in [-n, n+1]\}.$$

Inequalities (1.35) and (1.33) imply that

$$|\Delta_k - t_k| < 8\mathcal{L}. \tag{1.36}$$

Consider the finite sequence $\{b_k \in T_{x_k}M,\ k \in [-n, n+1]\}$ defined as follows:

$$b_{-n} = t_{-n}, \quad b_{k+1} = A_k b_k, \quad k \in [-n, n]. \tag{1.37}$$

Obviously, the following inequalities hold for $k \in [-n, n+1]$:

$$\operatorname{dist}(y_k, x_k) \le \operatorname{dist}(y_k, \xi_k) + \operatorname{dist}(x_k, \xi_k) \le 4\mathcal{L}d + 2d|\Delta_k| \le 2(Q + 2\mathcal{L})d.$$

These inequalities and inequalities (1.33) imply that

$$|t_k| \le 4(Q + 2\mathcal{L}). \tag{1.38}$$

Take $\mu_1 > 0$ such that

$$((N+1)^{2n} + (N+1)^{2n-1} + \cdots + 1)\mu_1 < 1, \tag{1.39}$$

where $N = \sup \|A_k\|$.

Set

$$\mu = \frac{\mu_1}{4(Q + 2\mathcal{L})}$$

and consider d so small that inequality (1.31) holds for $\delta = 4(Q + 2L)d$.

The definition of the vectors t_k implies that $dt_{k+1} = F_k(dt_k)$; since

$$|dt_k| \le 4d(Q + 2\mathcal{L})$$

by (1.38), we deduce from estimate (1.31) applied to $v = dt_k$ that

$$|dt_{k+1} - dA_k t_k| \le \mu d |t_k|.$$

Now we deduce from inequalities (1.38) that

$$|t_{k+1} - A_k t_k| \le 4\mu(Q + 2\mathcal{L}) = \mu_1, \quad k \in [-n, n]. \tag{1.40}$$

Consider the sequence $c_k = t_k - b_k$. Note that $c_{-n} = 0$ by (1.37). Estimates (1.40) imply that $|c_{k+1} - A_k c_k| \le \mu_1$. Hence,

$$|c_k| \le ((N+1)^{2n} + (N+1)^{2n-1} + \dots + 1)\mu_1 < 1, \quad k \in [-n, n].$$

Thus,

$$|t_k - b_k| < 1. \tag{1.41}$$

Consider the sequence $\{z_k \in T_{p_k}M, k \in \mathbb{Z}\}$ defined as follows:

$$\begin{cases} z_k = \Delta_k - b_k, k \in [-n, n+1], \\ z_k = 0, k \notin [-n, n+1]. \end{cases}$$

Inequalities (1.36) and (1.41) imply estimate (1.28), while equalities (1.34) and (1.37) imply relations (1.29). Lemma 1.21 is proved. □

Proof of Lemma 1.20. Fix $n > 0$ and consider the sequence

$$w_k^{(n)} = \begin{cases} w_k, k \in [-n, n], \\ 0, |k| > n. \end{cases}$$

By Lemma 1.21, there exists a sequence $\{z_k^{(n)} \in T_{p_k}M, k \in \mathbb{Z}\}$ such that

$$|z_k^{(n)}| \le 8\mathcal{L} + 1, \quad k \in \mathbb{Z}, \tag{1.42}$$

and

$$z_{k+1}^{(n)} = A_k z_k^{(n)} + w_k^{(n)}, \quad k \in [-n, n]. \tag{1.43}$$

Passing to a subsequence of $\{z_k^{(n)}\}$, we can find a sequence $\{v_k \in T_{p_k}M, k \in \mathbb{Z}\}$ such that

$$v_k = \lim_{n \to \infty} z_k^{(n)}, \quad k \in \mathbb{Z}.$$

(Let us note that we do not assume uniform convergence.) Passing to the limit in estimates (1.42) and equalities (1.43) as $n \to \infty$, we get relations (1.27). Lemma 1.20 and our theorem are proved. □

1.5 Hölder shadowing

In this paragraph we study pseudotrajectories of finite length. Note that currently such pseudotrajectories are almost not investigated. This problem is strongly related to the dependence between ε and d in the Definition 1.3. We study shadowing properties on finite intervals with polynomial dependence of ε and d and give an upper bound for the length of shadowable pseudotrajectories for non-hyperbolic systems.

Definition 1.10. We say that f has the *Finite Hölder shadowing property* with exponents $\theta \in (0,1)$, $\omega \geq 0$ (FinHolSh(θ,ω)) if there exist constants $d_0, L, C > 0$ such that for any $d < d_0$ and d-pseudotrajectory $\{y_k\}_{k\in[0,Cd^{-\omega}]}$ there exists a trajectory $\{x_k\}_{k\in[0,Cd^{-\omega}]}$ such that

$$\mathrm{dist}(x_k, y_k) < Ld^{\theta}, \quad k \in [0, Cd^{-\omega}].$$

Note that previously S. Hammel, J. Yorke and C. Grebogi based on results of numerical experiments conjectured the following [32, 33]:

Conjecture 1.1. *A typical dissipative map $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies* FinHolSh$(1/2, 1/2)$.

In this paragraph we prove the following theorem [104].

Theorem 1.22. *If a diffeomorphism $f \in C^2$ satisfies* FinHolSh(θ,ω) *with*

$$\theta > 1/2, \quad \theta + \omega > 1 \tag{1.44}$$

then f is structurally stable.

Conjecture 1.1 suggests that Theorem 1.22 cannot be improved.
Theorem 1.22 has an interesting consequence even for the case of infinite pseudotrajectories.

Definition 1.11. We say that f has *Hölder shadowing property* with exponent $\theta \in (0,1)$ (HolSh(θ)) if there exist constants $d_0, L > 0$ such that for any $d < d_0$ and d-pseudotrajectory $\{y_k\}_{k\in\mathbb{Z}}$ there exists a trajectory $\{x_k\}_{k\in\mathbb{Z}}$ such that inequalities (1.1) hold with $\varepsilon = Ld^{\theta}$.

It is easy to see that for any $\theta \in (0,1)$ and $\omega \geq 0$ the following inclusions hold

$$\mathrm{SS} = \mathrm{LipSh} \subset \mathrm{HolSh}(\theta) = \mathrm{FinHolSh}(\theta, +\infty) \subset \mathrm{FinHolSh}(\theta,\omega),$$

where SS denotes the set of structurally stable diffeomorphisms and LipSh, HolSh, FinHolSh denote sets of diffeomorphisms satisfying the corresponding shadowing properties.
The following theorem is a straightforward consequence of Theorem 1.22.

Theorem 1.23. *If a diffeomorphism $f \in C^2$ satisfies* HolSh(θ) *with $\theta > 1/2$ then f is structurally stable.*

Note that this theorem generalizes Theorem 1.18. Let us also mention a related work [42], where some consequences of Hölder shadowing for 1-dimensional maps were proved.
It is worth to mention a relation between Theorem 1.23 and a question suggested by Katok:

Question 1.1. *Is every diffeomorphism that is Hölder conjugate to an Anosov diffeomorphism itself Anosov?*

Recently it was shown that in general the answer to Question 1.1 is negative [27]. At the same time the following positive result was proved in [27].

Theorem 1.24. *A C^2-diffeomorphism that is conjugate to an Anosov diffeomorphism via Hölder conjugacy h is Anosov itself, provided that the product of Hölder exponents for h and h^{-1} is greater than $1/2$.*

It is easy to show that diffeomorphisms which are Hölder conjugate to a structurally stable one satisfy Hölder shadowing property. As a consequence of Theorem 1.23 we prove that a C^2-diffeomorphism that is conjugate to a structurally stable diffeomorphism via Hölder conjugacy h is structurally stable itself, provided that the product of Hölder exponents for h and h^{-1} is greater than $1/2$, which generalizes Theorem 1.24.

In order to prove Theorem 1.22 we prove the following relation between shadowing and sublinear growth properties.

Lemma 1.25. *If f satisfies assumptions of Theorem 1.22 then there exists $\gamma \in (0,1)$ such that for any trajectory $\{p_k\}_{k\in\mathbb{Z}}$ the sequence $\{A_k = \mathrm{D}\, f(p_k)\}$ satisfies* $\mathrm{SG}(\gamma)$.

Theorem 1.22 follows from this lemma and Theorem 1.17.

Proof of Lemma 1.25. Define exp, $\exp_x$, $B(r,x)$, $B_T(r,x)$ similarly to Section 1.4 and choose $\varepsilon > 0$ such that conditions (1.32) and (1.33) hold for balls $B_T(\varepsilon, x)$ and $B(\varepsilon, x)$.

Let $L, C, d_0 > 0$ and $\theta \in (1/2, 1)$, $\omega > 0$ be the constants from the definition of FinHolSh. Denote $\alpha = \theta - 1/2$. Inequalities (1.44) imply that

$$\alpha \in (0, 1/2), \quad 1/2 - \alpha < \omega. \tag{1.45}$$

Since M is compact and $f \in C^2$ there exists $S > 0$ such that

$$\mathrm{dist}(f(\exp_x(v)), \exp_{f(x)}(\mathrm{D}\, f(x)v)) \le S|v|^2, \quad x \in M,\ v \in T_xM,\ |v| < \varepsilon, \tag{1.46}$$

(we additionally decrease ε, if necessarily).

Fix $i \in \mathbb{Z}$ and $N > 0$. For an arbitrary sequence $\{w_k \in T_{p_k}M\}_{k\in[i+1,i+N+1]}$ with $|w_k| \le 1$ consider the following equations

$$v_{k+1} = A_k v_k + w_{k+1}, \quad k \in [i, i+N]. \tag{1.47}$$

For any sequence $\{v_k \in T_{p_k}M\}_{k\in[i,i+N+1]}$ denote $\|\{v_k\}\| = \max_{k\in[i,i+N+1]} |v_k|$. For any sequence $\{w_k \in T_{p_k}M\}_{k\in[i+1,i+N+1]}$ consider the set

$$E(i, N, \{w_k\}) = \left\{\{v_k\}_{k\in[i,i+N+1]} \text{ satisfies (1.47)}\right\}.$$

Denote

$$F(i, N, \{w_k\}) = \min_{\{v_k\}\in E(i,N,\{w_k\})} \|\{v_k\}\|. \tag{1.48}$$

Since $\|\cdot\| \ge 0$ is a continuous function on the linear space of sequences $\{v_k\}$ and the set $E(i, N, \{w_k\})$ is closed it follows that the value $F(i, N, \{w_k\})$ is well-defined. Note that a sequence $\{v_k\} \in E(i, N, \{w_k\})$ is determined by the value v_i. Consider the sequence $\{v_k\}$ corresponding to $v_i = 0$. It is easy to see that $|v_{i+k}| \le 1 + R + R^2 + \cdots + R^k$ for

$k \in [0, N+1]$, where $R = \max_{x\in M} \| \mathrm{D} f(x)\|$. Hence $F(i, N, \{w_k\}) \le 1 + R + R^2 + \cdots + R^{2N}$ for any $\{|w_k| \le 1\}$. It is easy to see that $F(i, N, \{w_k\})$ is continuous with respect to $\{w_k\}$ and hence

$$Q = Q(i, N) = \max_{\{w_k\},\, |w_k|\le 1} F(i, N, \{w_k\}) \tag{1.49}$$

is well defined.

Let us choose sequences $\{w_k\}$ and $\{v_k\} \in F(i, N, \{w_k\})$ such that

$$Q(i, N) = F(i, N, \{w_k\}), \quad F(i, N, \{w_k\}) = \|\{v_k\}\|.$$

The definition of Q and linearity of equation (1.47) imply the following two properties.

(Q1) For any sequence $\{w'_k\}_{k\in[i+1,i+N+1]}$ there exists a sequence $\{v'_k\}_{k\in[i,i+N+1]}$ satisfying

$$v'_{k+1} = A_k v'_k + w'_{k+1}, \quad \|\{v'_k\}\| \le Q(i, N)\|\{w'_k\}\|.$$

(Q2) For any sequence $\{v_k\}_{k\in[i,i+N+1]}$, satisfying (1.47) holds the following inequality

$$\|\{v_k\}\| \ge Q(i, N).$$

Relations (1.45) imply that there exists $\beta > 0$ such that the following conditions holds

$$0 < (2 + \beta)(1/2 - \alpha) < 1, \quad (2 + \beta)\omega > 1. \tag{1.50}$$

Denote

$$\gamma = \frac{1}{(2 + \beta)\omega} \in (0, 1), \quad \gamma' = 1 - (2 + \beta)(1/2 - \alpha) > 0,$$

$$d = \frac{\varepsilon}{Q^{2+\beta}}. \tag{1.51}$$

Let us prove that there exist $L' > 0$ independent of i and N such that

$$Q(i, N) \le L' N^{\gamma}. \tag{1.52}$$

Below we consider two cases.

Case 1. $C((S + 2)d)^{-\omega} < N$. Then $Q < (\varepsilon^{\omega}(S + 2)^{\omega}/C)^{\gamma} N^{\gamma}$ and inequality (1.52) is proved.

Case 2. $C((S + 2)d)^{-\omega} \ge N$. Below we prove even a stronger statement: there exists $L' > 0$ (independent of i and N) such that

$$Q(i, N) \le L'. \tag{1.53}$$

Considering the trajectory $\{p'_k = f^{-i}(p_k)\}$ we can assume without loss of generality that $i = 0$.

Consider the sequence

$$y_k = \exp_{p_k}(dv_k), \quad k \in [0, N].$$

Let us show that $\{y_k\}$ is an $(S+2)d$-pseudotrajectory. For $k \in [0, N]$ equations (1.32), (1.46) and inequalities $|dv_k| < \varepsilon$, $(dQ)^2 < d$ imply the following:

$$\begin{aligned} \operatorname{dist}(f(y_k), y_{k+1}) = \operatorname{dist}(f(\exp_{p_k}(dv_k)), \exp_{p_{k+1}}(d(A_k v_k + w_{k+1})) \leq \\ \leq \operatorname{dist}(f(\exp_{p_k}(dv_k)), \exp_{p_{k+1}}(dA_k v_k)) + \\ \operatorname{dist}(\exp_{p_{k+1}}(dA_k v_k), \exp_{p_{k+1}}(d(A_k v_k + w_k))) \leq \\ \leq S|dv_k|^2 + 2d \leq (S+2)d. \quad (1.54) \end{aligned}$$

We may assume that

$$Q > ((S+2)\varepsilon/d_0)^{1/(2+\beta)}. \tag{1.55}$$

Indeed, the righthand side of (1.55) does not depend on N, and if Q is smaller than the right side of (1.55) then we have already proved (1.52). In the text below we make similar remarks several times to ensure that Q is large enough.

Inequality (1.55) implies that $(S+2)d < d_0$. Since $f \in \text{FinHolSh}(1/2+\alpha, \omega)$ and the assumption of case 2 holds it follows that the pseudotrajectory $\{y_k\}_{k\in[0,N]}$ can be $L((S+2)d)^{1/2+\alpha}$-shadowed by a trajectory $\{x_k\}_{k\in[0,N]}$.

By reasons similar to (1.55) we may assume that $L((S+2)d)^{1/2+\alpha} < \varepsilon/2$. Inequalities (1.32) and (1.55) imply that for $k \in [0, N]$ the following inequalities hold

$$\operatorname{dist}(p_k, x_k) \leq \operatorname{dist}(p_k, y_k) + \operatorname{dist}(y_k, x_k) \leq 2d|v_k| + L((S+2)d)^{1/2+\alpha} < \varepsilon.$$

Hence $c_k = \exp_{p_k}^{-1}(x_k)$ is well-defined.

Denote $L_1 = L(S+2)^{1/2+\alpha}$. Since $\operatorname{dist}(y_k, x_k) < L_1 d^{1/2+\alpha}$, inequalities (1.33) imply that

$$|dv_k - c_k| < 2L_1 d^{1/2+\alpha}. \tag{1.56}$$

Hence

$$|c_k| < Qd + 2L_1 d^{1/2+\alpha}. \tag{1.57}$$

By the reasons similar to (1.55) we can assume that $|c_k| < \varepsilon$.

Since $f(x_k) = x_{k+1}$ inequalities (1.33) and (1.46) imply that for $k \in [0, N]$ the following relations hold

$$\begin{aligned} |c_{k+1} - A_k c_k| < 2\operatorname{dist}(\exp_{p_{k+1}}(c_{k+1}), \exp_{p_{k+1}}(A_k c_k)) = \\ = 2\operatorname{dist}(f(\exp_{p_k}(c_k)), \exp_{p_{k+1}}(A_k c_k)) \leq 2S|c_k|^2. \quad (1.58) \end{aligned}$$

Inequalities (1.50), (1.51), (1.57) imply that $|c_k| < L_2 Qd$ for some $L_2 > 0$ independent of N.

Let $t_{k+1} = c_{k+1} - A_k c_k$. By inequality (1.58) it follows that

$$|t_k| \leq 2S|c_k|^2 \leq L_3(Qd)^2$$

for some $L_3 > 0$ independent of N. Property (Q1) implies that there exists a sequence $\{\tilde{c}_k \in T_{p_k}M\}$ satisfying

$$\tilde{c}_{k+1} - A_k\tilde{c}_k = t_{k+1}, \quad |\tilde{c}_k| \leq QL_3(Qd)^2, \quad k \in [0, N].$$

Consider the sequence $r_k = c_k - \tilde{c}_k$. Obviously it satisfies the following conditions

$$r_{k+1} = A_k r_k, \quad |r_k - c_k| \le QL_3(Qd)^2, \quad k \in [0, N]. \tag{1.59}$$

Consider the sequence $e_k = \frac{1}{d}(dv_k - r_k)$. Equations (1.56) and (1.59) imply that

$$e_{k+1} = A_k e_k + w_k, \quad k \in [0, N] \tag{1.60}$$

and

$$|e_k| = \left| \frac{1}{d} \left((dv_k - c_k) - (r_k - c_k) \right) \right| \le L_1 d^{-1/2+\alpha} + L_3 Q^3 d, \quad k \in [0, N].$$

Property (Q2) implies that

$$L_1 d^{-1/2+\alpha} + L_3 Q^3 d \ge Q.$$

By (1.51) the last inequality is equivalent to

$$L_4 Q^{-(2+\beta)(-1/2+\alpha)} + L_5 Q^{1-\beta} \ge Q,$$

where $L_4, L_5 > 0$ do not depend on N. This inequality and (1.50) imply that

$$L_4 Q^{1-\gamma'} + L_5 Q^{1-\beta} \ge Q.$$

Hence

$$L_4 Q^{1-\gamma'} \ge Q/2 \quad \text{or} \quad L_5 Q^{1-\beta} \ge Q/2,$$

and

$$Q \le \max((2L_4)^{1/\gamma'}, (2L_5)^{1/\beta}).$$

We have proved that there exists $L' > 0$ such that (1.53) holds. This completes the proof of Case 2 and Lemma 1.25. $\square$

It is easy to see that the identity map satisfies $\mathrm{FinHolSh}(\theta, \omega)$ provided that $\theta + \omega \le 1$. To illustrate that Theorems 1.22, 1.23 are almost sharp we give not so pathological example. Consider a diffeomorphism $f : S^1 \to S^1$ constructed as follows.

(i) The nonwandering set of f consists of two fixed points $s, u \in S^1$.

(ii) In some neighborhood U_s of s there exists a coordinate system such that $f|_{U_s}(x) = x/2$.

(iii) In some neighborhood U_u of u there exists a coordinate system such that $f|_{U_u}(x) = x + x^3$.

(iv) In $S^1 \setminus (U_s \cup U_u)$ the map is chosen to be C^∞ and to satisfy the following condition: there exists $N > 2$ such that

$$f^N(S^1 \setminus U_u) \subset U_s, \quad f^{-N}(S^1 \setminus U_s) \subset U_u, \quad f^2(U_u) \cap U_s = \emptyset.$$

Theorem 1.26. *If $f : S^1 \to S^1$ satisfies the above properties (i)–(iv) then $f \in \mathrm{HolSh}(1/3)$ and $f \in \mathrm{FinHolSh}(1/2, 1/2)$.*

Proof. First let us prove a technical statement.

Lemma 1.27. *Denote $g(x) = x + x^3$. If $|x - y| \geq \varepsilon$ then*

$$|g(x) - g(y)| \geq \varepsilon + \varepsilon^3/4.$$

Proof. Using inequality $x^2 + xy + y^2 > (x - y)^2/4$ we deduce that

$$|g(x) - g(y)| = |x + x^3 - y - y^3| = |(x - y)(1 + x^2 + xy + y^2)| \geq \\ \geq |(x - y)||1 + (x - y)^2/4| \geq \varepsilon(1 + \varepsilon^2/4).$$

□

We divide the proof of Theorem 1.26 into several propositions.

Proposition 1.28. *Conditions (ii), (iii) imply that there exists $d_1 > 0$ such that*

$$B(d_1, f(U_s)) \subset U_s, \quad B(d_1, f^{-1}(U_u)) \subset U_u, \quad B(d_1, f(S^1 \setminus U_u)) \subset S^1 \setminus U_u. \tag{1.61}$$

Since $f|_{U_s}$ is hyperbolically contracting there exist $L > 0$ and $d_2 \in (0, d_1)$ such that for any d-pseudotrajectory $\{y_k\}$ with $d < d_2$ and $y_0 \in S^1 \setminus U_u$ the following conditions hold

- $\{y_k\}_{k \geq 0} \subset S^1 \setminus U_u$,
- $\operatorname{dist}(f^k(x_0), y_k) < Ld$, *for* $x_0 \in B(d, y_0),\ k \geq 0$,
- *if $\{y_k\}_{k\in\mathbb{Z}} \subset S^1 \setminus U_u$ then $\{y_k\}_{k\in\mathbb{Z}}$ can be Ld-shadowed by a trajectory.*

Proposition 1.29. *For any d-pseudotrajectory $\{y_k\}_{k\leq 0}$ with $d < d_1$ and $y_0 \in U_u$ the following inequality holds*

$$\operatorname{dist}(y_k, f^k(y_0)) < 2d^{1/3}, \quad k \leq 0. \tag{1.62}$$

Proof. Proposition 1.28 implies that $y_k \in U_u$ for $k < 0$. Assume (1.62) does not hold. Let

$$l = \max\{k \leq 0 : \operatorname{dist}(y_k, f^k(y_0)) \geq 2d^{1/3}\}.$$

Note that $l < 0$. Lemma 1.27 implies that

$$\operatorname{dist}(f(y_l), f^{l+1}(y_0)) > 2d^{1/3} + 2d.$$

Hence $\operatorname{dist}(y_{l+1}, f^{l+1}(y_0)) > 2d^{1/3}$, which contradicts to the choice of l. □

Proposition 1.30. *If $\{y_k\}_{k\in\mathbb{Z}} \subset U_u$ is a d-pseudotrajectory with $d < d_1$ then*

$$\operatorname{dist}(y_k, u) < 2d^{1/3}, \quad k \in \mathbb{Z}. \tag{1.63}$$

Proof. Let us identify y_k with its coordinate in the system introduced in (iii) above and consider $Y = \sup_{k\in\mathbb{Z}} |y_k|$. Assume that $Y > 2d^{1/3}$; then there exists $k \in \mathbb{Z}$ such that

$$|y_k| > \max(2d^{1/3}, Y - d/2).$$

Without loss of generality we may assume that $y_k > 0$. Since $y_k \in U_u$ the following holds

$$f(y_k) - y_k = y_k^3 > 2d.$$

Hence $y_{k+1} - y_k > (f(y_k) - y_k) - d > d$ and $y_{k+1} > Y + d/2$, which contradicts to the choice of Y. Inequalities (1.63) are proved. □

Proposition 1.31. *For any d-pseudotrajectory $\{y_k\}_{k\in[0,n]}$ with $d < d_1$ and $y_n \in U_u$ the following inequality holds*

$$\operatorname{dist}(y_{n-k}, f^{-k}(y_n)) \le dk, \quad k \in [0, n]. \tag{1.64}$$

Proof. Proposition 1.28 implies that $y_k \in U_u$ for $k \in [0, n]$. Assume that (1.64) does not hold. Denote

$$l = \min\{k \in [0, n] : \operatorname{dist}(y_{n-k}, f^{-k}(y_n)) > dk\}.$$

Note that $l > 0$. Lemma 1.27 implies that

$$\operatorname{dist}(f(y_{n-l}), f^{-l+1}(y_n)) > ld$$

and hence

$$\operatorname{dist}(y_{n-l+1}, f^{-l+1}(y_n)) > (l - 1)d,$$

which contradicts to the choice of l. □

Now we are ready to complete the proof of Theorem 1.26.

First let us prove that $f \in \mathrm{HolSh}(1/3)$. Consider an arbitrary d-pseudotrajectory $\{y_k\}_{k\in\mathbb{Z}}$ with $d < d_2$. Let us prove that it can be $Ld^{1/3}$-shadowed by a trajectory.

If $\{y_k\} \subset U_u$ then by Proposition 1.30 it can be $2d^{1/3}$-shadowed by $\{x_k = u\}$.

If $\{y_k\} \subset S^1 \setminus U_u$ then by Proposition 1.28 it can be Ld-shadowed.

In the other cases there exists l such that $y_l \in U_u$ and $y_{l+1} \notin U_u$. By Proposition 1.29

$$\operatorname{dist}(y_k, f^{k-l}(y_l)) < 2d^{1/3}, \quad k \le l.$$

By Proposition 1.28

$$\operatorname{dist}(y_k, f^{k-l}(y_l)) < Ld, \quad k \ge l + 1.$$

Hence $\{y_k\}$ is $Ld^{1/3}$-shadowed by the trajectory $\{x_k = f^{k-l}(y_l)\}$.

Now let us prove that $f \in \mathrm{FinHolSh}(1/2, 1/2)$. Consider an arbitrary d-pseudotrajectory $\{y_k\}_{k\in[0,1/d^{1/2}]}$ with $d < d_2$. Let us prove that it can be $Ld^{1/2}$-shadowed by a trajectory.

If $\{y_k\} \subset U_u$ then by Proposition 1.31 it can be $d^{1/2}$-shadowed by $\{x_k = f^{k-n}(y_n)\}$.

If $\{y_k\} \subset S^1 \setminus U_u$ then by Proposition 1.28 it can be Ld-shadowed.

In the other cases there exists l such that $y_l \in U_u$ and $y_{l+1} \notin U_u$. From Proposition 1.31 it is easy to conclude that

$$\operatorname{dist}(y_k, f^{k-l}(y_l)) < d^{1/2}, \quad k \le l.$$

Proposition 1.28 implies that

$$\operatorname{dist}(y_k, f^{k-l}(y_l)) < Ld, \quad k \ge l+1.$$

Hence $\{y_k\}$ is $Ld^{1/2}$-shadowed by the trajectory $\{x_k = f^{k-l}(y_l)\}$. $\square$

1.6 Periodic shadowing

The shadowing property means that, near a sufficiently precise approximate trajectory of a dynamical system, there is an exact trajectory. In this paragraph we study a similar question replacing arbitrary approximate and exact trajectories by periodic ones. The corresponding property is called periodic shadowing property and was introducted in [43].

In this paper, we study relations between periodic shadowing and structural stability,to be more precise, Ω-stability.

For a diffeomorphism $f \in \operatorname{Diff}^1(M)$ denote by $\Omega(f)$ the set of nonwondering points of f.

Definition 1.12. We say that a diffeomorphism $f \in \operatorname{Diff}^1(M)$ is Ω-*stable* if there exists a neighborhood $U \subset \operatorname{Diff}^1(M)$ of f such that for any $g \in U$ there exists a homeomorphism $h : \Omega(f) \to \Omega(g)$ such that

$$h \circ f(x) = g \circ h(x), \quad x \in \Omega(f).$$

Denote the set of Ω-stable diffeomorphisms by ΩS

It is well known that $f \in \Omega S$ if and only if f satisfies Axiom A and the no cycle condition, see, for example, [77].

For us will be important the following characterisation of Ω-stable diffeomorphisms. Let $\mathrm{HP} \subset \operatorname{Diff}^1(M)$ be the set of diffeomorphism f such that every periodic orbit of f is hyperbolic. The following was proved in [5, 34].

Theorem 1.32. $\operatorname{Int}^1(\mathrm{HP}) = \Omega S$.

It is easy to give an example of a diffeomorphism that is not structurally stable but has shadowing property (see [75], for example). Similarly to an example of not structurally stable diffeomorphism, which has shadowing property, there exist diffeomorphisms that are not Ω-stable but have periodic shadowing property. Thus, Ω-stability is not equivalent to periodic shadowing.

In this paragraph, we show that the C^1-interior of the set of diffeomorphisms having periodic shadowing property coincides with the set of Ω-stable diffeomorphisms. The second main result of this paper states that Lipschitz periodic shadowing property is equivalent to Ω-stability.

As in previous paragraphs let f be a diffeomorphism of a smooth closed manifold M with Riemannian metric dist.

Definition 1.13. We say that f has *periodic shadowing* property if for any positive ε there exists a positive d such that if $\xi = \{x_i\}$ is a periodic d-pseudotrajectory, then there exists a periodic point p such that

$$\operatorname{dist}(f^i(p), x_i) < \varepsilon, \quad i \in \mathbb{Z}. \tag{1.65}$$

Denote by PerSh the set of diffeomorphisms having periodic shadowing property.

Definition 1.14. We say that f has *Lipschitz periodic shadowing* property if there exist positive constants $\mathcal{L}, d_0$ such that if $\xi = \{x_i\}$ is a periodic d-pseudotrajectory with $d \leq d_0$, then there exists a periodic point p such that

$$\operatorname{dist}(f^i(p), x_i) \leq \mathcal{L}d, \quad i \in \mathbb{Z}. \tag{1.66}$$

Denote by LipPerSh the set of diffeomorphisms having Lipschitz periodic shadowing property.

The main result of this paragraph is stated as follows [63].

Theorem 1.33. *$Int^1(PerSh) = LipPerSh = \Omega S$.*

We divide the proof of this theorem into several sections. In section 1.6.1, we prove the inclusion $\Omega S \subset$ LipPerSh. Of course, this inclusion implies that $\Omega S \subset$ PerSh. Since the set ΩS is C^1-open, we conclude that $\Omega S \subset \operatorname{Int}^1$(PerSh). In section 1.6.2, we prove the inclusion Int^1(PerSh) $\subset \Omega S$. In section 1.6.3, we prove the inclusion LipPerSh $\subset \Omega S$.

1.6.1 Inclusion $\Omega S \subset$ LipPerSh

First we introduce some basic notation. Denote by $\operatorname{Per}(f)$ the set of periodic points of f and by $\Omega(f)$ the nonwandering set of f. Let $N = \sup_{x \in M} \|Df(x)\|$.

Let us formulate several auxiliary definitions and statements.

It is well known that if a diffeomorphism f satisfies Axiom A, then its nonwandering set can be represented as a disjoint union of a finite number of compact sets:

$$\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_m, \tag{1.67}$$

where the sets Ω_i are so-called basic sets (hyperbolic sets each of which contains a dense positive semi-trajectory).

We need the following two lemmas (see [80]).

Lemma 1.34. *Let f be a homeomorpism of a compact metric space (X, dist). For any neighborhood U of the nonwandering set $\Omega(f)$ there exist positive numbers B, d_1 such that if $\xi = \{x_i,\ i \in \mathbb{Z}\}$ is a d-pseudotrajectory of f with $d \leq d_1$ and*

$$x_k, x_{k+1}, \dots, x_{k+l} \notin U$$

for some $l > 0$ and $k \in \mathbb{Z}$, then $l \leq B$.

Let $\Omega_1, \dots, \Omega_m$ be the basic sets in decomposition (1.67) of the nonwandering set of an Ω-stable diffeomorphism f.

Lemma 1.35. *Let $U_1, \dots, U_m$ be disjoint neighborhoods of the basic sets $\Omega_1, \dots, \Omega_m$. There exist neighborhoods $V_j \subset U_j$ of the sets Ω_j and a number $d_2 > 0$ such that if $\xi = \{x_i,\ i \in \mathbb{Z}\}$ is a d-pseudotrajectory of f with $d \le d_2$ such that $x_0 \in V_j$ and $x_t \notin U_j$ for some $j \in \{1, \dots, m\}$ and some $t > 0$, then $x_i \notin V_j$ for $i \ge t$.*

Now we are ready to prove the following.

Lemma 1.36. $\Omega S \subset$ *LipPerSh.*

Proof. Applying Therem 1.1 we find disjoint neighborhoods $W_1, \dots, W_m$ of the basic sets $\Omega_1, \dots, \Omega_m$ in decomposition (1.67) such that (i) f has Lipschitz shadowing property on any of W_j with the same constants $\mathcal{L}, d_0^*$; (ii) f is expansive on any of W_j with the same expansivity constant a.

Find neighborhoods V_j, U_j of Ω_j (and reduce d_0^*, if necessary) so that the following properties are fulfilled:

- $V_j \subset U_j \subset W_j, \quad j = 1, \dots, m;$
- the statement of Lemma 2 holds for V_j and U_j with some $d_2 > 0$;
- the $\mathcal{L}d_0^*$-neighborhoods of U_j belong to W_j.

Apply Lemma 1.34 to find the corresponding constants B, d_1 for the neighborhood $V_1 \cup \dots \cup V_m$ of $\Omega(f)$.

We claim that f has the Lipschitz periodic shadowing property with constants $\mathcal{L}, d_0$, where

$$d_0 = \min\left(d_0^*, d_1, d_2, \frac{a}{2\mathcal{L}}\right).$$

Take a μ-periodic d-pseudotrajectory $\xi = \{x_i,\ i \in \mathbb{Z}\}$ of f with $d \le d_0$. Lemma 1.34 implies that there exists a neighborhood V_j such that $\xi \cap V_j \ne \emptyset$; shifting indices, we may assume that $x_0 \in V_j$.

In this case, $\xi \subset U_j$. Indeed, if $x_{i_0} \notin U_j$ for some i_0, then $x_{i_0+k\mu} \notin U_j$ for all k. It follows from Lemma 1.35 that if $i_0 + k\mu > 0$, then $x_i \notin V_j$ for $i \ge i_0 + k\mu$, and we get a contradiction with the periodicity of ξ and the inclusion $x_0 \in V_j$.

Thus, there exists a point p such that inequalities (1.66) hold. Let us show that $p \in \mathrm{Per}(f)$. By the choice of U_j and W_j, $f^i(p) \in W_j$ for all $i \in \mathbb{Z}$. Let $q = f^\mu(p)$. Inequalities (1.66) and the periodicity of ξ imply that

$$\mathrm{dist}(f^i(q), x_i) = \mathrm{dist}(f^i(q), x_{i+\mu}) \le \mathcal{L}d, \quad i \in \mathbb{Z}.$$

Thus,

$$\mathrm{dist}(f^i(q), f^i(p)) \le 2\mathcal{L}d \le a, \quad i \in \mathbb{Z},$$

which implies that $f^\mu(p) = q = p$. This completes the proof. ∎

1.6.2 Inclusion $\mathbf{Int}^1(\mathbf{PerSh}) \subset \Omega S$

By Theorem 1.32 it suffices for us to prove the following statement.

Lemma 1.37. *$Int^1(PerSh) \subset \mathrm{Int}^1(\mathrm{HP})$.*

Proof. To prove Lemma 1.37, it is enough for us to show that $\mathrm{Int}^1(\mathrm{PerSh}) \subset \mathrm{HP}$ and to note that the left-hand side of this inclusion is C^1-open.

To get a contradiction, let us assume that a diffeomorphism $f \in \mathrm{Int}^1(\mathrm{PerSh})$ has a nonhyperbolic periodic point p. Fix a C^1-neighborhood $\mathcal{N} \subset \mathrm{PerSh}$ of f.

For simplicity, let us assume that p is a fixed point and that the matrix $A_0 = Df(p)$ has an eigenvalue $\lambda = 1$ (the remaining cases are considered using a similar reasoning, see, for example, [74]).

Define exp, $\exp_x$, $B(r,x)$, $B_T(r,x)$ similarly to Section 1.4 and choose $\varepsilon > 0$ such that conditions (1.32) and (1.33) hold for balls $B_T(\varepsilon, x)$ and $B(\varepsilon, x)$.

In our case, an analog of mapping (1.30),

$$F = \exp_p^{-1} \circ f \circ \exp_p : T_pM \to T_pM,$$

has the form

$$F(v) = A_0 v + \phi(v).$$

Clearly, we can find a number $a \in (0, r)$ (recall that the number r was fixed above when properties of the exponential mapping were described), coordinates $v = (u, w)$ in T_pM with one-dimensional u, and a diffeomorphism $h \in \mathcal{N}$ such that if

$$H = \exp_p^{-1} \circ h \circ \exp_p$$

and $|v| \leq a$, then

$$H(v) = Av = (u, Bw),$$

where B is a matrix of size $(n-1)\times(n-1)$ (and n is the dimension of M). For this purpose, we take a matrix A, close to A_0 and having an eigenvalue $\lambda = 1$ of multiplicity one, and "annihilate" the C^1-small term $(A_0 - A)v + \phi(v)$ in the small ball $B_T(a, p)$.

Take a positive ε such that $8\varepsilon < a$. Since $h \in \mathcal{N}$, there exists a corresponding $d \in (0, \varepsilon)$ from the definition of periodic shadowing (for the diffeomorphism h). Take a natural number K such that $Kd > 8\varepsilon$. Reducing d, if necessary, we may assume that

$$8\varepsilon < Kd < 2a. \tag{1.68}$$

Let us construct a sequence $y_k \in T_pM,\ k \in \mathbb{Z}$, as follows:

$$y_0 = 0, \quad y_{k+1} = Ay_k + \left(\frac{d}{2}, 0\right), \quad 0 \leq k \leq K-1,$$

$$y_{k+1} = Ay_k - \left(\frac{d}{2}, 0\right), \quad K \leq k \leq 2K-1,$$

and $y_{k+2K} = y_k,\ k \in \mathbb{Z}$. Clearly,
$$y_K = \left(\frac{Kd}{2}, 0\right). \tag{1.69}$$
Let
$$x_k = \exp_p(y_k).$$
Since
$$\exp_p^{-1}(h(x_k)) = H(y_k) = Ay_k$$
and
$$|y_{k+1} - Ay_k| = \frac{d}{2},$$
the sequence $\xi = \{x_k\}$ is a $2K$-periodic d-pseudotrajectory of h.

By our assumption, there exists a periodic point p_0 of h such that
$$\operatorname{dist}(p_k, x_k) < \varepsilon, \quad k \in \mathbb{Z},$$
where $p_k = h^k(p_0)$. Let
$$p_k = \exp_p(q_k), \quad k \in \mathbb{Z},$$
where $q_k = (U_k, W_k)$, and let $y_k = (u_k, w_k)$; then
$$|U_k - u_k| \le |q_k - y_k| < 2\varepsilon, \quad k \in \mathbb{Z},$$
which implies that
$$|U_0| \le |q_0| < 2\varepsilon.$$
Since $q_{k+1} = H(q_k)$, $U_k = U_0$ for all k due to the structure of H. We conclude that $|U_K| < 2\varepsilon$ and get a contradiction with the inequalities $|U_K - u_K| < 2\varepsilon$, (1.68), and (1.69). The lemma is proved. □

1.6.3 Inclusion LipPerSh $\subset \Omega S$

In this section, we assume that $f \in$ LipPerSh (with constants $\mathcal{L} \ge 1, d_0 > 0$). Clearly, in this case $f^{-1} \in$ LipPerSh as well (and we assume that the constants $\mathcal{L}, d_0$ are the same for f and f^{-1}).

In the construction of pseudotrajectories, we apply the same linearization technique as in the previous section.

Lemma 1.38. *Every point $p \in \operatorname{Per}(f)$ is hyperbolic.*

Proof. To get a contradiction, let us assume that f has a nonhyperbolic periodic point p (to simplify notation, we assume that p is a fixed point; literally the same reasoning can be applied to a periodic point of period $m > 1$).

In this case, mapping (1.30) takes the form
$$F(v) = \exp_p^{-1} \circ f \circ \exp_p(v) = Av + \phi(v),$$

where A is a nonhyperbolic matrix. The following two cases are possible:

(Case 1): A has a real eigenvalue λ with $|\lambda| = 1$;

(Case 2): A has a complex eigenvalue λ with $|\lambda| = 1$.

We treat in detail only Case 1; we give a comment concerning Case 2. To simplify presentation, we assume that 1 is an eigenvalue of A; the case of eigenvalue -1 is treated similarly.

We can find coordinates v in T_pM such that, with respect to this coordinate, the matrix A has block-diagonal form,

$$A = \mathrm{diag}(B, P), \tag{1.70}$$

where B is a Jordan block of size $l \times l$:

$$B = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Of course, introducing new coordinates, we have to change the constants $\mathcal{L}, d_0, N$; we denote the new constants by the same symbols. In addition, we assume that $\mathcal{L}$ is integer.

We start considering the case $l = 2$; in this case,

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let

$$e_1 = (1, 0, 0, \dots, 0) \text{ and } e_2 = (0, 1, 0, \dots, 0)$$

be the first two vectors of the standard orthonormal basis.

Let $K = 25\mathcal{L}$.

Take a small $d > 0$ and construct a finite sequence $y_0, \dots, y_Q$ in T_pM (where Q is determined later) as follows: $y_0 = 0$ and

$$y_{k+1} = Ay_k + de_2, \quad k = 0, \dots, K-1. \tag{1.71}$$

Then

$$y_K = (Z_1(K)d, Kd, 0, \dots, 0),$$

where the natural number $Z_1(K)$ is determined by K (we do not write $Z_1(K)$ explicitly). Now we set

$$y_{k+1} = Ay_k - de_2, \quad k = K, \dots, 2K-1.$$

Then

$$y_{2K} = (Z_2(K)d, 0, 0, \dots, 0),$$

where the natural number $Z_2(K)$ is determined by K as well. Take $Q = 2K + Z_2(K)$; if we set

$$y_{k+1} = Ay_k - de_1, \quad k = 2K, \dots, Q-1,$$

then $y_Q = 0$. Let us note that both numbers Q and

$$Y := \frac{\max_{0 \le k \le Q-1} |y_k|}{d}$$

are determined by K (and hence, by $\mathcal{L}$).

Now we construct a Q-periodic sequence y_k, $k \in \mathbb{Z}$, that coincides with the above sequence for $k = 0, \dots, Q$.

We set $x_k = \exp_p(y_k)$ and claim that if d is small enough, then $\xi = \{x_k\}$ is a $4d$-pseudotrajectory of f (and this pseudotrajectory is Q-periodic by construction).

Indeed, we know that $|y_k| \le Yd$ for $k \in \mathbb{Z}$. Since $\phi(v) = o(|v|)$ as $|v| \to 0$,

$$|\phi(y_k)| < d, \quad k \in \mathbb{Z}, \tag{1.72}$$

if d is small enough.

The definition of $\{y_k\}$ implies that

$$|y_{k+1} - Ay_k| = d, \quad k \in \mathbb{Z}. \tag{1.73}$$

Note that

$$\exp_p^{-1}(f(x_k)) = F(y_k) = Ay_k + \phi(y_k);$$

thus, it follows from (1.72) and (1.73) that

$$|y_{k+1} - \exp_p^{-1}(f(x_k))| \le |y_{k+1} - Ay_k| + |\phi(y_k)| < 2d,$$

which implies that $\xi = \{x_k\}$ is a $4d$-pseudotrajectory of f if d is small enough.

Now we estimate the distances between points of trajectories of the mapping F and its linearization.

Let us take a vector $q_0 \in T_pM$ and assume that the sequence $q_k = F^k(q_0)$ belongs to the ball $|v| \le (Y + 8\mathcal{L})d$ for $0 \le k \le K$. Let $r_k = A^k q_0$ (we impose no conditions on r_k since below we estimate ϕ at points q_k only).

Take a small number $\mu \in (0, 1)$ (to be chosen later) and assume that d is small enough, so that the inequality

$$|\phi(v)| \le \mu|v|$$

holds for $|v| \le (Y + 8\mathcal{L})d$.

Then

$$|q_1| \le |Aq_0| + |\phi(q_0)| \le (N+1)|q_0|, \dots, |q_k| \le |Aq_{k-1}| + |\phi(q_{k-1})| \le (N+1)^k|q_0|$$

for $1 \le k \le K$, and

$$|q_1 - r_1| = |Aq_0 + \phi(q_0) - Aq_0| \le \mu|q_0|,$$

$$|q_2 - r_2| = |Aq_1 + \phi(q_1) - Ar_1| \le N|q_1 - r_1| + \mu|q_1| \le \mu(2N+1)|q_0|,$$

$$|q_3 - r_3| \le N|q_2 - r_2| + \mu|q_2| \le \mu(N(2N+1) + (N+1)^2)|q_0|,$$

and so on.

Thus, there exists a number $\nu = \nu(K, N)$ such that

$$|q_k - r_k| \le \mu\nu|q_0|, \quad 0 \le k \le K.$$

We take $\mu = 1/\nu$, note that $\mu = \mu(K, N)$, and get the inequalities

$$|q_k - r_k| \le |q_0|, \quad 0 \le k \le K, \tag{1.74}$$

for d small enough.

Since $f \in \mathrm{LipPerSh}$, for d small enough, the Q-periodic $4d$-pseudotrajectory ξ is $4\mathcal{L}d$-shadowed by a periodic trajectory. Let p_0 be a point of this trajectory such that

$$\mathrm{dist}(p_k, x_k) \le 4\mathcal{L}d, \quad k \in \mathbb{Z}, \tag{1.75}$$

where $p_k = f^k(p_0)$. Let $q_k = \exp_p^{-1}(p_k)$.

The inequalities $|y_k| \le Yd$ and (1.75) imply that

$$|q_k| \le |y_k| + 2\mathrm{dist}(p_k, x_k) \le (Y + 8\mathcal{L})d, \quad k \in \mathbb{Z}.$$

Note that $|q_0| \le 8\mathcal{L}d$.

Set $r_k = A^k q_0$; we deduce from estimate (1.74) that if d is small enough, then

$$|q_K - r_K| \le |q_0| \le 8\mathcal{L}d. \tag{1.76}$$

Denote by $v^{(2)}$ the second coordinate of a vector $v \in T_pM$.

It follows from the structure of the matrix A that

$$|r_K^{(2)}| = |q_0^{(2)}| \le 8\mathcal{L}d. \tag{1.77}$$

The relations

$$|y_K^{(2)}| = Kd \text{ and } |q_K - y_K| \le 8\mathcal{L}d$$

imply that

$$|q_K^{(2)}| \ge Kd - 8\mathcal{L}d = 17\mathcal{L}d \tag{1.78}$$

(recall that $K = 25\mathcal{L}$).

Estimates (1.76)–(1.78) are contradictory. Our lemma is proved in Case 1 for $l = 2$.

If $l = 1$, then the proof is simpler; the first coordinate of $A^k v$ equals the first coordinate of v, and we construct the periodic pseudotrajectory perturbing the first coordinate only.

If $l > 2$, the reasoning is parallel to that above; we first perturb the lth coordinate to make it Kd, and then produce a periodic sequence consequently making zero the lth coordinate, the $(l-1)$st coordinate, and so on.

If λ is a complex eigenvalue, $\lambda = a + bi$, we take a real 2×2 matrix

$$R = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and assume that in representation (1.70), B is a real $2l \times 2l$ Jordan block:

$$B = \begin{pmatrix} R & E_2 & 0 & \dots & 0 \\ 0 & R & E_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & R \end{pmatrix},$$

where E_2 is the 2×2 unit matrix.

After that, almost the same reasoning works; we note that $|Rv| = |v|$ for any 2-dimensional vector v and construct periodic pseudotrajectories replacing, for example, formulas (1.71) by the formulas

$$y_{k+1} = Ay_k + dw_k, \quad k = 0, \dots, K-1,$$

where jth coordinates of the vector w_k are zero for $j = 1, \dots, 2l-2, 2l+1, \dots, n$, while the 2-dimensional vector corresponding to $(2l-1)$st and $2l$th coordinates has the form $R^k w$ with $|w| = 1$, and so on. We leave details to the reader. The lemma is proved. $\square$

Lemma 1.39. *There exist constants $C > 0$ and $\lambda \in (0,1)$ depending only on N and $\mathcal{L}$ and such that, for any point $p \in \mathrm{Per}(f)$, there exist complementary subspaces $S(p)$ and $U(p)$ of the tangent space T_pM such that the following holds*

(H1) $Df(p)S(p) = S(f(p))$ *and* $Df(p)U(p) = U(f(p))$,

(H2.1) $|Df^j(p)v| \le C\lambda^j |v|, \quad v \in S(p), j \ge 0,$

(H2.2) $|Df^{-j}(p)v| \le C\lambda^j |v|, \quad v \in U(p), j \ge 0.$

Remark 1.40. Lemma 1.39 means that the set $\mathrm{Per}(f)$ has all the standard properties of a hyperbolic set, with the exception of compactness.

Proof. Take a periodic point $p \in \mathrm{Per}(f)$; let m be the minimal period of p.

Denote $p_i = f^i(p)$, $A_i = Df(p_i)$, and $B = Df^m(p)$. It follows from Lemma 5 that the matrix B is hyperbolic. Denote by $S(p)$ and $U(p)$ the invariant subspaces of B corresponding to parts of its spectrum inside and outside the unit disk, respectively. Clearly, $S(p)$ and $U(p)$ are invariant with respect to Df, $T_pM = S(p) \oplus U(p)$, and the following relations hold:

$$\lim_{n \to +\infty} B^n v_s = \lim_{n \to +\infty} B^{-n} v_u = 0, \quad v_s \in S(p), v_u \in U(p). \tag{1.79}$$

We prove that inequalities (H2.2) hold with $C = 16\mathcal{L}$ and $\lambda = 1 + 1/(8\mathcal{L})$ (inequalities (H2.1) are established by similar reasoning applied to f^{-1} instead of f).

Consider an arbitrary nonzero vector $v_u \in U(p)$ and an integer $j \ge 0$. Define sequences $v_i, e_i \in T_{p_i}M$ and $\lambda_i > 0$ for $i \ge 0$ as follows:

$$v_0 = v_u, \quad v_{i+1} = A_i v_i, \quad e_i = \frac{v_i}{|v_i|}, \quad \lambda_i = \frac{|v_{i+1}|}{|v_i|} = |A_i e_i|.$$

Let
$$\tau = \frac{\lambda_{m-1} \cdot \ldots \cdot \lambda_1 + \lambda_{m-1} \cdot \ldots \cdot \lambda_2 + \ldots + \lambda_{m-1} + 1}{\lambda_{m-1} \cdot \ldots \cdot \lambda_0}.$$
Consider the sequence $\{a_i \in \mathbb{R},\ i \geq 0\}$ defined by the following formulas:
$$a_0 = \tau, \quad a_{i+1} = \lambda_i a_i - 1. \tag{1.80}$$
Note that
$$a_m = 0 \quad \text{and} \quad a_i > 0, \quad i \in [0, m-1]. \tag{1.81}$$
Indeed, if $a_i \leq 0$ for some $i \in [0, m-1]$, then $a_k < 0$ for $k \in [i+1, m]$.

It follows from (1.79) that there exists $n > 0$ such that
$$|B^{-n} \tau e_0| < 1. \tag{1.82}$$

Consider the finite sequence $\{w_i \in T_{p_i}M,\ i \in [0, m(n+1)]\}$ defined as follows:
$$\begin{cases} w_i = a_i e_i, & i \in [0, m-1], \\ w_m = B^{-n} \tau e_0, & \\ w_{m+1+i} = A_i w_{m+i}, & i \in [0, mn-1]. \end{cases}$$
Clearly,
$$w_{km} = B^{k-1-n} \tau e_0, \quad k \in [1, n+1],$$
which means that we can consider $\{w_i\}$ as an $m(n+1)$-periodic sequence defined for $i \in \mathbb{Z}$.

Let us note that
$$A_i w_i = a_i A_i e_i = a_i \frac{v_{i+1}}{|v_i|}, \quad i \in [0, m-2],$$
$$w_{i+1} = (\lambda_i a_i - 1) \frac{v_{i+1}}{|v_{i+1}|} = a_i \frac{v_{i+1}}{|v_i|} - e_{i+1}, \quad i \in [0, m-2],$$
and
$$A_{m-1} w_{m-1} = a_{m-1} \frac{v_m}{|v_{m-1}|} = \frac{v_m}{\lambda_{m-1} |v_{m-1}|} = e_m$$
(in the last relation we take into account that $a_{m-1}\lambda_{m-1} = 1$ since $a_m = 0$).

The above relations and condition (1.82) imply that
$$|w_{i+1} - A_i w_i| < 2, \quad i \in \mathbb{Z}. \tag{1.83}$$

Now we take a small $d > 0$ and consider the $m(n+1)$-periodic sequence $\xi = \{x_i = \exp_{p_i}(dw_i),\ i \in \mathbb{Z}\}$.

We claim that if d is small enough, then ξ is a $4d$-pseudotrajectory of f.

Denote
$$\zeta_{i+1} = \exp^{-1}_{p_{k+1}}(f(x_i)) \ \text{ and } \ \zeta'_{i+1} = \exp^{-1}_{p_{k+1}}(x_{i+1}).$$
Then
$$\zeta_{i+1} = \exp^{-1}_{p_{k+1}} f(\exp_{p_k}(dw_i)) = F_i(dw_i) = A_i dw_i + \phi_i(dw_i),$$

where the mapping F_i is defined in (1.30) and $\phi_i(v) = o(|v|)$, and

$$\zeta'_{i+1} = \exp^{-1}_{p_{k+1}}(x_{i+1}) = dw_{i+1}.$$

It follows from estimates (1.83) that

$$|\zeta'_{i+1} - \zeta_{i+1}| \le 2d$$

for small d, and

$$\operatorname{dist}(f(x_i), x_{i+1}) \le 4d.$$

By Lemma 5, the m-periodic trajectory $\{p_i\}$ is hyperbolic; hence, $\{p_i\}$ has a neighborhood in which $\{p_i\}$ is a unique periodic trajectory. It follows that if d is small enough, then the pseudotrajectory $\{x_i\}$ is $4\mathcal{L}d$-shadowed by $\{p_i\}$.

The inequalities $\operatorname{dist}(x_i, p_i) \le 4\mathcal{L}d$ imply that $|a_i| = |w_i| \le 8\mathcal{L}$ for $0 \le i \le m-1$.

Now the equalities $\lambda_i = (a_{i+1}+1)/a_i$ imply that if $0 \le i \le m-1$, then

$$\begin{aligned}\lambda_0 \cdot \ldots \cdot \lambda_{i-1} = \frac{a_1+1}{a_0}\frac{a_2+1}{a_1}\ldots\frac{a_i+1}{a_{i-1}} &= \\ = \frac{a_i+1}{a_0}\left(1+\frac{1}{a_1}\right)\ldots\left(1+\frac{1}{a_{i-1}}\right) &\ge \\ \ge \frac{1}{8\mathcal{L}}\left(1+\frac{1}{8\mathcal{L}}\right)^{i-1} &> \frac{1}{16\mathcal{L}}\left(1+\frac{1}{8\mathcal{L}}\right)^{i} \qquad (1.84)\end{aligned}$$

(we take into account that $1 + 1/(8\mathcal{L}) < 2$ since $\mathcal{L} \ge 1$).

It remains to note that

$$|Df^i(p)v_u| = \lambda_{i-1}\cdots\lambda_0|v_u|, \quad 0 \le i \le m-1,$$

and that we started with an arbitrary vector $v_u \in U(p)$.

This proves our statement for $j \le m-1$. If $j \ge m$, we take an integer $k > 0$ such that $km > j$ and repeat the above reasoning for the periodic trajectory $p_0, \ldots, p_{km-1}$ (note that we have not used the condition that m is the minimal period). Lemma 1.39 is proved. □

Lemma 1.41. If $f \in \mathrm{LipPerSh}$, then f satisfies Axiom A.

Proof. Denote by P_l the set of points $p \in \mathrm{Per}(f)$ of index l (as usual, the index of a hyperbolic periodic point is the dimension of its unstable manifold).

Let R_l be the closure of P_l. Clearly, R_l is a compact f-invariant set. We claim that any R_l is a hyperbolic set. Let $n = \dim M$.

Consider a point $q \in R_l$ and fix a sequence of points $p_m \in P_l$ such that $p_m \to q$ as $m \to \infty$. By Lemma 6, there exist complementary subspaces $S(p_m)$ and $U(p_m)$ of $T_{p_m}M$ (of dimensions $n-l$ and l, respectively) for which estimates (H2.1) and (H2.2) hold.

Standard reasoning shows that, introducing local coordinates in a neighborhood of (q, T_qM) in the tangent bundle of M, we can select a subsequence p_{m_k} for which the sequences $S(p_{m_k})$

and $U(p_{m_k})$ converge (in the Grassmann topology) to subspaces of T_qM (let S_0 and U_0 be the corresponding limit subspaces).

The limit subspaces S_0 and U_0 are complementary in T_qM. Indeed, consider the "angle" β_{m_k} between the subspaces $S(p_{m_k})$ and $U(p_{m_k})$ which is defined (with respect to the introduced local coordinates in a neighborhood of (q, T_qM)) as follows:

$$\beta_{m_k} = \min |v^s - v^u|,$$

where the minimum is taken over all possible pairs of unit vectors $v^s \in S(p_{m_k})$ and $v^u \in U(p_{m_k})$.

Similarly to Remark 1.3 the values β_{m_k} are estimated from below by a positive constant $\alpha = \alpha(C, \lambda, N)$. Clearly, this implies that the subspaces S_0 and U_0 are complementary.

It is easy to show that the limit subspaces S_0 and U_0 are unique (which means, of course, that the sequences $S(p_m)$ and $U(p_m)$ converge). For the convenience of the reader, we prove this statement.

To get a contradiction, assume that there is a subsequence p_{m_i} for which the sequences $S(p_{m_i})$ and $U(p_{m_i})$ converge to complementary subspaces S_1 and U_1 different from S_0 and U_0 (for definiteness, we assume that $S_0 \setminus S_1 \neq \emptyset$).

Due to the continuity of Df, the inequalities

$$|Df^j(q)v| \leq C\lambda^j |v|, \quad v \in S_0 \cup S_1,$$

and

$$|Df^j(q)v| \geq C^{-1}\lambda^{-j} |v|, \quad v \in U_0 \cup U_1,$$

hold for $j \geq 0$.

Since

$$T_qM = S_0 \oplus U_0 = S_1 \oplus U_1,$$

our assumption implies that there is a vector $v \in S_0$ such that

$$v = v^s + v^u, \quad v^s \in S_1, v^u \in U_1, v^u \neq 0.$$

Then

$$|Df^j(q)v| \leq C\lambda^j |v| \to 0, \quad j \to \infty,$$

and

$$|Df^j(q)v| \geq C^{-1}\lambda^{-j}|v^u| - C\lambda^j|v^s| \to \infty, \quad j \to \infty,$$

and we get the desired contradiction.

It follows that there are uniquely defined complementary subspaces $S(q)$ and $U(q)$ for $q \in R_l$ with proper hyperbolity estimates; the Df-invariance of these subspaces is obvious. We have shown that each R_l is a hyperbolic set with $\dim S(q) = n - l$ and $\dim U(q) = l$ for $q \in R_l$.

If $r \in \Omega(f)$, then there exists a sequence of points $r_m \to r$ as $m \to \infty$ and a sequence of indices $k_m \to \infty$ as $m \to \infty$ such that $f^{k_m}(r_m) \to r$.

Clearly, if we continue the sequence

$$r_m, f(r_m), \dots, f^{k_m-1}(r_m)$$

periodically with period k_m, we get a periodic d_m-pseudotrajectory of f with $d_m \to 0$ as $m \to \infty$.

Since $f \in \text{LipPerSh}$, for large m there exist periodic points p_m such that $\text{dist}(p_m, r_m) \to 0$ as $m \to \infty$. Thus, periodic points are dense in $\Omega(f)$.

Since hyperbolic sets with different dimensions of the subspaces $U(q)$ are disjoint, we get the equality

$$\Omega(f) = R_0 \cup \dots \cup R_n,$$

which implies that $\Omega(f)$ is hyperbolic. The lemma is proved. $\square$

It was mentioned above that if a diffeomorphism f satisfies Axiom A, then its nonwandering set can be represented as a disjoint union of a finite number of basic sets (see representation (1.67)).

The basic sets Ω_i have stable and unstable "manifolds":

$$W^s(\Omega_i) = \{x \in M : \text{ dist}(f^k(x), \Omega_i) \to 0, \quad k \to \infty\}$$

and

$$W^u(\Omega_i) = \{x \in M : \text{ dist}(f^k(x), \Omega_i) \to 0, \quad k \to -\infty\}.$$

If Ω_i and Ω_j are basic sets, we write $\Omega_i \to \Omega_j$ if the intersection

$$W^u(\Omega_i) \cap W^s(\Omega_j)$$

contains a wandering point.

We say that f has a 1-cycle if there is a basic set Ω_i such that $\Omega_i \to \Omega_i$.

We say that f has a t-cycle if there are $t > 1$ basic sets

$$\Omega_{i_1}, \dots, \Omega_{i_t}$$

such that

$$\Omega_{i_1} \to \dots \to \Omega_{i_t} \to \Omega_{i_1}.$$

Lemma 1.42. *If $f \in LipPerSh$, then f has no cycles.*

Proof. To simplify presentation, we prove that f has no 1-cycles (in the general case, the idea is literally the same, but the notation is heavy).

To get a contradiction, assume that

$$p \in (W^u(\Omega_i) \cap W^s(\Omega_i)) \setminus \Omega(f).$$

In this case, there are sequences of indices $j_m, k_m \to \infty$ as $m \to \infty$ such that

$$f^{-j_m}(p), f^{k_m}(p) \to \Omega_i, \quad m \to \infty.$$

Since the set Ω_i is compact, we may assume that

$$f^{-j_m}(p) \to q \in \Omega_i \ \text{ and } f^{k_m}(p) \to r \in \Omega_i.$$

Since Ω_i contains a dense positive semi-trajectory, there exist points $s_m \to r$ and indices $l_m > 0$ such that $f^{l_m}(s_m) \to q$ as $m \to \infty$.

Clearly, if we continue the sequence

$$p, f(p), \dots, f^{k_m-1}(p), s_m, \dots, f^{l_m-1}(s_m), f^{-j_m}(p), \dots, f^{-1}(p)$$

periodically with period $k_m + l_m + j_m$, we get a periodic d_m-pseudotrajectory of f with $d_m \to 0$ as $m \to \infty$.

Since $f \in \text{LipPerSh}$, there exist periodic points p_m (for m large enough) such that $p_m \to p$ as $m \to \infty$, and we get the desired contradiction with the assumption that $p \notin \Omega(f)$. The lemma is proved. $\square$

Lemmas 1.38 – 1.42 show that $\text{LipPerSh} \subset \Omega S$.

Chapter 2

Partially hyperbolic diffeomorphisms

2.1 Central shadowing property

In Chapter 1 we studied relations between shadowing, hyperbolicity and structural stability. In the present chapter we study shadowing property for more general class of systems: partially hyperbolic diffeomorphisms.

As in the previous chapter let M be a compact C^∞ smooth manifold, with a Riemannian metric dist.

Definition 2.1. A diffeomorphism $f \in \mathrm{Diff}^1(M)$ is called *partially hyperbolic* if there exists $m \in \mathbb{N}$ such that the mapping f^m satisfies the following property. There exists a continuous invariant bundle

$$T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x), \qquad x \in M$$

and continuous positive functions $\nu, \hat{\nu}, \gamma, \hat{\gamma} : M \to \mathbb{R}$ such that

$$\nu, \hat{\nu} < 1, \qquad \nu < \gamma < \hat{\gamma} < \hat{\nu}^{-1}$$

and for all $x \in M$, $v \in T_xM$, $|v| = 1$

$$\begin{gathered} |Df^m(x)v| \leq \nu(x), \quad v \in E^s(x); \\ \gamma(x) \leq |Df^m(x)v| \leq \hat{\gamma}(x), \quad v \in E^c(x); \\ |Df^m(x)v| \geq \hat{\nu}^{-1}(x), \quad v \in E^u(x). \end{gathered} \tag{2.1}$$

Denote

$$E^{cs}(x) = E^s(x) \oplus E^c(x), \qquad E^{cu}(x) = E^c(x) \oplus E^u(x).$$

Note that due to [11] one cannot expect that in general shadowing holds for partially hyperbolic diffeomorphisms.

For further considerations we need the notion of dynamical coherence.

Definition 2.2. We say that a k – dimensional distribution E over TM is *uniquely integrable* if there exists a k – dimensional continuous foliation W of the manifold M, whose leaves are tangent to E at every point. Also, any C^1 – smooth path tangent to E is embedded to a unique leaf of W.

Definition 2.3. A partially hyperbolic diffeomorphism f is *dynamically coherent* if both the distributions E^{cs} and E^{cu} are uniquely integrable.

If f is dynamically coherent then distribution E^c is also uniquely integrable and corresponding foliation W^c is a subfoliation of both W^{cs} and W^{cu}. For a discussion how often partially hyperbolic diffeomorphisms are dynamically coherent see [15], [92].

In this paragraph below we always assume that f is dynamically coherent.

For $\tau \in \{s, c, u, cs, cu\}$ and $y \in W^\tau(x)$ let $\operatorname{dist}_\tau(x, y)$ be the inner distance on $W^\tau(x)$ from x to y. Note that

$$\operatorname{dist}(x, y) \le \operatorname{dist}_\tau(x, y), \quad y \in W^\tau(x). \tag{2.2}$$

Denote

$$W^\tau_\varepsilon(x) = \{y \in W^\tau(x),\ \operatorname{dist}_\tau(x, y) < \varepsilon\}.$$

We suggest the following generalization of the shadowing property for partially hyperbolic dynamically coherent diffeomorphisms.

Definition 2.4 (see for example [38])**.** An ε-pseudotrajectory $\{y_k\}$ is called *central* if for any $k \in \mathbb{Z}$ the inclusion $f(y_k) \in W^c_\varepsilon(y_{k+1})$ holds (see Fig. 2.1).

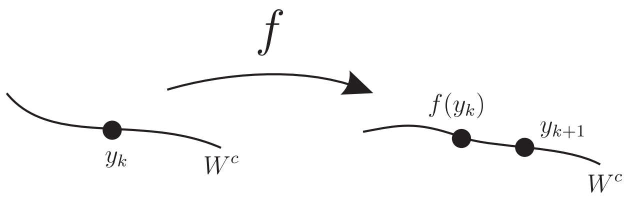

Figure 2.1: Central pseudotrajectory

Definition 2.5. A partially hyperbolic dynamically coherent diffeomorphism f satisfies the *central shadowing property* if for any $\varepsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory $\{x_k : k \in \mathbb{Z}\}$ there exists an ε-central pseudotrajectory $\{y_k\}$ of the diffeomorphism f, satisfying

$$\operatorname{dist}(x_k, y_k) \le \varepsilon, \quad k \in \mathbb{Z}. \tag{2.3}$$

Definition 2.6. A partially hyperbolic dynamically coherent diffeomorphism f satisfies the *Lipschitz central shadowing property* if there exist $d_0, \mathcal{L} > 0$ such that for any $d \in (0, d_0)$ and any d-pseudotrajectory $\{x_k : k \in \mathbb{Z}\}$ there exists an ε-central pseudotrajectory $\{y_k\}$, satisfying (2.3) with $\varepsilon = \mathcal{L}d$.

Note that the Lipschitz central shadowing property implies the central shadowing property.

We prove the following analogue of the shadowing lemma for partially hyperbolic diffeomorphisms [45].

Theorem 2.1. *Let diffeomorphism $f \in C^1$ be partially hyperbolic and dynamically coherent. Then f satisfies the Lipschitz central shadowing property.*

This result may be considered as a generalization of the classical shadowing lemma for the case of partially hyperbolic diffeomorphisms.

The proof of this theorem based on Tikhonov-Shauder fixed point theorem. Standard proofs of shadowing lemma [3], [13] are based on contracting mapping principle and cannot be repeated since foliations W^{cs}, W^{cu} have smooth leaves but the corresponding holonomies are only Hölder continuous (see for example [92] for exact statements).

Note that for Anosov diffeomorphisms any central pseudotrajectory is a true trajectory.

Remark 2.2. The statement of the classical shadowing lemma is valid for a neighborhood of a hyperbolic set Λ. For the central shadowing property we consider only the case $\Lambda = M$. The reason is that we need foliations

$$W^{cs}, W^{cu}, W^c = W^{cs} \cap W^{cu}, \tag{2.4}$$

which hardly can be defined in a neighborhood of Λ just by the partially hyperbolic structure of Λ. We expect that statement similar to Theorem 2.1 can be proved for $\Lambda \neq M$ with additionally given foliations (2.4), but this discussion is out of the scope of the dissertation.

Let us also mention the following related notion [38].

Definition 2.7. Partially hyperbolic, dynamically coherent diffeomorphism f is called *plaque expansive* if there exists $\varepsilon > 0$ such that for any ε-central pseudotrajectories $\{y_k\}$, $\{z_k\}$, satisfying

$$\operatorname{dist}(y_k, z_k) < \varepsilon, \quad k \in \mathbb{Z}$$

holds inclusion $z_k \in W^c(y_k)$ and z_k lies on the same connected component of $W^c(y_k) \cap B_\varepsilon(y_k)$ as y_k for all $k \in \mathbb{Z}$.

In the theory of partially hyperbolic diffeomorphisms the following conjecture plays important role [12], [38].

Conjecture 2.1 (Plague Expansivity Conjecture)**.** *Any partially hyperbolic, dynamically coherent diffeomorphism is plaque expansive.*

Let us note that if the diffeomorphism f in Theorem 2.1 is additionally plaque expansive then leaves $W^c(y_k)$ are uniquely defined (see Remark 2.7 below).

Among results related to Theorem 2.1 we would like to mention that partially hyperbolic dynamically coherent diffeomorphisms, satisfying plaque expansivity property are leaf stable (see [38, Chapter 7], [87] for details).

Proof. Proof of Theorem 2.1 In what follows below we will use the following statement, which is consequence of transversality and continuity of foliations W^s, W^{cu}.

Statement 2.3. *There exists $\delta_0 > 0$, $L_0 > 1$ such that for any $\delta \in (0, \delta_0]$ such that for any $x, y \in M$ satisfying* $\operatorname{dist}(x, y) < \delta$ *there exists unique point $z = W^s_\varepsilon(x) \cap W^{cu}_\varepsilon(y)$ for $\varepsilon = L_0\delta$.*

First we will prove the following lemma.

Lemma 2.4. *Let f satisfy assumptions of Theorem 2.1 and for some $m \geq 1$ diffeomorphism f^m has Lipschitz central shadowing property then diffeomorphism f also has Lipschitz central shadowing property.*

Proof. By assumptions of the theorem f^m has Lipschitz central shadowing property with constant $L > 0$. Note that foliations W^τ, $\tau \in \{s, u, c, cs, cu\}$ of f^m coincide with the corresponding foliations of f.

Let $\{y_j\}$ be a d-pseudotrajectory of f for some $d > 0$. Consider the sequence $\{z_k = y_{km}\}$, note that $\{z_k\}$ is a $C_1 d$-pseudotrajectory of f^m, where $C_1 = R^{m-1} + R^{m-2} + \dots R^1 + R^0$, where $R = \sup_{x\in M} \mathrm{D}\, f(x)$. Since f^m has Lipschitz central shadowing property then for small enough d pseudotrajectory $\{z_k\}$ can be $LC_1 d$ shadowed by a $LC_1 d$-central pseudotrajectory $\{q_k\}$ of f^m: the following holds

$$q_{k+1} \in W^c_{LC_1 d}(f^m(q_k)), \quad \operatorname{dist}(z_k, q_k) < LC_1 d, \quad k \in \mathbb{Z}.$$

Consider sequence $\{x_{j:=km+i} = f^i(q_k)\}$, where $k \in \mathbb{Z}$, $i \in [0, m-1]$. Note that $\{x_j\}$ is a $LC_1 d$-central pseudotrajectory for f and

$$\operatorname{dist}(x_{km+i}, y_{km+i}) \leq \operatorname{dist}(f^i(q_k), f^i(z_k)) + \operatorname{dist}(f^i(y_{km}), y_{km+i}) \leq R^{m-1} LC_1 d + C_1 d = L_1 d,$$

where $L_1 = R^{m-1} LC_1 + C^1$. The last inequality implies that $\{x_j\}$ is a desired central pseudotrajectory for $\{y_j\}$ and f satisfy Lipschitz central shadowing property with constant L_1. $\square$

Let us continue proof of Theorem 2.1. Taking into account Lemma 2.4 we can assume without loss of generality that conditions (2.1) hold for $m = 1$. Note that a similar claim can be done using adapted metric, see [30].

Denote

$$\lambda = \min_{x\in M}(\min(\hat{\nu}^{-1}(x), \nu^{-1}(x))) > 1.$$

Let us choose l so big that

$$\lambda^l > 2L_0.$$

By Lemma 2.4 it is sufficient to prove that f^l has the Lipschitz central shadowing property and hence, we can assume without loss of generality that $l = 1$.

Decreasing δ_0 if necessarily we conclude from inequalities (2.1) that

$$\operatorname{dist}_s(f(x), f(y)) \leq \frac{1}{\lambda} \operatorname{dist}_s(x, y), \quad y \in W^s_{\delta_0}(x) \tag{2.5}$$

and

$$\operatorname{dist}_u(f(x), f(y)) \geq \lambda \operatorname{dist}_u(x, y), \quad y \in W^u_{\delta_0}(x).$$

Denote

$$I^\tau_r(x) = \{z^\tau \in E^\tau(x),\ |z^\tau| \leq r\}, \quad \tau \in \{s, u, c, cs, cu\}, \quad r > 0,$$

$$I_r(x) = \{z \in T_xM,\ |z| \le r\}, \quad r > 0.$$

Consider standard exponential mappings $\exp_x : T_xM \to M$ and $\exp_x^\tau : T_xW^\tau(x) \to W^\tau(x)$, for $\tau \in \{s, c, u, cs, cu\}$. Standard properties of exponential mappings imply that there exists $\varepsilon_0 > 0$, such that for all $x \in M$ maps $\exp_x$, $\exp_x^\tau$ are well defined on $I_{\varepsilon_0}(x)$ and $I^\tau_{\varepsilon_0}(x)$ respectively and $\mathrm{D}\exp_x(0) = \mathrm{Id}$, $\mathrm{D}\exp_x^\tau(0) = \mathrm{Id}$. Those equalities imply the following.

Statement 2.5. *For $\mu > 0$ there exists $\varepsilon \in (0, \varepsilon_0)$ such that for any point $x \in M$, the following holds.*

A1 *For any $y, z \in B_\varepsilon(x)$ and $v_1, v_2 \in I_\varepsilon(x)$ the following inequalities hold*

$$\frac{1}{1+\mu}\operatorname{dist}(y, z) \le |\exp_x^{-1}(y) - \exp_x^{-1}(z)| \le (1+\mu)\operatorname{dist}(y, z),$$

$$\frac{1}{1+\mu}|v_1 - v_2| \le \operatorname{dist}(\exp_x(v_1), \exp_x(v_2)) \le (1+\mu)|v_1 - v_2|.$$

A2 *Conditions similar to **A1** hold for $\exp_x^\tau$ and dist_τ, $\tau \in \{s, c, u, cs, cu\}$.*

A3 *For $y \in W^\tau_\varepsilon(x)$, $\tau \in \{s, c, u, cs, cu\}$ the following holds*

$$\operatorname{dist}_\tau(x, y) \le (1+\mu)\operatorname{dist}(x, y).$$

A4 *If $\xi < \varepsilon$ and $y \in W^{cs}_\xi(x) \cap W^{cu}_\xi(x)$ then*

$$\operatorname{dist}_c(x, y) \le (1+\mu)\xi.$$

Consider small enough $\mu \in (0, 1)$ satisfying the following inequality

$$(1+\mu)^2 L_0/\lambda < 1. \tag{2.6}$$

Choose corresponding $\varepsilon > 0$ from Statement 2.5. Let $\delta = \min(\delta_0, \varepsilon/L_0)$.

For a pseudotrajectory $\{x_k\}$ consider maps $h^s_k : U_k \subset E^s(x_k) \to E^s(x_{k+1})$ defined as the following:

$$h^s_k(z) = (\exp^s_{x_{k+1}})^{-1}(p)$$

where

$$p = W^{cu}_{L_0\delta_0}(f(\exp^s_{x_k}(z))) \cap W^s_{L_0\delta_0}(x_{k+1}) \tag{2.7}$$

and U_k is the set of points for which map h^s_k is well-defined (see Fig. 2.2). Note that maps $h^s_k(z)$ are continuous. The following lemma plays a central role in the proof of Theorem 2.1.

Lemma 2.6. *There exists $d_0 > 0$, $L > 1$ such that for any $d < d_0$ and d-pseudotrajectory $\{x_k\}$ maps h^s_k are well-defined for $z \in I^s_{Ld}(x_k)$ and the following inequalities hold*

$$|h^s_k(z)| \le Ld, \quad k \in \mathbb{Z}. \tag{2.8}$$

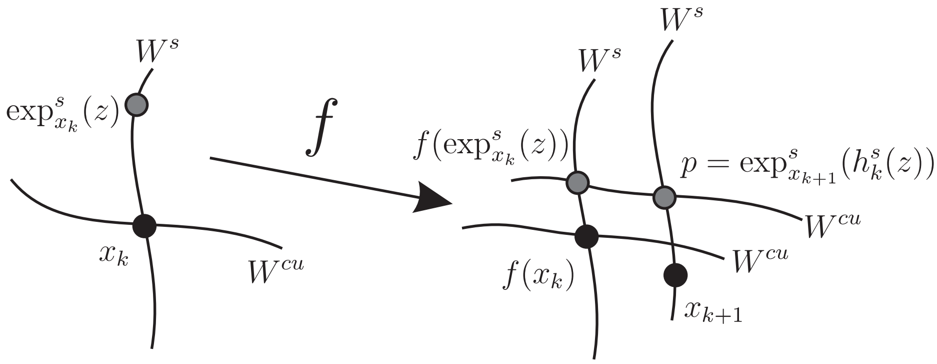

Figure 2.2: Definition of map h^s_k

Proof. Inequality (2.6) implies that there exists $L > 0$ such that

$$L_0(1 + L(1 + \mu)/\lambda)(1 + \mu) < L. \tag{2.9}$$

Let us choose $d_0 < \delta_0/2L$. Fix $d < d_0$, d-pseudotrajectory $\{x_k\}$, $k \in \mathbb{Z}$ and $z \in I^s_{Ld}(x_k)$.

Condition **A2** of Statement 2.5 implies that

$$\operatorname{dist}_s(x_k, \exp^s_{x_k}(z)) \le Ld(1 + \mu).$$

Inequality (2.5) implies the following

$$\operatorname{dist}_s(f(x_k), f(\exp^s_{x_k}(z))) \le \frac{1}{\lambda} Ld(1 + \mu).$$

Inequalities (2.2) and $\operatorname{dist}(f(x_k), x_{k+1}) < d$ imply (see Fig. 2.3 for illustration)

$$\operatorname{dist}(x_{k+1}, f(\exp^s_{x_k}(z))) \le \operatorname{dist}(x_{k+1}, f(x_k)) + \operatorname{dist}(f(x_k), f(\exp^s_{x_k}(z))) \le$$
$$d\left(1 + \frac{1}{\lambda} L(1 + \mu)\right) < Ld < \delta_0.$$

Statement 2.3 implies that point p from relation (2.7) is well-defined and inequality (2.9) implies the following

$$\operatorname{dist}_s(p, x_{k+1}), \operatorname{dist}_{cu}(p, f(\exp^s_{x_k}(z))) < dL_0(1 + \frac{1}{\lambda} L(1 + \mu)) < \frac{Ld}{1 + \mu}.$$

This inequality and Statement 2.5 imply

$$\operatorname{dist}_{cu}(f(\exp^s_{x_{k+1}}(z)), \exp^s_{x_k}(h^s_k(z))) < Ld, \tag{2.10}$$
$$|h^s_k(z)| < Ld,$$

which completes the proof.

$\square$

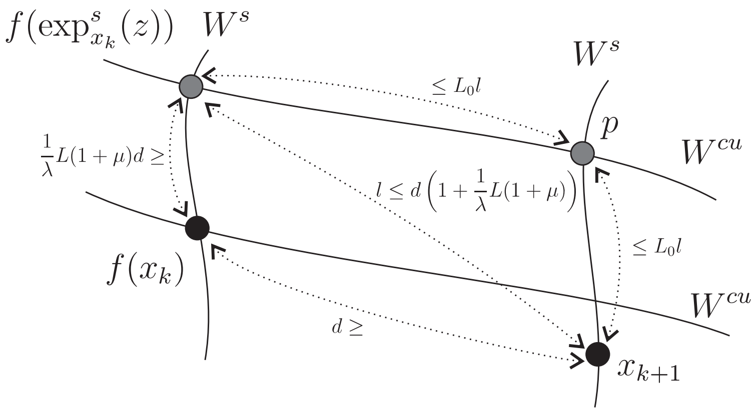

Figure 2.3: Illustration of the proof of Lemma 2.6

Let $d_0, L > 0$ are constants provided by Lemma 2.6. Let $d < d_0$ and $\{x_k\}$ is a d-pseudotrajectory. Denote

$$X^s = \prod_{k=-\infty}^{\infty} I^s_{Ld}(x_k).$$

This set endowed with the Tikhonov product topology is compact and convex.

Let us consider map $H : X^s \to X^s$ defined as following

$$H(\{z_k\}) = \{z'_{k+1}\}, \quad \text{where} \quad z'_{k+1} = h^s_k(z_k).$$

By Lemma 2.6 this map is well-defined. Since z'_{k+1} depends only on z_k map H is continuous. Due to the Tikhonov-Schauder theorem, the mapping H has a (maybe non-unique) fixed point $\{z^*_k\}$. Denote $y^s_k = \exp^s_{x_k}(z^*_k)$. Since $z^*_{k+1} = h^s_k(z^*_k)$, inequality (2.10) implies that

$$y^s_{k+1} \in W^{cu}_{Ld}(f(y^s_k)), \quad k \in \mathbb{Z}. \tag{2.11}$$

Since $|z^*_k| < Ld$ we conclude

$$\mathrm{dist}(x_k, y^s_k) \leq \mathrm{dist}_s(x_k, y^s_k) < (1+\mu)Ld < 2Ld, \quad k \in \mathbb{Z}.$$

Similarly (decreasing d_0 and increasing L if necessarily) one may show that there exists a sequence $\{y^u_k \in W^u_{2Ld}(x_k)\}$ such that

$$y^u_{k+1} \in W^{cs}_{Ld}(f(y^u_k)), \qquad k \in \mathbb{Z}.$$

Hence $\mathrm{dist}(y^s_k, y^u_k) < \mathrm{dist}(y^s_k, x_k) + \mathrm{dist}(x_k, y^u_k) < 4Ld$. Decreasing d_0 if necessarily we can assume that $4L_0Ld < \delta_0$. Then there exists an unique point $y_k = W^{cu}_{4L_0Ld}(y^s_k) \cap W^s_{4L_0Ld}(y^u_k)$ and inclusion (2.11) implies that for all $k \in \mathbb{Z}$ the following holds

$$\begin{aligned}
&\mathrm{dist}_{cu}(y_{k+1}, f(y_k)) < \\
&\qquad \mathrm{dist}_{cu}(y_{k+1}, y^s_{k+1}) + \mathrm{dist}_{cu}(y^s_{k+1}, f(y^s_k)) + \mathrm{dist}_{cu}(f(y^s_k), f(y_k)) < \\
&\qquad\qquad 4L_0Ld + Ld + 4RL_0Ld = L_{cu}d,
\end{aligned}$$

where $R = \sup_{x \in M} |\,\mathrm{D}\, f(x)|$ and $L_{cu} > 1$ do not depends on d. Similarly for some constant $L_{cs} > 1$ the following inequalities hold

$$\mathrm{dist}_{cs}(y_{k+1}, f(y_k)) < L_{cs} d, \quad k \in \mathbb{Z}.$$

Reducing d_0 if necessarily we can assume that points $y_{k+1},\ f(y_k)$ satisfy assumptions of condition **A4** of Statement 2.5, hence

$$\mathrm{dist}_c(y_{k+1}, f(y_k)) < (1+\mu)\max(L_{cs}, L_{cu})d, \quad k \in \mathbb{Z}$$

and sequence $\{y_k\}$ is an $L_1 d$-central pseudotrajectory with

$$L_1 = (1+\mu)\max(L_{cs}, L_{cu}).$$

To complete the proof let us note that

$$\mathrm{dist}(x_k, y_k) < \mathrm{dist}(x_k, y_k^s) + \mathrm{dist}(y_k^s, y_k) < 2Ld + 4L_0 Ld, \quad k \in \mathbb{Z}.$$

Taking $\mathcal{L} = \max(L_1, 2L + 4L_0)$ we conclude that $\{y_k\}$ is an $\mathcal{L}d$-central pseudotrajectory which $\mathcal{L}d$ shadows $\{x_k\}$. $\square$

Remark 2.7. Note that we do not claim uniqueness of such sequences $\{y_k^s\}$ and $\{y_k^u\}$. In fact it is easy to show (we leave details to the reader) that uniqueness of those sequences is equivalent to the plaque expansivity conjecture.

$\square$

2.2 Linear Skew Product

In this paragraph, we study shadowing property for a model class of partially hyperbolic diffeomorphisms: linear skew products. We give lower and upper bounds for the precision of shadowing in the spirit of Definition 1.10.

Let $\Sigma = \{0, 1\}^{\mathbb{Z}}$. Endow Σ with the standard probability measure ν and the following metric:

$$\mathrm{dist}(\{\omega^i\}, \{\tilde{\omega}^i\}) = 1/2^k, \quad \text{where } k = \min\{|i| : \omega^i \neq \tilde{\omega}^i\}.$$

For a sequence $\omega = \{\omega^i\} \in \Sigma$ denote by $t(\omega)$ the 0th element of the sequence: $t(\omega) = \omega^0$. Define the "shift map" $\sigma : \Sigma \to \Sigma$ as follows: $(\sigma(\omega))^i = \omega^{i+1}$.

Consider the space $Q = \Sigma \times \mathbb{R}$. Endow Q with the product measure $\mu = \nu \times \mathrm{Leb}$ and the maximum metric:

$$\mathrm{dist}((\omega, x), (\tilde{\omega}, \tilde{x})) = \max(\mathrm{dist}(\omega, \tilde{\omega}), \mathrm{dist}(x, \tilde{x})).$$

For $q \in Q$ and $a > 0$ denote by $B(a, q)$ the open ball of radius a centered at q.

Fix $\lambda_0, \lambda_1 \in \mathbb{R}$ satisfying the following conditions

$$0 < \lambda_0 < 1 < \lambda_1, \quad \lambda_0\lambda_1 \neq 1. \tag{2.12}$$

Consider the map $f : Q \to Q$ defined as follows:

$$f(\omega, x) = (\sigma(\omega), \lambda_{t(\omega)}x).$$

For $q \in Q$, $d > 0$, $N \in \mathbb{N}$ let $\Omega_{q,d,N}$ be the set of d-pseudotrajectories of length N starting at $q_0 = q$. If we consider q_{k+1} being chosen at random in $B(d, f(q_k))$ uniformly with respect to the measure μ, then $\Omega_{q,d,N}$ forms a finite time Markov chain. This naturally endows $\Omega_{q,d,N}$ with a probability measure P. See also [110] for a similar concept for infinite pseudotrajectories.

For $\varepsilon > 0$ let $p(q, d, N, \varepsilon)$ be the probability of a pseudotrajectory in $\Omega_{q,d,N}$ to be ε-shadowable. Note that this event is measurable since it forms an open subset of $\Omega_{q,d,N}$.

Lemma 2.8. *Let $q = (\omega, x)$, $\tilde{q} = (\omega, 0)$. For any $d, \varepsilon > 0$, $N \in \mathbb{N}$, the following equality holds:*

$$p(q, d, N, \varepsilon) = p(\tilde{q}, d, N, \varepsilon).$$

Proof. Consider $\{q_k = (\omega_k, x_k)\} \in \Omega_{q,d,N}$. Put $r_k := x_{k+1} - \lambda_{t(\omega_k)}x_k$. Consider a sequence $\{\tilde{q}_k = (\omega_k, \tilde{x}_k)\}$, where

$$\tilde{x}_0 = 0, \quad \tilde{x}_{k+1} = \lambda_{t(w_k)}x_k + r_k.$$

The following holds:

1. the correspondence $\{q_k\} \leftrightarrow \{\tilde{q}_k\}$ is one-to-one and preserves the probability measure;

2. for any $\varepsilon > 0$ pseudotrajectory $\{q_k\}$ is ε-shadowed by a trajectory of a point (ω, x) if and only if $\{\tilde{q}_k\}$ is ε-shadowed by a trajectory of a point $(\omega, x - x_0)$.

These statements complete the proof of the lemma. $\square$

For $d, \varepsilon > 0$, $N \in \mathbb{N}$ define

$$p(d, N, \varepsilon) := \int_{\omega \in \Sigma} p((\omega, 0), d, N, \varepsilon)\mathrm{d}\nu.$$

Note that the integral exists since for fixed d, N, ε, the value $p((\omega, 0), d, N, \varepsilon)$ depends only on a finite number of entries of ω. The quantity $p(d, N, \varepsilon)$ can be interpreted as the probability of a d-pseudotrajectory of length N to be ε-shadowed.

The main result of this paragraph is the following [105]:

Theorem 2.9. *For any $\lambda_0, \lambda_1 \in \mathbb{R}$ satisfying (2.12) there exist $\varepsilon_0 > 0$, $0 < c_0 < \infty$ such that for any $\varepsilon < \varepsilon_0$, the following holds:*

1. *If $c < c_0$, then $\lim_{N\to\infty} p(\varepsilon/N^c, N, \varepsilon) = 0$;*

2. *if $c > c_0$, then $\lim_{N\to\infty} p(\varepsilon/N^c, N, \varepsilon) = 1$.*

Remark 2.10. Later (Lemma 2.13) we prove that for any $N \in \mathbb{N}$, $L > 0$, $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0)$, the equality $p(\varepsilon_1/L, N, \varepsilon_1) = p(\varepsilon_2/L, N, \varepsilon_2)$ holds. Hence the result of Theorem 2.9 actually does not depend on the value of ε.

Remark 2.11. Due to Remark 2.10 analog of the Hammel-Grebogi-Yorke conjecture for map f suggests that $p(\varepsilon/N, N, \varepsilon)$ is close to 1. Hence, if $c_0 > 1$, then Hammel-Grebogi-Yorke conjecture is not satisfied. For an example of such parameters see Remark 2.16.

Remark 2.12. We expect that similarly to works [28, 29], such a skew product can be embedded into a diffeomorphism of a manifold of dimension 4. This would allow us to construct an open set of diffeomorphisms violating a high-dimensional analog of Conjecture 1.1. Similarly, we can construct an open set of diffeomorphisms satisfying this conjecture. However, we did not implement such a construction and leave it out of the scope of the present paper.

In order to prove Theorem 2.9 we formulate an auxilarily problem for random walks.

Let $a_0 = \ln \lambda_0$, $a_1 = \ln \lambda_1$. Consider the following random variable:

$$\gamma = \begin{cases} a_0 & \text{with probability } 1/2, \\ a_1 & \text{with probability } 1/2. \end{cases}$$

Fix $N > 0$. Consider the random walk $\{A_i\}_{i\in[0,\infty)}$ generated by γ and independent uniformly distributed in $[-1, 1]$ variables $\{r_i\}_{i\in[0,\infty)}$. Define a sequence $\{z_i\}_{i\in[0,N]}$ as follows:

$$z_0 = 0, \quad z_{i+1} = z_i + \frac{r_{i+1}}{e^{A_{i+1}}}. \tag{2.13}$$

For given sequences $(\{A_i\}_{i\in[0,N]}, \{r_i\}_{i\in[0,N]})$ define

$$B(k, n) := \frac{e^{A_k+A_n}}{e^{A_k} + e^{A_n}}|z_n - z_k| = \frac{e^{A_n}}{e^{A_k} + e^{A_n}} \left|e^{A_k} z_n - e^{A_k} z_k\right|,$$

$$K(\{A_i\}, \{r_i\}) := \max_{0\le k<n\le N} B(k, n),$$

$$s(N, L) := P(K(\{A_i\}_{i\in[0,N]}, \{r_i\}_{i\in[0,N]}) < L),$$

where $P(\cdot)$ is the probability of a certain event.

Below we prove the following lemma.

Lemma 2.13. *There exist $\varepsilon_0 > 0$, $L_0 > 0$ such that for any $d \ge 0$, $L > L_0$, $N \in \mathbb{N}$ satisfying $Ld < \varepsilon_0$ the following equality holds:*

$$p(d, N, Ld) = s(N, L).$$

Proof. Let us choose $\varepsilon_0, L_0 > 0$ such that if $\operatorname{dist}(\omega, \tilde{\omega}) < \varepsilon_0$, then $t(\omega) = t(\tilde{\omega})$ and the map σ satisfies the Lipschitz shadowing property with constants ε_0, L_0.

Fix $d < d_0$, $N > 0$ and $L > L_0$ satisfying $Ld < \varepsilon_0$. Let us choose ω at random according to the probability measure ν and a pseudotajectory $\{q_k\} = \{(\omega_k, x_k)\} \in \Omega_{(\omega,0),d,N}$ according to the measure P. Consider the sequences

$$\gamma_k = a_{t(\omega_k)}, \quad A_k = \sum_{i=0}^{k} \gamma_i, \quad r_k = \frac{1}{d}(x_k - \lambda_{t(\omega_{k-1})} x_{k-1}).$$

Note that r_k are independent uniformly distributed in $[-1, 1]$ and γ_k are independent and distributed according to γ.

Below we prove that the sequence $\{q_k\}$ can be Ld-shadowed if and only if

$$L \geq K(\{A_i\}, \{r_i\}). \tag{2.14}$$

Assume that the pseudotrajectory (ω_k, x_k) is Ld-shadowed by an exact trajectory (ξ_k, y_k). By the choice of ε_0, the following equality holds:

$$t(\omega_k) = t(\xi_k). \tag{2.15}$$

Now let us study the behavior of the second coordinate. Note that

$$y_{k+1} = \lambda_{t(\xi_k)} y_k = e^{\gamma_k} y_k, \quad y_n = e^{A_n - A_k} y_k, \tag{2.16}$$
$$x_n = e^{A_n - A_k} x_k + e^{A_k}(z_n - z_k),$$

where z_k are defined by (2.13). Hence,

$$(y_n - x_n) = e^{A_n - A_k}(y_k - x_k) + e^{A_k}(z_n - z_k).$$

From this equality it is easy to deduce that

$$\max(|y_k - x_k|, |y_n - x_n|) \geq B(k, n)$$

and the equality holds if $(y_k - x_k) = -(y_n - x_n)$. Hence, inequality (2.14) holds.

Now let us assume that (2.14) holds and prove that (w_k, x_k) can be Ld-shadowed. Let us choose a sequence $\{\xi_k\}$ which Ld-shadows $\{w_k\}$, then equalities (2.15) hold.

For $y_0 \in \mathbb{R}$ define y_k by relations (2.16) and consider function $F : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$F(y_0) = \max_{0 \leq k \leq N} |y_k - x_k|.$$

Since the function F is continuous, it is easy to show that it attains a minimum for some y_0. Denote $L' := \min_{y_0 \in \mathbb{R}} F(y_0)$ and let y_0 be such that $L' = F(y_0)$. Let $D = \{k \in [0, N] : |y_k - x_k| = F(y_0)\}$. Let us consider two cases.

Case 1. For all $k \in D$ the value $y_k - x_k$ has the same sign. Without loss of generality, we can assume that these values are positive. Then for small enough $\delta > 0$, the inequality $F(y_0 - \delta) < F(y_0)$ holds, which contradicts the choice of y_0.

Case 2. There exists indices $k, n \in D$ such that the values $y_k - x_k$ and $y_n - x_n$ have different signs. Then $(y_k - x_k) = -(y_n - x_n)$, and hence $L' = B(k, n) \leq K(\{A_i\}, \{z_i\})$. □

Now let us pass to the proof of Theorem 2.9.

Note that shadowing problems for the maps f and f^{-1} are equivalent (up to a constant multiplier at d). In what follows, we assume that $\lambda_0 \lambda_1 > 1$. Put

$$v := E(\gamma) = (a_0 + a_1)/2 > 0, \quad M := (\ln N)^2, \quad w := v/2.$$

In the proof of Theorem 2.9, we use the following statements.

Lemma 2.14 (Large Deviation Principle, [107, Secion 3])**.** *There exists an increasing function $h : (0,\infty) \to (0,\infty)$ such that for any $\varepsilon > 0$ and $\delta > 0$ and for large enough n, the following inequalities hold:*

$$P\left(\frac{A_n}{n} - E(\gamma) < -\varepsilon\right) < e^{-(h(\varepsilon)-\delta)n}.$$

$$P\left(\frac{A_n}{n} - E(\gamma) < -\varepsilon\right) > e^{-(h(\varepsilon)+\delta)n}.$$

Lemma 2.15 (Ruin Problem, [21, Chapter XII, §4, 5])**.** *Let b be the unique positive root of the equation*

$$\frac{1}{2}\left(e^{-ba_0} + e^{-ba_1}\right) = 1.$$

For any $\delta > 0$ and for large enough $C > 0$, the following inequalities hold:

$$P(\exists i \geq 0 : A_i \leq -C) \leq e^{-C(b-\delta)}, \tag{2.17}$$

$$P(\exists i \geq 0 : A_i \leq -C) \geq e^{-C(b+\delta)}, \tag{2.18}$$

Put $c_0 = 1/b$. Due to Lemma 2.13, it is enough to prove the following:

(S1) If $c < c_0$, then $\lim_{N\to\infty} s(N, N^c) = 0$.

(S2) If $c > c_0$, then $\lim_{N\to\infty} s(N, N^c) = 1$.

Remark 2.16. For $\lambda_0 = 1/2$, $\lambda_1 = 3$ the inequalities $b < 1$, $c_0 > 1$ hold, and hence by Remark 2.11 the statement of Conjecture 1.1 does not hold. Similarly, $c_0 > 1$ for $\lambda_0 = 1/3$, $\lambda_1 = 2$.

Below we prove items (S1) and (S2) separately.

Proof of (S1). Assume that $c < 1/b$. Let us choose $c_1 \in (c, 1/b)$ and $\delta > 0$ satisfying

$$c_1(b + \delta) < 1. \tag{2.19}$$

Consider the following events:

$$\begin{aligned} I &= \{\exists i \in [0, M] : A_i \leq -c_1 \ln N; \text{ and } A_{2M} \geq 0\}, \\ I_1 &= \{\exists i \in [0, M] : A_i \leq -c_1 \ln N\}, \\ I_2 &= \{\exists i \in [0, M] : A_i \leq -wM\}, \\ I_3 &= \{A_{2M} - A_M \leq wM\}. \end{aligned}$$

The following holds:

$$P(I) \geq P(I_1) - P(I_2) - P(I_3). \tag{2.20}$$

Lemmas 2.14, 2.15 imply the following

$$\begin{aligned}
P(I_1) &\geq P(\exists i \geq 0 : A_i \leq -c_1 \ln N) - P(\exists i > M : A_i \leq -c_1 \ln N) \\
&\geq e^{-c_1 \ln N(b+\delta)} - \sum_{i=M+1}^{N} P(A_i \leq 0) \geq N^{-c_1(b+\delta)} - \sum_{i=M+1}^{N} e^{-ih(v)} \\
&\geq N^{-c_1(b+\delta)} - \frac{1}{1-e^{-h(v)}} e^{-(M+1)h(v)} \geq N^{-c_1(b+\delta)} + o(N^{-2}). \qquad (2.21)
\end{aligned}$$

Similarly

$$P(I_2) \leq \sum_{i=M+1}^{\infty} P(A_i \leq 0) = o(N^{-2}), \qquad (2.22)$$

$$P(I_3) \leq e^{-Mh(v-w)} = o(N^{-2}). \qquad (2.23)$$

From inequalities (2.20)-(2.23) we conclude that

$$P(I) \geq N^{-c_1(b+\delta)} + o(N^{-2}). \qquad (2.24)$$

Assume that the event I has happened and let $i \in [0, M]$ be one of the indices satisfying the inequality $A_i < -c_1 \ln N$. Note that the following events are independent:

$$J_1 = \{r_i \in [1/2; 1]\}, \quad J_2 = \left\{ z_{2M} - z_0 \geq \frac{r_i}{e^{A_i}} \right\}.$$

Hence,

$$P\left(z_{2M} - z_0 \geq \frac{1}{2e^{A_i}} \right) \geq P(J_1)P(J_2) = 1/4 \cdot 1/2 = 1/8$$

and

$$P(B(0, 2M) > N^{c_1}/4) \geq \frac{1}{8} P(I) = \frac{1}{8} N^{-c_1(b+\delta)} + o(N^{-2}).$$

Note that for large enough N, the inequality $N^c < N^{c_1}/4$ holds, and hence

$$P(B(0, 2M) > N^c) \geq \frac{1}{8} N^{-c_1(b+\delta)} + o(N^{-2}).$$

Similarly, for any $k \in [0, N - 2M]$,

$$P(B(k, k + 2M) > N^c) \geq \frac{1}{8} N^{-c_1(b+\delta)} + o(N^{-2}).$$

Note that the events in the last expression for $k = 0, 2M, 2 \cdot 2M, \ldots ([N/(2M)] - 1)2M$ are independent, and hence

$$P\left(\exists k \in [0, N - 2M] : B(k, k + 2M) > N^c\right) \geq 1 - \left(1 - \left(\frac{1}{8} N^{-c_1(b+\delta)} + o(N^{-2})\right)\right)^{[N/(2M)]}. \qquad (2.25)$$

Using (2.19), we conclude that

$$\left(\frac{1}{8}N^{-c_1(b+\delta)}+o(N^{-2})\right)[N/(2M)] \geq \left(\frac{1}{8}N^{-c_1(b+\delta)}+o(N^{-2})\right)\left(\frac{N}{2(\ln N)^2}-1\right)$$
$$=\frac{1}{16(\ln N)^2}N^{1-c_1(b+\delta)}+o(N^{-1}) \xrightarrow[N\to\infty]{} \infty$$

and hence

$$\left(1-\left(\frac{1}{8}N^{-c_1(b+\delta)}+o(N^{-2})\right)\right)^{[N/(2M)]} \xrightarrow[N\to\infty]{} 0. \tag{2.26}$$

Relations (2.25), (2.26) imply that

$$P(K(\{A_i\}_{i\in[0,N]},\{r_i\}_{i\in[0,N]})>N^c) \xrightarrow[N\to\infty]{} 1.$$

Hence,

$$\lim_{N\to\infty} s(N,N^c)=0.$$

$\square$

Proof of (S2). Let $c>1/b$. Let us choose $c_1\in(1/b,c)$ and $\delta>0$ satisfying $c_1(b-\delta)>1$.

Note that for any $n>k$ the following inequalities hold:

$$e^{A_k}|z_n-z_k| \leq \sum_{i=k}^{n} e^{-(A_i-A_k)},$$

$$\frac{e^{A_n}}{e^{A_k}+e^{A_n}} \leq 1.$$

Hence,

$$K(\{A_i\},\{r_i\}) \leq \max_{0\leq k<n\leq N}\sum_{i=k}^{n} e^{-(A_i-A_k)} \leq \max_{0\leq k\leq N}\sum_{i=k}^{N} e^{-(A_i-A_k)} =: D(\{A_i\}). \tag{2.27}$$

The following holds:

$$P(D(\{A_i\})<N^c) \geq 1-P\left(\exists k\in[0,N]: \sum_{i=k}^{N} e^{-(A_i-A_k)}>N^c\right)$$
$$\geq 1-NP\left(\sum_{i=0}^{N} e^{-(A_i-A_k)}>N^c\right).$$

Note that if $\sum_{i=0}^{N} e^{-(A_i-A_k)}>N^c$, then one of the following inequalities holds:

$$\exists i\in[0,M]: e^{-A_i}>\frac{N^c}{2M},$$

$$\exists i \in [M, N] : e^{-A_i} > \frac{N^{c-1}}{2}.$$

Note that for large enough N, the following inequalities hold:

$$\frac{N^c}{2M} > N^{c_1}, \quad N^{c-1}/2 > e^{-wM},$$

and hence (arguing similarly to the previous section), for large enough N,

$$\begin{aligned} P\left(\sum_{i=0}^{N} e^{-(A_i - A_k)} > N^{c_1}\right) &\leq P(\exists i \in [0, M] : A_i < -c_1 \ln N) + P(\exists i \in [M, N] : A_i < wM) \\ &\leq e^{-(b-\delta)c_1 \ln N} + o(N^{-2}) = N^{-(b-\delta)c_1} + o(N^{-2}). \end{aligned}$$

Finally,

$$P(D(\{A_i\}) \leq N^c) \geq 1 - N(N^{-(b-\delta)c_1} + o(N^{-2})) \xrightarrow[N\to\infty]{} 1,$$

and hence relations (2.27) imply that

$$\lim_{N\to\infty} s(N, N^c) = 1.$$

$\square$

Chapter 3

Shadowing for vector fields in C^1-topology

3.1 Preliminaries

In this context, there is a real difference between the cases of discrete dynamical systems generated by diffeomorphisms and systems with continuous time (flows) generated by smooth vector fields. This difference is due to the necessity of reparametrizing shadowing trajectories in the latter case. One of the main goals of the present paper is to show that this difference is crucial, and the results for flows are essentially different from those for diffeomorphisms.

Let M be a smooth closed (i.e., compact and boundaryless) manifold with Riemannian metric dist and let $n = \dim M$. Consider a smooth ($\mathbf{C}^1$) vector field on X and denote by ϕ the flow of X. We denote by

$$O(x, \phi) = \{\phi(t, x) : \ t \in \mathbb{R}\}$$

the trajectory of a point x in the flow ϕ; $O^+(x, \phi)$ and $O^-(x, \phi)$ are the positive and negative semitrajectories, respectively. Denote by $\mathrm{Orb}(x) = \mathrm{Orb}_X(x) = \phi_{(-\infty,+\infty)}(x)$ the orbit of x. And denote by $\mathrm{Orb}^+(x) = \phi_{[0,+\infty)}(x)$, $\mathrm{Orb}^-(x) = \phi_{(-\infty,0]}(x)$ the positive, negative orbit of x respectively.

We consider the following $\mathbf{C}^1$ metric on the space of smooth vector fields: If X and Y are vector fields of class $\mathbf{C}^1$, we set

$$\rho_1(X, Y) = \max_{x \in M} \left(|X(x) - Y(x)| + \left\| \frac{\partial X}{\partial x}(x) - \frac{\partial Y}{\partial x}(x) \right\| \right),$$

where $|.|$ is the norm on the tangent space $T_x M$ generated by the Riemannian metric dist, and $\|.\|$ is the corresponding operator norm for matrices.

For a set A of vector fields, $\mathrm{Int}^1(A)$ denotes the interior of A in the $\mathbf{C}^1$ topology generated by the metric ρ_1.

As in the case of diffeomorphisms for us will be important notions of hyperbolicity and structural stability.

Definition 3.1. We say that a compact invariant set $\Lambda \subset M$ is *hyperbolic* if there exist numbers $C > 0$, $\lambda > 0$ and linear subspaces $E^s_x, E^u_x \subset T_xM$ such that for any $x \in \Lambda$ the following holds

1. $T_xM = E^s_x \oplus E^u_x \oplus < X(x) >$.

2. Let $\Phi(t)$ be the fundamental matrix of the variational systems

$$\frac{dy}{dy} = \frac{\partial X}{\partial x}(\phi(t,x))y$$

along the trajectory $\phi(t,p)$, satisfying $\Phi(0) = E$. Then

   (a) $\Phi(t)E^s_x = E^s_{\phi(t,x)}$, $\Phi(t)E^u_x = E^u_{\phi(t,x)}$,

   (b) $|\Phi(t)v^s| \le Ce^{-\lambda t}|v^s|$ for $v^s \in E^s_x$ and $t \ge 0$.

   (c) $|\Phi(-t)v^u| \le Ce^{-\lambda t}|v^u|$ for $v^u \in E^u_x$ and $t \ge 0$.

Definition 3.2. We say that a vector field $X \in \mathcal{F}(M)$ is *struclurally stable* if there exists a neighborhood $U \subset \mathcal{F}(M)$ of X such that for any $Y \in U$ there exists a homeomorphism $\alpha : M \to M$ which maps trajectories of X to trajectories of Y and preserves the direction of movement alomg trajectories. In other words there exists a map $\tau : \mathbb{R} \times M \to \mathbb{R}$ such that

- for any $x \in M$, the function $\tau(\cdot, x)$ increases and maps $\mathbb{R}$ into $\mathbb{R}$;
- $\tau(0, x) = x$ for any $x \in M$;
- $\alpha(\phi(t,x)) = \psi(\tau(t,x), \alpha(x))$ for any $t \in \mathbb{R}$, $x \in M$, where $\psi(\cdot,\cdot)$ is the flow generated by Y.

Let us denote by $\mathbf{S}$ and $\mathbb{N}$ the sets of structurally stable and nonsingular vector fields, respectively. For a vector field X denote by $\Omega(X)$ the set of nonwondering points of X.

Not Let us pass to the definition of the shadowing property.

Definition 3.3. Fix a number $d > 0$. We say that a mapping $g : \mathbb{R} \to M$ (not necessarily continuous) is a d-pseudotrajectory (both for the field X and flow ϕ) if

$$\operatorname{dist}(g(\tau + t), \phi(t, g(\tau))) < d \quad \text{for} \quad \tau \in \mathbb{R},\ t \in [0,1]. \tag{3.1}$$

Definition 3.4. A reparametrization is an increasing homeomorphism h of the line $\mathbb{R}$; we denote by Rep the set of all reparametrizations.

For $a > 0$, we denote

$$\operatorname{Rep}(a) = \left\{ h \in \operatorname{Rep} : \left| \frac{h(t) - h(s)}{t - s} - 1 \right| < a, \quad t, s \in \mathbb{R},\ t \ne s \right\}.$$

Definition 3.5. We say that a vector field X has the standard shadowing property ($X \in$ StSh) if for any $\varepsilon > 0$ we can find $d > 0$ such that for any d-pseudotrajectory $g(t)$ of X there exists a point $p \in M$ and a reparametrization $h \in \mathrm{Rep}(\varepsilon)$ such that

$$\mathrm{dist}(g(t), \phi(h(t), p)) < \varepsilon \quad \text{for} \quad t \in \mathbb{R}. \tag{3.2}$$

Definition 3.6. We say that a vector field X has the oriented shadowing property ($X \in$ OrientSh) if for any $\varepsilon > 0$ we can find $d > 0$ such that for any d-pseudotrajectory of X there exists a point $p \in M$ and a reparametrization $h \in \mathrm{Rep}$ such that inequalities (3.2) hold (thus, it is not assumed that the reparametrization h is close to identity).

Definition 3.7. We say that a vector field has the Lipschitz shadowing property if there exists $L, d_0 > 0$ such that for any $d \in (0, d_0)$ and d-pseudotrajectory g there exists $x_0 \in M$ and a reparametrisation $h \in \mathrm{Rep}(Ld)$ such that inequalities (3.2) hold for $\varepsilon = Ld$.

Let us note that the standard shadowing property is equivalent to the strong pseudo orbit tracing property (POTP) in the sense of Komuro [41]; the oriented shadowing property was called the normal POTP by Komuro [41] and the POTP for flows by Thomas [101].

Standard and oriented shadowing properties differs only in restrictions on reparametrizations. In case of standard shadowing reparametrization is asked to be close to identity and in case of the oriented shadowing it can be an arbitrarily increasing homeomorphism. Clearly

$$\mathrm{StSh}(M) \subset \mathrm{OrientSh}(M). \tag{3.3}$$

In paragraph 3.4 we show that the difference in the choice of reparametrization is essential, so inclusion (3.3) is strict [102]. However for vector fields without singularities standard and oriented shadowing properties are equivalent [41].

Definition 3.8. We say that a vector field X and the corresponding flow $\phi(t, x)$ are *expansive* if there exist constants $a, \delta > 0$ such that if

$$\mathrm{dist}(\phi(t, x), \phi(\alpha(t), y)) < a, \quad t \in \mathbb{R},$$

for points $x, y \in M$ and an increasing homeomorphism α of the real line, then $y = \phi(\tau, x)$ for some $|\tau| < \delta$.

As in the case of diffeomorphisms the following shadowing lemma holds.

Theorem 3.1. *If Λ is a hyperbolic set for a vector field X, then there exists a neighborhood V of Λ such that X has the Lipschitz shadowing property on V and is expansive on V.*

Moreover,

Theorem 3.2. *[71] Structurally stable vector fields satisfy the Lipschitz shadowing property.*

Remark 3.3. In fact, it is shown in [71] that if a structurally stable vector field does not have closed trajectories, then it has the Lipschitz shadowing property without reparametrization of shadowing trajectories: there exists $L > 0$ such that if $g(t)$ is a d-pseudotrajectory with small d, then there exists a point x such that

$$\mathrm{dist}(g(t), \phi(t, x)) \leq Ld, \quad t \in \mathbb{R}.$$

In the present chapter we study structure of the set of vector fields satisfying shadowing properties. Lee and Sakai [47] proved the following:

Theorem 3.4. $\mathrm{Int}^1(\mathrm{StSh} \cap \mathbb{N}) \subset \boldsymbol{S}$.

To formulate second result, we need one more notion.

We say that matrix A belong to class $\mathcal{K}$ if all its eigenvalues have nonzero real part. Let us note that fixed point p is hyperbolic iif $\mathrm{D}\, X(p) \in \mathcal{K}$. Let us denote by $\mathcal{K}_1^+$ the set of matrixes $A \in \mathcal{K}$ satisfying the following: there exists real eigenvalue $a_1 > 0$ such that if $c_1 + d_1 i$ is an eigenvalue of A with $c_1 > 0$ and $d_1 \neq 0$ then $c_1 > a_1$. Let us denote by $\mathcal{K}_2^+$ the set of matrixes $A \in \mathcal{K}$ satisfying the following: there exists a pair of complex conjugate eigenvalues $a_1 \pm b_1 i$ with $a_1 > 0$, such that if $c_1 > 0$ is an eigenvalue of A then $c_1 > a_1$. Note that $\mathcal{K}_1^+ \cap \mathcal{K}_2^+ = \emptyset$, but $\mathcal{K}_1^+ \cup \mathcal{K}_2^+ \neq \mathcal{K}$

Denote by $\mathcal{K}_1^-$ the set of matrixes A, satisfying $-A \in K_1^+$. Denote by $\mathcal{K}_2^-$ the set of matrixes A, satisfying $-A \in K_2^+$

Definition 3.9. Let us say that a vector field X belongs to the class $\mathcal{B}$ if X has two hyperbolic rest points p and q (not necessarily different) with the following properties:

1. $\mathrm{D}\, X(p) \in \mathcal{K}_2^+$,
2. $\mathrm{D}\, X(q) \in \mathcal{K}_2^-$,
3. the stable manifold $W^s(p)$ and the unstable manifold $W^u(q)$ have a trajectory of non-transverse intersection.

Condition (1) above means that the "weakest" contraction in $W^s(q)$ is due to the eigenvalues $\mu_{1,2}$ (condition (2) has a similar meaning).

In my Ph. D. Thesis among other results I have proved the following two theorems

Theorem 3.5. $\mathrm{Int}^1(\mathrm{OrientSh} \setminus \mathcal{B}) = \boldsymbol{S}$.

Theorem 3.6. *If $dimM \leq 3$, then* $\mathrm{Int}^1(\mathrm{OrientSh}) = \boldsymbol{S}$.

Note that Theorem 3.5 generalises above-mentioned result by Lee and Sakai.

3.2 Example of a not structurally stable vector field

In this section we show that exclusion from consideration of vector fields of class $\mathcal{B}$ was essential. More precisely we prove the following [83].

Theorem 3.7. $\mathrm{Int}^1(\mathrm{OrientSh}) \cap \mathcal{B} \neq \emptyset$.

3.2.1 Construction of the example

Consider a vector field X^* on the manifold $M = S^2 \times S^2$ that has the following properties (F1)-(F3) (ϕ^* denotes the flow generated by X^*).

(F1) The nonwandering set of ϕ^* is the union of four rest points p^*, q^*, s^*, u^*.

(F2) For some $\delta > 0$ we can introduce coordinates in the neighborhoods $B(\delta, p^*)$ and $B(\delta, q^*)$ such that

$$X^*(x) = J_p^*(x - p^*), \quad x \in B(\delta, p^*), \quad \text{and} \quad X^*(x) = J_q^*(x - q^*), \quad x \in B(\delta, q^*),$$

where

$$J_p^* = -J_q^* = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

(F3) The point s^* is an attracting hyperbolic rest point. The point u^* is a repelling hyperbolic rest point. The following condition holds:

$$W^u(p^*) \setminus \{p^*\} \subset W^s(s^*), \quad W^s(q^*) \setminus \{q^*\} \subset W^u(u^*). \tag{3.4}$$

The intersection of $W^s(p^*) \cap W^u(q^*)$ consists of a single trajectory α^*, and for any $x \in \alpha^*$, the condition

$$\dim T_x W^s(p^*) \oplus T_x W^u(q^*) = 3 \tag{3.5}$$

holds.

These conditions imply that the two-dimensional manifolds $W^s(p^*)$ and $W^u(q^*)$ intersect along a one-dimensional curve in the four-dimensional manifold M. Thus, $W^s(p^*)$ and $W^u(q^*)$ are not transverse; hence, $X^* \in \mathcal{B}$.

A construction of such a vector field is given in paragraph 3.2.5.

To prove Theorem 3.7, we show that $X^* \in \mathrm{Int}^1(\mathrm{OrientSh})$.

The vector field X^* satisfies Axiom A and the no-cycle condition; hence, X^* is Ω-stable. Thus, there exists a neighborhood V of X^* in the C^1-topology such that for any field $X \in V$, its nonwandering set consists of four hyperbolic rest points p, q, s, u which belong to small neighborhoods of p^*, q^*, s^*, u^*, respectively. We denote by ϕ the flow of any $X \in V$ and by $W^s(p), W^u(p)$ etc the corresponding stable and unstable manifolds.

Note that if the neighborhood V is small enough, then there exists a number $c > 0$ (the same for all $X \in V$) such that

$$B(c, s^*) \subset W^s(s) \quad \text{and} \quad B(c, u^*) \subset W^u(u).$$

Consider the set $\Theta = W^u(p^*) \cap \partial B(\delta, p^*)$ (where ∂A is the boundary of a set A). Condition (3.4) implies that there exists a neighborhood U_Θ of Θ and a number $T > 0$ such that

$$\phi^*(T, x) \in B(c/2, s^*), \quad x \in U_\Theta.$$

Reducing V, if necessary, we may assume that

$$W^u(p) \cap \partial B(\delta, p) \subset U_\Theta \quad \text{and} \quad \phi(T, x) \in B(c, s^*), \quad x \in U_\Theta.$$

Hence, $W^u(p) \setminus \{p\} \subset W^s(s)$, and

$$W^u(p) \cap W^s(q) = \emptyset. \tag{3.6}$$

Similarly, we may assume that $W^s(q) \setminus \{q\} \subset W^u(u)$.

The following two cases are possible for $X \in V$.

(S1) $W^s(p) \cap W^u(q) = \emptyset$.

(S2) $W^s(p) \cap W^u(q) \neq \emptyset$.

In case (S1), X is a Morse-Smale field; hence, $X \in \mathbf{S}$. Theorem 3.2 implies that $X \in$ OrientSh.

Thus, in the rest of the proof of Theorem 3.7, we consider case (S2). Our goal is to show that if the neighborhood V is small enough, then $X \in$ OrientSh.

3.2.2 Properties of small perturbations of X^*

Lemma 3.8. *If the neighborhood V is small enough, then the intersection $W^s(p) \cap W^u(q)$ consists of a single trajectory.*

Proof. Denote $x_p^* = \alpha^* \cap \partial B(\delta, p^*)$ and $x_q^* = \alpha^* \cap \partial B(\delta, q^*)$.

Consider sections Q_p and Q_q transverse to α at the points x_p^* and x_q^*, respectively, and the corresponding Poincaré map $F^* : Q_q \to Q_p$. Consider the curves $\xi_p^* = W^s(p^*) \cap Q_p \cap B(\delta/2, x_p^*)$ and $\xi_q^* = W^s(q^*) \cap Q_q \cap B(\delta/2, x_q^*)$. Note that ξ_p^* and $F^*(\xi_q^*)$ intersect at a single point x_p^*.

Let $\xi_p = W^s(p) \cap Q_p \cap B(\delta/2, x_p^*)$ and $\xi_q = W^u(q) \cap Q_q \cap B(\delta/2, x_q^*)$. Let F be the Poincaré transformation for X from Q_q to Q_p similar to F^*.

If the neighborhood V is small enough, then the curves ξ_p, ξ_q, and $F(\xi_q)$ are $\mathbf{C}^1$-close to ξ_p^*, ξ_q^*, and $F^*(\xi_q^*)$, respectively (hence, the intersection of ξ_p and $F(\xi_q)$ contains not more than one point).

The same reasoning as in the proof of (3.6) shows that if the neighborhood V is small enough, $x \in W^s(p) \setminus \{p\}$, and the trajectory of x does not intersect ξ_p, then $x \in W^u(u)$.

Thus, any trajectory in $W^s(p) \cap W^u(q)$ must intersect ξ_p; similarly, it must intersect ξ_q as well as $F(\xi_q)$.

It follows that the intersection $W^s(p) \cap W^u(q)$ (which is nonempty since we consider case (S2)) consists of a single trajectory containing the unique point x_p of intersection of ξ_p and $F(\xi_q)$ (we denote this trajectory by α). This completes the proof of Lemma 3.8.

Remark 3.9. Let us note an important property of intersection of $W^s(p)$ and $W^u(q)$ along α (see (3.8) below).

Let $x_q = F^{-1}(x_p)$; denote by i_p and i_q unit tangent vectors to the curves ξ_p and ξ_q at x_p and x_q, respectively. Our reasoning above and condition (3.5) show that if the neighborhood V is small enough, then the vectors i_p and $DF(x_q)i_q$ are not parallel:

$$DF(x_q)i_q \nparallel i_p. \tag{3.7}$$

Take any two points $y_p = \phi(t_1, x_p)$ and $y_q = \phi(t_2, x_q)$ with $t_1 \geq 0, t_2 \leq 0$; let S_p and S_q be smooth transversals to α at these points. Let e_p and e_q be tangent vectors of $S_p \cap W^s(p)$ and $S_q \cap W^u(q)$ at y_p and y_q, respectively. Denote by $f : S_q \to S_p$, $H_p : Q_p \to S_p$, and $H_q : S_q \to Q_q$ the corresponding Poincaré transformations for X. Then $f = H_p \circ F \circ H_q$,

$$e_p \parallel DH_p(x_p)i_p, \quad \text{and} \quad e_q \parallel DH_q^{-1}(x_q)i_q.$$

Hence, $Df(y_q)e_q \parallel DH_p \circ DF(x_q)i_q$, and it follows from (3.7) that

$$Df(y_q)e_q \nparallel e_p. \tag{3.8}$$

3.2.3 Oriented shadowing property for small perturabations

Now it remains to show that if V is small enough and $X \in V$, then $X \in \text{OrientSh}$ (recall that we consider case (S2)). This proof is rather complicated, and we first describe its scheme.

We fix two points $y_p, y_q \in \alpha$ in small neighborhoods U_p and U_q of p and q, respectively (the choice of U_p and U_q is specified later). We consider special pseudotrajectories (of type Ps): the "middle" part of such a pseudotrajectory is the part of α between y_q and y_p, while its "negative" and "positive" tails are parts of trajectories that start near y_q and y_p, respectively. We show that our shadowing problem is reduced to shadowing of pseudotrajectories of type Ps.

The key part of the proof is a statement "on four balls." It is shown that if $B_1, \dots, B_4$ are small balls such that B_1 and B_4 are centered at points of $W^s(q)$ and $W^u(p)$, while B_2 and B_3 are centered at y_q and y_p, respectively, then there exists an exact trajectory that intersects $B_1, \dots, B_4$ successfully as time grows. This statement (and its analog) allows us to prove that pseudotrajectories of type Ps can be shadowed.

Let us fix points $y_p, y_q \in \alpha$ (everywhere below, we assume that $y_p = \alpha(T_p)$ and $y_q = \alpha(T_q)$ with $T_p > T_q$) and a number $\delta > 0$. We say that $g(t)$ is a pseudotrajectory of type Ps(δ) if

$$g(t) = \begin{cases} \phi(t - T_p, x_p), & t > T_p, \\ \phi(t - T_q, x_q), & t < T_q, \\ \alpha(t), & t \in [T_q, T_p], \end{cases} \tag{3.9}$$

for some points

$$x_p \in B(\delta, y_p) \quad \text{and} \quad x_q \in B(\delta, y_q).$$

Fix an arbitrary $\varepsilon > 0$. We prove the following two statements (Propositions 3.10 and 3.11). In these statements, we say that a pseudotrajectory $g(t)$ can be ε-shadowed if there exists a reparametrization h and a point p such that (3.2) holds.

An Ω-stable vector field has a continuous Lyapunov function that strictly decreases along wandering trajectories (see [86]). Hence, there exist small neighborhoods U_p and U_q of points p and q, respectively, such that

$$\phi(t, x) \notin U_q, \quad x \in U_p,\ t \geq 0. \tag{3.10}$$

Proposition 3.10. *For any $\delta > 0$, $y_p \in \alpha \cap U_p$, and $y_q \in \alpha \cap U_q$ there exists $d > 0$ such that if $g(t)$ is a d-pseudotrajectory of X, then either $g(t)$ can be ε-shadowed or there exists a pseudotrajectory $g^*(t)$ of type $Ps(\delta)$ with these y_p and y_q such that* $\operatorname{dist}(g(t), g^*(t)) < \varepsilon/2, \quad t \in \mathbb{R}$.

Proposition 3.11. *There exists $\delta > 0$, $y_p \in \alpha \cap U_p$, and $y_q \in \alpha \cap U_q$ such that any pseudotrajectory of type $Ps(\delta)$ with these y_p and y_q can be $\varepsilon/2$-shadowed.*

Clearly, Propositions 3.10 and 3.11 imply that $X \in \mathrm{OrientSh}$.

To prove Proposition 3.10, we need an auxiliary statement.

Lemma 3.12. *For any $x \in \alpha$ and $\varepsilon, \varepsilon_1 > 0$ there exists $d > 0$ such that if*

$$\{g(t) :\ t \in \mathbb{R}\} \cap B(\varepsilon_1, x) = \emptyset, \tag{3.11}$$

for a d-pseudotrajectory $g(t)$, then one can find $x_0 \in M$ and $h(t) \in \mathrm{Rep}$ such that

$$\operatorname{dist}(g(t), \phi(h(t), x_0)) < \varepsilon, \quad t \in \mathbb{R}.$$

Proof. Take $\Delta < \varepsilon_1/2$ such that if $a_p = \phi(1, x)$ and $a_q = \phi(-1, x)$, then $a_p, a_q \notin B(\Delta, x)$. Let S_p and S_q be three-dimensional transversals to α at a_p and a_q, respectively. Let $f : S_q \to S_p$ be the corresponding Poincaré mapping. Note that the intersections $W^u(q) \cap S_q$ and $W^s(p) \cap S_p$ near a_q and a_p are one-dimensional, hence the curves $f(W^u(q) \cap S_q)$ and $W^s(p) \cap S_p$ in S_p are nontransverse.

It is shown in [58, 88] that there exists an arbitrarily small perturbation of the field X supported in $B(\Delta, x)$ and such that the Poincaré mapping $\tilde{f} : S_q \to S_p$ of the perturbed field $\tilde{X}$ satisfies the condition

$$\tilde{f}(W^u(q) \cap S_q) \cap (W^s(p) \cap S_p) = \emptyset.$$

Similarly to case (S1), we conclude that we can find $\tilde{X} \in \mathbf{S}$.

Set $\varepsilon_2 = \min(\varepsilon, \varepsilon_1/2)$ and find $d > 0$ such that any d-pseudotrajectory of the field $\tilde{X}$ can be ε_2-shadowed. We assume, in addition, that

$$\Delta + d < \varepsilon_1. \tag{3.12}$$

Consider an arbitrary d-pseudotrajectory $g(t)$ of X for which (3.11) holds. By (3.12), $g(t)$ is a d-pseudotrajectory of the field $\tilde{X}$. Due to the choice of d, there exists $x_0 \in M$ and $h(t) \in \mathrm{Rep}$ such that

$$\operatorname{dist}(g(t), \tilde{\phi}(h(t), x_0)) < \varepsilon_2,$$

where $\tilde{\phi}$ is the flow of $\tilde{X}$. Hence, $\{\tilde{\phi}(h(t), x_0),\ t \in \mathbb{R}\} \cap B(\varepsilon_1, x) = \emptyset$; it follows that $\tilde{\phi}(h(t), x_0) = \phi(h(t), x_0)$, which proves Lemma 3.12.

Proof of Proposition 3.10. Take $\delta > 0$, $y_p \in \alpha \cap U_p$, and $y_q \in \alpha \cap U_q$. Let $y_q = \alpha(T_q)$ and $y_p = \alpha(T_p)$. There exists $\delta_1 \in (0, \min(\delta, \varepsilon))$ such that $B(\delta_1, y_p) \subset U_p$, $B(\delta_1, y_q) \subset U_q$, and if $x_p \in B(\delta_1, y_p)$ and $x_q \in B(\delta_1, y_q)$, then

$$g^*(t) = \begin{cases} \phi(t - T_p, x_p), & t > T_p, \\ \alpha(t), & t \in [T_q, T_p], \\ \phi(t - T_q, x_q), & t < T_q, \end{cases} \tag{3.13}$$

is a pseudotrajectory of type $\mathrm{Ps}(\delta)$.

Take $x = \alpha(T)$, where $T \in (T_q, T_p)$. Applying Lemma 3.12, we can find $\varepsilon_1 > 0$ such that if d is small enough, then for any d-pseudotrajectory $g(t)$, one of the following two cases holds (after a shift of time):

(A1)

$$\{g(t),\ t \in \mathbb{R}\} \cap B(\varepsilon_1, x) = \emptyset,$$

and $g(t)$ can be ε-shadowed;

(A2)

$$g(T_p) \in B(\delta_1/2, y_p), \quad g(T_q) \in B(\delta_1/2, y_q),$$

and

$$\mathrm{dist}(g(t), \alpha(t)) < \varepsilon/2, \quad t \in [T_q, T_p].$$

To prove Proposition 3.10, it remains to consider case (A2).

Apply the same reasoning as in Lemma 3.12 to construct a field $\tilde{X} \in \mathbf{S}$ that coincides with X outside $B(\delta_1/2, y_q)$; let $\tilde{\phi}$ be the flow of $\tilde{X}$.

Note that $\tilde{X}$ does not have closed trajectories. Reducing d, if necessary, we may assume that any d-pseudotrajectory of $\tilde{X}$ can be $\delta_1/2$-shadowed in the sense of Remark 1.

Consider the mapping

$$\tilde{g}_p(t) = \begin{cases} \tilde{\phi}(t - T_p, g(T_p)), & t < T_p, \\ g(t), & t \in [T_p, T], \\ \tilde{\phi}(t - T, g(T)), & t > T, \end{cases}$$

where

$$T = \inf\{t > T_p : \tilde{g}_p(t) \in B(\delta_1, y_q)\}$$

(if $\{t > T_p : \tilde{g}_p(t) \in B(\delta_1, y_q)\} = \emptyset$, we set $T = +\infty$). Since

$$B(\delta_1/2, g(t)) \cap B(\delta_1/2, y_q) = \emptyset$$

for $t \in [T_p, T)$, $\tilde{g}_p(t)$ is a d-pseudotrajectory of $\tilde{X}$. Hence, there exists a point x_p such that

$$\mathrm{dist}(\tilde{g}_p(t), \tilde{\phi}(t - T_p, x_p)) < \delta_1/2, \quad t \in \mathbb{R}.$$

The first inclusion in (A2) implies that $x_p \in B(\delta, y_p)$.

Since trajectories of X and $\tilde{X}$ coincide outside $B(\delta_1/2, y_q)$, we deduce from (3.10) that $T = +\infty$; hence,

$$\operatorname{dist}(g(t), \phi(t - T_p, x_p)) < \delta_1/2, \quad t \geq T_p.$$

Similarly (reducing d, if necessary), we find $x_q \in B(\delta, y_q)$ such that

$$\operatorname{dist}(g(t), \phi(t - T_q, x_q)) < \delta_1/2, \quad t \leq T_q.$$

Clearly, the mapping (3.13) is a pseudotrajectory of type $\mathrm{Ps}(\delta)$ such that

$$\operatorname{dist}(g(t), g^*(t)) < \varepsilon/2, \quad t \in \mathbb{R}.$$

This completes the proof of Proposition 3.10.

In the remaining part of the paper, we prove Proposition 3.11. Let us recall that we consider a vector field X in a small neighborhood V of X^* for which $W^s(p) \cap W^u(q) \neq \emptyset$.

Without loss of generality, we may assume that

$$O^+(B(\varepsilon/2, s), \phi) \subset B(\varepsilon, s) \quad \text{and} \quad O^-(B(\varepsilon/2, u), \phi) \subset B(\varepsilon, u).$$

Take $m \in (0, \varepsilon/8)$ such that $B(m, p) \subset U_p$, $B(m, q) \subset U_q$ and the flow of the vector field X in the neighborhoods $B(2m, p)$ and $B(2m, q)$ is conjugate by a homeomorphism to the flow of a linear vector field.

We take points $y_p = \alpha(T_p) \in B(m/2, p) \cap \alpha$ and $y_q = \alpha(T_q) \in B(m/2, q) \cap \alpha$. Then $O^+(y_p, \phi) \subset B(m, p)$ and $O^-(y_q, \phi) \subset B(m, q)$. Take $\delta > 0$ such that if $g(t)$ is a pseudotrajectory of type $\mathrm{Ps}(\delta)$ (with y_p and y_q fixed above), $t_0 \in \mathbb{R}$, and $x_0 \in B(2\delta, g(t_0))$, then

$$\operatorname{dist}(\phi(t - t_0, x_0), g(t)) < \varepsilon/2, \quad |t - t_0| \leq T + 1, \tag{3.14}$$

where $T = T_p - T_q$.

Consider a number $\tau > 0$ such that if $x \in W^u(p) \setminus B(m/2, p)$, then $\phi(\tau, x) \in B(\varepsilon/8, s)$. Take $\varepsilon_1 \in (0, m/4)$ such that if two points $z_1, z_2 \in M$ satisfy the inequality $\operatorname{dist}(z_1, z_2) < \varepsilon_1$, then

$$\operatorname{dist}(\phi(t, z_1), \phi(t, z_2)) < \varepsilon/8, \quad |t| \leq \tau.$$

In this case, for any $y \in B(\varepsilon_1, x)$ (recall that we consider $x \in W^u(p) \setminus B(m/2, p)$), the following inequalities hold:

$$\operatorname{dist}(\phi(t, x), \phi(t, y)) < \varepsilon/4, \quad t \geq 0. \tag{3.15}$$

Reducing ε_1, if necessary, we may assume that if $x' \in W^s(q) \setminus B(m/2, q)$ and $y' \in B(\varepsilon_1, x')$, then

$$\operatorname{dist}(\phi(t, x'), \phi(t, y')) < \varepsilon/4, \quad t \leq 0.$$

Let $g(t)$ be a pseudotrajectory of type $\mathrm{Ps}(\delta)$, where δ, y_p, and y_q satisfy the above-formulated conditions. We claim that if δ is small enough, then $g(t)$ can be $\varepsilon/2$-shadowed (in fact, we have to reduce δ and to impose additional conditions on y_p and y_q). Below we denote $W^u_{loc}(p,m) = W^u(p) \cap B(m,p)$ etc.

Additionally decreasing δ, we may assume that for any points $z_p \in W^u_{loc}(p,m)$, $x_0 \in B(\delta, y_p)$, and $s > 0$ such that $\phi(s, x_0) \in B(\delta, z_p)$, the following inclusions hold:

$$\phi(t, x_0) \in B(2m, p), \quad t \in [0, s]. \tag{3.16}$$

Let us consider several possible cases.

Case (P1): $x_p \notin W^s(p)$ and $x_q \notin W^u(q)$. Let

$$T' = \inf\{t \in \mathbb{R} : \ \phi(t, x_p) \notin B(p, 3m/4)\}.$$

If δ is small enough, then $\operatorname{dist}(\phi(T', x_p), W^u(p)) < \varepsilon_1$. In this case, there exists a point $z_p \in W^u_{loc}(p,m) \setminus B(m/2, p)$ such that

$$\operatorname{dist}(\phi(T', x_p), z_p) < \varepsilon_1.$$

Applying a similar reasoning in a neighborhood of q (and reducing δ, if necessary), we find a point $z_q \in W^s_{loc}(q,m) \setminus B(m/2, q)$ and a number $T'' < 0$ such that $\operatorname{dist}(\phi(T'', x_q), z_q) < \varepsilon_1$.

Let us formulate a key lemma which we prove later (precisely this lemma is the above-mentioned statement "on four balls").

Lemma 3.13. *There exists $m > 0$ such that for any points*

$$y_p \in B(m,p) \cap \alpha, \quad z_p \in W^u_{loc}(p,m) \setminus \{p\},$$

$$y_q \in B(m,q) \cap \alpha, \quad z_q \in W^s_{loc}(q,m) \setminus \{q\},$$

and for any number $m_1 > 0$ there exists a trajectory of the vector field X that intersects successively the balls $B(m_1, z_q)$, $B(m_1, y_q)$, $B(m_1, y_p)$, and $B(m_1, z_p)$ as time grows.

We reduce m to satisfy Lemma 3.13 and apply this lemma with $m_1 = \min(\delta, \varepsilon_1)$. Find a point x_0 and numbers $t_1 < t_2 < t_3 < t_4$ such that

$$\phi(t_1, x_0) \in B(m_1, z_q), \quad \phi(t_2, x_0) \in B(m_1, y_q),$$

$$\phi(t_3, x_0) \in B(m_1, y_p), \quad \phi(t_4, x_0) \in B(m_1, z_p).$$

Inequalities (3.14) imply that if δ is small enough, then

$$\operatorname{dist}(\phi(t_3 + t, x_0), g(T_p + t)) < \varepsilon/2, \quad t \in [T_q - T_p, 0]. \tag{3.17}$$

Define a reparametrization $h(t)$ as follows:

$$h(t) = \begin{cases} h(T_q + T'' + t) = t_1 + t, & t < 0, \\ h(T_p + T' + t) = t_4 + t, & t > 0, \\ h(T_p + t) = t_3 + t, & t \in [T_q - T_p, 0], \\ h(t) \text{ increases}, & t \in [T_p, T_p + T'] \cup [T_q + T'', T_q]. \end{cases}$$

If $t \geq T_p + T'$, then inequality (3.15) implies that

$$\operatorname{dist}(\phi(h(t), x_0), \phi(t - (T_p + T'), z_p)) < \varepsilon/4$$

and

$$\operatorname{dist}(\phi(t - T_p, x_p), \phi(t - (T_p + T'), z_p)) < \varepsilon/4.$$

Hence, if $t \geq T_p + T'$, then

$$\operatorname{dist}(\phi(h(t), x_0), g(t)) < \varepsilon/2. \tag{3.18}$$

Inclusion (3.16) implies that for $t \in [T_p, T_p + T']$ the inclusions $\phi(h(t), x_0), g(t) \in B(m, p)$ hold, and inequality (3.18) holds for these t as well.

A similar reasoning shows that inequality (3.18) holds for $t \leq T_q$. If $t \in [T_q, T_p]$, then inequality (3.18) follows from (3.17). This completes the proof in case (P1).

Case (P2): $x_p \in W^s(p)$ and $x_q \notin W^u(q)$. In this case, Lemma 3.13 is replaced by the following statement.

Lemma 3.14. *There exists $m > 0$ such that for any points*

$$y_p \in B(m, p) \cap \alpha, \quad y_q \in B(m, q) \cap \alpha, \quad z_q \in W^s_{loc}(q, m) \setminus \{q\},$$

and a number $m_1 > 0$ there exists a trajectory of the vector field X that intersects successively the balls $B(m_1, z_q)$, $B(m_1, y_q)$, and $B(m_1, y_p) \cap W^s_{loc}(p, m)$ as time grows.

The rest of the proof uses the same reasoning as in case (P1).

Case (P3): $x_p \notin W^s(p)$ and $x_q \in W^u(q)$. This case is similar to case (P2).

Case (P4): $x_p \in W^s(p)$ and $x_q \in W^u(q)$. In this case, we take α as the shadowing trajectory; the reparametrization is constructed similarly to case (P1).

Thus, to complete the consideration of case (S2), it remains to prove Lemmas 3.13 and 3.14.

3.2.4 Proof of lemma "on four balls"

In this section we prove Lemmas 3.13 and 3.14.

Proof of Lemma 3.13. We first fix proper coordinates in small neighborhoods of the points p and q. Let us begin with the case of the point p.

Taking a small neighborhood V of the vector field X^*, we may assume that the Jacobi matrix $J_p = DX(p)$ is as close to J_p^* as we want.

Thus, we assume that $p = 0$ in coordinates $u_1 = (x_1, x_2)$, $u_2 = (x_3, x_4)$, and $J_p = \operatorname{diag}(A_p, B_p)$, where

$$A_p = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \quad B_p = \begin{pmatrix} a_p & -b_p \\ b_p & a_p \end{pmatrix}, \tag{3.19}$$

and

$$\lambda_1, \lambda_2, a_p, b_p > 4g, \tag{3.20}$$

where g is a small positive number to be chosen later (and a similar notation is used in U_q).

Then we can represent the field X in a small neighborhood U of the point p in the form

$$X(u_1, u_2) = \begin{pmatrix} A_p & 0 \\ 0 & B_p \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} X_{12}(u_1, u_2) \\ X_{34}(u_1, u_2) \end{pmatrix}, \tag{3.21}$$

where

$$X_{12}, X_{34} \in \mathbf{C}^1,\ |X_{12}|_{\mathbf{C}^1}, |X_{34}|_{\mathbf{C}^1} < g,\ X_{12}(0,0) = X_{34}(0,0) = (0,0). \tag{3.22}$$

Under these assumptions, $p = 0$ is a hyperbolic rest point whose two-dimensional unstable manifold in the neighborhood U is given by $u_2 = G(u_1)$, where $G : \mathbb{R}^2 \to \mathbb{R}^2$, $G \in \mathbf{C}^1$. We can find $g > 0$ such that if the functions X_{12} and X_{34} satisfy relations (3.22), then

$$\|DG(u_1)\| < 1 \quad \text{while} \quad (u_1, G(u_1)) \in U. \tag{3.23}$$

We introduce new coordinates in U by $v(u_1, u_2) = (u_1, u_2 - G(u_1))$ and use a smooth cut-off function to extend v to a $\mathbf{C}^1$ diffeomorphism w of M such that $w(x) = x$ outside a larger neighborhood U' of p. Denote by Y the resulting vector field in the new coordinates.

Remark 3.15. Note that Y is continuous but not necessary $\mathbf{C}^1$. Nevertheless, the following holds. Let S_1 and S_2 be small smooth three-dimensional disks transverse to a trajectory of Y and let f_Y be the corresponding Poincaré transformation generated by the vector field Y. Consider smooth disks $w^{-1}(S_1)$ and $w^{-1}(S_2)$ and let $f_X : w^{-1}(S_1) \to w^{-1}(S_2)$ be the corresponding Poincaré transformation. Since $f_X \in \mathbf{C}^1$ and $f_Y = w \circ f_X \circ w^{-1}$, we conclude that $f_Y \in \mathbf{C}^1$. We will use this fact below.

If $(v_1, v_2) = v(u_1, u_2)$, then

$$u_1 = v_1, \quad u_2 = v_2 + G(v_1). \tag{3.24}$$

Let $Y(v_1, v_2) = (Y_1(v_1, v_2), Y_2(v_1, v_2))$. Since the surface $u_2 = G(u_1)$ is a local stable manifold of the rest point 0 of the field X, the surface $v_2 = 0$ is a local stable manifold of the rest point 0 of the vector field Y. Hence,

$$Y_2(v_1, 0) = 0 \quad \text{for} \quad (v_1, 0) \in v(U).$$

Lemma 3.16. *The inequalities*

$$|Y_2(v_1, v_2) - (Y_2(v_1, 0) + B_p v_2)| \le 2g|v_2|, \quad (v_1, v_2) \in v(U), \tag{3.25}$$

hold.

Proof. Substitute equalities (3.24) into (3.21) to show that

$$Y_2(v_1, v_2) = B_p(v_2 + G(v_1)) + X_{34}(v_1, v_2 + G(v_1)) - \\ - DG(v_1)(A_p v_1 + X_{12}(v_1, v_2 + G(v_1))).$$

Relations (3.22) and (3.23) imply that

$$|X_{34}(v_1, v_2 + G(v_1)) - X_{34}(v_1, G(v_1))| \leq g|v_2|$$

and

$$|DG(v_1)(A_p v_1 + X_{12}(v_1, v_2 + G(v_1))) - DG(v_1)(A_p v_1 + X_{12}(v_1, G(v_1)))| \leq g|v_2|.$$

Hence,

$$|X_{34}(v_1, v_2 + G(v_1)) - X_{34}(v_1, G(v_1)) - \\ - (DG(v_1)(A_p v_1 + X_{12}(v_1, v_2 + G(v_1))) - DG(v_1)(A_p v_1 + X_{12}(v_1, G(v_1))))| \leq \\ \leq 2g|v_2|.$$

The left-hand side of the above inequality equals $|Y_2(v_1, v_2) - (Y_2(v_1, 0) + B_p v_2)|$, which proves inequality (3.25).

Note that if y_p, y_q, z_p, z_q, and $m_1 > 0$ are fixed, then there exists $m^* > 0$ such that if a trajectory β^* of the vector field Y intersects successfully the balls $B(m^*, v(z_q))$, $B(m^*, v(y_q))$, $B(m^*, v(y_p))$, and $B(m^*, v(z_p))$, then the trajectory $w^{-1}(\beta^*)$ of X has the property described in Lemma 3.13.

Thus, it is enough to prove Lemma 3.13 for the vector field Y. Since the mapping w is smooth, the vector field Y satisfies condition (3.8).

To simplify presentation, denote Y by X and its flow by ϕ. In this notation, there exists a neighborhood U_p of $p = 0$ in which

$$X(x) = \begin{pmatrix} A_p & 0 \\ 0 & B_p \end{pmatrix} x + X_p(x), \tag{3.26}$$

where $X_p \in \mathbf{C}^0$, and if $(x_1, x_2, x_3, x_4) \in U_p$, then

$$|P_{34}^p X_p(x_1, x_2, x_3, x_4)| < 2g \max(|x_3|, |x_4|) \quad \text{and} \quad P_{34}^p X_p(x_1, x_2, 0, 0) = 0 \tag{3.27}$$

(where we denote by P_{34}^p the projection in U_p to the plane of variables x_3, x_4 parallel to the plane of variables x_1, x_2). Conditions (3.27) imply that the plane $x_3 = x_4 = 0$ is a local stable manifold for the vector field X.

Introduce polar coordinates r, φ in the plane of variables x_3, x_4. In what follows (if otherwise is not stated explicitly), we use coordinates (x_1, x_2, r, φ). For $i \in \{1, 2, 3, 4, r, \varphi\}$, we denote by $P_i^p x$ the ith coordinate of a point $x \in U_p$.

Since the surface $W^u(p)$ is smooth and transverse to the plane $x_3 = x_4 = 0$, there exist numbers $K > 0$ and $m_2 > 0$ such that if points $x \in W^u_{loc}(p, m_2)$ and $y \in B(m_2, p)$ satisfy the equality $P^p_{34}x = P^p_{34}y$, then

$$\operatorname{dist}(x, y) \leq K \operatorname{dist}(y, W^u_{loc}(p, m_2)). \tag{3.28}$$

We reduce the neighborhood U_p so that $U_p \subset B(m_2, p)$.

Lemma 3.17. *Let $x(t) = (x_1(t), x_2(t), r(t), \varphi(t))$ be a trajectory of the vector field X. The relations*

$$\tfrac{\mathrm{d}}{\mathrm{d}\,t} r \in ((a_p - 4g)r, (a_p + 4g)r) \quad \textit{and} \quad \tfrac{\mathrm{d}}{\mathrm{d}\,t}\varphi \in (b_p - 4g, b_p + 4g) \tag{3.29}$$

hold while $x(t) \in U_p$.

Proof. Let $x_3(t) = P^p_3 x(t)$ and $x_4(t) = P^p_4 x(t)$. Relations (3.19), (3.26) and (3.27) imply that

$$\tfrac{\mathrm{d}}{\mathrm{d}\,t} x_3(t) = a_p x_3(t) - b_p x_4(t) + \Delta_3(t)$$

and

$$\tfrac{\mathrm{d}}{\mathrm{d}\,t} x_4(t) = b_p x_3(t) + a_p x_4(t) + \Delta_4(t),$$

where

$$|\Delta_3(t)|, |\Delta_4(t)| < 2gr(t). \tag{3.30}$$

Since $x_3(t) = r(t)\cos\varphi(t)$ and $x_4(t) = r(t)\sin\varphi(t)$, we obtain the equalities

$$r\tfrac{\mathrm{d}}{\mathrm{d}\,t}\varphi = rb_p + \Delta_4(t)\cos\varphi - \Delta_3(t)\sin\varphi$$

and

$$\tfrac{\mathrm{d}}{\mathrm{d}\,t} r = a_p r + \Delta_3(t)\cos\varphi + \Delta_4(t)\sin\varphi.$$

Inequalities (3.30) imply that

$$b_p - 4g < \tfrac{\mathrm{d}}{\mathrm{d}\,t}\varphi < b_p + 4g$$

and

$$(a_p - 4g)r < \tfrac{\mathrm{d}}{\mathrm{d}\,t} r < (a_p + 4g)r,$$

which proves our lemma.

A similar reasoning shows that there exists a neighborhood U_q of the point q in which we can introduce (after a smooth change of variables) coordinates (y_1, y_2, y_3, y_4) (and the corresponding polar coordinates (r, φ) in the plane of variables y_3, y_4) such that

$$W^u_{loc}(q, m) \subset \{y_3 = y_4 = 0\}$$

and for any trajectory $y(t) = (y_1(t), y_2(t), r(t), \varphi(t))$ of the vector field X, the relations

$$\tfrac{\mathrm{d}}{\mathrm{d}\,t} r \in ((a_q - 4g)r, (a_q + 4g)r) \quad \text{and} \quad \tfrac{\mathrm{d}}{\mathrm{d}\,t}\varphi \in (-b_q - 4g, -b_q + 4g)$$

hold while $y(t) \in U_q$.

Let us continue the proof of Lemma 3.13.

Let $S_p \subset U_p$ and $S_q \subset U_q$ be smooth three-dimensional disks that are transverse to the vector field X and contain the points y_p and y_q, respectively. Denote by $f : S_q \to S_p$ the corresponding Poincaré transformation (generated by the field X). We note that $f \in \mathbf{C}^1$ (see Remark 3.15) and $f(y_q) = y_p$.

Consider the lines $l_p = S_p \cap W^s_{loc}(p, m)$ and $l_q = S_q \cap W^u_{loc}(q, m)$ and unit vectors $e_p \in l_p$ and $e_q \in l_q$. Let P^p_{34} and P^q_{34} be the projections to the planes of variables x_3, x_4 and y_3, y_4 in the neighborhoods U_p and U_q, respectively. Relation (3.8) implies that

$$P^p_{34} Df(y_q)e_q \neq 0 \quad \text{and} \quad P^q_{34} Df^{-1}(y_p)e_p \neq 0. \tag{3.31}$$

Take $m_3 \in (0, m_1)$ such that

$$\phi(t, x) \in U_p, \quad x \in B(m_3, y_p),\ t \in (0, \tau_p(x)),$$

and

$$\phi(t, y) \in U_q, \quad y \in B(m_3, y_q),\ t \in (\tau_q(x), 0),$$

where

$$\tau_p(x) = \inf\{t > 0 :\ P^p_r(\phi(t, x)) \geq P^p_r z_p\},$$
$$\tau_q(x) = \sup\{t < 0 :\ P^q_r(\phi(t, y)) \geq P^q_r z_q\},$$

and z_p, z_q are the points mentioned in Lemma 3.13.

Consider the surface $L_p \subset S_p$ defined by

$$L_p = \{x + (y - y_p),\ x \in l_p, y \in f(l_q)\}.$$

Let $L_q = f^{-1}L_p \subset S_q$. The surfaces L_p and L_q are divided by the lines l_p and l_q into half-surfaces. Let L^+_p and L^+_q be any of these half-surfaces.

To any point $x \in L^+_p \cap f(L^+_q)$ there correspond numbers $r_p(x) = P^p_r x$ and $r_q(x) = P^q_r f^{-1}(x)$; consider the mapping $w : L^+_p \cap f(L^+_q) \to \mathbb{R}^2$ defined by $w(x) = (r_p(x), r_q(x))$. We claim that there exists a neighborhood $U_L \subset L^+_p \cap f(L^+_q)$ of the point y_p on which the mapping w is a homeomorphism onto its image.

Let r_0 and φ_0 be the polar coordinates of the vector $P^p_{34} Df(y_q)e_q$. Relation (3.31) implies that $r_0 \neq 0$. Hence, there exists a neighborhood V_q of the point y_q in S_q such that if $y \in V_q$, then

$$P^p_r Df(y)e_q \in [r_0/2, 2r_0] \quad \text{and} \quad P^p_\varphi Df(y)e_q \in [\varphi_0 - \pi/8, \varphi_0 + \pi/8]. \tag{3.32}$$

Take $c > 0$ such that $B(2c, y_q) \subset V_q$. Note that

$$f(y_q + \delta e_q) = f(y_q) + \int_0^\delta Df(y_q + se_q)e_q \, \mathrm{d}\, s, \quad \delta \in [0, c].$$

Conditions (3.32) imply that

$$P^p_\varphi\, (f(y_q + \delta e_q) - f(y_q)) \in [\varphi_0 - \frac{\pi}{8}, \varphi_0 + \frac{\pi}{8}],\ \delta \in [0, c], \tag{3.33}$$

and the mapping $Q_p(\delta) : [0, c] \to \mathbb{R}$ defined by $Q_p(\delta) = P_r^p f(y_q + \delta e_q)$ is a homeomorphism onto its image. Similarly (reducing g, if necessary), one can show that if $x \in B(g, y_p)$, then the mapping $Q_{q,x}(\delta) : [0, g] \to \mathbb{R}$ defined by $Q_{q,x}(\delta) = P_r^q f^{-1}(x + \delta e_p)$ is a homeomorphism onto its image.

Take $\delta_p, \delta_q \in [0, c]$ and let $x = \delta_p e_p + f(y_q + \delta_q e_q)$. Then $r_p(x) = Q_p(\delta_q)$ and $r_q(x) = Q_{q,f(y_q+\delta_q e_q)}(\delta_p)$. It follows that the mapping w is a homeomorphism onto its image. Indeed, if $g_1 > 0$ is small enough, then the mapping $w^{-1}(\xi, \eta) = (x(\xi), Q^{-1}_{q,x(\xi)}(\eta))$, where $x(\xi) = f(y_q + Q_p^{-1}(\xi)e_q)$, is uniquely defined and continuous for $(\xi, \eta) \in [0, g_1] \times [0, g_1]$.

We reduce m_3 so that the following relations hold:

$$m_3 < c, \quad B(m_3, y_p) \cap L_p^+ \subset U_L, \quad \text{and} \quad B(m_3, y_q) \cap L_q^+ \subset f^{-1}U_L.$$

Let us prove a statement which we use below.

Lemma 3.18. *For any $m_1 > 0$ there exist numbers $r_1, r_2 \in (0, m_1)$ and $T_1, T_2 > 0$ with the following property: if $\gamma(s) : [0, 1] \to L_p^+$ is a curve such that*

$$P_r^p \gamma(0) = r_1, \quad P_r^p \gamma(1) = r_2, \tag{3.34}$$

and

$$\gamma(s) \in L_p^+ \cap B(m_2, y_p), \quad s \in [0, 1], \tag{3.35}$$

then there exist numbers $\tau \in [T_2, T_1]$ and $s \in [0, 1]$ such that

$$\phi(\tau, \gamma(s)) \in B(m_1, z_p).$$

Proof. Let $r_p = P_r^p z_p$ and $\varphi_p = P_\varphi^p z_p$. For $r > 0$, denote

$$T_{\min}(r) = \frac{\log r_p - \log r}{a_p + 4g} \quad \text{and} \quad T_{\max}(r) = \frac{\log r_p - \log r}{a_p - 4g}.$$

Note that if $r < r_p$, then $T_{\max}(r) > T_{\min}(r)$ and that $T_{\min}(r) \to \infty$ as $r \to 0$. Take $T > 0$ such that if $\tau > T$, $x \in B(m_2, y_p)$, and

$$\phi(t, x) \subset U_p, \quad t \in [0, \tau],$$

then

$$\operatorname{dist}(W^u_{loc}(p, m), \phi(\tau, x)) < \frac{m_1}{2K}. \tag{3.36}$$

Take $r_1, r_2 \in (0, \min(m_2, r_p))$ such that

$$r_2 > r_1, \quad T_{\min}(r_2) > T,$$

and

$$(b_p - 4g)T_{\min}(r_1) - (b_p + 4g)T_{\max}(r_2) > 4\pi. \tag{3.37}$$

Set $T_1 = T_{\max}(r_1)$ and $T_2 = T_{\min}(r_2)$. Since the function $\gamma(s)$ is continuous, inclusions (3.29) and inequalities (3.20) imply that there exists a uniquely defined continuous function $\tau(s) : [0,1] \to \mathbb{R}$ such that

$$P_r^p \phi(\tau(s), \gamma(s)) = r_p.$$

It follows from inclusions (3.29) and equalities (3.34) that

$$\tau(0) \in [T_{\min}(r_1), T_{\max}(r_1)],\ \tau(1) \in [T_{\min}(r_2), T_{\max}(r_2)],\ \tau(s) \in [T_2, T_1].$$

Now we apply relations (3.20), (3.29), and (3.33) to show that

$$P_\varphi^p \phi(\tau(0), \gamma(0)) \geq (b_p - 4g) T_{\min}(r_1) + \varphi_0 - \pi/8$$

and

$$P_\varphi^p \phi(\tau(1), \gamma(1)) \leq (b_p + 4g) T_{\max}(r_2) + \varphi_0 + \pi/8.$$

Since the function $\tau(s)$ is continuous, the above inequalities and inequalities (3.37) imply the existence of $s \in [0,1]$ such that

$$P_\varphi^p \phi(\tau(s), \gamma(s)) = \varphi_p \mod 2\pi.$$

Hence, $P_{34}^p \phi(\tau(s), \gamma(s)) = P_{34}^p z_p$. It follows from this equality combined with relations (3.28), (3.36), and the inequality $\tau(s) > T$ that $\phi(\tau(s), \gamma(s)) \in B(m_1/2, z_p)$, which proves Lemma 3.18.

Let $r_1, r_2 \in (0, m_2)$ and $T_1, T_2 > 0$ be the numbers given by Lemma 3.18. Consider the set

$$A_p = \{\phi(t,x) :\ t \in [-T_1, -T_2], x \in \operatorname{Cl} B(m_2/2, z_p)\} \cap L_p^+.$$

Note that A_p is a closed set that intersects any curve $\gamma(s)$ satisfying conditions (3.34) and (3.35).

We apply a similar reasoning in the neighborhood U_q to the vector field $-X$ to show that there exist numbers $r_1', r_2' \in (0, m_2)$ and $T_1', T_2' > 0$ such that the set

$$A_q = \{\phi(t,x) :\ t \in [T_2', T_1'], x \in \operatorname{Cl} B(m_2/2, z_q)\} \cap L_q^+$$

is closed and intersects any curve $\gamma(s) : [0,1] \to L_q^+ \cap B(m_2, y_q)$ such that

$$P_r^q \gamma(0) = r_1' \quad \text{and} \quad P_r^q \gamma(1) = r_2'.$$

We claim that

$$A_p \cap f(A_q) \neq \emptyset, \tag{3.38}$$

which proves Lemma 3.13.

Consider the set $K \subset L_p^+ \cap f(L_q^+)$ bounded by the curves $k_1 = L_p^+ \cap \{P_r^p x = r_1\}$, $k_2 = L_p^+ \cap \{P_r^p x = r_2\}$, $k_1' = f(L_q^+ \cap \{P_r^q y = r_1'\})$, and $k_2' = f(L_q^+ \cap \{P_r^q y = r_2'\})$. Since $w(x)$ is a homeomorphism, the set K is homeomorphic to the square $[0,1] \times [0,1]$.

The following statement was proved in [79].

Lemma 3.19. *Introduce in the square $I = [0,1] \times [0,1]$ coordinates (u, v). Assume that closed sets $A, B \subset I$ are such that any curve inside I that joins the segments $u = 0$ and $u = 1$ intersects the set A and any curve inside I that joins the segments $v = 0$ and $v = 1$ intersects the set B. Then $A \cap B \neq \emptyset$.*

The set A_p is closed. By Lemma 3.18, A_p intersects any curve in K that joins the sides k_1 and k_2. Similarly, the set A_q is closed and intersects any curve that belongs to $f^{-1}(K)$ and joins the sides $f^{-1}(k_1')$ and $f^{-1}(k_2')$. Thus, the set $f(A_q)$ intersects any curve in K that joins the sides k_1' and k_2'. By Lemma 3.19 inequality (3.38) holds. Lemma 3.13 is proved. $\square$

Proof of Lemma 3.14. Similarly to the proof of Lemma 3.13, let us consider the subspaces L_p^+ and L_q^+ and a number $m_2 \in (0, m_1)$ and construct the set $A_q \subset L_q^+$. Note that the set $f^{-1}(B(m_1, y_p) \cap W^s(p) \cap L_p^+)$ contains a curve that satisfies conditions (3.34) and (3.35). Hence, $B(m_1, y_p) \cap W^s(p) \cap f(A_q) \neq \emptyset$. For any point in this intersection, its trajectory is the desired shadowing trajectory. $\square$

3.2.5 Embeding of X^* onto $S^2 \times S^2$

Consider two 2-dimensional spheres M_1 and M_2. Let us introduce coordinates (r_1, φ_1) and (r_2, φ_2) on M_1 and M_2, respectively, where $r_1, r_2 \in [-1, 1]$ and $\varphi_1, \varphi_2 \in \mathbb{R}/2\pi\mathbb{Z}$. We identify all points of the form $(-1, \cdot)$ as well as points of the form $(1, \cdot)$. Denote

$$M_1^+ = \{(r_1, \varphi_1), \quad r_1 \geq 0\} \text{ and } M_1^- = \{(r_1, \varphi_1), \quad r_1 \leq 0\}.$$

Consider a smooth vector field X_1 defined on M_1^+ such that its trajectories $(r_1(t), \varphi_1(t))$ satisfy the following conditions:

$$\tfrac{\mathrm{d}}{\mathrm{d}t} r_1 = 1, \tfrac{\mathrm{d}}{\mathrm{d}t} \varphi_1 = 0, \quad r_1 = 0;$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} r_1 > 0, \quad r_1 > 0;$$
$$\tfrac{\mathrm{d}}{\mathrm{d}t} r_1 = 0, \quad r_1 = 1.$$

We also assume that, in proper local coordinates in a neighborhood of the "North Pole" $(1, \cdot)$ of the sphere M_1, the vector field X_1 is linear, and

$$\mathrm{D}\, X_1(1, \cdot) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, $(1, \cdot)$ is an attracting hyperbolic rest point of X_1, and every trajectory of X_1 in M_1^+ tends to $(1, \cdot)$ as time grows.

Consider a smooth vector field X_2 on M_2 such that its nonwandering set $\Omega(X_2)$ consists of two rest points: a hyperbolic attractor $s_2 = (0, \pi)$ and a hyperbolic repeller $u_2 = (0, 0)$. Assume that, in proper coordinates, the vector field X_2 is linear in neighborhoods of s_2 and u_2, and

$$\mathrm{D}\, X_2(s_2) = -\,\mathrm{D}\, X_2(u_2) = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Consider the vector field X^+ defined on $M_1^+ \times M_2$ by the following formula

$$X^+(r_1, \varphi_1, r_2, \varphi_2) = (X_1(r_1, \varphi_1), r_1^2 X_2(r_2, \varphi_2)).$$

Consider infinitely differentiable functions $g_1 : M_1^+ \to \mathbb{R}$, $g_2, g_3 : [-1, 1] \to [-1, 1]$, and $g_4 : M_1^+ \to [0, 1]$ satisfying the following conditions:

$$g_1(0,0) = 0; \quad g_1(r_1, \varphi_1) \in (0, 2\pi), \quad (r_1, \varphi_1) \neq 0,$$

$$g_2'(r_2) \in (0, 2), \quad r_2 \in [-1, 1];$$
$$g_2(0) < 0,\ g_2(-1) = -1,\ g_2(1) = 1;$$
$$g_3(r_2) = 2r_2 - g_2(r_2), \quad r_2 \in [-1, 1];$$
$$g_4(0,0) = 1/2,\ \frac{\partial}{\partial \varphi_1} g_4(0,0) \neq 0.$$

Note that the functions g_2 and g_3 are monotonically increasing.

Consider a mapping $f^* : M_1^+ \times M_2 \to M_1^- \times M_2$ defined by the following formula:

$$f^*(r_1, \varphi_1, r_2, \varphi_2) = (-r_1, \varphi_1, g_4(r_1, \varphi_1) g_2(r_2) + (1 - g_4(r_1, \varphi_1)) g_3(r_2), \varphi_2 + g_1(r_1, \varphi_1)).$$

Clearly, f^* is surjective; the monotonicity of g_2 and g_3 implies that f^* is a diffeomorphism.

Using the standard technique with a "bump" function, one can construct a diffeomorphism $f : M_1^+ \times M_2 \to M_1^- \times M_2$ such that, for small neighborhoods $U_1 \subset U_2$ of $(1, \cdot, s_2)$, the following holds:

$$f(x) = f^*(x), \quad x \notin U_2,$$

and f is linear in U_1.

Consider the set $l = \{r_1 = 0, r_2 = 0, \varphi_2 = 0\}$. Simple calculations show that

$$f(l) \cap l = \{(0,0,0,0)\}, \tag{3.39}$$

and the tangent vectors to l and $f(l)$ at $(0,0,0,0)$ are parallel to the vectors $(0,1,0,0)$ and $(0, 1, (g_2(0) - g_3(0)) \frac{\partial}{\partial \varphi_1} g_4(0,0), \cdot)$, respectively. Hence,

$$\dim(T_{(0,0,0,0)} l \oplus T_{(0,0,0,0)} f(l)) = 2. \tag{3.40}$$

Define a vector field X^- on $M_1^- \times M_2$ by the formula

$$X^-(x) = -\operatorname{D} f(f^{-1}(x)) X^+(f^{-1}(x))$$

(and note that $x(t)$ is a trajectory of X^+ if and only if $f(x(-t))$ is a trajectory of X^-).

Finally, we define the following vector field X^* on $M_1 \times M_2$:

$$X^*(x) = \begin{cases} X^+(x), & x \in M_1^+ \times M_2, \\ X^-(x), & x \in M_1^- \times M_2 \end{cases}$$

Let us check that the vector field X^* is well-defined on the set $\{r_1 = 0\}$. Indeed, $X^+(0, \varphi_1, r_2, \varphi_2) = (1, 0, 0, 0)$ and $(\mathrm{D} f(0, \varphi_1, r_2, \varphi_2))^{-1}(1, 0, 0, 0) = (-1, 0, 0, 0)$. It is easy to see that $\mathrm{D} X^+(0, \varphi_1, r_2, \varphi_2) = \mathrm{D} X^-(0, \varphi_1, r_2, \varphi_2) = 0$. This implies that $X \in \mathbf{C}^1$.

Let us prove that the vector field X^* satisfies conditions (F1) – (F3). Let $(r_1(t), \varphi_1(t), r_2(t), \varphi_2(t))$ be a trajectory of X^*. The following inequalities hold:

$$\tfrac{\mathrm{d}}{\mathrm{d}\,t} r_1 > 0, \quad r_1 \neq \pm 1. \tag{3.41}$$

This implies the inclusion $\Omega(X^*) \subset \{r_1 = \pm 1\}$. By the construction of X^+, $\Omega(X^*) \cap \{r_1 = 1\} = \{(1, \cdot, s_2), (1, \cdot, u_2)\}$. Similarly, $\Omega(X^*) \cap \{r_1 = -1\} = \{f(1, \cdot, s_2), f(1, \cdot, u_2)\}$. Denote $s^* = (1, \cdot, s_2)$, $p^* = (1, \cdot, u_2)$, $q^* = f(p)$, and $u^* = f(s)$. Clearly, s^*, u^*, p^*, q^* are hyperbolic rest points, s^* is an attractor, u^* is a repeller, $\mathrm{D} X(p^*) = J_p^*$, and $\mathrm{D} X(q^*) = J_q^*$. In addition, in small neighborhoods of p^* and q^*, the vector field X^* is linear.

It is easy to see that

$$W^s(p^*) \cap \{r_1 = 1\} = \{p^*\} \text{ and } W^s(p^*) \cap \{r_1 = -1\} = \emptyset.$$

Inequality (3.41) implies that any trajectory in $W^s(p^*) \setminus \{p^*\}$ intersects the set $\{r_1 = 0\}$ at a single point. The definition of X^+ implies that $W^s(p^*) \cap \{r_1 = 0\} = l$. Similarly, any trajectory in $W^u(q^*) \setminus \{q^*\}$ intersects $\{r_1 = 0\}$ at a single point, and $W^u(q^*) \cap \{r_1 = 0\} = f(l)$. It follows from equality (3.39) that $W^s(p^*) \cap \{r_1 = 0\} \cap W^u(q^*)$ is a single point, and hence $W^s(p^*) \cap W^u(q^*)$ consists of a single trajectory.

Inequality (3.41) implies condition (3.4), and condition (3.40) implies (3.5).

3.3 Ω-stability

In the example in paragraph 3.2 the vector field is not structurally stable due to failure of the strong transversality condition. However it is not clear if one can construct a vector field with the robust shadowing property, which does not satisfy Axiom A'.

In present paragraph we prove that vector fields with the robust shadowing property are Ω-stable and hence satisfy Axiom A' [25]:

Theorem 3.20. *Every vector field satisfying the C^1-robustly oriented shadowing property is Ω-stable.*

The key role in the proof is played by the star condition (which means that one can not get non-hyperbolic singularities or closed trajectories via a C^1 small perturbation, see section 3.3.2 for the details). For diffeomorphisms, it is proved in [5, 34, 50, 55] that the star condition implies the Ω-stability. However, it is not true for vector fields [20, 48]. So we have to use additional arguments (Lemmas 3.29, 3.30) in order to prove Ω-stability.

3.3.1 Basic properties

Let $\mathrm{Per}(X)$ denote the set of rest points and closed orbits of a vector field X.

Definition 3.10. Let us recall that X is called a Kupka-Smale field ($X \in \mathrm{KS}$) if

(KS1) any trajectory in $\mathrm{Per}(X)$ is hyperbolic;

(KS2) stable and unstable manifolds of trajectories from $\mathrm{Per}(X)$ are transverse.

For us will be important the following result (see [24]):

Theorem 3.21. $\mathrm{Int}^1(\mathrm{KS}) = \boldsymbol{S}$.

Let $\mathcal{T}$ denote the set of vector fields X that have property (KS1).
In my Ph.D. Thesis the following lemmas was proved, see also [83].

Lemma 3.22.

$$Int^1(\mathrm{OrientSh}) \subset \mathcal{T}. \tag{3.42}$$

Lemma 3.23. *If* $X \in \mathrm{Int}^1(\mathrm{OrientSh})$*,* p *is a closed trajectory and* $q \in \mathcal{T}$ *then intersection* $W^s(p) \cap W^u(q)$ *and* $W^s(q) \cap W^u(p)$ *are transverse.*

Lemma 3.24. *If* $X \in \mathrm{Int}^1(\mathrm{OrientSh})$*,* p *and* q *are fixed points and either* $\mathrm{D}\,X(p) \in K_1^+$ *or* $\mathrm{D}\,X(q) \in K_1^-$ *then intersection* $W^s(p) \cap W^u(q)$ *is transvesre.*

Let

$$\mathrm{Sing}(X) = \{x \in M : X(x) = 0\}$$

be the set of singularities of X and

$$\mathrm{Per}'(X) = \{x \in M \setminus \mathrm{Sing}(X) : \exists\, T \text{ such that } \phi_T(x) = x\}$$

be the set of regular periodic points of X. (Here we say a point is *regular* if it is not a singularity.)

3.3.2 Star vector fields

In the proofs we need some results about star vector fields. Recall that a vector field $X \in \mathcal{X}^1(M)$ is a *star vector field* on M if X has a C^1 neighborhood $\mathcal{U}$ in $\mathcal{X}^1(M)$ such that, for every $Y \in \mathcal{U}$, every singularity of Y and every periodic orbit of Y is hyperbolic. Denote by $\mathcal{X}^*(M)$ the set of star vector fields on M.

It was proved in [83] and in Ph. D. Thesis of the author that every vector field satisfying the C^1-robustly oriented shadowing property is a star vector field.

Lemma 3.25. $\mathrm{Int}^1(\mathrm{OrientSh}(M)) \subset \mathcal{X}^*(M)$.

We say that a point $x \in M$ is *preperiodic* of X, if for any C^1 neighborhood $\mathcal{U}$ of X in $\mathcal{X}^1(M)$ and any neighborhood U of x in M, there exists $Y \in \mathcal{U}$ and $y \in U$ such that y is a regular periodic point of Y. Denote by $\mathrm{Per}_*(X)$ the set of preperiodic points of X. We will use the following result which is proved in [26].

Theorem 3.26. *Let $X \in \mathcal{X}^*(M)$. If $\mathrm{Sing}(X) \cap \mathrm{Per}_*(X) = \emptyset$, then X is Ω-stable.*

We also need some results about the dominated splitting in the tangent space of singularities. Let σ be a hyperbolic singularity of X. Denote by

$$\mathrm{Re}(\lambda_s) \leq \cdots \leq \mathrm{Re}(\lambda_2) \leq \mathrm{Re}(\lambda_1) < 0 < \mathrm{Re}(\gamma_1) \leq \mathrm{Re}(\gamma_2) \leq \cdots \leq \mathrm{Re}(\gamma_u),$$

the eigenvalues of $\mathrm{D}X(\sigma)$. The *saddle value* of σ is

$$\mathrm{SV}(\sigma) = \mathrm{Re}(\lambda_1) + \mathrm{Re}(\gamma_1).$$

We write $\mathrm{Ind}(\sigma)$ the *index* of a hyperbolic singularity $\sigma \in \mathrm{Sing}(X)$ which is the dimension of the stable manifold of σ. We write $\mathrm{Ind}(p)$ the *index* of a regular hyperbolic periodic point $p \in \mathrm{Per}'(X)$ which is the dimension of the strong stable manifold of p.

Recall that a *homoclinic connection* Γ of a singularity σ is the closure of a orbit of a regular point which is contained in both the stable and the unstable manifolds of σ. The following lemma is a simplified version of results in [98].

Lemma 3.27. *Let $\sigma \in \mathrm{Sing}(X)$ be a singularity of vector field $X \in \mathcal{X}^1(X)$ exhibiting a homoclinic connection Γ. If $\mathrm{SV}(\sigma) \geq 0$, then for any C^1 neighborhood $\mathcal{U}$ of X in $\mathcal{X}^1(M)$ and any neighborhood U of Γ in M, there exists $Y \in \mathcal{U}$ and $p \in \mathrm{Per}'(Y)$ such that $\mathrm{Orb}_Y(p) \subset U$ and $\mathrm{Ind}(p) = \mathrm{Ind}(\sigma) - 1$.*

By using the same argument in the proof of [49, Lemma 4.1], we can get:

Lemma 3.28. *Let $X \in \mathcal{X}^*(M)$ and $\sigma \in \mathrm{Sing}(X)$ be a singularity of X. If there exists a integer $1 \leq I \leq \mathrm{Ind}(\sigma) - 1$ such that, for any C^1 neighborhood $\mathcal{U}$ of X in $\mathcal{X}^*(M)$ and any neighborhood U of σ in M, there exists $Y \in \mathcal{U}$ and $p \in U \cap \mathrm{Per}'(Y)$ with $\mathrm{Ind}(p) = I$. Then E^s_σ splits into a dominated splitting*

$$E^s_\sigma = E^{ss}_\sigma \oplus E^c_\sigma,$$

where $\dim E^{ss}_\sigma = I$.

Combining Lemma 3.27 with Lemma 3.28, we obtain directly the following lemma about singularities of star vector fields exhibiting a homoclinic connection.

Lemma 3.29. *Let $X \in \mathcal{X}^*(M)$ be a star vector field and $\sigma \in \mathrm{Sing}(X)$ be a singularity of X exhibiting a homoclinic connection.*

- *If $\mathrm{SV}(\sigma) \geq 0$ and $\dim E^s_\sigma \geq 2$, then E^s_σ splits into a dominated splitting*

  $$E^s_\sigma = E^{ss}_\sigma \oplus E^c_\sigma,$$

  where $\dim E^c_\sigma = 1$;

- *If $\mathrm{SV}(\sigma) \leq 0$ and $\dim E^u_\sigma \geq 2$, then E^u_σ splits into a dominated splitting*

  $$E^u_\sigma = E^c_\sigma \oplus E^{uu}_\sigma,$$

  where $\dim E^c_\sigma = 1$.

3.3.3 Proof of Theorem 3.20

The key step of the proof is the following lemma, which will be proved in the next sections.

Lemma 3.30. *Let $X \in \mathcal{X}^*(M)$. If there exists a singularity $\sigma \in \mathrm{Sing}(X)$ exhibiting a homoclinic connection, then $X \notin \mathrm{Int}^1(\mathrm{OrientSh}(M))$.*

To create a homoclinic connection by C^1 perturbations, we need the following uniform C^1 connecting lemma.

Theorem 3.31. *[109] Let $X \in \mathcal{X}^1(M)$. For any C^1 neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X and any point $z \in M$ which is neither singular nor periodic of X, there exist three numbers $\rho > 1$, $T > 1$ and $\delta_0 > 0$, together with a C^1 neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ of X such that for any $X_1 \in \mathcal{U}_1$, any $0 < \delta < \delta_0$ and any two points x, y outside the tube $\Delta_{X_1,z} = \cup_{t\in[0,T]} B(\phi_{X_1,t}(z), \delta)$, if the positive X_1-orbit of x and the negative X_1-orbit of y both hit $B(z, \delta/\rho)$, then there exists $Y \in \mathcal{U}$ with $Y = X_1$ outside $\Delta_{X_1,z}$ such that y is on the positive Y-orbit of x.*

As a classical application of the connecting lemma, we have the following.

Lemma 3.32. *Let $X \in \mathcal{X}^1(M)$ and $\sigma \in \mathrm{Sing}(X)$ be a hyperbolic singularity of X which is preperiodic. Then for any C^1 neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X, there exists $Y \in \mathcal{U}$ such that $\sigma_Y \in \mathrm{Sing}(Y)$ exhibiting a homoclinic connection, where σ_Y is the continuation of σ.*

Proof. Since σ is preperiodic, there exists a sequence of vector fields $\{X_n \in \mathcal{X}^1(M)\}$ together with a sequence of periodic points $\{o_n \in \mathrm{Per}'(X_n)\}$ such that X_n C^1-approximate to X and o_n approximate to σ. By using a typical argument and taking converging subsequence if necessary, we have that there exist $p_n, q_n \in \mathrm{Orb}_{X_n}(o_n)$ and two points $a \neq \sigma$ and $b \neq \sigma$ on the local stable manifold $W^{s,X}_{\mathrm{loc}}(\sigma)$ and local unstable manifold $W^{u,X}_{\mathrm{loc}}(\sigma)$ of σ respectively such that p_n approximate to a and q_n approximate to b.

For a C^1 neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X and the point a, by Theorem 3.31, there exist numbers $\rho_a > 1$, $T_a > 1$, $\delta_{a,0} > 0$ and a C^1 neighborhood $\mathcal{U}_{a,1} \subset \mathcal{U}$ of X with the property of the connecting lemma. Similarly, for the neighborhood $\mathcal{U}_{a,1}$ of X and the point b, there exist numbers $\rho_b > 1$, $T_b > 1$, $\delta_{b,0} > 0$ and a C^1 neighborhood $\mathcal{U}_{b,1} \subset \mathcal{U}_{a,1}$ of X with the property of the connecting lemma. Choose $0 < \delta < \min\{\delta_{a,0}, \delta_{b,0}\}$ small enough such that the tubes $\Delta_{X,a} = \cup_{t\in[0,T_a]} B(\phi_{X,t}(a), \delta)$ and $\Delta_{X,b} = \cup_{t\in[0,T]} B(\phi_{X,t}(b), \delta)$ satisfy

$$\overline{\Delta_{X,a}} \cap \overline{\Delta_{X,b}} = \emptyset, \quad \overline{\Delta_{X,a}} \cap \overline{W^{u,X}_{\mathrm{loc}}(\sigma)} = \emptyset, \quad \text{and} \quad \overline{\Delta_{X,b}} \cap \overline{W^{s,X}_{\mathrm{loc}}(\sigma)} = \emptyset. \tag{3.43}$$

By the Invariant Manifold Theorem, there is a C^1 neighborhood $\mathcal{U}_1 \subset \mathcal{U}_{b,1}$ of X such that property (3.43) holds for any $X' \in \mathcal{U}_1$. Decrease $\mathcal{U}_1$ if necessary, we may also assume that for any $X' \in \mathcal{U}_1$ the local stable manifold $W^{s,X'}_{\mathrm{loc}}(\sigma_{X'})$ of $\sigma_{X'}$ hit $B(a, \delta/\rho)$ and the local unstable manifold $W^{u,X'}_{\mathrm{loc}}(\sigma_{X'})$ of $\sigma_{X'}$ hit $B(b, \delta/\rho)$.

Take n big enough such that $X_n \in \mathcal{U}_1$ and $p_n \in B(a, \delta/\rho)$, $q_n \in B(b, \delta/\rho)$. Thus we can take two points $a' \in W^{s,X_n}_{\mathrm{loc}}(\sigma_{X_n}) \cap B(a, \delta/\rho)$ and $b' \in W^{u,X_n}_{\mathrm{loc}}(\sigma_{X_n}) \cap B(b, \delta/\rho)$. Choose two points $x \in \mathrm{Orb}^-_{X_n}(b') \setminus \Delta_{X_n,b}$ and $y \in \mathrm{Orb}^+_{X_n}(a') \setminus \Delta_{X_n,a}$. Note that $x \in W^{u,X_n}_{\mathrm{loc}}(\sigma_{X_n})$ and $y \in W^{s,X_n}_{\mathrm{loc}}(\sigma_{X_n})$.

Since the positive X_n-orbit of x and the negative X_n-orbit of p_n both hit $B(b,\delta/\rho)$ (at b' and q_n), there exists $Z \in \mathcal{U}_{a,1}$ with $Z = X_n$ outside $\Delta_{X_n,b}$ such that $p_n \in \mathrm{Orb}^+_Z(x)$. Now we have the positive Z-orbit of x and the negative Z-orbit of y both hit $B(a,\delta/\rho)$ (at p_n and a'). Thus we can use the connecting lemma again and get a homoclinic connection of σ_Y for some $Y \in \mathcal{U}$. □

Now Theorem 3.20 is a consequence of Lemmas 3.25, 3.30, 3.32 and Theorem 3.26. Indeed,

Proof of Theorem 3.20. On the contrary, suppose that there exists a vector field $X \in \mathrm{Int}^1(\mathrm{OrientSh}(M))$ satisfying the C^1-robustly oriented shadowing property which is not Ω-stable. By Lemma 3.25 we know that $X \in \mathcal{X}^*(M)$ is a star vector field. Since X is not Ω-stable, we have that $\mathrm{Sing}(X) \cap \mathrm{Per}_*(X) \neq \emptyset$ according to Theorem 3.26. Suppose that $\sigma \in \mathrm{Sing}(X) \cap \mathrm{Per}_*(X)$ is a preperiodic singularity.

Then by Lemma 3.32, there exists a vector field Y arbitrarily C^1 close to X such that $\sigma_Y \in \mathrm{Sing}(Y)$ exhibiting a homoclinic connection, where σ_Y is the continuation of σ. Note that we have $Y \in \mathrm{Int}^1(\mathrm{OrientSh}(M))$ when Y close enough to X. It contradicts with Lemma 3.30. □

The rest part of the paper is devoted to the proof of Lemma 3.30. It follows the strategy similar to [83, Section 2, Case (B1)].

3.3.4 The proof of Lemma 3.30

Suppose on the contrary that there exists a star vector field $X \in \mathrm{Int}^1(\mathrm{OrientSh}(M))$ which has a singularity $\sigma \in \mathrm{Sing}(X)$ exhibiting a homoclinic connection.

Up to an arbitrarily C^1 small perturbation, we may assume that X is linear in a small neighborhood $U_r(\sigma)$ of σ on a proper chart, still exhibits a homoclinic connection $\Gamma \subset W^s(\sigma) \cap W^u(\sigma)$ (see [83] for more details on the perturbations). Without loss of generality, we can assume that $\mathrm{SV}(\sigma) \geq 0$.

Remark 3.33. Note that intersection $W^s(\sigma) \cap W^u(\sigma)$ is not transverse.

We consider the following two cases:

Case 1. $\dim E^s_\sigma = 1$.

In this case, take $\varepsilon = r/10$, then there exists $d > 0$ such that every d-pseudo orbit can be ε-oriented shadowed by a real orbit of X.

Take $p \in W^s(\sigma) \setminus \Gamma$ and $q \in W^u_{\mathrm{loc}}(\sigma) \cap \Gamma$ in a small neighborhood of σ such that the map

$$g(t) = \begin{cases} \phi_t(p), & t \leq 0; \\ \phi_t(q), & t > 0. \end{cases}$$

is a d-pseudo orbit. Thus g is ε-oriented shadowed by a real orbit $\mathrm{Orb}(x)$. Note that $q \in W^s(\sigma)$ implies that $x \in W^s(\sigma)$. But since $\dim E^s_\sigma = 1$ we have $x \in \Gamma$. It is a contradiction.

Case 2. $\dim E^s_\sigma \geq 2$.

In this case, by Lemma 3.29 we know that there exists a dominated splitting

$$E^s_\sigma = E^{ss}_\sigma \oplus E^c_\sigma,$$

where $\dim E^c_\sigma = 1$. This implies that $\sigma \in K_1^-$. Applying Lemma 3.24 we conclude that intersection $W^s(\sigma) \cap W^u(\sigma)$ is transverse, which contradicts to Remark 3.33.

3.4 The difference between the oriented and the standard shadowing propeties

In this paragraph we give an answer to the question posed by Komuro [41]: does the standard shadowing property is equivalent to the orinted shadowing property? Note that in the same work Komuro proved that for nonsingular vector fields those two shadowing properties are equivalent.

In this paragraph we give an example of a vector field on a 4-dimensional manifold which has the oriented shadowing property and do not have the standard shadowing property [103]. An example is a vector field with a nontransverse intersection of stable and unstable manifolds of two fixed points of a very special structure in their neighborhoods.

Theorem 3.34. *For $M = S^2 \times S^2$ there exists vector field $X \in \mathrm{OrientSh} \setminus \mathrm{StSh}$.*

3.4.1 Two dimensional vector field

Set $K = 5$. Consider $a, l > 0$, satisfying $Kl < 1$ and a continuous function $b : [0, +\infty) \to \mathbb{R}$, $b \in C^1(0, +\infty)$ defined as the following:

$$b(r) = \begin{cases} 0, & r \in \{0\} \cup ((K-1)l, +\infty), \\ -\frac{1}{\ln r}, & r \in (0, 2l), \\ b(r) \geq 0, & r \in [2l, (K-1)l]. \end{cases}$$

Let $\psi(t, x)$ be a flow on $\mathbb{R}^2$ generated by a vector field defined by the following formula

$$Y(x) = \begin{pmatrix} a & b(|x|) \\ b(|x|) & a \end{pmatrix} x,$$

which generates the following system of differential equations in polar coordinates

$$\begin{cases} \frac{\mathrm{d}\, r}{\mathrm{d}\, t} = ar, \\ \frac{\mathrm{d}\, \varphi}{\mathrm{d}\, t} = b(r). \end{cases}$$

For a point $x \in \mathbb{R}^2 \setminus \{0\}$ we denote by $\arg(x)$ the point $\frac{x}{|x|} \in \mathbf{S}^1$. If a point $x \in \mathbb{R}^2$ has polar coordinates (r, φ), and $r \neq 0$, we put $\arg(x) = \varphi$.

Lemma 3.35. *(i) For any $a, l > 0$, vector field Y is of class C^1.*

(ii) For any $a, l > 0$, and a point $x_0 \in \mathbb{R}^2 \setminus 0$, angle Θ and $T_0 < 0$ there exists $t < T_0$ such that $\arg(\psi(t, x_0)) = \Theta$.

(iii) There exists $a, l > 0$, such that the following condition holds. If for some points $x_0, x_1 \in \mathbb{R}^2$, $|x_0| < l$, $|x_1| < 2l$ and reparametrization $h \in \mathrm{Rep}(l)$, holds inequalities

$$\mathrm{dist}(\psi(h(t), x_1), \phi(t, x_0)) < l \tag{3.44}$$

provided that $|\psi(h(t), x_1)|, |\phi(t, x_0)| < 1$. Then $|arg(x_1) - arg(x_0)| < \pi/4$.

The proof of this lemma is quite technical, we give it in the Appendix.

Remark 3.36. Vector field Y is of class C^1 but not $C^{1+Hölder}$. We do not know if it is possible to construct a 2-dimensional vector field of class $C^{1+Hölder}$ satisfying items (ii), (iii) of Lemma 3.35. As the result our example of vector field X is not $C^{1+Hölder}$. We do not know if it is an essential restriction or drawback of our particular construction.

For the rest of the paper let us fix $a, l > 0$, $K > 3$ from item (iii) of Lemma 3.35.

We will also need the following statement, which we prove in the appendix.

Lemma 3.37. *Let S_1 and S_2 be three-dimensional vector spaces with coordinates (x_1, x_2, x_3) and (y_1, y_2, y_3) respectively. Let $Q : S_2 \to S_1$ be a linear map satisfying the following condition*

$$Q\{y_2 = y_3 = 0\} \neq \{x_2 = x_3 = 0\}.$$

Then for any $D > 0$ there exists $R > 0$ (depending on Q and D) such that for any two sets $\mathrm{Sp}_1 \subset S_1 \cap \{x_1 = 0\}$ and $\mathrm{Sp}_2 \subset S_2 \cap \{y_1 = 0\}$ satisfying

- $\mathrm{Sp}_1 \subset B(R, 0)$, $\mathrm{Sp}_2 \subset B(R, 0)$;
- Sp_1 *intersects any halfline in $S_1 \cap \{x_1 = 0\}$ starting at 0;*
- Sp_2 *intersects any halfline in $S_2 \cap \{y_1 = 0\}$ starting at 0;*

the sets

$$\mathrm{Cyl}_1 = \{(x_1, x_2, x_3), \quad |x_1| < D, \ (0, x_2, x_3) \in \mathrm{Sp}_1\},$$

$$\mathrm{Cyl}_2 = \{(y_1, y_2, y_3), \quad |y_1| < D, \ (0, y_2, y_3) \in \mathrm{Sp}_2\}$$

satisfy the condition $\mathrm{Cyl}_1 \cap Q\,\mathrm{Cyl}_2 \neq \emptyset$.

3.4.2 Construction of a 4-dimensional vector field

Consider a vector field X on the manifold $M = S^2 \times S^2$ that has the following properties (F1)-(F6) (ϕ denotes the flow generated by X).

(F1) The nonwandering set of ϕ is the union of four rest points p, q, s, u.

(F2) In the neighborhoods $U_p = B(1,p)$, $U_q = B(1,q)$ there exist systems of coordinates such that the following holds:

- $U_p - p$ and $U_q - q$ are 4-dimensional unit balls, where $p = (p_1, p_2, p_3, p_4)$ and $q = (q_1, q_2, q_3, q_4)$ are coordinates of points p and q respectively.
- Riemannian metric on M is equivalent to the Euclidian metric in those coordinate systems.
- In those coordinates the vector fields are given by the formulas

$$X(x) = J_p(x-p), \quad x \in U_p; \qquad X(x) = J_q(x-q), \quad x \in U_q,$$

where

$$J_p(x) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & a & -b(r(x_3,x_4)) \\ 0 & 0 & b(r(x_3,x_4)) & a \end{pmatrix} x,$$

$$J_q(x) = -\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a & 0 & -b(r(x_2,x_4)) \\ 0 & 0 & -2 & 0 \\ 0 & b(r(x_2,x_4)) & 0 & a \end{pmatrix} x.$$

For point $x \in U_p$ denote $P_1 x = x_1$, $P_{34} x = (x_3, x_4)$, where $x - p = (x_1, x_2, x_3, x_4)$, for point $x \in U_q$ denote $P_1 x = x_1$, $P_{24} x = (x_2, x_4)$, where $x - q = (x_1, x_2, x_3, x_4)$.

(F3) The point s is an attracting hyperbolic rest point. The point u is a repelling hyperbolic rest point. The following condition holds:

$$W^u(p) \setminus \{p\} \subset W^s(s), \quad W^s(q) \setminus \{q\} \subset W^u(u),$$

where $W^u(p)$ is the unstable manifold of p, $W^s(q)$ is the stable manifold of q, etc. For $m > 0$ we denote $W^u_{loc}(p,m) = W^u(p) \cap B(m,p)$ etc.

(F4) The intersection of $W^s(p) \cap W^u(q)$ consists of a single trajectory α, satisfying the following

$$\alpha \cap U_p \subset \{p + (t,0,0,0); t \in (0,1)\}; \quad \alpha \cap U_q \subset \{q - (t,0,0,0); t \in (0,1)\}$$

(F5) For some $\Delta \in (0,1)$, $T_a > 0$ the following holds

$$\phi(T_a, q + (-1, x_2, x_3, x_4)) = (p + (1, x_2, x_3, x_4)), \quad |x_2|, |x_3|, |x_4| < \Delta.$$

(F6) $\phi(t,x) \notin U_q$, for $x \in U_p,\ t \geq 0$.

The construction is similar to paragraph 3.2. We leave details to the reader.

Theorem 3.38. *Vector field X satisfies the oriented shadowing property.*

Theorem 3.39. *Vector field X does not satisfy the standard shadowing property.*

Trivially Theorem 3.34 follows from Theorems 3.38, 3.39.

3.4.3 Oriented Shadowing property

Fix $\varepsilon > 0$.

For points $y_p = \alpha(T_p) \in U_p, y_q = \alpha(T_q) \in U_q$ (note that $T_p > T_q$) and $\delta > 0$ we say that $g(t)$ is a pseudotrajectory of type $\mathrm{Ps}(\delta)$ if it has the form (3.9) for some points $x_p \in B(\delta, y_p)$ and $x_q \in B(\delta, y_q)$.

Proposition 3.40. *For any $\delta > 0$, $y_p \in \alpha \cap U_p$, and $y_q \in \alpha \cap U_q$ there exists $d > 0$ such that if $g(t)$ is a d-pseudotrajectory of X, then either $g(t)$ can be ε-oriented shadowed or there exists a pseudotrajectory $g^*(t)$ of type $Ps(\delta)$ with these y_p and y_q and $t_0 \in \mathbb{R}$ such that*

$$\mathrm{dist}(g(t), g^*(t + t_0)) < \varepsilon/2, \quad t \in \mathbb{R}.$$

Proposition 3.41. *There exists $\delta > 0$, $y_p \in \alpha \cap U_p$, and $y_q \in \alpha \cap U_q$ such that any pseudotrajectory of type $Ps(\delta)$ with these y_p and y_q can be $\varepsilon/2$-oriented shadowed.*

Clearly, Propositions 3.40 and 3.41 imply that $X \in \mathrm{OrientSh}$.

Proof of Proposition 3.40 is standard. Exactly the same statement was proved in paragraph 3.2 for a slightly different vector field (the only difference is in the structure of matrixes J_p, J_q). The proof can be literally repeated in our case.

The main idea of the proof is the following. In parts "far" from α vector field is structurally stable and hence have shadowing property according to Remark 3.3. This statement implies that if $g(t)$ does not intersect a small neighborhood of α it can be shadowed. If $g(t)$ intersects a small neighborhood of α then (after a shift of time) for $t > T_p$ points $g(t)$ also lies in a structurally stable part of X and can be shadowed by $\phi(t - T_p, x_p)$; similarly for $t < T_q$ points $g(t)$ can be shadowed by $\phi(t - T_q, x_q)$; for $t \in (T_q, T_p)$ points $g(t)$ are close to α. We omit details in the present paper.

Proof of Proposition 3.41. Defining the Riemannian metric in some neighborhoods of s and u in an appropriate way, we may assume that

$$O^+(B(\varepsilon/2, s), \phi) \subset B(\varepsilon, s) \quad \text{and} \quad O^-(B(\varepsilon/2, u), \phi) \subset B(\varepsilon, u).$$

Take $m \in (0, \varepsilon/8)$. We take points $y_p = \alpha(T_p) \in B(m/2, p) \cap \alpha$ and $y_q = \alpha(T_q) \in B(m/2, q) \cap \alpha$. Put $T = T_p - T_q$. Take $\delta > 0$ such that if $g(t)$ is a pseudotrajectory of type $\mathrm{Ps}(\delta)$ (with y_p and y_q fixed above), $t_0 \in \mathbb{R}$, and $x_0 \in B(2\delta, g(t_0))$, then inequalities (3.14) hold.

Consider a number $\tau > 0$ such that if $x \in W^u(p) \setminus B(m/2, p)$, then $\phi(\tau, x) \in B(\varepsilon/8, s)$. Take $\varepsilon_1 \in (0, m/4)$ such that if two points $z_1, z_2 \in M$ satisfy the inequality $\mathrm{dist}(z_1, z_2) < \varepsilon_1$, then

$$\mathrm{dist}(\phi(t, z_1), \phi(t, z_2)) < \varepsilon/8, \quad |t| \leq \tau.$$

In this case, for any $y \in B(\varepsilon_1, x)$ the following inequalities hold:

$$\mathrm{dist}(\phi(t, x), \phi(t, y)) < \varepsilon/4, \quad t \geq 0. \tag{3.45}$$

Decreasing ε_1, we may assume that if $x' \in W^s(q) \setminus B(m/2, q)$ and $y' \in B(\varepsilon_1, x')$, then

$$\mathrm{dist}(\phi(t, x'), \phi(t, y')) < \varepsilon/4, \quad t \leq 0.$$

Let $g(t)$ be a pseudotrajectory of type $\mathrm{Ps}(\delta)$, where y_p, y_q and δ satisfy the above-formulated conditions.

Let us consider several possible cases.

Case (P1): $x_p \notin W^s(p)$ and $x_q \notin W^u(q)$. Let

$$T' = \inf\{t \in \mathbb{R} : \ \phi(t, x_p) \notin B(p, 3m/4)\}.$$

If δ is small enough, then $\mathrm{dist}(\phi(T', x_p), W^u(p)) < \varepsilon_1$. In this case, there exists a point $z_p \in W^u_{loc}(p, m) \setminus B(m/2, p)$ such that

$$\mathrm{dist}(\phi(T', x_p), z_p) < \varepsilon_1.$$

Applying a similar reasoning in a neighborhood of q (and reducing δ, if necessary), we find a point $z_q \in W^s_{loc}(q, m) \setminus B(m/2, q)$ and a number $T'' < 0$ such that $\mathrm{dist}(\phi(T'', x_q), z_q) < \varepsilon_1$.

Consider hyperplanes $S_p := \{x_1 = P_1 y_p\}$, $S_q := \{x_1 = P_1 y_q\}$. Let us note that Poincaré map $Q : S_q \to S_p$ is linear, defined by $Q(x) = \phi(T, x)$ and satisfy $Q(\{x_2, x_4 = 0\}) \neq \{x_3, x_4 = 0\}$. Choose $R > 0$ from Lemma 3.37, applied to hyperplane S_p, S_q, mapping Q and $D = \varepsilon/8$. Note that for some $T_R > 0$ hold the inequalities

$$|\phi(t, P_{34} x_p)| < R, \ t < -T_R; \quad |\phi(t, P_{24} x_q)| < R, \ t > T_R.$$

Consider the sets

$$\mathrm{Sp}^- = \{\phi(t, P_{34} x_p), \ t < -T_R\}; \quad \mathrm{Sp}^+ = \{\phi(t, P_{24} x_q), \ t > T_R\}.$$

Due to Lemma 3.35 item (ii) sets $\mathrm{Sp}^\pm$ satisfy assumions of Lemma 3.37 and hence the sets

$$C^- = \{x \in S_p : \quad P_{34} x \in \mathrm{Sp}^-, |P_2 x| < D\},$$

$$C^+ = \{x \in S_q : \quad P_{24}x \in \mathrm{Sp}^+, |P_3 x| < D\}$$

satisfy $C^- \cap QC^+ \neq \emptyset$. Let us consider a point

$$x_0 \in C^- \cap QC^+ \tag{3.46}$$

and $t_p < -T_R$, $t_q > T_R$, such that $P_{34}x_0 = \phi(t_p, P_{34}x_s)$, $P_{24}Q^{-1}x_0 = \phi(t_q, P_{24}x_u)$. The following inclusions hold

$$\phi(-T_Q - T_R - T'', x_0) \in B(2\varepsilon_1, z_q); \quad \phi(-T_Q, x_0) \in B(D, y_q);$$

$$\phi(0, x_0) \in B(D, y_p); \quad \phi(T_R + T', x_0) \in B(2\varepsilon_1, z_p).$$

Inequalities (3.14) imply that if δ is small enough, then

$$\mathrm{dist}(\phi(t_3 + t, x_0), g(T_p + t)) < \varepsilon/2, \quad t \in [-T, 0]. \tag{3.47}$$

Define a reparametrization $h(t)$ as follows:

$$h(t) = \begin{cases} h(T_q + T'' + t) = -T_Q - T_R - T'' + t, & t < 0, \\ h(T_p + T' + t) = T_R + T' + t, & t > 0, \\ h(T_p + t) = t, & t \in [-T, 0], \\ h(t) \text{ increases}, & t \in [T_p, T_p + T'] \cup [T_q + T'', T_q]. \end{cases}$$

If $t \geq T_p + T'$, then inequality (3.45) implies that

$$\mathrm{dist}(\phi(h(t), x_0), \phi(t - (T_p + T'), z_p)) < \varepsilon/4;$$

$$\mathrm{dist}(\phi(t - T_p, x_p), \phi(t - (T_p + T'), z_p)) < \varepsilon/4.$$

Hence, if $t \geq T_p + T'$, then

$$\mathrm{dist}(\phi(h(t), x_0), g(t)) < \varepsilon/2. \tag{3.48}$$

For $t \in [T_p, T_p + T']$ the inclusions $\phi(h(t), x_0), g(t) \in B(m, p)$ hold, and inequality (3.48) holds for these t as well.

A similar reasoning shows that inequality (3.48) holds for $t \leq T_q$. If $t \in [T_q, T_p]$, then inequality (3.48) follows from (3.47). This completes the proof in case (P1).

Case (P2): $x_p \in W^s(p)$ and $x_q \notin W^u(q)$. In this case the proof uses the same reasoning as in case (P1). The only difference is that instead of (3.46) we construct a point $x_0 \in B(D, y_p) \cap W^s_{loc}(p, m)$ such that

$$\phi(-T - T'', x_0) \in B(2\varepsilon_1, z_q); \quad \phi(-T, x_0) \in B(\varepsilon/8, y_q).$$

The construction is straightforward and uses Lemma 3.35, item (ii).

Case (P3): $x_p \notin W^s(p)$ and $x_q \in W^u(q)$. This case is similar to case (P2).

Case (P4): $x_p \in W^s(p)$ and $x_q \in W^u(q)$. In this case, we take α as the shadowing trajectory; the reparametrization is constructed similarly to case (P1). $\square$

Remark 3.42. Proposition 3.41 can be easily generalised in order to prove that X satisfies Lipschitz oriented shadowing property (there exists $L, d_0 > 0$ such that for any $d < d_0$ and d-pseudotajectory $g(t)$ there exists x_0 and $h \in \mathrm{Rep}$ such that inequalities (3.2) hold for $\varepsilon = Ld$). Surprisingly we do not know how to prove Lipschitz analog of Proposition 3.40 and believe it is not correct. Moreover we conjecture the following:

Conjecture 3.1. *If a vector field X satisfies the Lipschitz oriented shadowing property then X is structurally stable.*

3.4.4 Standard Shadowing Property

Let us show that for small enough $\varepsilon < \min(l, \Delta/2)$ for any $d > 0$ there exists d-pseudotrajectory $g(t)$, which cannot be ε-shadowed.

Put $a_p = p + (1, 0, 0, 0)$, $a_q = q - (1, 0, 0, 0)$, $e_p = (0, 0, 0, 1)$ and $e_q = (0, 0, 0, -1)$. For any $d > 0$ consider pseudotrajectory

$$g(t) = \begin{cases} \phi(t, a_p + de_p), & t \geq 0, \\ \phi(t + T_a, a_q + de_q), & t \leq -T_a, \\ \phi(t, a_p), & t \in (-T_a, 0). \end{cases}$$

Note that for some $L_0 > 0$ map $g(t)$ is $L_0 d$ pseudotrajectory. Assume that for some $x_0 \in S^2 \times S^2$ and $h(t) \in \mathrm{Rep}(\varepsilon)$ hold the inequalities (3.2). Without loss of generality we can assume that $h(0) = 0$. Let us consider sets

$$S_p = \{(1, x_2, x_3, x_4) : \ |x_2|, |x_3|, |x_4| < \Delta\} \subset U_p;$$

$$S_q = \{(-1, x_2, x_3, x_4) : \ |x_2|, |x_3|, |x_4| < \Delta\} \subset U_q.$$

Inequalities (3.2) imply that $\mathrm{dist}(x_0, a_p + de_p) < \varepsilon$, and $\mathrm{dist}(\phi(h(-T_a), x_0), a_q + de_q) < \varepsilon$. Hence there exists $L_1 > 0$ and $H_p, H_q \in [-L_1\varepsilon, L_1\varepsilon]$ such that points $x_p = \phi(H_p, x_0)$ and $x_q = \phi(h(-T_a) + H_q, x_0)$ satisfy inclusions $x_p \in S_p$, $x_q \in S_q$.

Inequality (3.2) implies that for some $L_2 > 0$ the following holds

$$\begin{aligned} &|x_p - a_p|, |x_q - a_q| < L_2\varepsilon; \\ &\mathrm{dist}(\phi(h(t), x_p), g(t)) < L_2\varepsilon, \quad t > 0; \\ &\mathrm{dist}(\phi(h(t) - h(-T_a), x_q), g(t)) < L_2\varepsilon, \quad t \leq -T_a. \end{aligned} \tag{3.49}$$

Note that introduced above flow ψ satisfies $\psi(t, (x_3, x_4)) = P_{34}\phi(t, (0, 0, x_3, x_4))$. Hence inequalities (3.49) imply the following

$$\mathrm{dist}(\psi(h(t), P_{34}x_p), \psi(t, (0, d))) < L_2\varepsilon, \quad t > 0.$$

Let us choose $\varepsilon > 0$ satisfying the inequality $L_2\varepsilon < l$. Lemma 3.35 imply that $P_4 x_p > 0$. Similarly $P_4 x_q < 0$. This contradicts to the equality $x_p = \phi(T_a, x_q)$ and (F5). Hence $X \notin \mathrm{StSh}$.

3.4.5 Proof of Auxilarily statements

Proof of Lemma 3.35

Note that

$$\psi(t,(r,\varphi)) = (e^{at}r, \varphi + \int_0^t b(e^{a\tau}r)\,\mathrm{d}\,\tau). \tag{3.50}$$

Item (i). Let us show that $Y \in \mathbf{C}^1(\mathbb{R}^2)$. Since $b(r) \in \mathbf{C}^1(0,+\infty)$, it is enough to prove continuity of $DY(x)$ at $x=0$. Assume that $\sqrt{x_1^2+x_2^2} < 2l$. The following holds:

$$b'(r) = \frac{1}{r\ln^2 r}, \quad r \in (0, 2l);$$

$$\frac{\partial Y_1}{\partial x_1} = a + b'\left(\sqrt{x_1^2+x_2^2}\right)\frac{x_1x_2}{\sqrt{x_1^2+x_2^2}}; \quad \frac{\partial Y_1}{\partial x_2} = b'\left(\sqrt{x_1^2+x_2^2}\right)\frac{x_2^2}{\sqrt{x_1^2+x_2^2}}.$$

Since

$$\frac{|x_1x_2|}{\sqrt{x_1^2+x_2^2}}, \ \frac{x_2^2}{\sqrt{x_1^2+x_2^2}} < \sqrt{x_1^2+x_2^2}$$

and $rb'(r) \to 0$ as $r \to 0$, the following holds

$$\lim_{|x|\to 0}\frac{\partial Y_1}{\partial x_1}(x) = a, \quad \lim_{|x|\to 0}\frac{\partial Y_1}{\partial x_2}(x) = 0$$

Arguing similarly for $\frac{\partial Y_2}{\partial x_1}, \frac{\partial Y_2}{\partial x_2}$ we conclude that

$$\lim_{|x|\to 0} DY(x) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Note that

$$\left|Y(x) - \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}x\right| = \left|\begin{pmatrix} 0 & b(|x|) \\ b(|x|) & a \end{pmatrix}x\right| \le \frac{|x|}{|\ln(|x|)|},$$

which implies that

$$DY(0) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

and completes the proof of item (i).

Item (ii). By the equality (3.50) it is enough to show that for $r > 0$, $T_0 < 0$ holds the inequality

$$\int_{-\infty}^{T_0} b(e^{a\tau}r)\,\mathrm{d}\,\tau > 2\pi.$$

Without loss of generality we can assume that $r < 2l$. The following holds

$$\int_{-\infty}^{T_0} b(e^{a\tau}r)\,\mathrm{d}\,\tau = \int_{-\infty}^{T_0} -\frac{1}{a\tau+\ln r}\,\mathrm{d}\,\tau = -\frac{1}{a}\ln(|a\tau+\ln r|)\Big|_{-\infty}^{T_0} = +\infty.$$

Item (ii) is proved.

Item (iii). Note that

$$\ln \frac{K}{K+1} < 1, \quad K \sin \pi/8 > 1. \tag{3.51}$$

Fix $a > 0$. Choose small enough l, satisfying the following inequalities

$$Kl < 1, \quad l|\ln(Kl)| < 1, \quad \ln l > 4, \tag{3.52}$$

$$\frac{2}{a}\left(l - \frac{2}{\ln l}\right) < \frac{\pi}{8}. \tag{3.53}$$

Let $x_0 = (r_0, \varphi_0)$, $x_1 = (r_1, \varphi_1)$ and $h(t) \in \mathrm{Rep}(l)$ satisfy assumptions of the lemma. The following holds

$$r_0 < l, \quad r_1 < 2l. \tag{3.54}$$

Let us consider $T > 0$ and $\Delta \in \mathbb{R}$ such that

$$e^{aT} r_0 = Kl, \quad e^{a\Delta} r_0 = r_1. \tag{3.55}$$

Consider points $x_2 = \psi(T, x_0) = (r_2, \varphi_2)$ $x_3 = \psi(h(T), x_1) = (r_3, \varphi_3)$. Note that $r_2 = Kl$. Inequality (3.44) implies

$$\mathrm{dist}(x_2, x_3) < l.$$

Using inequalities (3.51) we conclude that

$$|\varphi_2 - \varphi_3| < \pi/8, \quad r_3 \in [(K-1)l, (K+1)l]. \tag{3.56}$$

Equality (3.50) implies that

$$r_3 = e^{ah(T)} r_1, \quad \varphi_2 = \varphi_0 + \int_0^T b(e^{a\tau} r_0)\,\mathrm{d}\tau, \quad \varphi_3 = \varphi_1 + \int_0^{h(T)} b(e^{a\tau} r_1)\,\mathrm{d}\tau.$$

Relations (3.44) and (3.55) implies

$$\frac{K}{K+1} e^{a(h(T)+\Delta)} r_0 = \frac{K}{K+1} e^{ah(T)} r_1 < e^{aT} r_0. \tag{3.57}$$

The following holds

$$\begin{aligned} \varphi_2 - \varphi_3 = (\varphi_0 - \varphi_1) + \int_0^T b(e^{a\tau} r_0)\,\mathrm{d}\tau - \int_0^{h(T)} b(e^{a\tau} r_1)\,\mathrm{d}\tau = \\ = (\varphi_0 - \varphi_1) + \int_0^T b(e^{a\tau} r_0)\,\mathrm{d}\tau - \int_\Delta^{h(T)+\Delta} b(e^{a\tau} r_0)\,\mathrm{d}\tau = \\ = (\varphi_0 - \varphi_1) + \int_0^\Delta b(e^{a\tau} r_0)\,\mathrm{d}\tau - \int_T^{h(T)+\Delta} b(e^{a\tau} r_0)\,\mathrm{d}\tau. \end{aligned} \tag{3.58}$$

Relations (3.44) and (3.55) imply that $e^{a(h(T)+\Delta)}r_0 = e^{ah(T)}r_1 > (K-1)l$ and hence

$$b(e^{a\tau}r_0) = 0, \quad \tau \in [T, h(T)+\Delta]. \tag{3.59}$$

Relations (3.57) imply inequalities

$$\ln\frac{K}{K+1} + a(h(T)+\Delta) < aT,$$

and hence

$$\Delta < (T - h(T)) - \frac{1}{a}\ln\frac{K}{K+1}.$$

Since $h(t) \in \mathrm{Rep}(l)$ and $T = (\ln(Kl) - \ln r_0)/a$ using inequalities (3.51), (3.52) we conclude that

$$\Delta < \frac{1}{a}\left(l|\ln(Kl) - \ln r_0| - \ln\frac{K}{K+1}\right) < \frac{1}{a}(2 - l\ln r_0). \tag{3.60}$$

Inequalities (3.54) imply that for $\tau \in [0, \Delta]$ holds the inequality $e^{a\tau}r_0 < 2l$, hence $b(e^{a\tau}r_0) = 1/\ln(e^{a\tau}r_0)$. Inequalities (3.52), (3.53), (3.60) imply that $a|\Delta| < -(\ln r_0)/2$, which implies $|b(e^{a\tau}r_0)| < 2b(r_0) = -2/\ln r_0$ and

$$\left|\int_0^\Delta b(e^{a\tau}r_0)\,\mathrm{d}\,\tau\right| < -\frac{2|\Delta|}{\ln r_0} < -\frac{2}{a}\,(2 - l\ln r_0)\,\frac{1}{\ln r_0} < \frac{2}{a}\left(l - \frac{2}{\ln l}\right) < \frac{\pi}{8}.$$

Combining this with relations (3.58), (3.59) we conclude that

$$|(\varphi_2 - \varphi_3) - (\varphi_0 - \varphi_1)| < \pi/8.$$

and hence (3.56) implies $|\varphi_0 - \varphi_1| < \pi/4$. Item (iii) is proved.

Proof of Lemma 3.37

Let us fix a linear map Q and a number $D > 0$. Consider the lines $l_1 \subset S_1$, $l_2 \subset S_2$ defined by $x_2 = x_3 = 0$, $y_2 = y_3 = 0$ respectively. Note that $Ql_2 \neq l_1$. Let us consider plane $V \subset S_1$ containing l_1 and Ql_2. Consider a parralelogram $P \subset V$, symmetric with respect to 0 with sides parralel to l_1 and Ql_2, satisfying the relation

$$P \subset \{|x_1| < D\} \cap Q(\{|y_1| < D\}). \tag{3.61}$$

Let us choose $R > 0$, such that the following inclusions hold

$$B(R,0) \cap V \subset P \quad \text{and} \quad Q(B(R,0) \cap Q^{-1}V) \subset P. \tag{3.62}$$

Let z_1 be a point of intersection Sp_1 and the line $V \cap \{x_1 = 0\}$. Condition (3.62) implies that $z_1 \in P$. Consider the line k_1, containing z_1 and parallel to l_1. Inclusion (3.61) implies that $k_1 \cap P \subset \mathrm{Cyl}_1$.

Similarly let z_2 be a point of intersection of Sp_2 and $V \cap \{y_1 = 0\}$. Condition (3.62) implies the inclusion $Qz_2 \in P$. Let k_2 be the line containing Qz_2 and parallel to Ql_2. Inclusion (3.61) implies that $Q^{-1}(k_2 \cap V) \subset \mathrm{Cyl}_2$.

Since $k_1 \nparallel k_2$, there exists a point $z \in k_1 \cap k_2$. Inclusions $z_1, z_2 \in P$ imply that $z \in P$. Hence $z \in \mathrm{Cyl}_1 \cap Q\,\mathrm{Cyl}_2$. Lemma 3.37 is proved.

Chapter 4

Lipschitz shadowing for vector fields

In the present chapter we study the Lipschitz shadowing and the Lipschitz periodic shadowing property. We prove that they the Lipschitz shadowing property is equivalent to structural stability and the Lipschitz periodic shadowing property is equivalent to Ω-stability.

The statements are similar to the case of diffeomorphisms, but the proofs are significantly more complicated due to necessity of reparametrisation of shadowing trajectories and presence of fixed points.

4.1 Preliminaries

Let us recall sevearl notions from the theory of structural stability for flows.

Definition 4.1. We say that vector field X and the corresponding flow ϕ satisfy the Axiom A' if

1. The nonwondering set $\Omega(X)$ is hyperbolic;
2. The set $\Omega(X)$ is the union of two disjoint compact ϕ-invariant sets Q_1, Q_2, where Q_1 consists of a finite number of fixed points, while Q_2 does not contain fixed points and points of closed trajectories are dense in Q_2.

For a hyperbolic trajectory $x(t)$ denote

$$W^s(x(t)) = \{y \in M : \exists t_0 \in \mathbb{R} \quad \text{such that} \quad \text{dist}(\phi(t, y), x(t - t_0)) \to_{t \to +\infty} 0\},$$

$$W^u(x(t)) = \{y \in M : \exists t_0 \in \mathbb{R} \quad \text{such that} \quad \text{dist}(\phi(t, y), x(t - t_0)) \to_{t \to -\infty} 0\},$$

Definition 4.2. We say that Axiom A' vector field X and the corresponding flow ϕ satisfy the strong transversality condition if for any two trajectories $x(t)$, $y(t)$ in the nonwondering set intersection $W^s(x(t)) \cap W^u(y(t))$ is transverse.

We will need the following characterisation of structural stability [91].

Theorem 4.1. *Vector field X is structurally stable if and only if it satisfies Axiom A' and the strong transversality condition.*

It is known that in this case, the nonwandering set of X can be represented as a disjoint union of a finite number of compact invariant sets (called basic sets):

$$\Omega(X) = \Omega_1 \cup \cdots \cup \Omega_m, \tag{4.1}$$

where each of the sets Ω_i is either a hyperbolic rest point of X or a hyperbolic set on which X does not vanish and which contains a dense positive semi-trajectory.

The basic sets Ω_i have stable and unstable sets:

$$W^s(\Omega_i) = \{x \in M : \operatorname{dist}(\phi(t,x), \Omega_i) \to 0, \quad t \to \infty\}$$

and

$$W^u(\Omega_i) = \{x \in M : \operatorname{dist}(\phi(t,x), \Omega_i) \to 0, \quad t \to -\infty\}.$$

If Ω_i and Ω_j are basic sets, we write $\Omega_i \to \Omega_j$ if the intersection

$$W^u(\Omega_i) \cap W^s(\Omega_j)$$

contains a wandering point.

We say that X has a 1-cycle if there is a basic set Ω_i such that $\Omega_i \to \Omega_i$.

We say that X has a k-cycle if there are $k > 1$ basic sets

$$\Omega_{i_1}, \dots, \Omega_{i_k}$$

such that

$$\Omega_{i_1} \to \cdots \to \Omega_{i_k} \to \Omega_{i_1}.$$

We use the following characterisation of Ω-stable vector fields (see [85] and [35])

Theorem 4.2. *Vector field X is Ω-stable if and only if X satisfies Axiom A' and the no-cycle condition.*

In what follows we will also use the following notion.

Definition 4.3. For $\varepsilon > 0$ we say that sequence of points $x_0, x_1, \dots, x_n$ is an *ε-chain* if there exists times $t_0, \dots, t_{n-1} > 1$ such that

$$\operatorname{dist}(x_{k+1}, \phi(t_k, x_k)) < \varepsilon, \quad k = 0, \dots, n-1.$$

We write $x \rightsquigarrow y$ if for any $\varepsilon > 0$ there exists an ε-chain $x = x_0, x_1, \dots, x_n = y$.

We say that point x in *chain-recurrent* if $x \rightsquigarrow x$. Denote set of all chain-recurrent points by $\mathcal{CR}$.

4.2 The Lipschitz shadowing property

In this section we prove the following theorem [69]

Theorem 4.3. *A vector field X satisfies the Lipschitz shadowing property if and only if X is structurally stable.*

In paragraph 1.4 we proved that expansive diffeomorphisms having the Lipschitz shadowing property are Anosov.

We show, as a consequence of Theorem 4.3, that expansive vector fields having the Lipschitz shadowing property are Anosov.

Theorem 4.4. *An expansive vector field X having the Lipschitz shadowing property is Anosov.*

Proof. By Theorem 4.3, a vector field X having the Lipschitz shadowing property is structurally stable. Hence, there exists a neighborhood $\mathcal{N}$ of X in the C^1-topology such that any vector field in $\mathcal{N}$ is expansive (this property of X is sometimes called robust expansivity).

By Theorem B of [59], robustly expansive vector fields having the shadowing property are Anosov. □

The rest of the section is devoted to the proof of Theorem 4.3.

As was mentioned in Theorem 3.2 structurally stable vector fields have the Lipschitz shadowing property. Our goal here is to show that vector fields satisfying Lipschitz shadowing are structurally stable. Due to Theorem 4.1 it is enough to show that such a vector field satisfies Axiom A$'$ and the strong transversality condition.

4.2.1 Discrete Lipschitz shadowing property

First we show that Lipschitz shadowing implies discrete Lipschitz shadowing. Define a diffeomorphism f on M by setting $f(x) = \phi(1, x)$.

Definition 4.4. The vector field X has the *discrete Lipschitz shadowing property* if there exist d_0, $L > 0$ such that if $y_k \in M$ is a sequence with

$$\operatorname{dist}(y_{k+1}, f(y_k)) \le d, \quad k \in \mathbb{Z}$$

for $d \le d_0$, then there exist sequences $x_k \in M$ and $t_k \in \mathbb{R}$ satisfying

$$|t_k - 1| \le Ld, \quad \operatorname{dist}(x_k, y_k) \le Ld, \quad x_{k+1} = \phi(t_k, x_k)$$

for all k.

Lemma 4.5. *Lipschitz shadowing implies discrete Lipschitz shadowing.*

Proof. Let y_k be a sequence with

$$\operatorname{dist}(y_{k+1}, f(y_k)) = \operatorname{dist}(y_{k+1}, \phi(1, y_k)) \leq d, \quad k \in \mathbb{Z}.$$

Then we define

$$y(t) = \phi(t-k, y_k) \quad k \leq t < k+1, \quad k \in \mathbb{Z}.$$

Assume that $k \leq \tau < k+1$. If $0 \leq t \leq 1$ and $\tau + t < k+1$, then

$$\operatorname{dist}(y(\tau+t), \phi(t, y(\tau)) = \operatorname{dist}(\phi(\tau+t-k, y_k), \phi(t, \phi(\tau-k, y_k))) = 0$$

and if $k+1 \leq \tau + t$, then

$$\begin{aligned}
&\operatorname{dist}(y(\tau+t), \phi(t, y(\tau))) \\
&= \operatorname{dist}(\phi(\tau+t-k-1, y_{k+1}), \phi(t+\tau-k, y_k)) \\
&= \operatorname{dist}(\phi(\tau+t-k-1, y_{k+1}), \phi(\tau+t-k-1, \phi(1, y_k))) \\
&\leq \nu d,
\end{aligned}$$

where ν is a constant such that

$$\operatorname{dist}(\phi(t,x), \phi(t,y)) \leq \nu \operatorname{dist}(x,y) \quad \text{for} \quad x, y \in M,\ 0 \leq t \leq 1. \tag{4.2}$$

Then if $d \leq d_0/\nu$, there exists a trajectory $x(t)$ of X and a function $\alpha(t)$ satisfying

$$\left| \frac{\alpha(t_2) - \alpha(t_1)}{t_2 - t_1} - 1 \right| \leq \mathcal{L}\nu d$$

for $t_2 \neq t_1$ and

$$\operatorname{dist}(y(t), x(\alpha(t))) \leq \mathcal{L}\nu d$$

for all t. Then if we define

$$x_k = x(\alpha(k)), \quad t_k = \alpha(k+1) - \alpha(k),$$

we see that

$$x_{k+1} = x(\alpha(k+1)) = \phi(\alpha(k+1) - \alpha(k), x(\alpha(k))) = \phi(t_k, x_k),$$

$$\operatorname{dist}(x_k, y_k) = \operatorname{dist}(x(\alpha(k)), y(k)) \leq \mathcal{L}\nu d$$

and

$$|t_k - 1| = \left| \frac{\alpha(k+1) - \alpha(k)}{k+1-k} - 1 \right| \leq \mathcal{L}\nu d.$$

Taking $L = \mathcal{L}\nu$ and d_0 in Definition 4.4 as d_0/ν, we complete the proof of the lemma. $\square$

4.2.2 Inhomogenious linear equation

Our main tool in the proof is the following lemma which relates the shadowing problem to the problem of existence of bounded solutions of certain difference equations. To "linearize" our problem, we apply the standard technique of exponential mappings.

Similarly to the case of diffeomorphisms we consider the following. There exists $r > 0$ such that, for any $x \in M$, $\exp_x$ is a diffeomorphism of $B_T(r, x)$ onto its image, and $\exp_x^{-1}$ is a diffeomorphism of $B(r, x)$ onto its image. In addition, we may assume that r has the following property:

If v, $w \in B_T(r, x)$, then

$$\operatorname{dist}(\exp_x(v), \exp_x(w)) \leq 2|v - w|; \tag{4.3}$$

if y, $z \in B(r, x)$, then

$$|\exp_x^{-1}(y) - \exp_x^{-1}(z)| \leq 2 \operatorname{dist}(y, z). \tag{4.4}$$

Let $x(t)$ be a trajectory of X; set $p_k = x(k)$ for $k \in \mathbb{Z}$. Denote $A_k = Df(p_k)$ and $\mathcal{M}_k = T_{p_k}M$. Clearly, A_k is a linear isomorphism between $\mathcal{M}_k$ and $\mathcal{M}_{k+1}$.

In the sequel whenever we construct d-pseudotrajectories of the diffeomorphism f, we always take d so small that the points of the pseudotrajectories under consideration, the points of the associated shadowing trajectories, their lifts to tangent spaces, etc. belong to the corresponding balls $B(r, p_k)$ and $B_T(r, p_k)$.

We consider the mappings

$$F_k = \exp_{p_{k+1}}^{-1} \circ f \circ \exp_{p_k} : B_T(\rho, p_k) \to \mathcal{M}_{k+1} \tag{4.5}$$

with $\rho \in (0, r)$ small enough, so that

$$f \circ \exp_{p_k}(B_T(\rho, p_k)) \subset B(r, p_{k+1}).$$

It follows from standard properties of the exponential mapping that $D\exp_x(0) = \mathrm{Id}$; hence,

$$DF_k(0) = A_k.$$

Since M is compact, for any $\mu > 0$ we can find $\delta = \delta(\mu) > 0$ such that if $|v| \leq \delta$, then

$$|F_k(v) - A_k v| \leq \mu |v|. \tag{4.6}$$

Lemma 4.6. *Assume that X has the discrete Lipschitz shadowing property with constant L. Let $x(t)$ be an arbitrary trajectory of X, let $p_k = x(k)$, $A_k = Df(p_k)$ and let $b_k \in \mathcal{M}_k$ be a bounded sequence (denote $b = \|b\|_\infty$). Then there exists a sequence s_k of scalars with $|s_k| \leq b' = L(2b + 1)$ such that the difference equation*

$$v_{k+1} = A_k v_k + X(p_{k+1}) s_k + b_{k+1}$$

has a solution v_k such that

$$\|v\|_\infty \leq 2b'.$$

Proof. Fix a natural number N and define $\Delta_k \in \mathcal{M}_k$ as the solution of

$$v_{k+1} = A_k v_k + b_{k+1}, \quad k = -N, \dots, N-1$$

with $\Delta_{-N} = 0$. Then

$$|\Delta_k| \leq C, \tag{4.7}$$

where C depends on N, b and an upper bound on $|A_k|$.

Fix a small number $d > 0$ and fix μ in (4.6) so that $\mu < 1/(2C)$. Then consider the sequence of points $y_k \in M$, $k \in \mathbb{Z}$, defined as follows: $y_k = p_k$ for $k \leq -N$, $y_k = \exp_{p_k}(d\Delta_k)$ for $-N+1 \leq k \leq N$, and $y_{N+k} = f^k(y_N)$ for $k > 0$.

By definition, $y_{k+1} = f(y_k)$ for $k \leq -N$ and $k \geq N$. If $-N-1 \leq k \leq N-1$, then

$$y_{k+1} = \exp_{p_{k+1}}(d\Delta_{k+1}) = \exp_{p_{k+1}}(dA_k\Delta_k + db_{k+1}),$$

and it follows from estimate (4.3) that if d is small enough, then

$$\operatorname{dist}\Big(y_{k+1}, \exp_{p_{k+1}}(dA_k\Delta_k)\Big) \leq 2d|b_{k+1}| \leq 2db. \tag{4.8}$$

On the other hand,

$$f(y_k) = \exp_{p_{k+1}}(F_k(d\Delta_k))$$

(see the definition (4.5) of the mapping F_k), and we deduce from (4.3), (4.6) and (4.7) that if $Cd \leq \delta(\mu)$

$$\begin{aligned}\operatorname{dist}\Big(f(y_k), \exp_{p_{k+1}}(dA_k\Delta_k)\Big) &\leq 2|F_k(d\Delta_k) - dA_k\Delta_k| \\ &\leq 2\mu|d\Delta_k| \\ &\leq 2C\mu d \\ &< d.\end{aligned} \tag{4.9}$$

Estimates (4.8) and (4.9) imply that

$$\operatorname{dist}(y_{k+1}, f(y_k)) < d(2b+1), \quad k \in \mathbb{Z},$$

if d is small enough (let us emphasize here that the required smallness of d depends on b, N and estimates on A_k). By hypothesis, there exist sequences x_k and t_k such that

$$|t_k - 1| \leq b'd, \ \operatorname{dist}(x_k, y_k) \leq b'd, \ x_{k+1} = \phi(t_k, x_k), \quad k \in \mathbb{Z}.$$

If we write

$$x_k = \exp_{p_k}(dc_k), \ t_k = 1 + ds_k,$$

then it follows from estimate (4.4) that

$$|dc_k - d\Delta_k| \leq 2\operatorname{dist}(x_k, y_k) \leq 2b'd.$$

Thus,

$$|c_k - \Delta_k| \leq 2b', \ k \in \mathbb{Z}. \tag{4.10}$$

Clearly,
$$|s_k| \le b',\ k \in \mathbb{Z}. \tag{4.11}$$

We may assume that the value ρ fixed above is small enough, so that the mappings

$$G_k : (-\rho, \rho) \times B_T(\rho, p_k) \to \mathcal{M}_{k+1}$$

given by
$$G_k(t, v) = \exp_{p_{k+1}}^{-1}(\phi(1 + t, \exp_{p_k}(v))).$$
are defined. Then $G_k(0, 0) = 0$,

$$D_t G_k(t, v)|_{t=0,v=0} = X(p_{k+1}), \quad D_v G_k(t, v)|_{t=0,v=0} = A_k. \tag{4.12}$$

We can write the equality
$$x_{k+1} = \phi(1 + ds_k, x_k)$$
in the form
$$\exp_{p_{k+1}}(dc_{k+1}) = \phi(1 + ds_k, \exp_{p_k}(dc_k)),$$
which is equivalent to
$$dc_{k+1} = G_k(ds_k, dc_k). \tag{4.13}$$
Now let $d = d_m$, where $d_m \to 0$. Note that the corresponding $c_k = c_k^{(m)}$, $t_k = t_k^{(m)}$, and $s_k = s_k^{(m)}$ depend on m.

Since $|c_k^{(m)}| \le 2b' + C$ and $|s_k^{(m)}| \le b'$ for all $m \ge 1$ and $-N \le k \le N - 1$, by taking a subsequence if necessary, we can assume that $c_k^{(m)} \to \tilde{c}_k$, $t_k^{(m)} \to \tilde{t}_k$, and $s_k^{(m)} \to \tilde{s}_k$ for $-N \le k \le N - 1$ as $m \to \infty$.

Applying relations (4.13) and (4.12), we can write

$$d_m c_{k+1}^{(m)} = G_k(d_m s_k^{(m)}, d_m c_k) = A_k d_m c_k^{(m)} + X(p_{k+1}) d_m s_k^{(m)} + o(d_m).$$

Dividing by d_m, we get the relations

$$c_{k+1}^{(m)} = A_k c_k^{(m)} + X(p_{k+1}) s_k^{(m)} + o(1),\ -N \le k \le N - 1.$$

Letting $m \to \infty$, we arrive at

$$\tilde{c}_{k+1} = A_k \tilde{c}_k + X(p_{k+1}) \tilde{s}_k,\ -N \le k \le N - 1,$$

where
$$|\Delta_k - \tilde{c}_k| \le 2b', \quad |\tilde{s}_k| \le b', \quad -N \le k \le N - 1$$
due to (4.10) and (4.11).

Denote the obtained $\tilde{s}_k$ by $s_k^{(N)}$. Then $v_k^{(N)} = \Delta_k - \tilde{c}_k$ is a solution of the system

$$v_{k+1}^{(N)} = A_k v_k^{(N)} + X(p_{k+1}) s_k^{(N)} + b_{k+1},\ -N \le k \le N - 1,$$

such that $|v_k^{(N)}| \le 2b'$.

There exist subsequences $s_k^{(j_N)} \to s'_k$ and $v_k^{(j_N)} \to v'_k$ as $N \to \infty$ (we do not assume uniform convergence) such that $|s'_k| \le b'$, $|v'_k| \le 2b'$, and

$$v'_{k+1} = A_k v'_k + X(p_{k+1})s'_k + b_{k+1}, \; k \in \mathbb{Z}.$$

Thus, the lemma is proved. □

In the following three sections we assume that X has the Lipschitz shadowing property (and, consequently, the discrete Lipschitz shadowing property).

4.2.3 Hyperbolicity of the rest points

Let x_0 be a rest point. We apply Lemma 4.6 with $p_k = x_0$. Noting that $X(p_k) = 0$, we conclude that the difference equation

$$v_{k+1} = Df(x_0)v_k + b_{k+1}$$

has a bounded solution v_k for all bounded sequences $b_k \in \mathcal{M}_{x_0}$. Then it follows from Theorem 1.5 that

$$v_{k+1} = Df(x_0)v_k$$

is hyperbolic on both $\mathbb{Z}_+$ and $\mathbb{Z}_-$. In particular, this implies that any solution bounded on $\mathbb{Z}_+$ tends to 0 as $k \to \infty$. However if $Df(x_0)$ had an eigenvalue on the unit circle, the equation would have a nonzero solution with constant norm. Hence the eigenvalues of $Df(x_0)$ lie off the unit circle. So x_0 is hyperbolic.

4.2.4 The rest points are isolated in the chain recurrent set

Lemma 4.7. *If a rest point x_0 is not isolated in the chain recurrent set $\mathcal{CR}$, then there is a homoclinic orbit $x(t)$ associated with it.*

Proof. We choose $d > 0$ so small that $\mathrm{dist}(\phi(t, y), x_0) \le \mathcal{L}d$ for $|t|$ large implies that $\phi(t, y) \to x_0$ as $|t| \to \infty$.

Assume that there exists a point $y \in \mathcal{CR}$ such that $y \ne x_0$ is arbitrarily close to x_0. Since y is chain recurrent, given any ε_0 and $\theta > 0$ we can find points $y_1, \dots, y_N$ and numbers $T_0, \dots, T_N > \theta$ such that

$$\mathrm{dist}(\phi(T_0, y), y_1) < \varepsilon_0,$$
$$\mathrm{dist}(\phi(T_i, y_i), y_{i+1}) < \varepsilon_0, \quad i = 1, \dots, N,$$
$$\mathrm{dist}(\phi(T_N, y_N), y) < \varepsilon_0.$$

Set $T = T_0 + \dots + T_N$ and define g^* on $[0, T]$ by

$$g^*(t) = \begin{cases} \phi(t, y), & 0 \le t < T_0, \\ \phi(t, y_i), & T_0 + \dots + T_{i-1} \le t < T_0 + \dots + T_i, \\ y, & t = T. \end{cases}$$

Clearly, for any $\varepsilon > 0$ we can find ε_0 depending only on ε and ν (see (4.2)) such that $g^*(t)$ is an ε-pseudotrajectory on $[0, T]$.

Then we define

$$g(t) = \begin{cases} x_0, & t \leq 0, \\ g^*(t), & 0 < t \leq T, \\ x_0, & t > T. \end{cases}$$

We want to choose y and ε in such a way that $g(t)$ is a d-pseudotrajectory. We need to show that for all τ and $0 \leq t \leq 1$

$$\mathrm{dist}(\phi(t, g(\tau)), g(t+\tau)) \leq d. \tag{4.14}$$

Clearly this holds for (i) $\tau \leq -1$, (ii) $\tau \geq T$, (iii) $\tau, \tau+t \in [-1, 0]$, and (iv) $\tau, \tau+t \in [0, T]$. If $-1 \leq \tau \leq 0$, $\tau + t > 0$, then with ν as in (4.2)

$$\begin{aligned} \mathrm{dist}(\phi(t, g(\tau)), g(\tau+t)) &= \mathrm{dist}(x_0, g^*(\tau+t)) \\ &\leq \mathrm{dist}(x_0, \phi(\tau+t, y)) + \mathrm{dist}(\phi(\tau+t, y), g^*(\tau+t)) \\ &\leq \nu \mathrm{dist}(x_0, y) + \varepsilon \\ &\leq d, \end{aligned}$$

if $\mathrm{dist}(y, x_0)$ and ε are sufficiently small. Note that, for the fixed y, we can decrease ε and increase $N, T_0, \ldots, T_N$ arbitrarily so that $g(t)$ remains a d-pseudotrajectory.

Similarly, (4.14) holds if $\tau \in [0, T]$ and $\tau + t > T$.

Thus $g(t)$ is $\mathcal{L}d-$shadowed by a trajectory $x(t)$ so that in particular $\mathrm{dist}(x(t), x_0) \leq \mathcal{L}d$ if $|t|$ is sufficiently large so that $x(t) \to x_0$ as $|t| \to \infty$.

We must also be sure that $x(t) \neq x_0$. If y is not on the local stable manifold of x_0, then there exists $\varepsilon_1 > 0$ independent of y such that $\mathrm{dist}(\phi(t_0, y), x_0) \geq \varepsilon_1$ for some $t_0 > 0$. We can choose $T_0 > t_0$. Now we know that $\mathrm{dist}(x(t), \phi(t_0, y)) \leq \mathcal{L}d$. So provided $\mathcal{L}d < \varepsilon_1$, we have $x(t_0) \neq x_0$.

If y is on the local stable manifold of x_0, then provided $\mathrm{dist}(y, x_0)$ is sufficiently small, it is not on the local unstable manifold of x_0. Then, applying the same argument to the flow with time reversed noting that the chain recurrent set is also the chain recurrent set for the reversed flow and also that the reversed flow will have the Lipschitz shadowing property also, we show that $x(t) \neq x_0$.

Now we show the existence of this homoclinic orbit $x(t)$ leads to a contradiction. Set $p_k = x(k)$. Since $A_k X(p_k) = X(p_{k+1})$, it is easily verified that if

$$\beta_{k+1} = \beta_k + s_k, \quad k \in \mathbb{Z}$$

then $v_k = \beta_k X(p_k)$ is a solution of

$$v_{k+1} = A_k v_k + X(p_{k+1}) s_k, \quad k \in \mathbb{Z}. \tag{4.15}$$

Also if s_k is bounded then $\beta_k X(p_k)$ is also bounded, since $X(p_k) \to 0$ exponentially as $|k| \to \infty$ and $|\beta_k|/|k|$ is bounded.

By Lemma 4.6, for all bounded $b_k \in \mathcal{M}_k$ there exists a bounded scalar sequence s_k such that

$$v_{k+1} = A_k v_k + X(p_{k+1}) s_k + b_{k+1}$$

has a bounded solution. But we know (4.15) has a bounded solution. It follows that

$$v_{k+1} = A_k v_k + b_{k+1}$$

has a bounded solution for arbitrary $b_k \in \mathcal{M}_k$. Then it follows from Theorem 1.5 that

$$v_{k+1} = A_k v_k$$

is hyperbolic on both $\mathbb{Z}_+$ and $\mathbb{Z}_-$ and that the spaces $B^+(A)$ and $B^-(A)$ are transverse. This is a contradiction since $\dim B^+(A) + \dim B^-(A) = n$ (because $B^+(A)$ has the same dimension as the stable manifold of x_0 and $B^-(A)$ has the same dimension as the unstable manifold of x_0) but they contain $X(p_0) \neq 0$ in their intersection.

So we conclude that the rest points are isolated in the chain recurrent set. $\square$

4.2.5 Hyperbolicity of the chain recurrent set

We have shown that the rest points of X are hyperbolic and form a finite, isolated subset of the chain recurrent set $\mathcal{CR}$. Let Σ be the chain recurrent set minus the rest points. We want to show this set is hyperbolic. To this end we use the following lemma. Let us first introduce some notation.

Let $x(t)$ be a trajectory of X in Σ. Put $p_k = x(k)$ and denote by P_k the orthogonal projection in $\mathcal{M}_k$ with kernel spanned by $X(p_k)$ and by V_k the orthogonal complement to $X(p_k)$ in $\mathcal{M}_k$. Introduce the operators $B_k = P_{k+1} A_k : V_k \mapsto V_{k+1}$.

Lemma 4.8. *For every bounded sequence $b_k \in V_k$ (denote $b = \|b\|_\infty$) there exists a solution $v_k \in V_k$ of the system*

$$v_{k+1} = B_k v_k + b_{k+1}, \quad k \in \mathbb{Z}, \tag{4.16}$$

such that for all k,

$$|v_k| \leq 2L(2b+1).$$

Proof. By Lemma 4.6, there exists a bounded sequence s_k such that the system

$$w_{k+1} = A_k w_k + X(p_{k+1}) s_k + b_{k+1}, \quad k \in \mathbb{Z} \tag{4.17}$$

has a solution w_k with $|w_k| \leq 2L(2b+1)$.

Note that $A_k X(p_k) = X(p_{k+1})$. Since $(\mathrm{Id} - P_k)v \in \{X(p_k)\}$ for $v \in \mathcal{M}_k$, we see that $P_{k+1} A_k (\mathrm{Id} - P_k) = 0$, which gives us the equality

$$P_{k+1} A_k = P_{k+1} A_k P_k. \tag{4.18}$$

Multiplying (4.17) by P_{k+1}, taking into account the equalities $P_{k+1} X(p_{k+1}) = 0$ and $P_{k+1} b_{k+1} = b_{k+1}$, and applying (4.18), we see that $v_k = P_k w_k$ is the required solution.

Thus, the lemma is proved. $\square$

Now we prove Σ is hyperbolic. Let $x(t)$ be a trajectory in Σ with the same notation as given before Lemma 4.8. Then by Lemma 4.8 and Theorem 1.5,

$$v_{k+1} = B_k v_k, \quad v_k \in V_k \tag{4.19}$$

is hyperbolic on both $\mathbb{Z}_+$ and $\mathbb{Z}_-$ and $B^+(B)$ and $B^-(B)$ are transverse. It follows that the adjoint system

$$v_{k+1} = (B_k)^{*-1} v_k, \quad v_k \in V_k$$

is hyperbolic on both $\mathbb{Z}_+$ and $\mathbb{Z}_-$ and has no nontrivial bounded solution.

Now we consider the discrete linear skew product flow on the normal bundle V over Σ generated by the map defined for $p \in \Sigma$, $v \in V_p$ (where V_p is the orthogonal complement to $X(p)$ in T_pM) by

$$(p, v) \mapsto (\phi(1, p), B_p v), \tag{4.20}$$

where $B_p = P_{\phi(1,p)} D\phi(1, p)$, P_p being the orthogonal projection of T_pM onto V_p. Its adjoint flow is generated by the map defined by

$$(p, v) \mapsto (\phi(1, p), (B_p^*)^{-1} v).$$

Now we want to apply the Corollary on page 492 in Sacker and Sell [93]. What we have shown above is that the adjoint flow has the no nontrivial bounded solution property. It follows from the Sacker and Sell corollary that the adjoint flow is hyperbolic and hence the original skew product flow

$$(p, v) \mapsto (\phi(1, p), B_p v)$$

is also. However then it follows from Theorem 3 in Sacker and Sell [94] that Σ is hyperbolic.

4.2.6 Strong Transversality

To verify strong transversality, let $x(t)$ be a trajectory that belongs to the intersection of the stable and unstable manifolds of two trajectories, $x_+(t)$ and $x_-(t)$, respectively, lying in the chain recurrent set. Denote $p_0 = x(0)$ and $p_k = x(k)$, $k \in \mathbb{Z}$; let $W^s(p_0)$ and $W^u(p_0)$ denote the stable manifold of $x_+(t)$ and the unstable manifold of $x_-(t)$, respectively. Denote by E^s and E^u the tangent spaces of $W^s(p_0)$ and $W^u(p_0)$ at p_0.

By Lemma 4.8 (using the same notation as in the previous section), for all bounded $b_k \in V_k$, there exists a bounded solution $v_k \in V_k$ of (4.16). By Theorem 1.5 again, this implies that

$$\mathcal{E}^s + \mathcal{E}^u = V_0, \tag{4.21}$$

where

$$\begin{aligned} \mathcal{E}^s &= \{w_0 : w_{k+1} = B_k w_k,\ \sup\nolimits_{k \geq 0} |w_k| < \infty\}, \\ \mathcal{E}^u &= \{w_0 : w_{k+1} = B_k w_k,\ \sup\nolimits_{k \leq 0} |w_k| < \infty\} \end{aligned}$$

Moreover (4.19) is hyperbolic on both $\mathbb{Z}_+$ and $\mathbb{Z}_-$.

We are going to use the following folklore result, which for completeness we prove after showing it implies the strong transversality:

$$\mathcal{E}^s \subset E^s, \quad \mathcal{E}^u \subset E^u. \tag{4.22}$$

Combining equality (4.21) with the inclusions (4.22) and the trivial relations

$$E^s = V_0 \cap E^s + \{X(p_0)\}, \quad E^u = V_0 \cap E^u + \{X(p_0)\},$$

we conclude that

$$E^s + E^u = T_{p_0}M,$$

and so the strong transversality holds.

Let us now prove the first relation in (4.22); the second one can be proved in a similar way.

Case 1: The limit trajectory in $\mathcal{CR}$ is a rest point. In this case, the stable manifold of the rest point coincides with its stable manifold as a fixed point of the time-one map $f(x) = \phi(1, x)$. By the theory for diffeomorphisms, if p_k is a trajectory on the stable manifold, the tangent space to the stable manifold at p_0 is the subspace E^s of initial values of bounded solutions of

$$v_{k+1} = A_k v_k, \quad k \geq 0. \tag{4.23}$$

Let us prove that $\mathcal{E}^s \subset E^s$. Fix an arbitrary sequence w_k satisfying $w_{k+1} = B_k w_k$ with $w_0 \in \mathcal{E}^s$. Consider the sequence

$$v_k = \lambda_k X(p_k)/|X(p_k)| + w_k$$

with λ_k satisfying

$$\lambda_{k+1} = \frac{|X(p_{k+1})|}{|X(p_k)|}\lambda_k - \frac{X(p_{k+1})^*}{|X(p_{k+1})|}A_k w_k \tag{4.24}$$

and $\lambda_0 = 0$. It is easy to see that v_k satisfy (4.23).

Since $x(t)$ is on the stable manifold of a hyperbolic rest point, there are positive constants K and α such that

$$|\dot{x}(t)| \leq Ke^{-\alpha(t-s)}|\dot{x}(s)|$$

for $0 \leq s \leq t$. From this it follows that

$$|X(p_k)| \leq Ke^{-\alpha(k-m)}|X(p_m)|$$

for $0 \leq m \leq k$ so that the scalar difference equation

$$\lambda_{k+1} = \frac{|X(p_{k+1})|}{|X(p_k)|}\lambda_k$$

is hyperbolic on $\mathbb{Z}_+$ and is, in fact, stable. Since the second term on the right-hand side of equation (4.24) is bounded as $k \to \infty$, it follows that λ_k are bounded for any choice of λ_0.

This fact implies that v_k is a bounded solution of (4.23), and we conclude that $v_0 = w_0 \in E^s$, hence $\mathcal{E}^s \subset E^s$.

The proof in Case 1 is complete.

Case 2: Assume that the limit trajectory is in Σ, the chain recurrent set minus the fixed points which we know to be hyperbolic. We want to find the intersection of its stable manifold near $p_0 = x(0)$ with the cross-section at p_0 orthogonal to the vector field (in local coordinates generated by the exponential mapping). To do this, we discretize the problem and note that there exists a number $\sigma > 0$ such that a point $p \in M$ close to p_0 certainly belongs to $W^s(p_0)$ if and only if the distances of consecutive points of intersections of the positive semitrajectory of p with the sets $\exp_{p_k}(\mathcal{M}_k)$ to the points p_k do not exceed σ.

For suitably small $\mu > 0$, we find all sequences t_k and $z_k \in V_k$, the subspace of $T_{p_k}M$ orthogonal to $X(p_k)$, such that for $k \geq 0$

$$|t_k - 1| \leq \mu, \quad |z_k| \leq \mu, \quad y_{k+1} = \phi(t_k, y_k),$$

where $y_k = \exp_{p_k}(z_k)$. Thus we have to solve the equation

$$\exp_{p_{k+1}}(z_{k+1}) = \phi(t_k, \exp_{p_k}(z_k)), \quad k \geq 0$$

for t_k and $z_k \in V_k$ such that $|t_k - 1| \leq \mu$ and $|z_k| \leq \mu$.

We set it up as a problem in Banach spaces. By Theorem 1.5 and Lemma 4.8 the difference equation

$$z_{k+1} = B_k z_k, \quad z_k \in V_k$$

(recall that $B_k = P_{k+1}A_k$ and P_k is the orthogonal projection on $\mathcal{M}_k$ with range V_k), has an exponential dichotomy on $\mathbb{Z}_+$ with projection (say) $Q_k : V_k \mapsto V_k$. Denote by $\mathcal{R}(Q_0)$ the range of Q_0 and note that $\mathcal{R}(Q_0) = \mathcal{E}^s$. Fix a positive number μ_0 and denote by V the space of sequences

$$\{z_k \in V_k, |z_k| \leq \mu_0, k \in \mathbb{Z}_+\}$$

and by $l^\infty(\mathbb{Z}_+, \{\mathcal{M}_{k+1}\})$ the space of sequences $\{\zeta_k \in \mathcal{M}_{k+1},\ k \in \mathbb{Z}_+\}$ with the usual norm.

Then the C^1 function

$$G : [1 - \mu_0, 1 + \mu_0]^{\mathbb{Z}_+} \times V \times \mathcal{R}(Q_0) \mapsto \ell^\infty(\mathbb{Z}_+, \{\mathcal{M}_{k+1}\}) \times \mathcal{R}(Q_0)$$

given by

$$G(t, z, \eta) = (\{z_{k+1} - \exp_{p_{k+1}}^{-1}(\phi(t_k, \exp_{p_k}(z_k))\}_{k \geq 0}, Q_0 z_0 - \eta)$$

is defined if μ_0 is small enough.

We want to solve the equation

$$G(t, z, \eta) = 0$$

for (t, z) as a function of η. It is clear that

$$G(1, 0, 0) = 0,$$

where the first argument of G is $\{1, 1, \ldots\}$, the second argument is $\{0, 0, \ldots\}$ and the right-hand side is $(\{0, 0, \ldots\}, 0)$.

To apply the implicit function theorem, we must verify that

$$T = \frac{\partial G}{\partial(t, z)}(1, 0, 0)$$

is invertible. Note that if $(s, w) \in \ell^\infty(\mathbb{Z}_+, \mathbb{R}) \times V$, then

$$T(s, w) = (\{w_{k+1} - X(p_{k+1})s_k - A_k w_k\}_{k\geq 0}, Q_0 w_0).$$

To show that T is invertible, we must show that

$$T(s, w) = (g, \eta)$$

has a unique solution (s, w) for all $(g, \eta) \in l^\infty(\mathbb{Z}_+, \{\mathcal{M}_{k+1}\}) \times \mathcal{R}(Q_0)$. So we need to solve the equations

$$w_{k+1} = A_k w_k + X(p_{k+1})s_k + g_k, \quad k \geq 0$$

subject to

$$Q_0 w_0 = \eta.$$

If we multiply the difference equation by $X(p_{k+1})^*$ and solve for s_k, we obtain

$$s_k = -\frac{X(p_{k+1})^*}{|X(p_{k+1})|^2}[A_k w_k + g_k], \quad k \geq 0$$

and if we multiply it by P_{k+1}, we obtain

$$w_{k+1} = P_{k+1}A_k w_k + P_{k+1}g_k = B_k w_k + P_{k+1}g_k, \quad k \geq 0.$$

Now we know this last equation has a unique bounded solution $w_k \in V_k$, $k \geq 0$, satisfying $Q_0 w_0 = \eta$. Then the invertibility of T follows.

Thus we can apply the implicit function theorem to show that there exists $\mu > 0$ such that provided $|\eta|$ is sufficiently small, the equation $G(t, z, \eta) = 0$ has a unique solution $(t(\eta), z(\eta))$ such that $\|t - 1\|_\infty \leq \mu$, $\|z\|_\infty \leq \mu$. Moreover, $t(0) = 1$, $z(0) = 0$ and the functions $t(\eta)$ and $z(\eta)$ are C^1.

The points $\exp_{p_0}(z_0(\eta))$ with small $|\eta|$ form a submanifold containing p_0 and contained in $W^s(p_0)$. Thus, the range of the derivative $z_0'(0)$ is contained in E^s.

Take an arbitrary vector $\xi \in \mathcal{E}^s$ and consider $\eta = \tau\xi, \xi \in \mathbb{R}$. Differentiating the equalities

$$z_{k+1}(\tau\xi) = \exp^{-1}_{p_{k+1}}(\phi(t_k(\tau\xi), \exp_{p_k}(z_k(\tau\xi)))), \quad k \geq 0,$$

and

$$Q_0 z_0(\tau\xi) = \tau\xi$$

with respect to τ at $\tau = 0$, we see that

$$s_k = \frac{\partial t_k}{\partial \eta}|_{\eta=0}\xi, \quad w_k = \frac{\partial z_k}{\partial \eta}|_{\eta=0}\xi \in V_k$$

are bounded sequences satisfying

$$w_{k+1} = A_k w_k + X(p_{k+1})s_k, \quad Q_0 w_0 = \xi.$$

Multiplying by P_{k+1}, we conclude that

$$w_{k+1} = B_k w_k, k \geq 0, \quad Q_0 w_0 = \xi.$$

It follows that $w_0 \in \mathcal{E}^s = \mathcal{R}(Q_0)$. Then $w_0 = Q_0 w_0 = \xi$. We have shown that the range of $z_0'(0)$ is exactly $\mathcal{E}^s$, and thus $\mathcal{E}^s \subset E^s$.

4.3 Lipschitz periodic shadowing

Definition 4.5. We say that the vector field X has the *Lipschitz periodic shadowing property* ($X \in$ LipPerSh) if there exist d_0 and $\mathcal{L} > 0$ such that if $y : \mathbb{R} \mapsto M$ is a periodic *d-pseudotrajectory* for $d \leq d_0$, then $y(t)$ is *$\mathcal{L}d-$shadowed by a periodic trajectory*, that is, there exists a trajectory $x(t)$ of X and a reparametrisation $\alpha \in \text{Rep}(\mathcal{L}d)$ satisfying inequalities

$$\text{dist}(y(t), x(\alpha(t))) < \mathcal{L}d, \quad t \in \mathbb{R} \tag{4.25}$$

and such that

$$x(t + \omega) = x(t)$$

for some $\omega > 0$. The last equality implies that $x(t)$ is either a closed trajectory or a rest point of the flow ϕ.

In order to characterise the set of vector fields satisfying the Lipschitz periodic shadowing property we need the following notion.

Definition 4.6. We say that a vector field $X \in \mathcal{F}(M)$ is *Ω-stable* if there exists a neighborhood $U \subset \mathcal{F}(M)$ of X such that for any $Y \in U$ there exists a homeomorphism $\alpha : \Omega(X) \to \Omega(Y)$ which maps trajectories of X to trajectories of Y and preserves the direction of movement along trajectories.

The main results of this paragraph is the following theorem [69].

Theorem 4.9. *A vector field X satisfies the Lipschitz periodic shadowing property if and only if X is Ω-stable.*

Taking into account Theorem 4.2, to prove Theorem 4.9, it is sufficient to prove the following two lemmas.

Lemma 4.10. *If a vector field X has the Lipschitz periodic shadowing property, then X satisfies Axiom A' and the no-cycle condition.*

Lemma 4.11. *If X satisfies Axiom A' and the no-cycle condition, then X has the Lipschitz periodic shadowing property.*

Lemma 4.10 is proved in Sections 4.3.1-4.3.5; Lemma 4.11 is proved in Section 4.3.6.

The proof of Lemma 4.10 is divided into several steps.

We assume that X has the Lipschitz periodic shadowing property and establish the following statements.

1. Closed trajectories are uniformly hyperbolic.

2. Rest points are hyperbolic.

3. The chain-recurrent set coincides with the closure of the set of rest points and closed trajectories; rest points are isolated in the chain-recurrent set.

4. The hyperbolic structure on the set of closed trajectories can be extended to the chain-recurrent set.

5. The no-cycle condition holds.

4.3.1 Uniform hyperbolicity of closed trajectories

Without loss of generality we can assume that $\mathcal{L} > 1$.

Let $x(t)$ be a nontrivial closed trajectory of period ω. Choose $n_1, n \in \mathbb{N}$ such that $\tau = n_1\omega/n \in [1/2, 1]$. Let $x_k = x(k\tau)$, $f(x) = \phi(\tau, x)$, and $A_k = \mathrm{D} f(x_k)$. Note that $A_{k+n} = A_k$. Below we prove a statement similar to Lemma 4.6.

Lemma 4.12. *If $X \in \mathrm{LipPerSh}$, then for any $b > 0$ there exists a constant K (the same for all closed trajectories $x(t)$ of X) such that for any sequence $b_k \in T_{x_k}M$ with $|b_k| < b$ there exist sequences $s_k \in \mathbb{R}$ and $v_k \in T_{x_k}M$ with the following properties:*

$$v_{k+1} = A_k v_k + X(x_{k+1})s_{k+1} + b_{k+1} \tag{4.26}$$

and

$$|s_k|, |v_k| \le K. \tag{4.27}$$

Before we go to the proof of Lemma 4.12, we need to generalize the notion of discrete Lipschitz shadowing property. Let $d, \tau > 0$; we say that a sequence y_k is a τ-discrete d-pseudotrajectory if $\mathrm{dist}(y_{k+1}, \phi(\tau, y_k)) < d$.

Let $\varepsilon > 0$; we say that a sequence x_k ε-shadows y_k if there exists a sequence $t_k > 0$ such that

$$\mathrm{dist}(x_k, y_k) < \varepsilon, \quad |t_k - \tau| < \varepsilon, \quad x_{k+1} = \phi(t_k, x_k).$$

The following lemma can be proved similarly to Lemma 4.5.

Lemma 4.13. *If $X \in \mathrm{LipPerSh}$, then there exist constants $d_0, L > 0$ such that for any $\tau \in [1/2, 1]$ and $d > 0$ and any periodic τ-discrete d-pseudotrajectory y_k with $d \le d_0$ there exists a sequence x_k (not necessarily periodic) that Ld-shadows y_k.*

In the proof of Lemma 4.12, we use the following technical statement which is well-known in control theory [100].

Lemma 4.14. *Let $B : \mathbb{R}^m \to \mathbb{R}^m$ be a linear operator such that the absolute values of its eigenvalues equal 1. Then for any $\Delta_0 \in \mathbb{R}^m$ and $\delta > 0$ there exists a number $R \in \mathbb{N}$ and a sequence $\delta_k \in \mathbb{R}^m$, $k \in [1, R]$, such that $|\delta_k| < \delta$ and the sequence $\Delta_k \in \mathbb{R}^m$ defined by*

$$\Delta_{k+1} = B\Delta_k + \delta_{k+1}, \quad k \in [0, R-1], \tag{4.28}$$

satisfies $\Delta_R = 0$.

Proof of Lemma 4.12. Fix an arbitrary sequence b_k with $|b_k| < b$ and a number $l \in \mathbb{N}$.

First we will find a number $l_1 > l$ and sequences c_k and Δ_k defined for $k \in [-ln, l_1 n]$ such that $|c_k| < b$ and

$$\begin{gathered} c_k = b_k, \quad k \in [-ln, ln], \\ \Delta_{k+1} = A_k\Delta_k + c_{k+1}, \quad k \in [-ln, l_1 n - 1], \\ \Delta_{-ln} = \Delta_{l_1 n}. \end{gathered} \tag{4.29}$$

Consider the operator $A : T_{x_0} \to T_{x_0}$ defined by $A = A_{n-1} \cdots A_0$.

The tangent space T_{x_0} can be represented in the form

$$T_{x_0} = E_0^s \oplus E_0^c \oplus E_0^u \tag{4.30}$$

so that the subspace E_0^s corresponds to the eigenvalues λ_j of A such that $|\lambda_j| < 1$, the subspace E_0^c corresponds to the eigenvalues λ_j such that $|\lambda_j| = 1$, and the subspace E_0^u corresponds to the eigenvalues λ_j such that $|\lambda_j| > 1$.

For any index k consider the decomposition $T_{x_k} = E_k^s \oplus E_k^c \oplus E_k^u$ as the image of decomposition (4.30) under the mapping $A_{k-1} \cdots A_0$.

In the coordinates corresponding to these decompositions, the matrices A_k can be represented in the following form:

$$A_k = \mathrm{diag}(A_k^s, A_k^c, A_k^u).$$

Set $A_\sigma = A_{n-1}^\sigma \cdots A_0^\sigma$ for $\sigma = s, c, u$. Consider the corresponding coordinate representations $b_k = (b_k^s, b_k^c, b_k^u)$, $c_k = (c_k^s, c_k^c, c_k^u)$, and $\Delta_k = (\Delta_k^s, \Delta_k^c, \Delta_k^u)$ (and note that the values $|b_k^s|, |b_k^c|, |b_k^u|$ are not necessarily less than b).

Equations (4.29) are equivalent to the system

$$\Delta_{k+1}^s = A_k^s\Delta_k^s + c_{k+1}^s, \tag{4.31}$$

$$\Delta_{k+1}^c = A_k^c\Delta_k^c + c_{k+1}^c,$$

$$\Delta_{k+1}^u = A_k^u\Delta_k^u + c_{k+1}^u. \tag{4.32}$$

Set $c_k = b_k$ for $k \in [-ln, ln-1]$.

Consider the sequence satisfying (4.31) with initial data $\Delta^s_{-ln} = 0$ and denote Δ^s_{ln} by a^s; Consider the sequence satisfying (4.32) with initial data $\Delta^u_{ln} = 0$ and denote Δ^u_{-ln} by a^u.

There exist numbers $l_s, l_u > 1$ such that

$$|A^{-l}a^u| < b,\ l \geq l_u, \quad |A^l a^s| < b,\ l \geq l_s.$$

Set $\Delta_{-ln} = (0, 0, a^u)$; then the definition of a^s and a^u implies that $\Delta_{ln} = (a^s, C_1, 0)$ for some $C_1 \in E^c_0$.

Set $c_k = 0$ for $k \in [ln+1, (l+l_s)n]$; then $\Delta_{(l+l_s)n} = (A_s^{l_s} a^s, C_2, 0)$ for some $C_2 \in E^c_0$.

Set $c_k = 0$ for $k \in [(l+l_s)n+1, (l+l_s+1)n-1]$ and $c_k = (-A_s^{l_s+1}a^s, 0, 0)$ for $k = (l+l_s+1)n$. Then $\Delta_{(l+l_s+1)n} = (0, C_3, 0)$ for some $C_3 \in E^c_0$.

Applying Lemma 4.14 to $A^c : E^c_0 \to E^c_0$, we find a number R and a sequence δ_k with $|\delta_k| \leq b$ such that if

$$x_{i+1} = A^c x_i + \delta_{i+1}, \quad x_0 = \Delta^c_{(l+l_s+1)n},$$

then $x_R = 0$. Then if we set for $i = 0, \ldots, R-1$, $c_k = 0$ for $(l+l_s+i+1)n+1 \leq k \leq (l+l_s+i+2)n-1$ and $c_{(l+l_s+i+2)n} = (0, \delta_{i+1}, 0)$, we see that

$$\Delta^c_{(l+l_s+i+2)n} = A^c \Delta^c_{(l+l_s+i+1)n} + \delta_{i+1}, \quad i = 0, \ldots, R-1,$$

so that $\Delta^c_{(l+l_s+R+1)n} = 0$; of course, the other two components of $\Delta_{(l+l_s+R+1)n}$ remain zero.

Set $c_k = 0$ for $k \in [(l+l_s+R+1)n+1, (l+l_s+R+2)n-1]$ and $c_k = (0, 0, A_u^{-l_u}a^u)$ for $k = (l+l_s+R+2)n$; then $\Delta_{(l+l_s+R+2)n} = (0, 0, A_u^{-l_u}a^u)$. Finally, we set $c_k = 0$ for $k \in [(l+l_s+R+2)n+1, (l+l_s+R+2+l_u)n]$ and see that $\Delta_{(l+l_s+2+R+l_u)n} = (0, 0, a^u) = \Delta_{-ln}$. Thus, we have constructed the sequences mentioned in the beginning of the proof.

Taking d small enough, considering the periodic τ-discrete pseudotrajectory $y_k = \exp_{x_k}(d\Delta_k)$, and repeating the reasoning similar to that in the proof of Lemma 4.6, we can prove that relations (4.26) and (4.27) hold with $K = L(2b+1)$ for $k \in [-ln, ln-1]$.

After that, we repeat the reasoning used in the last two paragraphs of the proof of Lemma 4.6 to complete the proof of Lemma 4.12. □

As in Sec. 4.2.5, we define $\mathcal{M}_k, V_k, P_k$, and $B_k = P_{k+1}A_k : V_k \to V_{k+1}$. Note that $B_k^{-1} = P_k A_k^{-1}$. Since M is compact, there exists a constant $N > 0$ such that $\| \mathrm{D}\,\phi(\tau, x)\| < N$ for any $\tau \in [-1, 1]$ and $x \in M$. Hence, $\|A_k\|, \|A_k^{-1}\| < N$, and

$$\|B_k\|, \|B_k^{-1}\| < N. \tag{4.33}$$

The same reasoning as in the proof of Lemma 6 establishes the following statement.

Lemma 4.15. *There exists a constant $K > 0$ (the same for all closed trajectories $x(t)$) such that for every sequence $b_k \in V_k$ with $|b_k| \leq 1$ there exists a solution $v_k \in V_k$ of the system*

$$v_{k+1} = B_k v_k + b_{k+1}$$

such that

$$\|v_k\| \leq K.$$

A remark on page 26 of [18], Lemma 4.15 and the inequalities (4.33) imply that there exist constants $C_1 > 0$ and $\lambda_1 \in (0,1)$ (the same for all closed trajectories) and a representation $V_k = E^s(x_k) \oplus E^u(x_k)$ such that

$$B_k E^s(x_k) = E^s(x_{k+1}), \quad B_k E^u(x_k) = E^u(x_{k+1}),$$

$$|B_{l+k} \cdots B_k v^s| \leq C_1 \lambda_1^l |v_s|, \quad v^s \in E^s(x_k),\ l > 0, k \in \mathbb{Z}$$
$$|B_{-l+k}^{-1} \cdots B_k^{-1} v^u| \leq C_1 \lambda_1^l |v_u|, \quad v^u \in E^u(x_k),\ l > 0, k \in \mathbb{Z}.$$

Remark 4.16. In fact in [18] exponential dichotomy with uniform constants was proved only on $\mathbb{Z}^+$. However we can extend the corresponding inequalities to the whole of $\mathbb{Z}$ by the periodicity of B_k.

Since $\tau \in [1/2, 1]$ and $\| \mathrm{D}\, \phi(\tau, x)\| \leq N$ the above conditions imply that there exist constants $C_2 > 0$ and $\lambda_2 \in (0,1)$ such that if $x(t)$ is a closed trajectory, then

$$|P_{\phi(t,x_0)} \mathrm{D}\, \phi(t, x(t_0)) v^s| \leq C_2 \lambda_2^t |v^s|, \quad v^s \in E^s(x(t_0)),\ t > 0,\ t_0 \in \mathbb{R}, \tag{4.34}$$

$$|P_{\phi(-t,x_0)} \mathrm{D}\, \phi(-t, x(t_0)) v^u| \leq C_2 \lambda_2^t |v^u|, \quad v^u \in E^u(x(t_0)),\ t > 0,\ t_0 \in \mathbb{R}, \tag{4.35}$$

where $P_{y \in M}$ is the orthogonal projection of $T_y M$ with kernel $X(y)$ and

$$E^{s,u}(x(t_0)) = P_{\phi(t_0,x)} \mathrm{D}\, \phi(t_0, x) E^{s,u}(x_0).$$

Remark 4.17. In particular, the above inequalities imply that $x(t)$ is a hyperbolic closed trajectory.

4.3.2 Hyperbolicity of the rest points

Let x_0 be a rest point. As in subsection 4.2.3 (using Lemma 4.12), we conclude that $\mathrm{D}\, \phi(1, x_0)$ is hyperbolic; hence, x_0 is a hyperbolic rest point.

4.3.3 The rest points are isolated in the chain-recurrent set

Denote by $\mathrm{Per}(X)$ the set of rest points and points belonging to closed trajectories of a vector field X; let $\mathcal{CR}(X)$ be the set of its chain-recurrent points. For a set $A \subset M$ denote by $\mathrm{Cl}\, A$ the closure of A and by $B(a, A)$ its a-neighborhood.

Lemma 4.18. *If* $X \in \mathrm{LipPerSh}$, *then* $\mathrm{Cl}\,\mathrm{Per}(X) = \mathcal{CR}(X)$.

Proof. If $y_0 \in \mathcal{CR}(X)$, then for any $d > 0$ there exists a periodic d-pseudotrajectory $g(t)$ such that $g(0) = y_0$.

Since $X \in \mathrm{LipPerSh}$, there exists a point $x_d \in \mathrm{Per}(X)$ such that $\mathrm{dist}(x_d, y_0) < \mathcal{L}d$. Hence, $B(\mathcal{L}d, y_0) \cap \mathrm{Per}(X) \neq \emptyset$ for arbitrary $d > 0$, which proves our lemma. ☐

Lemma 4.19. *Let* $X \in \mathrm{LipPerSh}$ *and let* p *be a rest point of* X. *Then* $p \notin \mathrm{Cl}(\mathcal{CR}(X) \setminus p)$.

Proof. It has already been proved that all rest points of a vector field $X \in$ LipPerSh are hyperbolic; hence the set of rest points is finite. Assume that $p \in \mathrm{Cl}(\mathcal{CR}(X) \setminus p)$. Then Lemma 4.18 implies that $p \in \mathrm{Cl}(\mathrm{Per}(X) \setminus p)$.

Denote by $W^s_{loc,a}(p)$ and $W^u_{loc,a}(p)$ the local stable and unstable manifolds of size a.

Since the rest point p is hyperbolic, there exists $\varepsilon \in (0, 1/2)$ such that if $x \in M$ and $\phi(t,x) \subset B(4\varepsilon, p)$, $t \geq 0$, then $x \in W^s_{loc,4\varepsilon}(p)$; if $\phi(t,x) \subset B(4\varepsilon, p)$, $t \leq 0$, then $x \in W^u_{loc,4\varepsilon}(p)$; and if $\phi(t,x) \subset B(4\varepsilon, p)$, $t \in \mathbb{R}$, then $x = p$.

Let $d_1 = \min(d_0, \varepsilon/\mathcal{L})$, where d_0 and $\mathcal{L}$ are the constants from the definition of LipPerSh. Take a point $x_0 \in \mathrm{Per}(X)$ (let the period of the trajectory of x_0 equal ω) and a number $T > 0$ and define the mapping

$$g_{x_0,T}(t) = \begin{cases} p, & t \in [-T, T], \\ \phi(t-T, x_0), & t \in (T, T+\omega), \end{cases}$$

for $t \in [-T, T+\omega)$. Continue this mapping periodically to the line $\mathbb{R}$.

There exists $d_2 < d_1$ depending only on d_1 and ν (see (4.2)) such that if $x_0 \in B(d_2, p)$, then $g_{x_0,T}(t)$ is a d_1-pseudotrajectory for any $T > 0$. We fix such a point $x_0 \in B(d_2, p)$ and consider below pseudotrajectories $g_{x_0,T}$ with this fixed x_0 and with increasing numbers T.

By our assumptions, the pseudotrajectory $g_{x_0,T}$ can be ε-shadowed by the trajectory of a point $z_T \in \mathrm{Per}(X)$ with reparametrization $\alpha_T(t)$:

$$\mathrm{dist}(g_{x_0,T}(t), \phi(\alpha_T(t), z_T)) < \varepsilon. \tag{4.36}$$

Our choice of ε implies that there exist times $t_1, t_2 > 0$ such that

$$\mathrm{dist}(p, \phi(t_1, x_0)) \in [2\varepsilon, 3\varepsilon], \quad \phi(t, x_0) \in B(4\varepsilon, p), \quad t \in [0, t_1],$$

$$\mathrm{dist}(p, \phi(-t_2, x_0)) \in [2\varepsilon, 3\varepsilon], \quad \phi(t, x_0) \in B(4\varepsilon, p), \quad t \in [-t_2, 0].$$

We emphasize that the numbers t_1, t_2 depend on our choice of the point x_0 but not on our choice of T. Let

$$r_T = \phi(\alpha_T(T + t_1), z_T), \quad q_T = \phi(\alpha_T(-T - t_2), z_T).$$

Inequalities (4.36) and the following two relations imply that

$$\phi(t, q_T) \in B(5\varepsilon, p), \quad t \in [0, -\alpha_T(-T - t_2)], \tag{4.37}$$

$$\phi(t, r_T) \in B(5\varepsilon, p), \quad t \in [-\alpha_T(T + t_1), 0]. \tag{4.38}$$

Since $\mathcal{L} d_2 \leq \varepsilon < 1/2$ and t_1, t_2 are fixed, inequality (4.25) implies that if T is large enough, then

$$-\alpha_T(-T - t_2) \geq T/2, \quad \alpha_T(T + t_1) \geq T/2. \tag{4.39}$$

Since (4.37)-(4.39) imply that $\mathrm{dist}(\phi(t, q_T), p) \leq 4\varepsilon$ for $0 \leq t \leq T/2$ and $\mathrm{dist}(\phi(t, r_T), p) \leq 4\varepsilon$ for $0 \geq t \geq -T/2$ it follows that

$$\mathrm{dist}(q_T, W^s_{loc,4\varepsilon}(p)), \mathrm{dist}(r_T, W^u_{loc,4\varepsilon}(p)) \to 0, \quad T \to +\infty.$$

Since $q_T, r_T \in B(4\varepsilon, p) \setminus B(\varepsilon, p)$, we can choose sequences $q_n = q_{T_n} \to q$ and $r_n = r_{T_n} \to r$ such that $q, r \neq p$, $q \in W^s_{loc,4\varepsilon}(p)$, and $r \in W^u_{loc,4\varepsilon}(p)$.

Denote by $O(q_n)$ the (closed) trajectory of the point q_n.

From Remark 4.17 we know that $O(q_n)$ is a hyperbolic closed trajectory.

Passing to a subsequence, if necessary, we may assume that the values $\dim W^s(O(q_n))$ are the same for all n. Since

$$\dim W^s(O(q_n)) + \dim W^u(O(q_n)) = \dim M + 1$$

and

$$\dim W^s(p) + \dim W^u(p) = \dim M,$$

we see that at least one of the following inequalities holds:

$$\dim W^s(O(q_n)) > \dim W^s(p)$$

or

$$\dim W^u(O(q_n)) > \dim W^u(p).$$

Without loss of generality, we can assume that the first inequality holds (in the other case we note that $O(q_n) = O(r_n)$ and consider the vector field $-X$).

Denote $\sigma = \dim W^s(p)$. Consider the space $E^s_n = E^s(q_n)$ corresponding to inequalities (4.34), (4.35). Then the following holds

$$\dim E^s_n = \dim W^s(O(q_n)) - 1 \geq \sigma.$$

Passing to a subsequence, if necessary, we may assume that $E^s_n \to F^s \subset V_q$, where V_q is the subspace in T_qM orthogonal to $X(q)$ (here and below, we consider convergence of linear spaces in the Grassman topology). Passing to the limit in inequalities (4.34), we conclude that

$$|P_{\phi(t,q)}\, \mathrm{D}\, \phi(t,q)v^s| \leq C_2\lambda_2^t|v^s|, \quad v^s \in F^s,\ t > 0.$$

This inequality implies the inclusion $F^s \subset TW^s_q(p)$. Hence,

$$F^s \oplus \langle X(q) \rangle \subset T_qW^s(q),$$

and $\dim W^s(q) \geq \sigma + 1$. We get a contradiction which proves Lemma 4.15. $\square$

4.3.4 Hyperbolicity of the chain-recurrent set

Consider a point $y \in \mathcal{CR}(X)$ that is not a rest point. Lemma 4.18 implies that there exists a sequence $x_n \in \mathrm{Per}(X)$ such that $x_n \to y$.

Consider the decomposition $V_{x_n} = E^s(x_n) + E^u(x_n)$ corresponding to inequalities (4.34), (4.35). Denote $E^{s,u}_n = E^{s,u}(x_n)$. Passing if necessary to a subsequence, we may assume that the dimensions $\dim E^s_n$ and $\dim E^u_n$ are the same for all n. Since y is not a rest point, $V_{x_n} \to V_y$.

Since inequalities (4.34) and (4.35) hold for all closed trajectories with the same constants C_2 and λ_2, standard reasoning implies that the “angles” between E_n^s and E_n^u are uniformly separated from 0 (see, for instance, [70]). So passing if necessary to a subsequence, we may assume that $E_n^s \to E^s$ and $E_n^u \to E^u$.

Hence, $E^s \cap E^u = \{0\}$, $\dim(E^s + E^u) = \dim E^s + \dim E^u = \dim V_y$, and $E^s + E^u = V_y$. Estimates (4.34) and (4.35) for the points x_n imply similar estimates for y. Hence, the skew product flow (4.20) is hyperbolic, and Theorem 3 in Sacker and Sell [94] implies that $\mathcal{CR}(X)$ is hyperbolic.

4.3.5 No-cycle condition

In the previous two subsections we have proved that the vector field X (and its flow ϕ) satisfies Axiom A′.

Lemma 4.20. *If* $X \in \text{LipPerSh}$, *then* X *has no cycles.*

Proof. To simplify the presentation, we prove that X has no 1-cycles (in the general case, the idea is essentially the same, but the notation is heavy).

To get a contradiction, assume that

$$p \in (W^u(\Omega_i) \cap W^s(\Omega_i)) \setminus \Omega(X).$$

Then there are sequences of times $j_m, k_m \to \infty$ as $m \to \infty$ such that

$$\phi(-j_m, p), \phi(k_m, p) \to \Omega_i, \quad m \to \infty.$$

Since the set Ω_i is compact, we may assume that

$$\phi(-j_m, p) \to q \in \Omega_i \;\text{ and }\; \phi(k_m, p) \to r \in \Omega_i.$$

Since Ω_i contains a dense positive semi-trajectory, there exist points $s_m \to r$ and times $l_m > 0$ such that $\phi(l_m, s_m) \to q$ as $m \to \infty$.

Clearly, if we continue the mapping

$$g(t) = \begin{cases} \phi(t, p), & t \in [0, k_m], \\ \phi(t - k_m, s_m), & t \in [k_m, k_m + l_m], \\ \phi(t - j_m - k_m - l_m, p), & t \in [k_m + l_m, k_m + l_m + j_m], \end{cases}$$

periodically with period $k_m + l_m + j_m$, we get a periodic d_m-pseudotrajectory of X with $d_m \to 0$ as $m \to \infty$.

Since $X \in \text{LipPerSh}$, there exist points $p_m \in \text{Per}(X)$ (for m large enough) such that $p_m \to p$ as $m \to \infty$, and we get the desired contradiction with the assumption that $p \notin \Omega(X)$. The lemma is proved. ∎

4.3.6 Ω-stability implies Lipschitz periodic shadowing

The proof of Lemma 4.11 is similar to the corresponding proof in section 1.6.1, where the case of diffeomorphisms is considered. In this section we give the most important steps and leave the details to the reader.

Proof of Lemma 4.11. Let us formulate several auxiliary definitions and statements.

Let us say that a vector field X has the Lipschitz shadowing property on a set U if there exist positive constants $\mathcal{L}, d_0$ such that if $g(t)$ with $\{g(t) : t \in \mathbb{R}\} \subset U$ is a d-pseudotrajectory (in our standard sense:

$$\operatorname{dist}(g(\tau + t), \phi(t, g(\tau))) < d, \quad \tau \in \mathbb{R}, t \in [0, 1])$$

with $d \leq d_0$, then there exists a point $p \in U$ and a reparametrization α satisfying inequality (4.25) such that

$$\operatorname{dist}(g(t), \phi(\alpha(t), p)) < \mathcal{L}d, \quad t \in \mathbb{R}. \tag{4.40}$$

We say that a vector field X is expansive on a set U if there exist positive numbers a (expansivity constant) and δ such that if two trajectories $\{\phi(t, p) : t \in \mathbb{R}\}$ and $\{\phi(t, q) : t \in \mathbb{R}\}$ belong to U and there exists a continuous real-valued function $\alpha(t)$ such that

$$\operatorname{dist}(\phi(\alpha(t), q), \phi(t, p)) \leq a, \quad t \in \mathbb{R},$$

then $p = \phi(\tau, q)$ for some real $\tau \in (-\delta, \delta)$.

Let X be an Ω-stable vector field. Consider the decomposition (4.1) of $\Omega(X)$. We will refer to the following well-known statement [65].

Theorem 4.21. *If Ω_i is a basic set, then there exists a neighborhood U of Ω_i such that X has the Lipschitz shadowing property on U and is expansive on U.*

We also need the following two lemmas. Analogs of these lemmas were proved for diffeomorphisms in [80]; the proofs for flows are the same.

Lemma 4.22. *For any neighborhood U of the nonwandering set $\Omega(X)$ there exist positive numbers B, d_1 such that if $g(t)$ is a d-pseudotrajectory of ϕ with $d \leq d_1$ and*

$$g(t) \notin U, \quad t \in [\tau, \tau + l],$$

for some $l > 0$ and $\tau \in \mathbb{R}$, then $l \leq B$.

Lemma 4.23. *Assume that the vector field X is Ω-stable. Let $U_1, \dots, U_m$ be disjoint neighborhoods of the basic sets $\Omega_1, \dots, \Omega_m$. There exist neighborhoods $V_j \subset U_j$ of the sets Ω_j and a number $d_2 > 0$ such that if $g(t)$ is a d-pseudotrajectory of X with $d \leq d_2$, $g(\tau) \in V_j$ and $g(\tau + t_0) \notin U_j$ for some $j \in \{1, \dots, m\}$, some $\tau \in \mathbb{R}$ and some $t_0 > 0$, then $g(\tau + t) \notin V_j$ for $t \geq t_0$.*

Now we pass to the proof itself.

Apply Theorem 4.21 and Lemmas 4.22, 4.23 and find disjoint neighborhoods $W_1, \dots, W_m$ of the basic sets $\Omega_1, \dots, \Omega_m$ such that

(i) X has the Lipschitz shadowing property on each W_j with the same constants $\mathcal{L}, d_0^*$;

(ii) X is expansive on each W_j with the same expansivity constants a, δ.

Find neighborhoods V_j, U_j of Ω_j (and reduce d_0^*, if necessary) so that the following properties are fulfilled:

- $V_j \subset U_j \subset W_j, \quad j = 1, \dots, m;$
- the statement of Lemma 4.23 holds for V_j and U_j with some $d_2 > 0$;
- the $\mathcal{L}d_0^*$-neighborhoods of U_j belong to W_j.

Apply Lemma 4.22 to find the corresponding constants B, d_1 for the neighborhood $V_1 \cup \dots \cup V_m$ of $\Omega(X)$.

We claim that X has the Lipschitz periodic shadowing property with constants $\mathcal{L}, d_0$, where

$$d_0 = \min\left(d_0^*, d_1, d_2, \frac{a}{2\mathcal{L}}\right).$$

Take a μ-periodic d-pseudotrajectory $g(t)$ of X with $d \le d_0$. Without loss of generality we can assume that $\mu > \delta$ (since μ is not necessarily the minimal period). Lemma 4.22 implies that there exists a neighborhood V_j such that the pseudotrajectory $g(t)$ intersects V_j; shifting time, we may assume that $g(0) \in V_j$.

In this case, $\{g(t) : t \in \mathbb{R}\} \subset U_j$. Indeed, if $g(t_0) \notin U_j$ for some t_0, then $g(t_0 + k\mu) \notin U_j$ for all k. It follows from Lemma 4.23 that if $t_0 + k\mu > 0$, then $g(t) \notin V_j$ for $t \ge t_0 + k\mu$, and we get a contradiction with the periodicity of $g(t)$ and the inclusion $g(0) \in V_j$.

Thus, there exists a point p such that inequalities (4.40) hold for some reparametrization α satisfying inequality (4.25). Let us show that either p is a rest point or the trajectory of p is closed. By the choice of U_j and W_j, $\phi(t, p) \in W_j$ for all $t \in \mathbb{R}$. Let $q = \phi(\mu, p)$.

Inequalities (4.40) and the periodicity of $g(t)$ imply that

$$\operatorname{dist}(g(t), \phi(\alpha(t+\mu) - \mu, q)) =$$

$$\operatorname{dist}(g(t+\mu), \phi(\alpha(t+\mu), p)) \le \mathcal{L}d, \quad t \in \mathbb{R}.$$

Thus,

$$\operatorname{dist}(\phi(\alpha(t), p), \phi(\alpha(t+\mu) - \mu, q)) \le 2\mathcal{L}d \le a, \quad t \in \mathbb{R},$$

which implies that

$$\operatorname{dist}(\phi(\theta, p), \phi(\beta(\theta), q)) \le 2\mathcal{L}d \le a, \quad \theta \in \mathbb{R},$$

where $\beta(\theta) = \alpha(\alpha^{-1}(\theta) + \mu) - \mu$.

Since $\phi(t, p) \in W_j$ for all $t \in \mathbb{R}$, our expansivity condition on W_j implies that $q = \phi(\tau, p)$ for some $\tau \in (-\delta, \delta)$.

This completes the proof. $\square$

Chapter 5

Group Actions

In parallel with a classical theory of dynamical systems (which studies actions of $\mathbb{Z}$ and $\mathbb{R}$), global qualitative properties of actions of more complicated groups were studied (see the book [22] and the review [23]). The shadowing property for actions of abelian groups $\mathbb{Z}^n$ for nonnegative integer n and m was introduced by Pilyugin, Tikhomirov in 2003 [82].

Since that many mathematicians contributed to the shadowing theory of actions of abelian groups in various contexts. Kościelnak established the periodic shadowing property and strong tolerance stability for generic $\mathbb{Z}^2$-actions of an interval [44]. Oprocha studied topologically Anosov $\mathbb{Z}^d$-actions that are not topologically hyperbolic in any direction and proved an analog of the Spectral Decomposition Theorem [61, 62]. Maczyńska and Tabor studied the shadowing property for linear $\mathbb{Z}^d$-actions [51]. Kulczycki and Kwietnak studied relations between the shadowing property and distality for actions of $\mathbb{R}^n$ [46]. Begun and Pilyugin established analogs of Takens theorems for actions of $\mathbb{Z}^\infty$ [10].

In the present chapter we introduce and study the shadowing property for actions of finitely generated, not necessarily abelian groups.

For the case of finitely generated nilpotent groups we prove that an action of the whole group has the shadowing property (and expansivity) if the action of at least one element has the shadowing property and expansivity (Theorem 5.5). For linear actions of Abelian groups we also prove that this condition is also necessarily (Theorem 5.9). This result can be viewed as a shadowing lemma for actions of nilpotent groups, since it implies that if an action of one element is hyperbolic, then the group action has the shadowing property. Note that in some cases an action of a group is called hyperbolic if there exists an element which action is hyperbolic (see [7, 22, 39]).

We show that our result cannot be directly generalized to the case of solvable groups. We consider a particular linear action of a solvable Baumslag-Solitar group (Theorem 5.13) and demonstrate that the shadowing property has a more complicated nature, in particular, it depends on quantitative characteristics of hyperbolicity of the action.

We also consider actions of "big groups" (free groups, groups with infinitely many ends). In particular we show that there is no linear action of a non-abelian free group that has the shadowing property. This statement leads us to a question: which groups admit an action on a manifold satisfying the shadowing property?

These three results illustrate that the shadowing property depends not only on hyperbolic properties of actions of its elements but on the group structure as well.

5.1 Finitely generated groups.

In this section for completeness of exposition we will outline basic notions from theory of finitely generated groups, give relevant definitions, and formulate statements that we use in the sequel. We refer the interested reader to the following books on group theory: [9,14,19].

We say that a subset H of a group G is a *subgroup* and write $H \leq G$ if it contains e, the identity element of G, and together with the operation $\cdot$ of the group G satisfies the group axioms.

We say that a subgroup $H \leq G$ is a *normal subgroup* and write $H \lhd G$ ($G \rhd H$) if $gH = Hg$ for all $g \in G$ (where as usual $gH := \{gh \mid h \in H\}$, $Hg := \{hg \mid h \in H\}$).

Let H be a normal subgroup of G. A *factor group* or a *quotient group* is a group that as a set is the set of all *left cosets* of H in G, i.e. $\{gH \mid g \in G\}$, and has the group operation given by $(g_1H)(g_2H) = g_1g_2H$ for all $g_1, g_2 \in G$.

One of the possible ways of defining a group is by defining a generating set. A subset S of a group G is called a *generating set* if for any $g \in G$ there exists a finite number of elements $s_1, \dots, s_j \in S \cup S^{-1}$ (where as usual $S^{-1} := \{s^{-1} \mid s \in S\}$) such that $g = s_j \dots s_1$. Naturally, in most cases a generating set is not unique. A group G is called *finitely generated* if it has a finite generating set.

Another possible way of defining a group is by giving a system of generators and relations. Thus $\mathbb{Z}^2$ can be defined as: $< a, b \mid ab = ba >$ (first the generating set is indicated, then the system of relations on this generators).

Fix a group G. For any $g, h \in G$ the element $[g, h] := ghg^{-1}h^{-1}$ is called the commutator of g and h. Let G_1 and G_2 be two subgroups of G. We denote by $[G_1, G_2]$ the subgroup of G given by the generating set $\{[g_1, g_2] \mid g_1, g_2 \in G\}$.

A group G is called *abelian* or *commutative* if $[g, h] = e$ for any $g, h \in G$.

Definition 5.1. Any abelian group is called a *nilpotent group of class 1.* A group G is called *nilpotent of class* n if it has the *lower central series* of length n, i.e. there exist subgroups $G_1, \dots, G_{n+1} \leq G$ such that

$$G = G_1 \rhd \dots \rhd G_{n+1} = e, \qquad \text{where } G_n \neq e, \quad G_{i+1} = [G_i, G] \quad \forall i \in \{1, \dots, n\}.$$

G_i).

The simplest example of a nonabelian nilpotent group is a so-called *Heisenberg group* (see [19]): $< a, b, c \mid c = [a, b], ac = ca, bc = cb >$.

Definition 5.2. A group is called *virtually nilpotent* if it has a nilpotent normal subgroup of a finite index (i.e. the corresponding factor group is finite).

Remark 5.1. Note that any subgroup of a finitely generated virtually nilpotent group is finitely generated. In fact the similar statement holds for a more general class of polycyclic groups (see [14, 97] for the details).

Virtually nilpotent groups are important due to the celebrated theorem of Gromov: Any group of polynomial growth is virtually nilpotent. We refer the reader to [31] for the precise statement.

Definition 5.3. A group is called *solvable* or *soluble* if it has finite *subnormal series*, i.e. there exist $G_0, \dots, G_n \leq G$ (not necessarily finitely generated groups) such that $e = G_n \lhd \dots \lhd G_1 \lhd G_0 = G$ and G_i/G_{i+1} is an abelian group for any $i \in \{0, \dots, n-1\}$.

We study Baumslag-Solitar groups (see [19]):

$$BS(m,n) = < a, b \mid ba^m = a^n b >,$$

(where $m, n \in \mathbb{Z}$ are fixed) which are solvable for $m = 1$. These groups are well known in group theory as a source of numerous counterexamples.

We study actions of $F_n = < a_1, \dots, a_n \mid \cdot >$, the *free group* with n generators, which is obviously not solvable.

5.2 Shadowing for actions of finitely generated groups.

Let G be a finitely generated (not necessarily abelian) group. Let Ω be a metric space with a metric dist. For any $x \in \Omega$, $U \subset \Omega$, $\delta > 0$ denote

$$B(\delta, x) = \{y \in \Omega : \text{dist}(x, y) < \delta\}, \quad B(\delta, U) = \cup_{x \in U} B(\delta, x).$$

We say that a map $\Phi : G \times \Omega \to \Omega$ is a (left) action of a group G if the following holds:

(G1) the map $f_g = \Phi(g, \cdot)$ is a homeomorphism of Ω for any $g \in G$;

(G2) $\Phi(e, x) = x$ for any $x \in \Omega$, where $e \in G$ is the identity element of the group G;

(G3) $\Phi(g_1 g_2, x) = \Phi(g_1, \Phi(g_2, x))$ for any $g_1, g_2 \in G$, $x \in \Omega$.

We say that an action Φ is *uniformly continuous* if for some symmetric generating set S (a generating set is called symmetric if together with any element $s \in S$ it contains s^{-1}) of a group G the maps f_s are uniformly continuous for all $s \in S$. Note that if Ω is compact, then any action of a finitely generated group is uniformly continuous.

Let us fix some finite symmetric generating set S of a group G.

Definition 5.4. For any $d > 0$ we say that a sequence $\{y_g\}_{g \in G}$ is a d-pseudotrajectory of an action Φ (with respect to the generating set S) if

$$\text{dist}(y_{sg}, f_s(y_g)) < d \quad \forall s \in S, g \in G. \tag{5.1}$$

Definition 5.5. We say that an uniformly continuous action Φ has the *shadowing property* on a set $V \subset \Omega$ if for any $\varepsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory $\{y_g\}_{g \in G} \subset V$ there exists a point $x_e \in \Omega$ such that

$$\operatorname{dist}(y_g, f_g(x_e)) < \varepsilon \quad \forall g \in G. \tag{5.2}$$

In this case we say that $\{y_g\}_{g \in G}$ is ε-shadowed by the exact trajectory $\{x_g\}_{g \in G} = \{f_g(x_e)\}_{g \in G}$. If $V = \Omega$, we simply say that Φ has the shadowing property.

This notion is a natural generalization of the concept of the shadowing property introduced in [82] for actions of $\mathbb{Z}^n$.

Let us also note that the definition of a pseudotrajectory depends on a choice of the generating set S. However in Section 5.3 we show that if an uniformly continuous action has the shadowing property for one finite symmetric generating set, then it has the shadowing property for any finite symmetric generating set.

The proof of Proposition 5.4 is straightforward, see Appendix.

The following notion of expansivity is important for our results:

Definition 5.6. An action Φ is *expansive* (or has *expansivity*) on a set $U \subset \Omega$ if there exists $\Delta > 0$ such that if for some $x_1, x_2 \in U$

$$\Phi(g, x_1), \Phi(g, x_2) \in U, \quad \operatorname{dist}(\Phi(g, x_1), \Phi(g, x_2)) < \Delta \quad \forall g \in G,$$

then $x_1 = x_2$.

Remark 5.2. Note that if $G_1 \leq G$ is a subgroup of G and $\Phi|_{G_1}$ is expansive, then Φ is expansive too.

Any homeomorphism $f : \Omega \to \Omega$ induces an action $\Phi_f : \mathbb{Z} \times \Omega \to \Omega$ of the group $\mathbb{Z}$ defined as $\Phi_f(k, x) = f^k(x)$ for any $k \in \mathbb{Z}$, $x \in \Omega$. We say that

1. a homeomorphism f has the shadowing property on a set $V \subset \Omega$;
2. a homeomorphism f is expansive on a set $U \subset \Omega$;

if the corresponding action Φ_f has this properties. Note that these definitions are equivalent to classical definitions of these notions.

Definition 5.7. Consider two sets $U, V \subset \Omega$. We say that an uniformly continuous action Φ is topologically Anosov with respect to the pair (U, V) if the following conditions are satisfied:

(TA1) there exists $\gamma > 0$ such that $B(\gamma, V) \subset U$;

(TA2) Φ has the shadowing property on V;

(TA3) Φ is expansive on U.

Remark 5.3. This definition generalises the notion of topologically Anosov actions for homeomorphisms [6] and abelian groups [82]. For the case of homeomorphisms it was studied by many authors, see for example [2, 4, 6]. Let us mention remarkable results by Hiraide [36, 37], where it was proved that topologically Anosov homeomorphisms on surfaces are conjugated to linear Anosov automorphisms.

5.3 Correctness of the definition

In this section we prove correwctness of the definition of the shadowing property, more precisely we prove that this notion does not depend on the choice of generators.

Proposition 5.4. *Let S and S' be two finite symmetric generating sets for a group G. An uniformly continuous action Φ has the shadowing property on a set $V \subset \Omega$ with respect to the generating set S, if and only if it has the shadowing property on a set $V \subset \Omega$ with respect to the generating set S'.*

Proof of Proposition 5.4. The generating set S induces on G a so-called *word norm* defined as length of the shortest representation of an element in terms of elements from S. We define by $|g|_S$ (or simply by $|g|$) the word norm of an element g with respect to S.

It is well known that two word norms corresponding to two finite generating sets S and S' are bilipschitz equivalent, i.e. there exists a constant $C \geq 1$ such that

$$|g|_{S'}/C \leq |g|_S \leq C|g|_{S'} \quad \forall g \in G. \tag{5.3}$$

Fix an $\epsilon > 0$. Let d be the number from the definition of shadowing of Φ with respect to S corresponding to ϵ. By uniform continuity of Φ, there exists a constant $d_1 < d/C$ such that

$$\operatorname{dist}(f_g(\omega_1), f_g(\omega_2)) < d/C, \tag{5.4}$$

for any $g \in G$, $\omega_1, \omega_2 \in \Omega$ provided $|g|_{S'} \leq C$ and $\operatorname{dist}(\omega_1, \omega_2) < d_1$.

Let $\{y_g\}_{g\in G} \subset V$ be a d_1-pseudotrajectory of Φ with respect to the generating set S', i.e. by (5.1)

$$\operatorname{dist}(y_{s'g}, f_{s'}(y_g)) < d_1 \quad \forall s' \in S', g \in G$$

Consequently, by (5.4),

$$\begin{aligned}\operatorname{dist}(y_{s'_C\dots s'_1 g}, f_{s'_C\dots s'_1}(y_g)) \leq{}& \operatorname{dist}(y_{s'_C\dots s'_1 g}, f_{s'_C}(y_{s'_{C-1}\dots s'_1 g}))+\\ &+ \operatorname{dist}(f_{s'_C}(y_{s'_{C-1}\dots s'_1 g}), f_{s'_C}(f_{s'_{C-1}}(y_{s'_{C-2}\dots s'_1 g}))) + \ldots +\\ &+ \operatorname{dist}(f_{s'_C\dots s'_2}(y_{s'_1 g}), f_{s'_C\dots s'_2}(f_{s'_1}(y_g))) < d_1 + d/C + \ldots + d/C < d\end{aligned}$$

for any $s'_1, \ldots, s'_C \in S'$, $g \in G$. To summarize,

$$\operatorname{dist}(y_{hg}, f_h(y_g)) < d \tag{5.5}$$

for all $g \in G$ and $h \in G$ such that $|h|_{S'} \leq C$. It follows from (5.3) that any element $s \in S$ satisfies $|s|_{S'} \leq C$. Thus it follows from (5.5) that the sequence $\{y_g\}_{g\in G}$ is a d-pseudotrajectory of Φ with respect to the generating set S.

It follows from our assumptions that $\{y_g\}_{g\in G}$ is ϵ-shadowed by some point x_e. However inequalities (5.2) do not depend on the choice of the generating set. Thus Φ has the shadowing property with respect to S'. Clearly Φ is an uniformly continuous action with respect to S' too.

□

5.4 Actions of nilpotent groups.

The following theorem is the main result this chapter [64].

Theorem 5.5. *Let Φ be an uniformly continuous action of a finitely generated virtually nilpotent group G on a metric space Ω. Assume that there exists an element $g \in G$ such that f_g is topologically Anosov with respect to a pair (U, V). Then the action Φ is topologically Anosov with respect to the pair (U, V).*

The main step of the proof is the following lemma, which is interesting by itself:

Lemma 5.6. *Let G be a finitely generated group and H be a finitely generated normal subgroup of G. Let Φ be an uniformly continuous action on Ω. If $\Phi|_H$ is topologically Anosov with respect to a pair (U, V), then Φ is topologically Anosov with respect to the pair (U, V) too.*

Proof of Lemma 5.6. Fix a finite symmetric generating set S_H in H and continue it to a finite symmetric generating set S in G. By Proposition 5.4, we can assume that our initial generating set S was chosen in this way.

Let $\Delta, \gamma > 0$ be the constants from the definitions of a topologically Anosov action and expansivity for $\Phi|_H$. Since the maps $\{f_s\}_{s\in S}$ are uniformly continuous, there exists $\delta < \min(\Delta/3, \gamma)$ such that

$$\mathrm{dist}(f_s(\omega_1), f_s(\omega_2)) < \Delta/3 \tag{5.6}$$

for any $s \in S$ and any two points $\omega_1, \omega_2 \in \Omega$ satisfying $\mathrm{dist}(\omega_1, \omega_2) < \delta$.

Fix $\varepsilon \in (0, \delta)$ and choose $d < \varepsilon$ from the definition of shadowing for $\Phi|_H$ for the generating set S_H. Fix a d-pseudotrajectory $\{y_g\}_{g\in G} \subset V$ of Φ.

For any element $q \in G$ consider the sequence $\{z_h\}_{h\in H} = \{y_{hq}\}_{h\in H}$. Note that this sequence is a d-pseudotrajectory of $\Phi|_H$. Since $\Phi|_H$ is topologically Anosov with respect to (U, V), there exists a unique point $x_q \in U$ such that

$$\mathrm{dist}(z_h, \Phi(h, x_q)) = \mathrm{dist}(y_{hq}, f_h(x_q)) < \varepsilon \quad \forall h \in H. \tag{5.7}$$

Existence of such x_q follows from (TA2), uniqueness follows from (TA1), (TA3), and the inequality $\varepsilon < \gamma$.

Let us prove that $\{x_q\}_{q\in G}$ is an exact trajectory.

Fix $s \in S$ and $q \in G$. Consider an arbitrary element $h \in H$. Since H is a normal subgroup of G, there exists an element $h' \in H$ such that

$$sh' = hs. \tag{5.8}$$

It follows from (5.6)–(5.8) that

$$\mathrm{dist}(y_{sh'q}, f_h(x_{sq})) < \epsilon, \tag{5.9}$$

$$\mathrm{dist}(f_s(y_{h'q}), f_s(f_{h'}(x_q))) < \Delta/3. \tag{5.10}$$

Since $\{y_g\}_{g\in G}$ is a d-pseudotrajectory for Φ, it follows from (5.8)–(5.10) that

$$\begin{aligned}&\operatorname{dist}(f_h(x_{sq}), f_h(f_s(x_q))) \le \\ &\qquad \operatorname{dist}(f_h(x_{sq}), y_{sh'q}) + \operatorname{dist}(y_{sh'q}, f_s(y_{h'q})) + \operatorname{dist}(f_s(y_{h'q}), f_{hs}(x_q))) \le \\ &\qquad\qquad \epsilon + d + \Delta/3 < \Delta.\end{aligned}$$

Due to expansivity of $\Phi|_H$ on U, we conclude that

$$x_{sq} = f_s(x_q) \quad \forall s \in S, q \in G.$$

Since S is a generating set for G, these equalities imply that $x_q = f_q(x_e)$ for all $q \in G$, and hence by (5.7) x_e satisfies inequalities (5.2).

Expansivity of Φ is trivial, because of Remark 5.2. □

Next we prove Theorem 5.5 for the case of nilpotent groups.

Lemma 5.7. *Let G be a finitely generated nilpotent group of class n and Φ be an uniformly continuous action of G on a metric space Ω. Assume that there exists an element $g \in G$ such that f_g is topologically Anosov with respect to (U, V). Then the action Φ is topologically Anosov with respect to (U, V).*

Proof. Let us prove this lemma by induction on n.

For $n = 1$ the group G is abelian and hence the group $P = \langle g\rangle$ generated by g is a normal subgroup of G. Since f_g is topologically Anosov, applying Lemma 5.6 we conclude that Φ is topologically Anosov.

Let $n > 1$ and assume that we have proved the lemma for all nilpotent groups of class less or equal $n - 1$. Denote $Q = [G, G]$ and $P = \langle Q, g\rangle$ (i.e. P is the minimal subgroup of G that contains Q and g).

Proposition 5.8. *(N1) The group P is a normal subgroup of G.*

(N2) The group P is nilpotent of class at most $n - 1$.

Proof of Proposition 5.8. Let us start from Item (N1). Fix arbitrary $p \in P$, $h \in G$. Note that $hph^{-1} \in [G, G]p = Qp \subset P$, which proves the claim.

Let us prove Item (N2). It is clear that any subgroup of a nilpotent group of class n is a nilpotent group of class at most n. However we need a stronger result for the subgroup P. As the analysis of simple examples shows (e.g. the direct product of the Heisenberg group and $\mathbb{Z}$), a nilpotent group of class n may have proper subgroups of class n. So item (N2) is not trivial.

Denote

$$R = [Q, G] = [[G, G], G].$$

Clearly, in order to prove (N2) it is sufficient to prove that

$$[P, P] \subset [[G, G], G] = R$$

(since it implies $[[P,P],P] \subset [[[G,G],G],G]$ and etc.).

Since $Qg = gQ$, any element $p \in P$ has a representation as qg^k for some $q \in Q$, $k \in \mathbb{Z}$. Fix $p_1, p_2 \in P$ and put $p_1 = q_1 g^{k_1}, p_2 = q_2 g^{k_2}$. Note that

$$p_1 p_2 = q_1 g^{k_1} q_2 g^{k_2} = q_1 r_1 q_2 g^{k_1+k_2} = r_2 q_1 q_2 g^{k_1+k_2},$$

$$p_2 p_1 = q_2 g^{k_2} q_1 g^{k_1} = q_2 r_3 q_1 g^{k_1+k_2} = r_4 q_2 q_1 g^{k_1+k_2} = r_5 q_1 q_2 g^{k_1+k_2}$$

for some $r_1, \dots, r_5 \in R$, and hence $[p_1, p_2] = r_2 r_5^{-1} \in R$.

□

Let us note that these properties strongly use nilpotency of Q, and their analogs do not hold, for example, for solvable groups.

Let us continue the proof of Lemma 5.7. Since P is a finitely generated (due to Remark 5.1) nilpotent group of class at most $n-1$, $g \in P$, and f_g is topologically Anosov, by the induction assumptions we conclude that $\Phi|_P$ is topologically Anosov. Combining this property, (N1) and Lemma 5.6 we conclude that Φ has the shadowing property. □

Proof of Theorem 5.5. Since G is virtually nilpotent, there exists a nilpotent normal subgroup H of G of finite index. Due to Remark 5.1 the group H is finitely generated. Consider $g \in G$ from the assumptions of the theorem. Since H is a subgroup of finite index, there exists $k > 0$ such that $g^k \in H$. Since f_g is topologically Anosov, the map $f_g^k = f_{g^k}$ is also topologically Anosov. Hence, by Lemma 5.7, the action $\Phi|_H$ is topologically Anosov. Applying Lemma 5.6 we conclude that Φ is topologically Anosov too. □

5.5 Linear actions of Abelian groups

Consider a linear action of $\mathbb{Z}^p$ on $\mathbb{C}^m$. In this case we fix p non-singular $m \times m$ matrixes $A_1, \dots, A_p$. Assuming that they pairwise commute, we get the action

$$\Phi : \mathbb{Z}^p \times \mathbb{C}^m \to \mathbb{C}^m \tag{5.11}$$

defined by

$$\Phi(n, x) = A_1^{n_1} \dots A_p^{n_p} x \tag{5.12}$$

for $n = (n_1, \dots, n_p) \in \mathbb{Z}^p$ and $x \in \mathbb{C}^m$.

It is known [57] that for any family of pairwise commuting matrixes A_i there exists a unitary matrix U such that each matrix $T_i = U^{-1} A_i U$ is upper triangular. Obviously, the change of variables $x = Uy$ preserves any shadowing and expansivity properties. Hence, we may assume taht the matrixes A_i are upper triangular.

Denote by λ_{ij} the j-th diagonal element of matrix A_i.

Theorem 5.9. *[82] Under the above conditions, the following statements are equivalent:*

(1) action (5.11) *has the shadowing property;*

(2) for any $j \in \{1, \dots, m\}$ there exists $i \in \{1, \dots, p\}$ such that $|\lambda_{ij}| \neq 1$;

(3) there is no vector $v \neq 0$ such that

$$A_i v = \mu_i v, \quad i = 1, \dots, p, \quad \textit{where} \quad |\mu_i| = 1. \tag{5.13}$$

Proof. Denote by (n1), (n2), and (n3) the negotiations of (1), (2), and (3) respectevily. We prove the implications

$$(\mathrm{n1}) \Rightarrow (\mathrm{n2}) \Rightarrow (\mathrm{n3}) \Rightarrow (\mathrm{n1})$$

First we prove (n1) $\Rightarrow$ (n2).

Note that if a matrix A is hyperbolic then it satisfies shadowing and expansivity properties. Thus it follows from Theorem 5.5 that to establish (1) it is enough to show that there exists $n = (n_1, \dots, n_p) \in \mathbb{Z}^p$ such that the matrix

$$A = A_1^{n_1} \dots A_p^{n_p} \tag{5.14}$$

is hyperbolic.

For contradiction, assume that condition (2) is satisfied while any matrix (5.14) has an eigenvalue λ with $|\lambda| = 1$. Since the matrixes A_i are upper triangular the set of eigenvalues of (5.14) is

$$\{\lambda_{1j}^{n_1} \dots \lambda_{pj}^{n_p}, \quad j = 1, \dots, m\}. \tag{5.15}$$

By our assumption, for any $n = (n_1, \dots, n_p)$ there is j such that

$$|\lambda_{1j}^{n_1} \dots \lambda_{pj}^{n_p}| = 1. \tag{5.16}$$

To proceed, we need the following auxiliarly statement.

Lemma 5.10. *Assume that numbers μ_{ij}, where $i = 1, \dots, p$ and $j = 1, \dots, m$, satisfy the following condition: for any $n_1, \dots, n_p \in \mathbb{Z}$ there exists j such that*

$$n_1\mu_{1j} + \dots + n_p\mu_{pj} = 0. \tag{5.17}$$

Then there exists j such that

$$\mu_{ij} = 0, \quad i = 1, \dots, p. \tag{5.18}$$

Proof. We prove the statement by induction on p. For $p = 1$ the statement is obvious.

Assume that the statement is proved for $p - 1$. Fix $n_2, \dots, n_p \in \mathbb{Z}$ and consider $u_j := n_2\mu_{2j} + \dots + n_p\mu_{pj}$. Assumptions of the lemma imply that for any $n_1 \in \mathbb{Z}$ there exists j such that

$$n_1\mu_{1j} + u_j = 0. \tag{5.19}$$

Hence the euqality (5.19) holds for infinitely many n_1 with the same j. Note that for this j the equalities $u_j = 0$ and $\mu_{1j} = 0$ hold.

Denote by $J \subset \{1, \dots, m\}$ the set of such j that $\mu_{1j} = 0$. We proved that that $J \neq \emptyset$ and for any $n_2, \dots, n_p \in \mathbb{Z}$ there exists $j \in J$ satisfying the euqlity

$$n_2\mu_{2j} + \dots + n_p\mu_{pj} = 0.$$

By the induction assumtion there exisst $j' \in J$ such that $\mu_{2j'} = \mu_{3j'} = \dots = \mu_{pj'} = 0$. Hence j' satisfies equalities (5.18). Lemma 5.10 is proved. $\square$

Setting $\mu_{ij} := \log |\lambda_{ij}|$ we reduce conditions (5.16) to (5.17). By Lemma 5.10 there exists j such that $|\lambda ij| = 1$ for $i = 1, \dots, p$. This proves implication (n1) $\Rightarrow$ (n2).

Before proving (n2) $\Rightarrow$ (n3), we establish an auxilarly statement.

Lemma 5.11. *Let $A'_1, \dots, A'_p$ be pairwise commuting linear operators on $\mathbb{C}^m$ such that*

$$\ker(a_1 A'_1 + \cdots + a_p A'_p) \neq \{0\} \tag{5.20}$$

for any real numbers $a_1, \dots, a_p$. Then

$$\cap_{i=1}^{p} \ker A'_i \neq \{0\}. \tag{5.21}$$

Proof. In the proof, we often use the following simple statement. Let A and B be commuting linear operators and let $\ker B = Y$. Then $A(Y) \subset Y$. Indeed if $y \in Y$ then $By = 0$, so $ABy = 0$ and $B(Ay) = 0$. Thus $Ay \in Y$.

We prove the lemma by induction on p. The case $p = 1$ is trivial. Let $p = 2$. Define

$$C_k = A'_1 + kA'_2 \quad \text{and} \quad X_k = \ker C_k$$

for nonnegative integer k. Obviously C_i and C_j commute for any i and j. Our statement above implies that

$$C_i(X_j) \subset X_j. \tag{5.22}$$

Let $\mathrm{Lin}(Y_1, \dots, Y_r)$ be the linear hull of the linear subspaces $Y_1, \dots, Y_r$.

We claim that there exist n such that

$$X_{n+1} \cap \mathrm{Lin}(X_0, \dots, X_n) \neq \{0\}. \tag{5.23}$$

Indeed if $X_{n+1} \cap \mathrm{Lin}(X_0, \dots, X_n) = \{0\}$ for any n, then

$$\dim \mathrm{Lin}(X_0, \dots, X_n) \geq n + 1,$$

which is impossible if $n \geq m$.

Let n be minimal satisfying (5.23). Consider $x \neq 0$ such that

$$x \in X_{n+1} \cap \mathrm{Lin}(X_0, \dots, X_n). \tag{5.24}$$

represent x in the form $x_0 + \cdots + x_n$, where $x_i \in X_i$. Note that

$$C_{n+1}x = C_{n+1}x_0 + \cdots + C_{n+1}x_n = 0. \tag{5.25}$$

It follows from (5.22) $y_i := C_{n+1}x_i \in X_i$. Relation (5.25) implies that $y_0 + \cdots + y_n = 0$. If we assume that $y_i \neq 0$ for some i, and consider the maximal i with this property, then

$$y_i = -(y_0 + \cdots + y_{i-1}) \in \mathrm{Lin}(X_0, \dots X_{i-1}),$$

contradicting the choice of n and the inequality $i - 1 < n$. Thus,

$$y_i = 0, \quad \text{for } 0 \leq i \leq n. \tag{5.26}$$

Since $x \neq 0$, there exists $k \in \{1, \ldots, n\}$ such that $x_k \neq 0$. The equalities

$$y_k = C_{n+1}x_k = 0 \quad \text{and} \quad C_k x_k = 0$$

imply that $x_k \in \ker A'_1 \cap \ker A'_2$. Thus our lemma is proved for $p = 2$.

Now assume that our statement holds for p. Define $B_{i,k} = A'_1 + kA'_i$ and $X_{i,k} = \ker B_{i,k}$. Since operators A'_i pairwise commute, so do $B_{i,k}$ and A'_i has a nonempty kernel.

By the induction assumption applied to $A'_{p+1}, B_{2,k}, \ldots, B_{p,k}$ with any k,

$$Y_k := \ker A'_{p+1} \cap X_{2,k} \cap \cdots \cap X_{p,k} \neq \{0\}.$$

The same reasoning as above shows that there exists n such that

$$Y_{n+1} \cap \mathrm{Lin}(Y_0, \ldots, Y_n) \neq \{0\}.$$

Consider minimal n with this property. Take $y_i \in Y_i$ such that

$$y_{n+1} = y_0 + \cdots + y_n \tag{5.27}$$

and $y_{n+1} \neq 0$. Applying the operator $B_{2,n+1}$ to (5.27) and taking into account that $y_{n+1} \in Y_{n+1} \subset X_{2,n+1}$, we see that

$$0 = B_{2,n+1}y_0 + \cdots + B_{2,n+1}y_n \tag{5.28}$$

Since $B_{2,n+1}y_l \in Y_l$, the reasoning applied to establish (5.26) shows that $B_{2,n+1}y_l = 0$ for any $l \in \{0, \ldots, n\}$.

Consider $y_l \neq 0$. We claim that

$$y_l \in \ker A'_1 \cap A'_2 \cap \cdots \cap \ker A'_{p+1}. \tag{5.29}$$

Since $y_l \in Y_l$, we have $y_l \in \ker A'_{p+1}$. The relations $B_{2,n+1}y_l = 0$ and $y_l \in Y_i \subset X_{2,l}$ imply that

$$(A'_1 + (n+1)A'_2)y_l = 0 \quad \text{and} \quad (A'_1 + lA'_2)y_l = 0.$$

It follows that $A'_1 y_l = 0$ and $A'_2 y_l = 0$. Since $y_l \in X_{i,l} = \ker B_{i,l}$ for any $l \in \{3, \ldots, p\}$ we see in addition, that $(A'_1 + lA'_i)y_l = 0$. Thus, $A'_i y_l = 0$ for these l, relation (5.29) holds, and lemma is proved. $\square$

This lemma implies an important property of pairwise commuting matrixes $A_1, \ldots, A_n$ generating action (5.11). Obviously the desired implication (n2)$\Rightarrow$(n3) follows from this property.

Corollary 5.12. *For any $j \in \{1, \ldots, m\}$ there exists a vector $v \neq 0$ such that*

$$A_i v = \lambda_{ij} v, \quad i = 1, \ldots, p. \tag{5.30}$$

Proof. Fix j and consider matrixes $A'_i = A_i - \lambda_{ij} E_m$, where E_m is the identity $m \times m$ matrix. This matrixes are triangular and pairwise commute. Their jth diagonal element are zero.

Hence condition (5.20) of Lemma 5.11 is satisfies. By Lemma 5.11, there exists a vector $v \neq 0$ such that $A'_i v = 0$ for $i = 1, \ldots, p$. Obviously, v satisfies (5.30). □

Now let us prove the implication (n3)⇒(n1). Fix a vector v with $|v| = 1$ satisfying (5.13). We claim that action (5.11) does not have shadowing property. Let us construct a pseudotrajectory as follows. Fix a positive number d and a sequence $\{c_l : l \in \mathbb{Z}\}$ of integers with the following properties: $|c_{l+1} - c_l| = 1$ for any l, the sequence $|c_l|$ is unbounded, and the limits $\lim_{|l| \to \infty} c_l$ do not exists. Define $\Lambda = \mu_1$ and set $a_l = d c_l \Lambda^l v$ and $x_n = A_2^{n_2} \ldots A_p^{n_p}$ for $n = (n_1, \ldots, n_p)$. Obviously $\{x_n\}$ is a $2d$-pseudotrajectory of (5.11).

To complete the proof we claim that $\sup_n |\Phi(n, y) - x_n| = \infty$ for any $y \in \mathbb{C}^m$. To see this, it is enough to show that

$$\sup_{(n_1, 0, \ldots, 0)} |\Phi((n_1, 0, \ldots, 0), y) - x_{(n_1, 0, \ldots, 0)}| = \infty \tag{5.31}$$

for any $y \in \mathbb{C}^m$. Fix a basis $e_1, \ldots, e_m$ in $\mathbb{C}^m$ as follows:

$$e_1 = v, \quad A_1 e_i = \Lambda e_i + e_{i-1}, \quad \text{for} 2 \leq i \leq k,$$

and

$$A_1 = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

in this basis, where B and C are $k \times k$ and $(m-k) \times (m-k)$ matrixes, respectively.

If $a^{(1)}$ is the first corrdinate of a vector $a \in \mathbb{C}^m$ in the chosen basis, then

$$x^{(1)}_{(l, 0, \ldots, 0)} = d c_l \Lambda^l.$$

For $y = (y_1, \ldots, y_m) \in \mathbb{C}^m$, write $y' = (y_1, \ldots, y_k)$.

The matrix B has the form $\Lambda E_k + J$, where $J^i = 0$ for $i \geq k$. Hence for any $y \in \mathbb{C}^m$,

$$(A_1^l y)^{(1)} = (B^l y')^{(1)} = \left(\sum_{i=0}^{k-1} \frac{l!}{(l-i)! i!} \Lambda^{l-i} J^i y' \right)^{(1)} = \Lambda^l P(l),$$

where $P(l)$ is a polynomial in l of degree not exceeding $k-1$ (determined by the fixed vector y).

If (5.31) does not hold for some $y \in \mathbb{C}^m$, then the expression

$$|(A_1^l y)^{(1)} = x^{(1)}_{(l, 0, \ldots, 0)}| = |d c_l - P(l)|$$

is bounded in l. This contradicts the choice of the sequence c_l since either $P(l)$ is constant (while c_l is unbounded) or $|P(l)| \to \infty$ as $|l| \to \infty$ (while c_l does not have limits as $|l| \to \infty$).

The proof is complete. □

5.6 An action of a Baumslag-Solitar group.

It turns out that Theorem 5.5 cannot be generalized to the case of solvable groups. Consider a solvable group $G = BS(1,n) = < a, b|ba = a^n b >$, where $n > 1$. For any $\lambda > 0$ consider the action $\Phi : G \times \mathbb{R}^2 \to \mathbb{R}^2$ generated by the maps

$$f_a(x) = Ax, \quad f_b(x) = Bx,$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \lambda & 0 \\ 0 & n\lambda \end{pmatrix}.$$

Note that $BA = A^n B$, and hence the action Φ is well defined.

For any $\lambda > 1$ the map f_b is hyperbolic, however the following holds: show that for any linear one-dimensional action of group $BS(1,n)$ holds relation $f_a^{n-1} = \mathrm{Id}$.

Theorem 5.13. *[64]*

(i) For $\lambda \in (1, n]$ the action Φ does not have the shadowing property.

(ii) For $\lambda > n$ the action Φ has the shadowing property.

Proof. Without loss of generality, by Proposition 5.4, we consider the group $BS(1,n) = < a, b \mid ba = a^n b >$ with the standard generating set $S = \{a, b, a^{-1}, b^{-1}\}$. Denote by P_1 and P_2 the natural projections on the coordinate axes in $\mathbb{R}^2$. As before denote

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \lambda & 0 \\ 0 & n\lambda \end{pmatrix}.$$

Note that

$$A^r = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \quad B^r = \begin{pmatrix} \lambda^r & 0 \\ 0 & (n\lambda)^r \end{pmatrix} \qquad \forall r \in \mathbb{Z}. \tag{5.32}$$

Proof of Item (i). To derive a contradiction assume that Φ has the shadowing property and choose $d > 0$ from the definition of the shadowing property applied to $\epsilon = 1$.

Consider an auxiliary action $\Psi : G \times (\mathbb{R} \times \mathbb{Z}) \to (\mathbb{R} \times \mathbb{Z})$ generated by the maps

$$g_a(x,k) = (x + n^{-k}, k), \quad g_b(x,k) = (x, k+1).$$

It is easy to check that $g_b \circ g_a = g_a^n \circ g_b$, and hence the action Ψ is well defined.

Consider the map $F : (\mathbb{R} \times \mathbb{Z}) \to \mathbb{R}$ defined as follows

$$F(x,k) = \left((1+\beta)\lambda^k |x|^\beta ; (n\lambda)^k |x|^{1+\beta}\right),$$

where $\beta = \frac{\ln \lambda}{\ln n} \in (0, 1]$.

Finally, consider the sequence

$$y_g = \frac{d}{3} \cdot F(\Psi(g, (0,0))) \quad \forall g \in G.$$

We claim that $\{y_g\}_{g\in G}$ is a d-pseudotrajectory for the action Φ, i.e. inequalities (5.1) hold for all $s \in \{a, b, a^{-1}, b^{-1}\}$.

Indeed, fix $g \in G$. Denote $(x, k) = \Psi(g, (0, 0))$.

If $s = b^{\pm 1}$, then it is easy to see that $y_{sg} = f_s(y_g)$.

If $s = a$, then

$$P_1 y_{sg} = \frac{d}{3}(1+\beta)\lambda^k \left|x + n^{-k}\right|^{\beta}, \quad P_2 y_{sg} = \frac{d}{3}(n\lambda)^k \left|x + n^{-k}\right|^{1+\beta}.$$

Denote $\Delta = n^{-k}$. Then $\lambda^k = \Delta^{-\beta}$. In such notation

$$P_1\left(y_{ag} - f_a(y_g)\right) = \frac{d}{3}(1+\beta)\Delta^{-\beta}\left(|x+\Delta|^{\beta} - |x|^{\beta}\right),$$

$$P_2\left(y_{ag} - f_a(y_g)\right) = \frac{d}{3}\left(\Delta^{-(1+\beta)}|x+\Delta|^{1+\beta} - \left(\Delta^{-(1+\beta)}|x|^{1+\beta} + (1+\beta)\Delta^{-\beta}|x|^{\beta}\right)\right).$$

We use the following inequalities, which hold for all $\beta \in (0, 1]$ and $x, \Delta \in \mathbb{R}$:

$$|x+\Delta|^{\beta} \le |x|^{\beta} + |\Delta|^{\beta},$$

$$|x+\Delta|^{1+\beta} \le |x|^{1+\beta} + (1+\beta)|\Delta||x|^{\beta} + |\Delta|^{1+\beta}.$$

From these inequalities it is easy to conclude that

$$|P_1(y_{ag} - f_a(y_g))| \le (\beta+1)d/3, \quad |P_2(y_{ag} - f_a(y_g))| \le d/3,$$

which implies

$$|y_{ag} - f_a(y_g)| < d.$$

Similarly

$$|y_{a^{-1}g} - f_{a^{-1}}(y_g)| < d.$$

And hence $\{y_g\}_{g\in G}$ is a d-pseudotrajectory.

Since by our assumptions the action Φ has the shadowing property, there exists $x_e \in \mathbb{R}$ such that (5.2) holds.

Note that $y_{b^k} = 0$ for any $k \ge 0$. Substituting $g = b^k$ into (5.2), we conclude that $|B^k x_e| \le 1$ and hence, by expansivity of f_b, $x_e = (0, 0)$.

Now substituting $g = b^k a$ into (5.2) and looking on the first coordinate we conclude that:

$$|\lambda^k d/2 - 0| < 1 \quad \forall k \ge 0,$$

which is impossible for sufficiently large k. The derived contradiction finishes the proof of Item (i).

Proof of Item (ii). Fix $\varepsilon > 0$. Note that the map f_b has the shadowing property and is expansive. Let us choose $d \in (0, \varepsilon)$ such that any d-pseudotrajectory of f_b can be ε-shadowed by an exact trajectory of f_b. Consider an arbitrary d-pseudotrajectory $\{y_g\}_{g\in G}$ of the action Φ.

For any element $q \in G$ consider the sequence $\{z_k\}_{k\in\mathbb{Z}}$, defined by $z_k = y_{b^k q}$. Note that this sequence is a d-pseudotrajectory for f_b. Since f_b has the shadowing property and is expansive, there exists a unique point $x_q \in \mathbb{R}$ such that

$$|z_k - f_b^k(x_q)| = |y_{b^k q} - f_b^k(x_q)| < \varepsilon \quad \forall k \in \mathbb{Z}. \tag{5.33}$$

We claim that $x_q = \Phi(q, x_e)$. To prove this, it is enough to show that

$$x_{bq} = Bq, \quad x_{aq} = Ax_q \qquad \forall q \in G. \tag{5.34}$$

The first equality follows directly from expansivity of f_b. Let us prove the second one.

Note that the relation $ba = a^n b$ implies that

$$b^k a = a^{(n^k)} b^k \quad \forall k > 0. \tag{5.35}$$

Fix an arbitrary $q \in G$. Note that since $\{y_t\}_{t\in G}$ is a d-pseudotrajectory,

$$|P_1 y_{a^{n^k} b^k q} - P_1 y_{b^k q}| < dn^k. \tag{5.36}$$

By a straightforward induction it is easy to show that for all $j \in [1, n^k]$ the inequality

$$|P_2 y_{a^{n^k} b^k q} - P_2 y_{(a^{n^k - j}) b^k q} - jP_1 y_{(a^{n^k - j}) b^k q}| < \frac{j(j+1)}{2} d$$

holds. In particular

$$|P_2 y_{a^{n^k} b^k q} - P_2 y_{b^k q} - n^k P_1 y_{b^k q}| < \frac{n^k(n^k+1)}{2} d. \tag{5.37}$$

Relations (5.33), (5.35), and the definition of a pseudotrajectory imply that for any $k > 0$ the following relations hold:

$$\left|B^k x_{aq} - y_{b^k aq}\right| < \varepsilon,$$

$$y_{b^k aq} = y_{a^{(n^k)} b^k q},$$

$$\left|y_{b^k q} - B^k x_q\right| < \varepsilon,$$

and hence by (5.32), (5.36) and (5.37)

$$|\lambda^k (P_1 x_{aq} - P_1 x_q)| < 2\varepsilon + dn^k \quad \forall k > 0,$$

$$|(n\lambda)^k P_2 x_{aq} - (n\lambda)^k P_2 x_q - (n\lambda)^k P_1 x_q| < 2\varepsilon + \frac{n^k(n^k+1)}{2} d \quad \forall k > 0.$$

Since $\lambda > n$,

$$P_1 x_{aq} = P_1 x_q, \quad P_2 x_{aq} = P_2 x_q + P_1 x_q,$$

which implies (5.34) and finishes the proof of Item (ii).

$\square$

5.7 Actions of free groups.

For actions of free groups we prove the following theorem [64]:

Theorem 5.14. *Any linear action of a finitely generated free group with at least two generators on an Euclidean space does not have the shadowing property.*

This theorem leads us to the following conjecture and question:

Conjecture 5.1. *Any uniformly continuous action of the finitely generated free group with at least two generators on a manifold does not have the shadowing property.*

Question 5.1. *Which groups admit an action on a manifold satisfying the shadowing property?*

However the following obvious remark holds:

Remark 5.15. Let X be a discrete two-point space. Trivially the identity action of any finitely generated group on X has the shadowing property. A similar statement is true when X is a Cantor set.

We derive Theorem 5.14 from the following more general, but more technical statement:

Theorem 5.16. *Let G be a finitely generated free group with at least two generators. Let Φ be an uniformly continuous action of G on a non-discrete metric space Ω.*

1. *If for some $g \in G$ the map f_g is expansive, then Φ does not have the shadowing property.*

2. *If for some $g \in G$, $g \neq e$, the map f_g does not have the shadowing property, then Φ does not have the shadowing property too.*

Remark 5.17. Item 1 of Theorem 5.16 holds for a more general class of groups with infinitely many ends (we refer the reader to [14] for the precise definition).

Proof of Theorem 5.14. Since for linear actions of $\mathbb{Z}$ both the shadowing property and expansivity are equivalent to hyperbolicity, Theorem 5.14 follows from Theorem 5.16. $\square$

Without loss of generality, by Proposition 1, we consider a free group $G == < a_1, \dots, a_n | \cdot >$ with the standard generating set $S = \{a_1^{\pm 1}, \dots, a_n^{\pm 1}\}$. It means that any element $g \in G$ has a normal form $g = s_r \dots s_1$ (where $s_j \in S$), i.e. the unique shortest representation in terms of elements of S.

Proof of Theorem 5.16. **Proof of Item 1**. To derive a contradiction, suppose that Φ has the shadowing property. Let d be the number that corresponds to $\epsilon = \Delta$ (the constant of expansivity of f_g) in the definition of shadowing for Φ.

Consider the normal form of g: $g = s_r \dots s_1$. Fix any $q \in S \backslash \{s_1, s_1^{-1}\}$. Since f_q^{-1} is uniformly continuous, there exists a number $d_1 < d$ such that

$$\mathrm{dist}(f_q^{-1}(w_1), f_q^{-1}(w_2)) < d, \tag{5.38}$$

for any $w_1, w_2 \in \Omega$ satisfying $\operatorname{dist}(w_1, w_2) < d_1$.

Since Ω is non-discrete, we can fix two distinct points $\omega_0, \omega \in \Omega$ such that $\operatorname{dist}(\omega_0, \omega) < d_1$. We construct a pseudotrajectory $\{y_t\}_{t\in G}$ in the following way:

$$y_t = \begin{cases} \Phi(t, f_q^{-1}(\omega)), & \text{if the normal form of } t \in G \text{ starts with } q, \\ \Phi(t, f_q^{-1}(\omega_0)), & \text{otherwise.} \end{cases}$$

Note that, by (5.38),

$$\operatorname{dist}(y_q, f_q(y_e)) = \operatorname{dist}(\omega, \omega_0) < d_1 < d,$$
$$\operatorname{dist}(y_e, f_q^{-1}(y_q)) = \operatorname{dist}(f_q^{-1}(\omega_0), f_q^{-1}(\omega)) < d,$$

and the equality $y_{st} = f_s(y_t)$ holds for all other $s \in S$, $t \in G$. Hence $\{y_t\}_{t\in G}$ is a d-pseudotrajectory.

Our assumptions imply the existence of a point x_e such that inequalities (5.2) hold. Consequently,

$$\operatorname{dist}(y_{g^k}, \Phi(g^k, x_e)) = \operatorname{dist}(f_g^k(f_q^{-1}(\omega_0)), f_g^k(x_e)) < \Delta, \quad \forall k \in \mathbb{Z},$$

which, by expansivity, implies that

$$x_e = f_q^{-1}(\omega_0). \tag{5.39}$$

Since the normal form of $\{g^k q\}_{k\in\mathbb{Z}}$ starts from q,

$$\operatorname{dist}(y_{g^k q}, \Phi(g^k q, x_e)) = \operatorname{dist}(f_g^k(\omega), f_g^k(f_q(x_e))) < \Delta, \quad \forall k \in \mathbb{Z}.$$

Hence, by expansivity, $\omega = f_q(x_e)$, which together with (5.39) contradicts to the choice of ω and ω_0. Thus Φ does not have shadowing, which proves Item 1.

Proof of Item 2. Let ϵ be any number such that for any $d < \epsilon$ the map f_g has a d-pseudotrajectory that cannot be ϵ-shadowed by any exact trajectory of f_g. Consider the normal form of $g = s_r \dots s_1$. Fix any $d < \epsilon$. There exists a number $d_1 < d$ such that for any ϕ that has a form $\phi = f_{s_j} \dots f_{s_1}$ or $\phi = f_{s_j^{-1}} \dots f_{s_r^{-1}}$ for some $1 \le j \le r$ we have

$$\operatorname{dist}(\phi(w_1), \phi(w_2)) \le d, \tag{5.40}$$

for all $w_1, w_2 \in \Omega$ such that $\operatorname{dist}(w_1, w_2) \le d_1$.

Consider a d_1-pseudotrajectory $\{x_k\}_{k\in\mathbb{Z}}$ for f_g that cannot be ϵ-shadowed and the sequences $\{z_k\}_{k\in\mathbb{Z}}$, $\{y_t\}_{t\in G}$ defined as follows

$$\begin{cases} z_{rk} = x_k, & k \in \mathbb{Z}, \\ z_{rk+j+1} = f_{s_{j+1}}(z_{rk+j}), & 0 \le j < r-1, \quad k \in \mathbb{Z}; \end{cases}$$

and

$$y_t = \begin{cases} \Phi(v, z_{rk+j}), & \text{for } w = tv^{-1} = s_j \dots s_1 (s_r \dots s_1)^k,\ k \ge 0,\ 1 \le j \le r; \\ \Phi(v, z_{-rk-j}), & \text{for } w = tv^{-1} = s_{r-j+1}^{-1} \dots s_r^{-1} (s_r \dots s_1)^{-k},\ k \ge 0,\ 1 \le j \le r; \end{cases}$$

where v is the element of minimal length such that $t = vw$ for some $w = tv^{-1}$ of the form defined above.

By (5.40) the sequence $\{y_t\}_{t\in G}$ is a d-pseudotrajectory. If it is ε-shadowed by the trajectory of a point u_e, then $\{x_k\}_{k\in\mathbb{Z}}$ is ε-shadowed by $\{f_g^k(u_e)\}_{k\in\mathbb{Z}}$, which leads to a contradiction. $\square$

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Summary of the Habilitation Thesis “Quantitative properties of infinite and finite pseudotrajectories”

This Thesis is devoted to the study relation between shadowing properties of dynamical systems generated by diffeomorphisms, vector field and actions of more complicated groups with such forms of hyperbolicity as structural stability, Ω-stability and partial hyperbolicity.

Let M be a smooth compact manifold of class C^∞ with the Riemannian metric dist and $f : M \to M$ be a diffeomorphism on M.

Definition 1. For an interval $I = (a, b)$ with $a = \mathbb{Z} \cup \{-\infty\}$, $b = \mathbb{Z} \cup \{+\infty\}$ and $d > 0$ we say that a sequence of points $\{y_k\}_{k\in\mathbb{Z}}$ is a *d-pseudotrajectory*, if the following holds:

$$\operatorname{dist}(y_{k+1}, f(y_k)) < d, \quad k \in \mathbb{Z},\ k, k+1 \in I.$$

Usually we will consider pseudotrajectories defined on $\mathbb{Z}$.

Initially pseudotrajectories were introduced in the theory of chain-recurrent sets and in structural stability theory. Pseudotrajectories also naturally appear in numerical simulations of dynamical systems.

The shadowing problem in the most general setting is related to the following question: under which conditions for any pseudotrajectory of a dynamical system there exists a close exact trajectory? The problem of shadowing was initiated in works of Anosov [1] and Bowen [2]. Current state of the shadowing theory is reflected in monographs [14, 18] and review [20].

Definition 2. We say that f has the *standard shadowing property* (StSh) if for any $\varepsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory $\{y_k\}_{k\in\mathbb{Z}}$ there exists an exact trajectory $\{x_k\}_{k\in\mathbb{Z}}$, satisfying

$$\operatorname{dist}(x_k, y_k) < \varepsilon, \quad k \in \mathbb{Z}. \tag{1}$$

In that case we say that pseudotrajectory $\{y_k\}$ is ε-shadowed by an exact trajectory $\{x_k\}$.

Shadowing property plays important role in the smooth dynamical system theory. Indeed, if diffeomorphisms f_1, f_2 are close then exact trajectories of f_2 are pseudotrajectories for f_1, hence the shadowing property is a weak form of stability. From the point of view numerical simulations if a diffeomorphism f (or vector field X) has the shadowing property, then approximate trajectories, attained as a result of numerical simulation of a corresponding dynamical system, reflects the behaviour of the system on infinite time interval.

In this Thesis we study quantitative characteristics of the shadowing property: dependence between ε and d, and shadowing of pseudotrajectories of finite length.

Shadowing theory is strongly related to the notions of hyperbolicity and structural stability. Let us introduce the following notions.

Denote by T_xM the tangent bundle of M at point $x \in M$. Let $|v|$ be the norm of a vector $v \in T_xM$, corresponding to the metric dist. Denote by $B(r, x)$ an open ball in M of radius r centered at $x \in M$ and by $B_T(r,x)$ an open ball in T_xM of radius r centered in the origin. For a subset A of a metric space, denote by $B(r,A)$ the union of all balls of radius r centered in the points of A. Denote by $\mathrm{Cl}\,A$ the closure of a set A.

Denote by $\mathrm{Diff}^1(M)$ the space of diffeomorphisms on M, endowed with the C^1-topology. For a set $P \subset \mathrm{Diff}^1(M)$ denote by $\mathrm{Int}^1(P)$ its C^1-interior.

Definition 3. We say that a compact invariant set $\Lambda \subset M$ is *hyperbolic* if there exist $C > 0$, $\lambda \in (0,1)$ and a decomposition of a tangent bundle $T_xM = E^s_x \oplus E^u_x$ for $x \in \Lambda$ such that

1. $Df(x)E^{s,u}_x = E^{s,u}_{f(x)}$ for $x \in \Lambda$;
2. $|Df^k(x)v^s| \le C\lambda^k|v^s|$ for $x \in \Lambda$, $v^s \in E^s_x$, $k \ge 0$.
3. $|Df^{-k}(x)v^u| \le C\lambda^k|v^u|$ for $x \in \Lambda$, $v^u \in E^u_x$, $k \ge 0$.

If $\Lambda = M$ is a hyperbolic set then we say that f is an *Anosov* diffeomorphism.

It is well-known that dynamical systems have shadowing property in a neighborhood of a hyperbolic set [1, 2]. This statement is often called the shadowing lemma.

Definition 4. We say that a diffeomorphism $f \in \mathrm{Diff}^1(M)$ is *structurally stable* if there exists a neighborhood $U \subset \mathrm{Diff}^1(M)$ of f such that for any $g \in U$ there exists a homeomorphism $h : M \to M$ such that $h \circ f = g \circ h$.

For a diffeomorphism $f \in \mathrm{Diff}^1(M)$ denote by $\Omega(f)$ the set of nonwondering points of f.

Definition 5. We say that a diffeomorphism $f \in \mathrm{Diff}^1(M)$ is Ω-*stable* if there exists a neighborhood $U \subset \mathrm{Diff}^1(M)$ of f such that for any $g \in U$ there exists a homeomorphism $h : \Omega(f) \to \Omega(g)$ such that

$$h \circ f(x) = g \circ h(x), \quad x \in \Omega(f).$$

Denote the set of Ω-stable diffeomorphisms by ΩS

Notions of structural stability and hyperbolicity are strongly related. It is known that a diffeomorphism f is structurally stable iif it satisfies Axiom A (hyperbolicity of the nonwondering set and density of periodic orbits in the nonwondering set) and the strong transversality condition [11, 27]. A diffeomorphism $f \in \Omega S$ if and only if f satisfies Axiom A and the no cycle condition, see, for example [21].

In Chapter 1 we study quantitative characteristics of shadowing for diffeomorphisms.

Definition 6. We say that f has the *Lipschitz shadowing property* (LipSh) if there exists L, d_0 such that for any $d < d_0$ and d-pseudotrajectory $\{y_k\}_{k\in\mathbb{Z}}$ there exists an exact trajectory $\{x_k\}_{k\in\mathbb{Z}}$, such that inequalities (1) hold with $\varepsilon = Ld$.

In [28, 31] the following theorem was proved (see also [15], Appendix A).

Theorem 1. *Structurally stable diffeomorphisms have the Lipschitz shadowing property.*

It is easy to see that the Lipschitz shadowing property implies the standard shadowing property. Let us note that the standard shadowing property persists under topological conjugacy, and hyperbolicity, transversality, and structural stability do not. Hence, there is no chance to characterise the set of diffeomorphisms satisfying the standard shadowing property in terms of hyperbolicity, transversality and structural stability. However in the modern theory of dynamical systems it is believed that shadowing and hyperbolicity are almost equivalent. At the same time numerical simulations show good results for much broader class of systems.

The situation completely changes if we consider C^1-interior of the sets, satisfying the shadowing property. Sakai proved the following theorem [30].

Theorem 2. *C^1-interior of the set of diffeomorphisms, satisfying the standard shadowing property coincide with the set of structurally stable diffeomorphisms.*

At the same time there exists plenty of non structurally stable examples satisfying the standard shadowing property [18, 19, 22].

In this thesis we introduce the following notion.

Definition 7. Let us say that f has the Hölder shadowing property on finite intervals with exponents $\theta \in (0, 1)$, $\omega > 0$ (FinHolSh(θ, ω)), if there exists $d_0, L, C > 0$, such that for any $d < d_0$ and d-pseudotrajectory $\{y_k\}_{k \in [0, Cd^{-\omega}]}$ there exists an exact trajectory $\{x_k\}_{k \in [0, Cd^{-\omega}]}$, satisfying

$$\operatorname{dist}(x_k, y_k) < Ld^{\theta}, \quad k \in [0, Cd^{-\omega}].$$

It is easy to show that for $\theta \in (0, 1)$ and $\omega > 0$ the following inclusions hold

$$\mathrm{SS} \subset \mathrm{LipSh} \subset \mathrm{HolSh}(\theta) := \mathrm{FinHolSh}(\theta, +\infty) \subset \mathrm{FinHolSh}(\theta, \omega) \subset \mathrm{StSh},$$

where SS denote the set of structurally stable diffeomorphisms.

In paragraph 1.2 we introduce the notion of inhomogeneous linear equation, in paragraph 1.3 we introduce notion of slow growth property for inhomogeneous linear equation and relate it to the notion of exponential dichotomy. Those notions will be the main tool in Chapter 1 and essentially used in Chapter 4.

In paragraph 1.4 we prove the following theorem [23].

Theorem 3. *The following statements are equivalent:*

- *diffeomorphism f has the Lipschitz shadowing property;*
- *diffeomorphism f is structurally stable.*

Let us note the following corollary from Theorem 3.

Definition 8. Recall that we say that diffeomorphism f has the expansivity property if there exists $a > 0$, such that if $x, y \in M$ and

$$\operatorname{dist}(f^k(x), f^k(y)) < a, \quad k \in M.$$

then $x = y$.

Corollary 4. *The following statements are equivalent:*

- *diffeomorphism f has the Lipschitz shadowing property and is expansive;*
- *f is an Anosov diffeomorphism.*

In paragraph 1.5 we prove the following theorem [34].

Theorem 5. *Diffeomorphism $f \in C^2$, satisfying* $\mathrm{FinHolSh}(\theta, \omega)$ *with $\theta > 1/2$, $\theta + \omega > 1$ is structurally stable.*

This theorem gives an upper bound for the length of shadowable pseudotrajctories for not structurally stable diffeomorphisms. Note that previously Hammel, Grebogi and York [6, 7] based on results of numerical simulation conjectured the following.

Conjecture 1. *Typical dissipative map $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy* $\mathrm{FinHolSh}(1/2, 1/2)$.

This conjecture allows us to assume that Theorem 5 cannot be improved.

Let us describe connection of those results with other problems in the dynamical system's theory. Theorem 5 has an interesting consequence even in the case of infinite pseudotrajectories.

Theorem 6. *Diffeomorphism $f \in C^2$, satisfying* $\mathrm{HolSh}(\theta) = \mathrm{FinHolSh}(\theta, +\infty)$ *with $\theta > 1/2$, is structurally stable.*

Note relation between Theorem 6 and the question suggested by Katok:

Question 1. *Is it true that any diffeomorphism Hölder conjugated to Anosov, is Anosov itself?*

Recently it was shown that in general the answer to this question is negative, however the following statement is correct [5].

Theorem 7. *A C^2-diffeomorphism, conjugated to an Anosov diffeomorphism via Hölder homeomorphism h, is Anosov itself provided that the product of Hölder exponents of h and h^{-1} greater* $1/2$.

It is easy to show that diffeomorphism Hölder conjugate to a structurally stable satisfy the Hölder shadowing property. Therefore using Theorem 6 we can conclude the following statement generalising Theorem 7.

Corollary 8. *A C^2-diffeomorphism, conjugated to a structurally stable diffeomorphism via Hölder homeomorphism h, is structurally stable itself provided that product of Hölder exponents of h and h^{-1} greater* $1/2$.

Important role in the theory of dynamical systems plays the structure of the set of periodic orbits. In this context it is natural to consider so-called periodic shadowing property. In paragraph 1.5 we consider the following notion.

Definition 9. We say that diffeomorphism f has the periodic shadowing property if for any $\varepsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory $\{y_k\}$ there exists periodic trajectory $\{x_k\}$ such that the following inequalities hold

$$\operatorname{dist}(x_k, y_k) < \varepsilon, \quad k \in \mathbb{Z}.$$

Denote the set of diffeomorphisms satisfying the periodic shadowing property by PerSh.

Definition 10. We say that diffeomorphism f satisfies the *Lipschitz periodic shadowing property* (LipPerSh) if there exists $L, d_0 > 0$, such that for any periodic d-pseudotrajectory $\{y_k\}$ with $d < d_0$ there exists a periodic trajectory $\{x_k\}$ such that the following inequalities hold

$$\operatorname{dist}(x_k, y_k) < Ld, \quad k \in \mathbb{Z}.$$

In paragraph 1.6 we prove the following theorem [12].

Theorem 9. $\operatorname{Int}^1(\text{PerSh}) = \text{LipPerSh} = \Omega S$.

In Chapter 2 we consider shadowing property for partially hyperbolic diffeomorphisms.

In paragraph 2.1 we introduce notion of the central shadowing property and prove analogue of the shadowing lemma for partially hyperbolic diffeomorphisms.

Let f be a partially hyperbolic diffeomorphism with the corresponding decomposition of the tangent bundle

$$T_x M = E_x^s \oplus E_x^c \oplus E_x^u.$$

Denote

$$E_x^{cs} := E_x^s \oplus E_x^c, \quad E_x^{cu} := E_x^u \oplus E_x^c.$$

Definition 11. A partially hyperbolic diffeomorphism f is dynamically coherent, if both distributions E^{cs} and E^{cu} are uniquely integrable. In that case distribution E^c is also uniquely integrable, and corresponding foliation W^c is a subfoliation of W^{cs} and W^{cu}.

See [3, 29] for the detailed discussion on the notion of dynamical coherence.

In paragraph 2.1 we assume that f is dynamically coherent.

Denote by $\operatorname{dist}_c(\cdot, \cdot)$ the distance in the internal metric of the manifold W^c. Denote by $W_\varepsilon^c(x) = \{y \in W^c(x) : \operatorname{dist}_c(x, y) < \varepsilon\}$.

We consider the following notion for the shadowing property of dynamically coherent diffeomorphisms.

Definition 12. (see for instance [8]) We say that an ε-pseudotrajectory $\{y_k\}$ is *central*, if for any $k \in \mathbb{Z}$ the inclusion $y_{k+1} \in W_\varepsilon^c(y_k)$ hold.

Definition 13. We say that partially hyperbolic, dynamically coherent diffeomorphism f satisfy the *central shadowing property*, if for any $\varepsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory $\{y_k\}$ there exists an ε-central pseudotrajectory $\{x_k\}$, satisfying the inequalities

$$\operatorname{dist}(x_k, y_k) < \varepsilon, \quad k \in \mathbb{Z}. \tag{2}$$

Definition 14. We say that partially hyperbolic dynamically coherent diffeomorphism f satisfy the *Lipschitz central shadowing property*, if there exists $L, d_0 > 0$, such that for any $d < d_0$ and d-pseudotrajectory $\{y_k\}$ there exists an Ld-central pseudotrajectory, satisfying the inequalities (2), with $\varepsilon = Ld$.

Note that the Lipschitz central shadowing property implies the central shadowing property. In the Thesis we prove the following analogue of the shadowing lemma [10].

Theorem 10. *A partially hyperbolic dynamically coherent diffeomorphism f satisfies the Lipschitz central shadowing property.*

The proof is based on the Schauder fixed-point theorem. The classical proofs of the shadowing lemma [1, 2] are based on the contracting mapping principle and cannot be used in our context, since the holonomy maps corresponding to foliations W^{cs}, W^{cu}, are Hölder continuous but not Lipschitz (see for instance [29]).

In paragraph 2.2 we consider the shadowing problem in a special case.

Consider the space $\Sigma = \{0, 1\}^{\mathbb{Z}}$, endowed with the standard metric dist and probability measure ν. For a sequence $\omega = \{\omega^i\} \in \Sigma$ denote by $t(\omega)$ the 0-th element of the sequence: $t(\omega) = \omega^0$. Define "the shift map" $\sigma : \Sigma \to \Sigma$ as follows: $(\sigma(\omega))^i = \omega^{i+1}$.

Consider the space $Q = \Sigma \times \mathbb{R}$. Endow Q with the maximal metric and the product measure $\mu = \nu \times \text{Leb}$.

Fix $\lambda_0, \lambda_1 \in \mathbb{R}$, satisfying the following

$$0 < \lambda_0 < 1 < \lambda_1, \quad \lambda_0\lambda_1 \neq 1. \tag{3}$$

Consider map $f : Q \to Q$, defined as follows

$$f(\omega, x) = (\sigma(\omega), \lambda_{t(\omega)}x).$$

For $q \in Q$, $d > 0$, $N \in \mathbb{N}$ denote by $\Omega_{q,d,N}$ the set of d-pseudotrajectories of length N, starting at $q_0 = q$. Assuming that q_{k+1} is chosen randomly in $B(d, f(q_k))$ according to measure μ, the set $\Omega_{q,d,N}$ is endowed by the structure of a Markov process. This construction endow $\Omega_{q,d,N}$ with a probability measure P. Similar construction for infinite pseudotrajectories were described in [37]. For $\varepsilon > 0$ denote by $p(q, d, N, \varepsilon)$ the probability of a pseudotrajectory from $\Omega_{q,d,N}$ to be ε-shadowable. Note that this event is measurable since it is open.

Let $q = (\omega, x)$, $\tilde{q} = (\omega, 0)$. For any $d, \varepsilon > 0$, $N \in \mathbb{N}$ the following equality holds $p(q, d, N, \varepsilon) = p(\tilde{q}, d, N, \varepsilon)$. Set

$$p(d, N, \varepsilon) := \int_{\omega \in \Sigma} p((\omega, 0), d, N, \varepsilon) d\nu.$$

The number $p(d, N, \varepsilon)$ is the probability of a d-pseudotrajectory to be ε-shadowable.

The main result of this paragraph is the following theorem [36].

Theorem 11. *For any $\lambda_0, \lambda_1 \in \mathbb{R}$, satisfying relations* (3) *there exist $\varepsilon_0 > 0$ and $c_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ the following hold:*

1. *If* $c < c_0$, *then* $\lim_{N\to\infty} p(\varepsilon/N^c, N, \varepsilon) = 0$*;*

2. *If* $c > c_0$, *then* $\lim_{N\to\infty} p(\varepsilon/N^c, N, \varepsilon) = 1$*;*

Analogue of Conjecture 1 for map f allows to suggest that the value $p(\varepsilon/N, N, \varepsilon)$ is almost 1. So if $c_0 > 1$ then Conjecture 1 does not hold. In the Thesis (Remark 2.16) we give example of such parameters.

In Chapters 3, 4 we study various shadowing properties for vector fields.

We consider not only the set of vector fields satisfying some shadowing property but its interiors in the C^1-topology, i.e. the set of vector fields satisfying some shadoiwng property together with all its C^1-small perturbation. Denote by $F(M)$ the space of C^1 vector fields on a manifold M, endowed with the C^1-topology. For a vector field X denote by $\phi(t, x)$ the flow generated by X.

For a set $P \subset F(M)$ denote by $\mathrm{Int}^1(P)$ its C^1-interior. For avector field X denote by $\mathrm{Per}(X)$ the set of fixed points and closed trajectories of X. For a hyperbolic trajectory p let us denote by $W^s(p)$ and $W^u(p)$ its stable and unstable manifolds respectively.

Let us pass to the definition of the shadowing property for vector fields.

Definition 15. We say that map $g : \mathbb{R} \to M$ (not necessarily continuous) is a *d-pseudotrajectory*, if the following inequalities hold

$$\mathrm{dist}(g(t+\tau), \phi(\tau, g(t))), \quad t \in \mathbb{R}, |\tau| < 1.$$

To define shadowing properties for vector fields we need the notion of reparametrisation.

Definition 16. We call a *reparametrisation* an increasing homeomorphism of the real line $h : \mathbb{R} \to \mathbb{R}$. Denote set of all reparametrisation by Rep. For $\varepsilon > 0$ denote by $\mathrm{Rep}(\varepsilon)$ the set of reparametrisations, satisfying the following inequalities

$$\left| \frac{h(t_1) - h(t_2)}{t_1 - t_2} - 1 \right| < \varepsilon.$$

Definition 17. Let us say that a vector field X and the corresponding flow ϕ satisfy the *standard shadowing property*, if for any $\varepsilon > 0$ there exists $d > 0$, such that for any d-pseudotrajectory $g(t)$ there exists a reparametrisation $h \in \mathrm{Rep}(\varepsilon)$ and a point $x \in M$, such that the following inequalities hold

$$\mathrm{dist}(g(t), \phi(h(t), x)) < \varepsilon, \quad t \in \mathbb{R}. \tag{4}$$

Denote by StSh the set of vector fields satisfying the standard shadowing property.

We use the same notation StSh as in the case of diffeomorphisms. In what follows it would be clear from the context if we consider the case of diffeomorphisms or vector fields.

Let us note that the notion of reparametrisation is necessarily in the definition of shadowing property. Indeed if we replace inequalities (4) in the Definition 17 by the inequalities

$$\mathrm{dist}(g(t), \phi(t, x)) < \varepsilon, \quad t \in \mathbb{R},$$

then a lot of “good” vector fields do not satisfy the shadowing property. As an example we can consider a vector field on a manifold M, which has a hyperbolic closed trajectory [18].

Let us introduce various shadowing properties.

Definition 18. We say that a vector field X and the corresponding flow ϕ satisfy the *Lipschitz shadowing property* if there exist constants $L, d_0 > 0$ such that for any $d < d_0$ and d-pseudotrajectory $g(t)$ there exists a point $x \in M$ and reparametrisation $h \in \mathrm{Rep}(Ld)$ such that the following inequalities hold

$$\mathrm{dist}(g(t), \phi(h(t), x)) < Ld, \quad t \in \mathbb{R}.$$

Denote by LipSh the set of vector fields satisfying the Lipschitz shadowing property.

Definition 19. We say that vector field X and corresponding flow ϕ satisfy the *oriented shadowing property*, if for any $\varepsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory $g(t)$ there exists a reparametrisation $h \in \mathrm{Rep}$ and a point $x \in M$ such that the inequalities (4) hold. Note that we do not assume closeness of reparametrisation h to the identity map. Denote by OrientSh the set of vector fields satisfying the oriented shadowing property.

Clearly the following inclusions hold

$$\mathrm{LipSh} \subset \mathrm{StSh} \subset \mathrm{OrientSh}\,.$$

The notion of the standard shadowing property is equivalent to the strong pseudo orbit tracing property (POTP), introduced by Komuro [9]; the oriented shadowing property is equivalent to the normal POTP introduced by Komuro [9] and the pseudo orbit tracing property, introduced by Thomas [32].

Note that all three introduced above notions of the shadowing property are not equivalent. Examples of vector fields lying in the set $\mathrm{StSh} \setminus \mathrm{LipSh}$ are well-known and relatively easy to construct. In paragraph 3.4 we construct an example of a vector field lying in the set $\mathrm{OrientSh} \setminus \mathrm{StSh}$. Earlier Komuro showed that the oriented and the standard shadowing properties are equivalent for the case of vector fields without fixed points [9]. In the same paper Komuro posed a question if those two notions are equivalent in general [9, Remark 5.1]?

As in the case of diffeomorphisms the following notions play an important role in the shadowing theory.

Definition 20. We say that a compact invariant set $\Lambda \subset M$ is *hyperbolic* if there exist numbers $C > 0$, $\lambda > 0$ and linear subspaces $E^s_x, E^u_x \subset T_xM$ such that for any $x \in \Lambda$ the following holds

1. $T_xM = E^s_x \oplus E^u_x \oplus < X(x) >$.
2. Let $\Phi(t)$ be the fundamental matrix of the variational systems

$$\frac{dy}{dy} = \frac{\partial X}{\partial x}(\phi(t, x))y$$

along the trajectory $\phi(t, p)$, satisfying $\Phi(0) = E$. Then

(a) $\Phi(t)E^s_x = E^s_{\phi(t,x)}$, $\Phi(t)E^u_x = E^u_{\phi(t,x)}$,

(b) $|\Phi(t)v^s| \leq Ce^{-\lambda t}|v^s|$ for $v^s \in E^s_x$ and $t \geq 0$.

(c) $|\Phi(-t)v^u| \leq Ce^{-\lambda t}|v^u|$ for $v^u \in E^u_x$ and $t \geq 0$.

If $\Lambda = M$ is a hyperbolic set then we say that X is an Anosov vector field.

Definition 21. We say that a vector field $X \in \mathcal{F}(M)$ is *structurally stable* if there exists a neighborhood $U \subset \mathcal{F}(M)$ of X such that for any $Y \in U$ there exists a homeomorphism $\alpha : M \to M$ which maps trajectories of X to trajectories of Y and preserves the direction of movement alomg trajectories. In other words there exists a map $\tau : \mathbb{R} \times M \to \mathbb{R}$ such that

- for any $x \in M$, the function $\tau(\cdot, x)$ increases and maps $\mathbb{R}$ into $\mathbb{R}$;
- $\tau(0, x) = x$ for any $x \in M$;
- $\alpha(\phi(t, x)) = \psi(\tau(t, x), \alpha(x))$ for any $t \in \mathbb{R}$, $x \in M$, where $\psi(\cdot, \cdot)$ is the flow generated by Y.

Denote by SS the set of structurally stable vector fields.

For a vector field X denote by $\Omega(X)$ the set of nonwondering points of X.

Definition 22. We say that a vector field $X \in \mathcal{F}(M)$ is Ω-*stable* if there exists a neighborhood $U \subset \mathcal{F}(M)$ of X such that for any $Y \in U$ there exists a homeomorphism $\alpha : \Omega(X) \to \Omega(Y)$ which maps trajectories of X to trajectories of Y and preserves the direction of movement along trajectories. Denote by ΩS the set of Ω-stable vector fields.

Pilyugin proved the following theorem [17].

Theorem 12. SS $\subset$ LipSh

In Chapter 3 we study the set OrientSh in the C^1-topology.

The following notion plays an important role in this chapter. We say that matrix A is of class K, if all its eigenvalues has nonzero real parts. Denote by K_2^+ the set of matrixes K, with pair of complex conjugate eigenvalues $a_1 \pm b_1 i$ with $a_1 > 0$, such that if $c_1 > 0$ is an eigenvalue of A, then $c_1 > a_1$. Denote by K_2^- the set of matrixes A, such that $-A \in K_2^+$.

Definition 23. We say that a vector field X is of class B, if there exists two fixed points p_1 and p_2 (not necessarily distinct) of X, satisfying the following properties:

1. $DX(p_1) \in K_2^+$,
2. $DX(p_2) \in K_2^-$,
3. invariant manifolds $W^s(p_1)$ and $W^u(p_2)$ has a trajectory of non transverse intersection.

In my PhD thesis I have proved the following theorems [25, 26, 35].

Theorem 13. $\mathrm{Int}^1(\mathrm{OrientSh}) \setminus B = \mathrm{SS}$.

Theorem 14. *If* $\dim M \le 3$, *then* $\mathrm{Int}^1(\mathrm{OrientSh}) = \mathrm{SS}$.

In paragraph 3.1 we construct an example of a vector field, which shows that exclusions of vector field of class B in Theorem 13 is essential [26].

Theorem 15. *There exists a vector field* $X \in \mathrm{Int}^1(\mathrm{OrientSh}) \setminus \mathrm{SS}$ *on manifold* $S^2 \times S^2$.

This theorem is the main result of chapter 3. Note that Theorem 13 implies that $X \in B$.
In paragraph 3.2 we show that [4]

Theorem 16. $\mathrm{Int}^1(\mathrm{OrientSh}) \subset \Omega S$.

This result implies that example from Theorem 15 must satisfy Axiom A' and violate the strong transversality condition.
In paragraph 3.3 we show that sets StSh and OrientSh do not coincide [33].

Theorem 17. *There exists a vector field* $X \in \mathrm{OrientSh} \setminus \mathrm{StSh}$ *on manifold* $S^2 \times S^2$.

In Chapter 4 we consider vector fields with the Lipschitz shadowing property and the Lipschitz periodic shadowing property.
In paragraph 4.1 we prove the following theorem [16].

Theorem 18. *The vector field X has the Lipschitz shadowing property if and only if X is structurally stable.*

Definition 24. We say that a vector field X and the corresponding flow ϕ has the expansivity property, if there exists constants $a, \delta > 0$ such that if the inequalities

$$\mathrm{dist}(\phi(t, x), \phi(\alpha(t), x)) < a, \quad t \in \mathbb{R}$$

hold for some $x, y \in M$ and increasing homeomorphism $\alpha \in \mathrm{Rep}$, satisfying $\alpha(0) = 0$, then $x = \phi(\tau, y)$, where $|\tau| < \delta$.

We prove the following statement as a corollary from Theorem 18.

Corollary 19. *If a vector field X satisfy Lipschitz shadowing property and expansivity then X is an Anosov vector field.*

In paragraph 4.2 we study connection between Ω-stability and periodic shadowing property [16].

Definition 25. We say that a vector field X satisfies the *Lipschitz periodic shadowing property* (LipPerSh) if there exists $L, d_0 > 0$ such that any periodic d-pseudotrajectory $g(t)$ with $d < d_0$, can be Ld-shadowed by closed trajectory.

Theorem 20. *Vector field X has the Lipschitz periodic shadowing property if and only if it is Ω-stable.*

	Diffeomorphisms	Vector fields
C^1	$\mathrm{Int}^1(\mathrm{StSh}) = \mathrm{SS}$ $\mathrm{Int}^1(\mathrm{PerSh}) = \Omega S$	$\mathrm{Int}^1(\mathrm{OrientSh}) \neq \mathrm{SS}$ $(\dim M > 3)$ $\mathrm{Int}^1(\mathrm{OrientSh}) \subset \Omega S$ $\mathrm{OrientSh} \neq \mathrm{StSh}$ $\mathrm{Int}^1(\mathrm{OrientSh} \setminus B) = \mathrm{SS}$ $\mathrm{Int}^1(\mathrm{OrientSh}) = \mathrm{SS}$ $(\dim M \leq 3)$
Lipschitz	$\mathrm{LipSh} = \mathrm{SS}$ $\mathrm{LipPerSh} = \Omega S$ $f \in C^2, \mathrm{FinHolSh}(\alpha, \omega)$ $\alpha, \omega > 1/2 \Rightarrow f \in \mathrm{SS}$	$\mathrm{LipSh} = \mathrm{SS}$ $\mathrm{LipPerSh} = \Omega S$

Table 1: Relations between shadowing properties and structural stability

Results of the chapters 1, 3, 4 are represented in a short form in the Table 1.

Equality $\mathrm{Int}^1(\mathrm{StSh}) = \mathrm{SS}$ was proved by Sakai in 1994; equalities $\mathrm{Int}^1(\mathrm{OrientSh} \setminus B) = \mathrm{SS}$ and $\mathrm{Int}^1(\mathrm{OrientSh}) = \mathrm{SS}$ $(\dim M \leq 3)$ were obtained in the PhD Thesis of the author. The rest results were achieved by the author (some of results are co-authored) in 2010 – 2015.

In Chapter 5 we consider actions of finitely generated groups. We introduce notion of the shadowing property for groups actions. We consider in details shadowing for actions of nilpotent, solvable and free groups.

Consider a finitely generated (not necessarily Abelian) group G and a metric space Ω with metric dist.

We say that the map $\Phi : G \times \Omega \to \Omega$ is a (left) action of a group G if the following conditions hold:

(G1) for any $g \in G$ the map $f_g = \Phi(g, \cdot)$ is a homeomorphism of Ω;

(G2) $\Phi(e, x) = x$ for any $x \in \Omega$, where e is the identity element of G;

(G3) $\Phi(g_1, \Phi(g_2, x)) = \Phi(g_1 g_2, x)$ for any $g_1, g_2 \in G$ and $x \in \Omega$.

We say that an action is uniformly continuous, if there exists a finite symmetric generating set $S \subset G$ of G such that maps f_s are uniformly continuous for all $s \in S$. Note that if Ω is compact, then any action of the finitely generated group is uniformly continuous.

Fix finite symmetric generated set S of G.

Definition 26. For $d > 0$ we say that a sequence $\{y_g\}_{g \in G}$ is a *d-pseudotrajectory* of action Φ (with respect to a generating set S) if

$$\mathrm{dist}(f_s(y_g), y_{sg}) < d, \quad g \in G, s \in S.$$

Definition 27. We say that uniformly continuous action Φ satisfies the *shadowing property* on a set $V \subset \Omega$, if for any $\varepsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory $\{y_g\}$ there exists a point $x_e \in \Omega$, such that

$$\mathrm{dist}(f_g(x_e), y_g) < \varepsilon, \quad g \in G.$$

In that case we say that $\{y_g\}$ is ε-shadowed by $\{x_g = f_g(x_e)\}$. If $V = \Omega$, we say that Φ satisfies the shadowing property.

Note that the definition of the shadowing property depends on the choice of a generating set S. However the following statement shows that if an uniformly continuous action satisfies the shadowing property for one symmetric generating set then it satisfies the shadowing property for all other symmetric generating sets.

Statement 21. *Consider two finite symmetric generating sets S and S' of G. If a uniformly continuous action Φ satisfies the shadowing property on a set $V \subset \Omega$ with respect to a generating set S, then Φ satisfies the shadowing property on a set V with respect to a generating set S'.*

Definition 28. We say that action Φ satisfies the expansivity property on a set $U \subset \Omega$, if there exists $a > 0$, such that if some $x, y \in U$ the inequalities

$$\mathrm{dist}(f_g(x), f_g(y)) < a,$$

hold then $x = y$.

Note that if for a subgroup $G_1 \leq G$ action $\Phi|_{G_1}$ satisfies the expansivity property then Φ satisfies the expansivity property as well.

Definition 29. Consider sets $U, V \subset \Omega$. We say that uniformly continuous action Φ is *topologically Anosov* with respect to a pair (U, V), if the following conditions hold:

1. there exists $\gamma > 0$, such that $B(\gamma, V) \subset U$;
2. Φ satisfies the shadowing property on V;
3. Φ has the expansivity property on U.

In paragraph 5.1 we introduce necessarily notions from the group theory.
In paragraph 5.2 we introduce the shadowing property for group actions.
In paragraph 5.3 we prove correctness of the definition of the shadowing property.
In paragraph 5.4 we consider actions of nilpotent groups. The main result of this paragraph is the following [13].

Theorem 22. *Consider a uniformly continuous action Φ of finitely generated virtually nilpotent group G of a metric space Ω. Assume that there exists an element $g \in G$, such that homeomorphism f_g is topologically Anosov with respect to a pair (U, V). Then action Φ is topologically Anosov with respect to (U, V).*

In paragraph 5.5 we prove that for linear actions of Abelian groups of C^m assumptions of Theorem 22 are also necessarily, more precisely we prove the following [24]:

Theorem 23. *A linear action Φ of an Abelian group $G = \mathbb{Z}^n$ satisfies the shadowing property on C^m if and only if there exists $g \in G$ such that linear map f_g is hyperbolic.*

In paragraph 5.6 we consider actions of solvable groups. We show that Theorem 22 cannot be generalised for the case of solvable groups. Consider a solvable group $BS(1,n) =< a, b | ba = a^n b >$, with $n > 1$. For any $\lambda > 1$ consider an action Φ, generated by the maps

$$f_a(x) = Ax, \quad f_b(x) = Bx$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \lambda & 0 \\ 0 & n\lambda \end{pmatrix}.$$

Note that $BA = A^n B$, hence action Φ defined correctly. For $\lambda > 1$ the map f_b is hyperbolic, however the following os correct [13]:

Theorem 24.
- *If $\lambda \in (1, n]$, then action Φ does not satisfy the shadowing property.*
- *If $\lambda > n$, then actionΦ satisfies the shadowing property.*

In paragraph 5.7 we consider actions of the free groups. We prove the following [13]:

Theorem 25. *Any linear action on Euclidian space of a non Abelian finitely generated group does not satisfy the shadowing property.*

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