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with λ -curvature term in higher
dimensions**

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REGULARITY OF DIRAC-HARMONIC MAPS WITH λ -CURVATURE TERM IN HIGHER DIMENSIONS

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ABSTRACT. In this paper, we will study the partial regularity for stationary Dirac-harmonic maps with λ -curvature term. For a weakly stationary Dirac-harmonic map with λ -curvature term (ϕ, ψ) from a smooth bounded open domain $\Omega \subset \mathbb{R}^m$ with $m \geq 2$ to a compact Riemannian manifold N , if $\psi \in W^{1,p}(\Omega)$ for some $p > \frac{2m}{3}$, we prove that (ϕ, ψ) is smooth outside a closed singular set whose $(m-2)$ -dimensional Hausdorff measure is zero. Furthermore, if the target manifold N does not admit any harmonic sphere S^l , $l = 2, \dots, m-1$, then (ϕ, ψ) is smooth.

1. INTRODUCTION

Variational problems from fields of theoretical physics, like quantum field or string theory, usually come in some particular dimension, with some finite dimensional, but non-compact symmetry group. These include harmonic maps coming from the nonlinear sigma model in dimension 2 or Yang-Mills fields in dimension 4. Typically, they then represent borderline cases of the Palais-Smale condition, and therefore, standard PDE methods for proving the regularity of solutions may not apply. In those dimensions, geometric analysis can usually identify a particular blow-up behavior, that is, a special scheme for the emergence and the control of singularities. That is, minimizing sequences can develop singularities, but in the limit, these singularities can be described as regular solutions on some blown-up domain.

The mathematical aspects, however, are also of much interest and subtlety in higher dimensions. In those dimensions, solutions can really become singular. Again, this has been widely explored in geometric analysis. For instance, the equations for minimal submanifolds in Euclidean or Riemannian spaces lose the conformal invariance, and completely new phenomena emerge, in particular around the Bernstein problem, and this has been a key trigger for the development of geometric measure theory. For harmonic mappings, see [18, 15, 16, 35], and for Yang-Mills, Rivière has carried out the systematic investigation in dimensions larger than 4, see [32] and the references therein. In those cases, the best analytical results that can be obtained are usually partial regularity results, that is, one can control the Hausdorff dimension of the singular set and often also the structure of the singularities.

Here, we engage in such an investigation for Dirac-harmonic maps, a variational problem motivated by the supersymmetric non-linear sigma model of quantum field theory. They arise again

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naturally in dimension 2, where we again find conformal invariance and can perform a – rather subtle – blow-up analysis. Dirac-harmonic maps were first introduced and studied in [9, 8] in dimension 2. In light of the above, it seems worthwhile to also investigate them in higher dimensions, and again, we expect that the analytical behavior will be rather different. Such an analysis has been started by Wang and Xu [40]. In particular, they derived a monotonicity formula and controlled the singular set as for harmonic maps. In fact, since Dirac-harmonic maps generalize harmonic maps in the sense that they couple a harmonic map type field with a nonlinear Dirac field, one should naturally expect that the structure of harmonic map regularity theory can serve as a guideline. Nevertheless, as it turns out already in dimension 2, while the results are indeed roughly similar to those known for harmonic maps, their proofs can become considerably more difficult. This forces the development of new techniques, some of which then in turn also lead to deeper insights for harmonic maps. Here, we take a step further by implementing the important analysis of Lin [26] who could show regularity in the absence of obstructions, represented by harmonic spheres in a certain range of dimensions. Also, we consider a model that is more general than that in [40], but which is important from the original perspective of quantum field theory, that of Dirac-harmonic maps with curvature term. While the curvature term usually only comes with a constant factor in the literature, we find that we can also admit a field-dependent, variable factor, without impeding the analysis.

We now recall the technical details of the models, and then state our main results at the end of this introduction. Let (M, g) be an m -dimensional compact spin Riemannian manifold, ΣM the spinor bundle over M and $\langle \cdot, \cdot \rangle_{\Sigma M}$ the metric on ΣM . Choosing a local orthonormal basis $e_\alpha, \alpha = 1, \dots, m$ on M , the usual Dirac operator is defined as $\not{D} := e_\alpha \cdot \nabla_{e_\alpha}$, where ∇ is the spin connection on ΣM and \cdot is the Clifford multiplication. For more details on spin geometry and Dirac operators, one can refer to [25].

Let ϕ be a smooth map from M to another compact Riemannian manifold (N, h) with dimension $n \geq 2$. If ϕ^*TN is the pull-back bundle of TN by ϕ , we get the twisted bundle $\Sigma M \otimes \phi^*TN$. Naturally, there is a metric $\langle \cdot, \cdot \rangle_{\Sigma M \otimes \phi^*TN}$ on $\Sigma M \otimes \phi^*TN$ which is induced from the metrics on ΣM and ϕ^*TN . Also we have a natural connection $\widetilde{\nabla}$ on $\Sigma M \otimes \phi^*TN$ which is induced from the connections on ΣM and ϕ^*TN . Let ψ be a section of the bundle $\Sigma M \otimes \phi^*TN$. In local coordinates $\{y^i\}$, it can be written as

$$\psi = \psi^i \otimes \partial_{y^i}(\phi),$$

where each $\psi^i \in \Gamma(\Sigma M)$ is a usual spinor and $\{\partial_{y^i}\}$ is a local basis on N . Then $\widetilde{\nabla}$ becomes

$$(1.1) \quad \widetilde{\nabla} \psi = \nabla \psi^i \otimes \partial_{y^i}(\phi) + (\Gamma_{jk}^i \nabla \phi^j) \psi^k \otimes \partial_{y^i}(\phi),$$

where Γ_{jk}^i are the Christoffel symbols of the Levi-Civita connection of N . The Dirac operator along the map ϕ is defined by

$$\not{D}\psi := e_\alpha \cdot \widetilde{\nabla}_{e_\alpha} \psi.$$

Now, consider the action functional introduced in [8, 9]

$$(1.2) \quad L(\phi, \psi) = \int_M (|d\phi|^2 + \langle \psi, \not{D}\psi \rangle_{\Sigma M \otimes \phi^*TN}) dvol_g.$$

Critical points (ϕ, ψ) of L are called *Dirac-harmonic maps* from M to N .

In local coordinates, the Euler-Lagrange equations of the functional L are given as follows

$$(1.3) \quad \left(\Delta \phi^i + \Gamma_{jk}^i g^{\alpha\beta} \phi_\alpha^j \phi_\beta^k \right) \frac{\partial}{\partial y^i}(\phi(x)) = R(\phi, \psi),$$

$$(1.4) \quad \not{D}\psi = 0,$$

where $R(\phi, \psi)$ is defined by

$$R(\phi, \psi) = \frac{1}{2} R_{lij}^m(\phi(x)) \operatorname{Re} \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle \frac{\partial}{\partial y^m}(\phi(x)).$$

Here $\operatorname{Re}(z)$ denotes the real part of $z \in \mathbb{C}$ and R_{lij}^m stands for the Riemann curvature tensor of the target manifold (N, g) . See [8, 9] for details.

Dirac-harmonic maps are motivated from the supersymmetric nonlinear sigma model from quantum field theory [12, 19]. They have been investigated extensively in recent years. This subject generalizes the theory of harmonic maps and harmonic spinors. The regularity problem for harmonic maps has been extensively studied in the literature, see e.g. [28, 35, 17, 13, 3, 6] for the classical regularity theory of minimizing harmonic maps and stationary harmonic maps. Based on the geometric analysis techniques developed for harmonic maps and more generally critical elliptic systems with an antisymmetric structure [31, 33], regularity issues for Dirac-harmonic maps in dimension two were systematically studied in [8, 43, 40, 11, 37]. In higher dimensions, Wang-Xu [40] introduced the notion of stationary Dirac-harmonic maps and derived a monotonicity formula for stationary Dirac-harmonic maps, based on which some partial regularity results were obtained. They proved the singular set has Hausdorff dimension at most $m - 2$. In this paper, we give conditions on the target manifold under which the dimension can be reduced further. Moreover, we prove these properties hold for a general case, i.e. Dirac-harmonic maps with λ -curvature term. The blow-up analysis for Dirac-harmonic maps has been investigated in [8, 42, 44, 27]. To study the existence problem, a heat flow approach was introduced in [10] and further explored in [21, 22, 23].

Usually, the supersymmetric nonlinear sigma model of quantum field theory includes an additional curvature term in addition to (1.2). This leads us to consider the following functional

$$(1.5) \quad L_c(\phi, \psi) = \frac{1}{2} \int_M \left(|d\phi|^2 + \langle \psi, \not{D}\psi \rangle_{\Sigma M \otimes \phi^* TN} - \frac{1}{6} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle \right) d\operatorname{vol}_g.$$

Critical points (ϕ, ψ) of L_c are called *Dirac-harmonic maps with curvature term* from M to N . They were first proposed and studied by Chen-Jost-Wang [7], where a type of Liouville theorem was proved. The regularity for weak solutions in dimension two was considered in [4]. The blow-up theory, including the energy identity and bubble tree convergence, for a sequence of Dirac-harmonic maps with curvature term from a closed Riemann surface with uniformly bounded energy has been systematically investigated in [20]. For the regularity problem of a similar model with a different type of curvature term, i.e., Dirac-harmonic maps with Ricci type spinor potential, we refer to Xu-Chen [41].

In this paper, we shall consider the following functional:

$$L_\lambda(\phi, \psi) = \frac{1}{2} \int_M \left(|d\phi|^2 + \langle \psi, \not{D}\psi \rangle_{\Sigma M \otimes \phi^* TN} - \frac{\lambda(\phi)}{6} R_{ikjl}(\phi) \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle \right) d\operatorname{vol}_g$$

where λ is a smooth function on N . Since (N, h) is a compact Riemannian manifold, we define a nonnegative constant:

$$(1.6) \quad \Lambda_1 := \max_{y \in N} |\lambda(y)| + |\nabla \lambda(y)|.$$

The critical points (ϕ, ψ) of L_λ are called *Dirac-harmonic maps with λ -curvature term* from M to N . Thus, (ϕ, ψ) is a Dirac-harmonic map iff $\lambda \equiv 0$ and it is a Dirac-harmonic map with curvature term iff $\lambda \equiv 1$.

By the Nash embedding theorem, we embed N isometrically into \mathbb{R}^K . Following Wang-Xu's analysis set up for Dirac-harmonic maps in higher dimensions in [40], we denote

$$W^{1,2}(M, N) := \{\phi \in W^{1,2}(M, \mathbb{R}^K) | \phi(x) \in N, \text{ a.e. } x \in M\},$$

$$S^{1,4/3}(\Sigma M \otimes \phi^* TN) := \{\psi \in \Gamma(\Sigma M \otimes \phi^* TN) | \int_M (|\psi|^4 + |\nabla \psi|^{\frac{4}{3}}) < \infty\}.$$

Here $\psi \in \Gamma(\Sigma M \otimes \phi^* TN)$ should be understood as a K -tuple of spinors (ψ^1, \dots, ψ^K) satisfying

$$\sum_{i=1}^K \psi^i v_i = 0$$

for any normal vector $v = (v_1, \dots, v_K) \in \mathbb{R}^K$.

In the sequel, for simplicity, we shall consider the case that $M = \Omega$ is a bounded open domain of \mathbb{R}^m with smooth boundary and equipped with the Euclidean metric. Then, the spinor bundle ΣM over M can be identified with $\Sigma = \Omega \times \mathbb{C}^L$, $L = \text{rank}_{\mathbb{C}} \Sigma$. See [25].

Definition 1.1. We call $(\phi, \psi) \in W^{1,2}(\Omega, N) \times S^{1,4/3}(\mathbb{C}^L \otimes \phi^* TN)$ a weakly Dirac-harmonic map with λ -curvature term if it is a critical point of L_λ over the Sobolev space $W^{1,2}(\Omega, N) \times S^{1,4/3}(\mathbb{C}^L \otimes \phi^* TN)$.

Our first main result is the following small regularity theorem.

Theorem 1.2. For $m \geq 2$, there exists an $\epsilon_0 = \epsilon_0(m, \Lambda_1, N) > 0$ such that if $(\phi, \psi) \in W^{1,2}(\Omega, N) \times S^{1,4/3}(\mathbb{C}^L \otimes \phi^* TN)$ is a weakly Dirac-harmonic map with λ -curvature term satisfying

$$(1.7) \quad \sup_{x \in B_{r_0}(x_0), 0 < r \leq r_0} r^{2-m} \int_{B_r(x)} (|\nabla \phi|^2 + |\psi|^4) d\text{vol}_g \leq \epsilon_0^2,$$

then $(\phi, \psi) \in C^\infty(B_{\frac{r_0}{2}}(x_0))$, and it satisfies

$$(1.8) \quad \|\nabla \phi\|_{L^\infty(B_{r_0/2}(x_0))} + \|\psi\|_{L^\infty(B_{r_0/2}(x_0))}^2 \leq C r_0^{-\frac{m}{2}} (\|\nabla \phi\|_{L^2(B_{r_0}(x_0))} + \|\psi\|_{M^{4,2}(B_{r_0}(x_0))}^2),$$

where $C = C(m, \Lambda_1, N) > 0$ and Λ_1 is as in (1.6).

When $\lambda = 0$, the conclusion in the above theorem has been proven in [40]. When $m = 2$ and $\lambda = 1$, one can refer to [4].

Similarly to the classical regularity theory of harmonic maps, in order to study the partial regularity in higher dimensions, we need to introduce the notion of stationary solutions.

Definition 1.3. A weakly Dirac-harmonic map with λ -curvature term $(\phi, \psi) \in W^{1,2}(\Omega, N) \times S^{1, \frac{4}{3}}(\mathbb{C}^L \otimes \phi^*TN)$ is called *stationary* if it is also a critical point of L_λ with respect to the domain variations, i.e. for any $Y \in C_0^\infty(\Omega, \mathbb{R}^n)$, it holds

$$\frac{d}{dt}\bigg|_{t=0} \int_{\Omega} \left(|d\phi_t|^2 + \langle \psi_t, \not{D}\psi_t \rangle_{\Sigma M \otimes \phi_t^*TN} - \frac{\lambda}{6} R_{ijkl} \langle \psi_t^i, \psi_t^j \rangle \langle \psi_t^k, \psi_t^l \rangle \right) d\text{vol} = 0,$$

where $\phi_t(x) = \phi(x + tY(x))$ and $\psi_t(x) = \psi(x + tY(x))$.

We would like to remark that for the cases $\lambda = 0, 1$, the above definition has been introduced in [40, 5], respectively, where the following monotonicity formula was derived: for any $x_0 \in \Omega$ and $0 < r_1 \leq r_2 < \text{dist}(x_0, \partial\Omega)$,

$$\begin{aligned} & r_2^{2-m} \int_{B_{r_2}(x_0)} (|\nabla\phi|^2 + \frac{\lambda}{6} R_{ijkl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle) dx - r_1^{2-m} \int_{B_{r_1}(x_0)} (|\nabla\phi|^2 + \frac{\lambda}{6} R_{ijkl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle) dx \\ &= \int_{r_1}^{r_2} r^{2-m} \int_{\partial B_r} (2|\frac{\partial\phi}{\partial r}|^2 + \text{Re}\langle \psi, \partial_r \cdot \psi_r \rangle) dH^{n-1} dr \end{aligned}$$

where $\partial_r = \frac{\partial}{\partial r} = \frac{\partial}{\partial |x-x_0|}$ and $\psi_r = \widetilde{\nabla}_{\partial_r} \psi$. The second term of the right hand side of the above equation does not have a fixed sign, which makes the use of this formula difficult. This is why in [40], some additional condition on the spinors was imposed in order to get the partial regularity of stationary Dirac-harmonic maps.

In this paper, we shall impose the same extra condition for the spinor as in Wang-Xu [40] and get the following partial regularity theorem for stationary Dirac-harmonic maps with λ -curvature term. For similar results for stationary harmonic maps and stationary Dirac-harmonic maps, we refer to [3, 13, 40].

Theorem 1.4. For $m \geq 2$, let $(\phi, \psi) \in W^{1,2}(\Omega, N) \times S^{1, \frac{4}{3}}(\mathbb{C}^L \otimes \phi^*TN)$ be a weakly stationary Dirac-harmonic map with λ -curvature term. Suppose $\|\psi\|_{W^{1,p}(\Omega)} \leq \Lambda$ for some $p > \frac{2m}{3}$, then there exists a closed subset $S(\phi) \subset \Omega$, with $H^{m-2}(S(\phi)) = 0$, such that $(\phi, \psi) \in C^\infty(\Omega \setminus S(\phi))$.

Furthermore, we have

Theorem 1.5. Under the same assumption as in the above theorem, if N does not admit harmonic spheres, S^l , $l = 2, \dots, m-1$, then (ϕ, ψ) is smooth.

To prove Theorem 1.2, we firstly use the idea of Wang in [39] to improve the regularity of the spinor ψ and then apply regularity results for elliptic system with an antisymmetric structure (see e.g. Theorem 5.2 in Appendix) to handle the map ϕ . For Theorem 1.4, since $\nabla\psi \in L^p$ for some $p > \frac{2m}{3}$, it follows from using Theorem 1.2, the monotonicity formula and applying similar arguments as in Wang-Xu [40]. As for our last Theorem 1.5, thanks to the observation in Proposition 4.5 for some formulas of the spinors, following Lin's scheme in [26], we consider the concentration set of a blow-up sequence of Dirac-harmonic maps with λ -curvature term. The proof is based on the analysis of defect measures by geometric measure theory.

The rest of the paper is organized as follows. In Section 2, we first derive the Euler-Lagrange equation for stationary Dirac-harmonic maps with λ -curvature term. Secondly, we establish the

monotonicity formula crucial to prove Theorem 1.4 and Theorem 1.5. In Section 3, we prove the small regularity Theorem 1.2 and then Theorem 1.4 follows immediately by applying some monotonicity formula argument. In Section 4, we use the blow-up analysis to prove Theorem 1.4. For the reader's convenience, we will state some well-known regularity results and estimates for some first and second order elliptic systems in Section 5.

2. EULER-LAGRANGE EQUATIONS AND MONOTONICITY FORMULA

In this section, we will derive the Euler-Lagrange equation and the monotonicity formula for Dirac-harmonic maps with λ -curvature term.

First, similarly to the cases $\lambda = 0, 1$ considered in [9] and [7], respectively, the Euler-Lagrange equations of the functional L_λ can be derived in terms of local coordinates as follows:

Lemma 2.1. *Let (ϕ, ψ) be a Dirac-harmonic map with λ -curvature term from M to N . Then, in local coordinates, (ϕ, ψ) satisfies*

$$(2.1) \quad \begin{aligned} \tau(\phi) &= \frac{1}{2} R_{li}^m(\phi) \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle \frac{\partial}{\partial y^m}(\phi) - \frac{\lambda}{12} h^{mp} R_{ikjl;p} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle \frac{\partial}{\partial y^m}(\phi) \\ &\quad - \frac{\nabla^N \lambda(\phi)}{12} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle, \end{aligned}$$

$$(2.2) \quad \not{D}\psi = \frac{\lambda}{3} R_{jkl}^m \langle \psi^j, \psi^l \rangle \psi^k \frac{\partial}{\partial y^m}(\phi),$$

where $\tau(\phi) = (-\Delta \phi^i + \Gamma_{jk}^i g^{\alpha\beta} \phi_\alpha^j \phi_\beta^k) \frac{\partial}{\partial y^i}(\phi(x))$ is the tension field of ϕ and $\nabla^N \lambda = h^{mp} \frac{\partial \lambda}{\partial y^p} \frac{\partial}{\partial y^m}$ is the gradient vector field on N .

Proof. By the computation of Section II in [7], we obtain the ψ -equation for L_λ ,

$$\not{D}\psi = \frac{\lambda}{3} R_{jkl}^m \langle \psi^j, \psi^l \rangle \psi^k \frac{\partial}{\partial y^m}(\phi)$$

and

$$\begin{aligned} &\frac{d}{dt} \Big|_{t=0} \frac{1}{2} \int_M (|d\phi_t|^2 + \langle \psi, \not{D}\psi \rangle) d\text{vol}_g \\ &= \frac{1}{2} \int_M \left(-2h_{im} \tau^i(\phi) + \frac{2}{3} \Gamma_{mi,p} R_{jkl}^p \langle \psi^i, \psi^k \rangle \langle \psi^j, \psi^l \rangle + \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle R_{mlij} \right) \xi^m d\text{vol}_g, \end{aligned}$$

where ϕ_t is the variation of ϕ with $\phi_0 = \phi$ and $\frac{d}{dt} \Big|_{t=0} = \xi$.

We just need to compute the last term:

$$\begin{aligned} &-\frac{d}{dt} \Big|_{t=0} \int_M \frac{\lambda(\phi_t)}{12} R_{ijkl} \langle \psi^i, \psi^k \rangle \langle \psi^j, \psi^l \rangle d\text{vol}_g \\ &= - \int_M \frac{\lambda(\phi)}{12} R_{ijkl,m} \langle \psi^i, \psi^k \rangle \langle \psi^j, \psi^l \rangle \xi^m d\text{vol}_g - \frac{1}{12} \int_M R_{ijkl} \langle \psi^i, \psi^k \rangle \langle \psi^j, \psi^l \rangle \frac{\partial \lambda}{\partial y^m} \xi^m d\text{vol}_g. \end{aligned}$$

Thus,

$$\begin{aligned} & -\frac{dL_\lambda(\phi_t)}{dt}\bigg|_{t=0} \\ &= \frac{1}{2} \int_M (-2h_{im}\tau^i(\phi) + \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle R_{mlij} - \frac{\lambda(\phi)}{6} R_{ijkl;m} \langle \psi^i, \psi^k \rangle \langle \psi^j, \psi^l \rangle \\ & \quad - \frac{1}{6} R_{ijkl} \langle \psi^i, \psi^k \rangle \langle \psi^j, \psi^l \rangle \frac{\partial \lambda}{\partial y^m}) \xi^m. \end{aligned}$$

The conclusion of the lemma follows immediately. \square

By the Nash embedding theorem, we embed N isometrically into \mathbb{R}^N , denoted by $f : N \rightarrow \mathbb{R}^K$. Since $\lambda \in C^\infty(N)$, there exists an extended function $\lambda \in C_0^\infty(\mathbb{R}^K)$ (for simplicity, we still denote it by λ), such that

$$\|\lambda\|_{C^2(\mathbb{R}^K)} \leq C(N) \|\lambda\|_{C^2(N)}.$$

Set

$$\phi' = f \circ \phi \quad \text{and} \quad \psi' = f_* \psi.$$

If we identify ϕ with ϕ' and ψ with ψ' , similarly to the case of $\lambda = 1$ and $\dim M = 2$ considered in [4, 20], we can get the following extrinsic form of the Euler-Lagrange equation:

Lemma 2.2. *Let $(\phi, \psi) \in W^{1,2}(\Omega, N) \times S^{1, \frac{4}{3}}(\mathbb{C}^L \otimes \phi^* TN)$ be a weakly Dirac-harmonic map with λ -curvature term. Then, (ϕ, ψ) satisfies*

$$(2.3) \quad -\Delta \phi = A(d\phi, d\phi) + \operatorname{Re}(P(\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi)) - G(\psi),$$

$$(2.4) \quad \not\partial \psi = \mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi) + F(\psi, \psi)\psi$$

where

$$\begin{aligned} \operatorname{Re}(P(\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi)) &= P(A(\partial_{y^j}, \partial_{y^l}); \partial_{y^i}) \operatorname{Re} \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle; \\ G(\psi) &= \frac{\lambda}{6} (\langle \nabla A_{ij}, A_{kl} \rangle - \langle \nabla A_{il}, A_{jk} \rangle) \operatorname{Re}(\langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle) \\ &\quad - \frac{B^\top}{12} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle; \\ F(\psi, \psi)\psi &= \frac{\lambda}{3} \left(P(A(\partial_{y^j}, \partial_{y^l}); \partial_{y^k}) - P(A(\partial_{y^j}, \partial_{y^k}); \partial_{y^l}) \right) \langle \psi^j, \psi^l \rangle \psi^k, \end{aligned}$$

and $B := (\frac{\partial \lambda}{\partial z^1}, \dots, \frac{\partial \lambda}{\partial z^K})$, B^\top is the tangential part of B along the map ϕ , $P(\cdot; \cdot)$ is the shape operator, i.e.

$$\langle P(\xi; X), Y \rangle = \langle A(X, Y), \xi \rangle$$

for any $X, Y \in \Gamma(TN)$, $\xi \in \Gamma(T^\perp N)$, A is the second fundamental form of N in \mathbb{R}^K and

$$\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi) = (\nabla \phi^i \cdot \psi^j) \otimes A(\partial_{y^i}, \partial_{y^j}).$$

Proof. The proof here is almost the same as the computations in the case of $\lambda = 1$ (see Section 3 in [20] where the inner product for the spinors was taken to be Hermitian as in this paper and hence one needs to take the real parts for certain terms. See also Lemma 3.5 in [4]). We omit the details here. \square

Secondly, we will derive some useful formulae (i.e. Lemma 2.3 and Lemma 2.4) for stationary Dirac-harmonic maps with λ -curvature term which are just Lemma 4.2 and Lemma 4.4 in [40] for $\lambda = 0$ and Proposition 5.3 and Proposition 5.5 in [5] for $\lambda = 1$.

Lemma 2.3. *Let $(\phi, \psi) \in W^{1,2}(\Omega, N) \times S^{1, \frac{4}{3}}(\mathbb{C}^L \otimes \phi^*TN)$ be a weakly Dirac-harmonic map with λ -curvature term. Then (ϕ, ψ) is stationary if and only if for any $Y \in C_0^\infty(\Omega, \mathbb{R}^n)$, there holds*

$$(2.5) \quad \int_{\Omega} \left(\left\langle \frac{\partial \phi}{\partial x^\alpha}, \frac{\partial \phi}{\partial x^\beta} \right\rangle - \frac{1}{2} |\nabla \phi|^2 \delta_{\alpha\beta} + \frac{1}{2} \operatorname{Re} \langle \psi, \frac{\partial}{\partial x^\alpha} \cdot \widetilde{\nabla}_{\frac{\partial}{\partial x^\beta}} \psi \rangle - \frac{\lambda}{12} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle \delta_{\alpha\beta} \right) \frac{\partial Y^\beta}{\partial x^\alpha} = 0.$$

Proof. Let $t \in \mathbb{R}$ small enough and $y = F_t(x) := x + tY(x)$ and $x = F_t^{-1}(y)$. On one hand, by Lemma 4.2 in [40], we have

$$(2.6) \quad \begin{aligned} & \frac{d}{dt} \Big|_{t=0} \frac{1}{2} \int_{\Omega} (|d\phi_t|^2 + \langle \psi_t, \not{D}\psi_t \rangle_{\Sigma M \otimes \phi_t^* TN}) dx \\ &= \int_{\Omega} \left(\left\langle \frac{\partial \phi}{\partial x^\alpha}, \frac{\partial \phi}{\partial x^\beta} \right\rangle - \frac{1}{2} |\nabla \phi|^2 \delta_{\alpha\beta} + \frac{1}{2} \operatorname{Re} \langle \psi, \frac{\partial}{\partial x^\alpha} \cdot \widetilde{\nabla}_{\frac{\partial}{\partial x^\beta}} \psi \rangle \right) \frac{\partial Y^\beta}{\partial x^\alpha} dx. \end{aligned}$$

On the other hand, we have

$$(2.7) \quad \begin{aligned} & \frac{d}{dt} \Big|_{t=0} \frac{1}{2} \int_{\Omega} \frac{\lambda}{6} R_{ikjl} \langle \psi_t^i, \psi_t^j \rangle \langle \psi_t^k, \psi_t^l \rangle dx \\ &= \frac{d}{dt} \Big|_{t=0} \frac{1}{2} \int_{\Omega} \frac{\lambda}{6} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle \operatorname{Jac} F_t^{-1} dx \\ &= -\frac{\lambda}{12} \int_{\Omega} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle \operatorname{div}(Y) dx, \end{aligned}$$

where we used the fact that

$$\frac{d}{dt} \Big|_{t=0} \operatorname{Jac} F_t^{-1} = -\operatorname{div}(Y).$$

Combining (2.6) with (2.7), we will get the conclusion of the lemma. \square

Now, we can derive the monotonicity formula for weakly stationary Dirac-harmonic maps with λ -curvature term (see [40, 5] for the cases of $\lambda = 0, 1$)

Lemma 2.4. *Let $(\phi, \psi) \in W^{1,2}(\Omega, N) \times S^{1, \frac{4}{3}}(\mathbb{C}^L \otimes \phi^*TN)$ be a weakly stationary Dirac-harmonic map with λ -curvature term. Then for any $x_0 \in \Omega$ and $0 < r_1 \leq r_2 < \operatorname{dist}(x_0, \partial\Omega)$, there holds*

$$\begin{aligned} & r_2^{2-m} \int_{B_{r_2}(x_0)} (|\nabla \phi|^2 + \frac{\lambda}{6} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle) dx - r_1^{2-m} \int_{B_{r_1}(x_0)} (|\nabla \phi|^2 + \frac{\lambda}{6} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle) dx \\ &= \int_{B_{r_2}(x_0) \setminus B_{r_1}(x_0)} |x - x_0|^{2-m} (2 \left| \frac{\partial \phi}{\partial r} \right|^2 + \operatorname{Re} \langle \psi, \partial_r \cdot \psi_r \rangle) dx \end{aligned}$$

where $\partial_r = \frac{\partial}{\partial r} = \frac{\partial}{\partial |x-x_0|}$ and $\psi_r = \widetilde{\nabla}_{\partial_r} \psi$.

Proof. For simplicity, we assume $x_0 = 0 \in \Omega$. For any $\epsilon > 0$ and $0 < r < \operatorname{dist}(0, \partial\Omega)$, let $\varphi_\epsilon(x) = \varphi_\epsilon(|x|) \in C_0^\infty(B_r)$ be such that $0 \leq \varphi_\epsilon(x) \leq 1$ and $\varphi_\epsilon(x)|_{B_{(1-\epsilon)r}} = 1$. Taking $Y(x) = x\varphi_\epsilon(x)$ into

the formula (2.5) and noting that

$$\frac{\partial Y^\beta}{\partial x^\alpha} = \varphi_\epsilon(x) \delta_{\alpha\beta} + \frac{x^\alpha x^\beta}{|x|} \varphi'_\epsilon(x),$$

we have

$$\begin{aligned} & (1 - \frac{m}{2}) \int_{B_r} |\nabla \phi|^2 \varphi_\epsilon(x) + \int_{B_r} \frac{1}{2} \operatorname{Re} \langle \psi, \mathbb{D} \psi \rangle \varphi_\epsilon(x) - \int_{B_r} \frac{\lambda m}{12} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle \varphi_\epsilon(x) \\ &= \int_{B_r} \left(-|\frac{\partial \phi}{\partial r}|^2 + \frac{1}{2} |\nabla \phi|^2 - \frac{1}{2} \operatorname{Re} \langle \psi, \partial r \cdot \widetilde{\nabla}_{\partial r} \psi \rangle + \frac{\lambda}{12} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle \right) |x| \varphi'_\epsilon(x). \end{aligned}$$

Using the equation (2.2) and letting $\epsilon \rightarrow 0$, we get

$$\begin{aligned} & (2 - m) \int_{B_r} (|\nabla \phi|^2 + \frac{\lambda}{6} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle) + r \int_{\partial B_r} (|\nabla \phi|^2 + \frac{\lambda}{6} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle) \\ &= r \int_{\partial B_r} (2|\frac{\partial \phi}{\partial r}|^2 + \operatorname{Re} \langle \psi, \partial r \cdot \widetilde{\nabla}_{\partial r} \psi \rangle), \end{aligned}$$

which yields

$$\begin{aligned} & \frac{d}{dr} \left(r^{2-m} \int_{B_r} (|\nabla \phi|^2 + \frac{\lambda}{6} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle) dx \right) \\ &= r^{2-m} \int_{\partial B_r} (2|\frac{\partial \phi}{\partial r}|^2 + \operatorname{Re} \langle \psi, \partial r \cdot \widetilde{\nabla}_{\partial r} \psi \rangle). \end{aligned}$$

The conclusion of the lemma follows by integrating r from r_1 to r_2 . □

The following corollary is a small extension of the case of $\lambda = 0$ considered in [40]:

Corollary 2.5. *Let $(\phi, \psi) \in W^{1,2}(\Omega, N) \times S^{1, \frac{4}{3}}(\mathbb{C}^L \otimes \phi^* TN)$ be a weakly stationary Dirac-harmonic map with λ -curvature term. If we assume*

$$\|\psi\|_{W^{1,p}(\Omega)} \leq \Lambda$$

for some $\frac{2m}{3} < p < m$, then for any $x_0 \in \Omega$ and $0 < r_1 \leq r_2 < \min\{\operatorname{dist}(x_0, \partial\Omega), 1\}$, there holds

$$\begin{aligned} r_1^{2-m} \int_{B_{r_1}(x_0)} |\nabla \phi|^2 dx &\leq r_2^{2-m} \int_{B_{r_2}(x_0)} |\nabla \phi|^2 dx + C(m) \|\psi\|_{L^{\frac{mp}{m-p}}(B_{r_2}(x_0))} \|\nabla \psi\|_{L^p(B_{r_2}(x_0))} r_2^{3-\frac{2m}{p}} \\ &\quad + C(m, N) \Lambda_1 \|\psi\|_{L^{\frac{mp}{m-p}}(B_{r_2}(x_0))}^4 r_2^{6-\frac{4m}{p}}, \end{aligned}$$

where Λ_1 is as defined in (1.6).

Proof. By Lemma 2.4, we know

$$\begin{aligned} r_1^{2-m} \int_{B_{r_1}(x_0)} |\nabla \phi|^2 dx &\leq r_2^{2-m} \int_{B_{r_2}(x_0)} (|\nabla \phi|^2 + \frac{\lambda}{6} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle) dx \\ &\quad + \int_{B_{r_2}(x_0) \setminus B_{r_1}(x_0)} |x - x_0|^{2-m} |\psi| |\nabla \psi| dx \\ &\quad - r_1^{2-m} \int_{B_{r_1}(x_0)} \frac{\lambda}{6} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle dx, \end{aligned}$$

which implies

$$\begin{aligned} r_1^{2-m} \int_{B_{r_1}(x_0)} |\nabla \phi|^2 dx &\leq r_2^{2-m} \int_{B_{r_2}(x_0)} |\nabla \phi|^2 dx + C(N) \Lambda_1 r_2^{2-m} \int_{B_{r_2}(x_0)} |\psi|^4 dx \\ (2.8) \quad &\quad + \int_{B_{r_2}(x_0) \setminus B_{r_1}(x_0)} |x - x_0|^{2-m} |\psi| |\nabla \psi| dx + C(N) \Lambda_1 r_1^{2-m} \int_{B_{r_1}(x_0)} |\psi|^4 dx. \end{aligned}$$

By Sobolev's embedding and Young's inequality, we have

$$\begin{aligned} \int_{B_r(x_0)} |x - x_0|^{2-m} |\psi| |\nabla \psi| dx &\leq \|\psi\|_{L^{\frac{mp}{m-p}}(B_r(x_0))} \|\nabla \psi\|_{L^p(B_r(x_0))} \| |x - x_0|^{2-m} \|_{L^{\frac{mp}{mp-2m+p}}(B_r(x_0))} \\ (2.9) \quad &\leq C(m) \|\psi\|_{L^{\frac{mp}{m-p}}(B_r(x_0))} \|\nabla \psi\|_{L^p(B_r(x_0))} r^{3-\frac{2m}{p}} \end{aligned}$$

and

$$(2.10) \quad r_1^{2-m} \int_{B_{r_1}(x_0)} |\psi|^4 dx + r_2^{2-m} \int_{B_{r_2}(x_0)} |\psi|^4 dx \leq C(m) \|\psi\|_{L^{\frac{mp}{m-p}}(B_{r_2}(x_0))}^4 r_2^{6-\frac{4m}{p}}.$$

□

Then the conclusion of the corollary follows immediately from (2.8).

3. PROOF OF THEOREM 1.2 AND THEOREM 1.4

In this section, we will prove our main results: Theorem 1.2 and Theorem 1.4.

For Theorem 1.2, we will firstly use the idea in [39] to raise the integrability of ψ . Let us recall the definition of Morrey spaces (see [29]). For $p \geq 1$, $0 < \mu \leq m$ and a domain $U \subset \mathbb{R}^m$, the Morrey space $M^{p,\mu}(U)$ is defined by

$$M^{p,\mu}(U) := \{f \in L_{loc}^p(U) \mid \|f\|_{M^{p,\mu}(U)} < \infty\}$$

where

$$\|f\|_{M^{p,\mu}(U)}^p := \sup_{B_r \subset U} r^{\mu-m} \int_{B_r} |f|^p.$$

Lemma 3.1. *For any $4 < p < \infty$ and $m \geq 2$, there exists a positive constant $\epsilon_1 = \epsilon_1(p, m, N) > 0$ and $C = C(m, p, N) > 0$, such that if (ϕ, ψ) is a weak solution of (2.4) and*

$$\|\nabla \phi\|_{M^{2,2}(B_1)} + \| |\lambda| |\psi|^2 \|_{M^{2,2}(B_1)} \leq \epsilon_1,$$

then $\psi \in L^p(B_{1/2})$ and satisfies the estimate

$$(3.1) \quad \|\psi\|_{L^p(B_{1/2})} \leq C(m, p, N) \|\psi\|_{M^{4,2}(B_1)}.$$

The idea of proving this lemma is similar to Lemma 2.2 in [39] which has been applied to some other Dirac type equation in dimension 2 in [37, 4]. Recently, [24] (Lemma 6.1) proved a similar lemma for a more general equation in higher dimensions which can be used in our case.

Proof. By (2.4), it is easy to see that ψ satisfies the equation of the form (5.1) in the Appendix with

$$|A| \leq C(N)(|\nabla\phi| + |\lambda||\psi|^2), \quad B \equiv 0,$$

the conclusion of the lemma follows from Lemma 5.1 in the Appendix (or Lemma 6.1 in [24]) immediately. \square

Combining Lemma 3.1 with Theorem 5.2, we can now prove Theorem 1.2.

Proof of Theorem 1.2. Without loss of generality, we may assume $r_0 = 1$. By assumption (1.7), it is easy to see that

$$\|\nabla\phi\|_{M^{2,2}(B_1(x_0))} + \| |\psi|^2 \|_{M^{2,2}(B_1(x_0))} \leq \epsilon_0.$$

If $\epsilon_0 \leq \frac{\epsilon_1}{1+\Lambda_1}$, by Lemma 3.1, we have $\psi \in L^{\frac{4mq}{2+q}}(B_{\frac{3}{4}}(x_0))$ for any $2 < q < \infty$ and

$$\|\psi\|_{L^{\frac{4mq}{2+q}}(B_{\frac{3}{4}}(x_0))} \leq C(m, q, N) \|\psi\|_{M^{4,2}(B_1(x_0))}.$$

Thus

$$G(\psi) \in L^{\frac{mq}{2+q}}(B_{\frac{3}{4}}(x_0)).$$

By slightly modifying the extrinsic equations for Dirac-harmonic maps (i.e., the case of $\lambda = 0$) considered in [43, 11, 37] (see equations (3.6) and (3.8) in [37]), it is easy to see that the equation (2.3) for the map can be written as the following form

$$\Delta\phi = \widehat{\Omega} \cdot \nabla\phi + f$$

with an antisymmetric potential $\widehat{\Omega}$ satisfying

$$|\widehat{\Omega}| \leq \widetilde{C}(N)(|\nabla\phi| + |\psi|^2),$$

and with an error term f satisfying

$$|f| = |G(\psi)| \leq C(\Lambda_1, N)|\psi|^4.$$

Take $\epsilon_0 = \min\{\frac{\epsilon_1}{1+\Lambda_1}, \frac{\epsilon}{\widetilde{C}(N)}\}$, where ϵ is the constant in Theorem 5.2 in the Appendix. By Theorem 5.2, we know $\nabla\phi \in M^{q,2}(B_{\frac{5}{8}}(x_0))$ for any $2 < q < \infty$ and

$$\begin{aligned} \|\nabla\phi\|_{M^{q,2}(B_{\frac{5}{8}}(x_0))} &\leq C(m, q, N)(\|\nabla\phi\|_{L^2(B_{\frac{3}{4}}(x_0))} + \|G(\psi)\|_{L^{\frac{mq}{2+q}}(B_{\frac{3}{4}}(x_0))}) \\ &\leq C(m, q, \Lambda_1, N)(\|\nabla\phi\|_{L^2(B_1(x_0))} + \|\psi\|_{M^{4,2}(B_1(x_0))}), \end{aligned}$$

which implies $|\Delta\phi| \in L^q(B_{\frac{5}{8}}(x_0))$ and for some $q > m$. The elliptic theory tells us $\phi \in W^{2,q}(B_{\frac{1}{2}}(x_0))$. Thus $\phi \in C^{1,\alpha}(B_{\frac{1}{2}}(x_0))$ for some $\alpha > 0$. Then by (2.4) and the standard first order elliptic estimates Lemma 5.3, we get $\psi \in W^{1,q}(B_{\frac{1}{2}}(x_0))$ which yields $\psi \in C^{1,\alpha}(B_{\frac{1}{2}}(x_0))$ and (1.8) holds. The higher order regularity then follows from the classical Schauder estimates for the Laplace and Dirac equation (see Lemma 5.4 in the Appendix) and a standard bootstrap argument. \square

Now, we prove our main Theorem 1.4.

Proof of Theorem 1.4. Without loss of generality, we assume $\lambda \neq 0$ (for $\lambda = 0$, one can see [40]). Let $\epsilon_0 > 0$ be the constant in Theorem 1.2. Define

$$(3.2) \quad S(\phi) := \{x \in \Omega : \liminf_{r \searrow 0} r^{2-m} \int_{B_r(x)} |\nabla \phi|^2 \geq \frac{\epsilon_0^2}{2^m}\}.$$

It is well known that $H^{n-2}(S(\phi)) = 0$. Next, we show $S(\phi)$ is a closed set and $(\phi, \psi) \in C^\infty(\Omega \setminus S(\phi))$.

For any $x_0 \in \Omega \setminus S(\phi)$ and $\epsilon > 0$, there exists $0 < r_0 < \epsilon$ such that,

$$(3.3) \quad (2r_0)^{2-m} \int_{B_{2r_0}(x_0)} |\nabla \phi|^2 dx < \frac{\epsilon_0^2}{2^m}.$$

Therefore,

$$(3.4) \quad \sup_{z \in B_{r_0}(x_0)} r_0^{2-m} \int_{B_{r_0}(z)} |\nabla \phi|^2 dx < \frac{\epsilon_0^2}{4}.$$

By Corollary 2.5, for any $0 < r_0 < \frac{1}{2} \min\{\text{dist}(x_0, \partial\Omega), 1\}$, we have

$$(3.5) \quad \begin{aligned} & \sup_{z \in B_{r_0}(x_0), 0 < r \leq r_0} r^{2-m} \int_{B_r(z)} (|\nabla \phi|^2 + |\psi|^4) dx \\ & \leq \sup_{z \in B_{r_0}(x_0)} r_0^{2-m} \int_{B_{r_0}(z)} |\nabla \phi|^2 dx + C(m) \|\psi\|_{L^{\frac{mp}{m-p}}(B_{2r_0}(x_0))} \|\nabla \psi\|_{L^p(B_{2r_0}(x_0))} r_0^{3-\frac{2m}{p}} \\ & \quad + C(m, N)(1 + \Lambda_1) \|\psi\|_{L^{\frac{mp}{m-p}}(B_{2r_0}(x_0))}^4 r_0^{6-\frac{4m}{p}} \\ & \leq \frac{\epsilon_0^2}{4} + C(m, p, \Omega, N)(\Lambda^2 + (1 + \Lambda_1)\Lambda^4) r_0^{3-\frac{2m}{p}}, \end{aligned}$$

where the last inequality follows from Sobolev's embedding $W^{1,p}(\Omega) \hookrightarrow L^{\frac{mp}{m-p}}(\Omega)$.

Taking $\epsilon \leq \left(\frac{\epsilon_0^2}{4C(m,p,\Omega,N)(\Lambda^2+(1+\Lambda_1)\Lambda^4)}\right)^{\frac{2m}{p}-3}$, we get

$$(3.6) \quad \sup_{z \in B_{r_0}(x_0), 0 < r \leq r_0} r^{2-m} \int_{B_r(z)} (|\nabla \phi|^2 + |\psi|^4) dx \leq \frac{\epsilon_0^2}{2}.$$

Then Theorem 1.2 tells us that $(\phi, \psi) \in C^\infty(B_{r_0/2}(x_0))$ which implies $B_{r_0/4}(x_0) \subset \Omega \setminus S(\phi)$. We finished the proof. \square

4. PROOF OF THEOREM 1.5

In this section, we consider a weakly converging sequence of stationary Dirac-harmonic maps with λ -curvature term.

Let $\{(\phi_n, \psi_n)\}$ be a sequence of stationary Dirac-harmonic maps with λ -curvature term with bounded energy

$$E(\phi_n, \psi_n) = \int_{\Omega} (|\nabla \phi_n|^2 + |\psi_n|^4) \leq \Lambda.$$

Additionally, we assume

$$\|\psi_n\|_{W^{1,p}(\Omega)} \leq \Lambda$$

for some $p > \frac{2m}{3}$. Similar to harmonic maps [34], define the energy concentration set Σ as follows

$$(4.1) \quad \Sigma = \{x \in \Omega \mid \liminf_{r \searrow 0} \liminf_{n \rightarrow \infty} r^{2-m} \int_{B_r(x)} |\nabla \phi_n|^2 dx \geq \epsilon_0\}.$$

Suppose $(\phi_n, \psi_n) \rightharpoonup (\phi, \psi)$ weakly in $W^{1,2}(\Omega, N) \times L^4(\mathbb{C}^L \otimes \phi_n^* TN)$ and

$$\mu_n := |\nabla \phi_n|^2 dx \rightarrow \mu = |\nabla \phi|^2 dx + \nu$$

in the sense of Radon measures.

Without loss of generality, we assume $B_1(0) \subseteq \Omega$. Then, we have

Lemma 4.1. *Let $\{(\phi_n, \psi_n)\}$ be a sequence of stationary Dirac-harmonic maps with λ -curvature term with bounded energy and $\|\psi_n\|_{W^{1,p}} \leq \Lambda$ for some $p > \frac{2m}{3}$. Denote*

$$(4.2) \quad \Sigma = \{x \in B_1 \mid \liminf_{r \searrow 0} \liminf_{n \rightarrow \infty} r^{2-m} \int_{B_r(x)} |\nabla \phi_n|^2 dx \geq \frac{\epsilon_0^2}{2^m}\},$$

where ϵ_0 is the constant in Theorem 1.2, then Σ is closed in B_1 and $H^{m-2}(\Sigma) \leq C(\epsilon_0, m, \Lambda)$. Moreover,

$$(4.3) \quad \Sigma = \text{spt}(\nu) \cup \text{sing}(\phi),$$

where $\text{sing}(\phi)$ denoted the singular set of ϕ , i.e. for any $x_0 \in \text{sing}(\phi)$, ϕ is not smooth at x_0 .

Proof. For $x_0 \in B_1 \setminus \Sigma$, by the proof of Theorem 1.4, there exists a positive constant $r_0 > 0$ and a subsequence of $\{n\}$ (also denoted by $\{n\}$), such that, for any n , there holds

$$(2r_0)^{2-m} \int_{B_{2r_0}(x)} |\nabla \phi_n|^2 dx < \frac{\epsilon_0^2}{2^m},$$

which implies (similar to deriving (3.6))

$$\sup_{z \in B_{r_0}(x), 0 < r \leq r_0} r^{2-m} \int_{B_r(z)} (|\nabla \phi_n|^2 + |\psi_n|^4) dx < \frac{\epsilon_0^2}{2}.$$

By Theorem 1.2, we know

$$(4.4) \quad r_0 \|\nabla \phi_n\|_{L^\infty(B_{r_0/2}(x_0))} + \sqrt{r_0} \|\psi_n\|_{L^\infty(B_{r_0/2}(x_0))} \leq C(m, r_0, \epsilon_0, \Lambda_1, N).$$

Then, it is easy to see that there exists a small positive constant $r_1 = r_1(m, r_0, \epsilon_0, \lambda_1, N)$, such that, whenever $r \leq r_1$,

$$\sup_{x \in B_{r_0/4}(x_0)} r^{2-m} \int_{B_r(x)} |\nabla \phi_n|^2 dx < \frac{\epsilon_0^2}{2^m}.$$

Thus, $B_{r_0/4}(x_0) \subset B_1 \setminus \Sigma$. So, Σ is a closed set.

It is standard to get $H^{m-2}(\Sigma) \leq C(\epsilon_0, m, \Lambda)$ by a covering lemma (cf. [26]).

For (4.3), on the one hand, let $x_0 \in B_1 \setminus \Sigma$. Then (4.4) holds and by standard elliptic estimates, we have

$$(4.5) \quad \|\phi_n\|_{C^{1+\alpha}(B_{r_0/4}(x_0))} + \|\psi_n\|_{C^\alpha(B_{r_0/2}(x_0))} \leq C,$$

for some $0 < \alpha < 1$. So, there exists a subsequence of $\{\phi_n, \psi_n\}$ (also denoted by $\{\phi_n, \psi_n\}$) such that $\phi_n \rightarrow \phi$ strongly in $W^{1,2}$ and $\phi \in C^\infty(B_{r_0/8}(x_0))$ which imply that $x_0 \notin \text{sing}(\phi)$ and $x_0 \notin \text{spt} \nu$ since $\nu \equiv 0$ on $B_{r_0/8}(x_0)$.

On the other hand, let $x_0 \in \Sigma$, by the definition, for any $r > 0$ small enough, when n is sufficient large, we have

$$\frac{\mu_n(B_r(x_0))}{r^{m-2}} \geq \frac{\epsilon_0^2}{2^{m+1}}.$$

Letting $n \rightarrow \infty$,

$$\frac{\mu(B_r(x_0))}{r^{m-2}} \geq \frac{\epsilon_0^2}{2^{m+1}}$$

for a.e. $r > 0$. Suppose $x_0 \notin \text{sing}(\phi)$, then

$$r^{2-m} \int_{B_r(x_0)} |\nabla \phi|^2 dx \leq \frac{\epsilon_0^2}{2^{m+2}}$$

whenever $r > 0$ is small enough. Then we have

$$\frac{\nu(B_r(x_0))}{r^{m-2}} \geq \frac{\epsilon_0^2}{2^{m+2}}$$

for all small positive $r > 0$ and $x_0 \in \text{spt} \nu$. This finishes the proof of lemma. \square

Lemma 4.2. *Under the assumption of the preceding lemma, the limit*

$$(4.6) \quad \lim_{r \rightarrow 0} \frac{\nu(B_r(x))}{r^{m-2}}$$

exists for H^{m-2} a.e. $x \in \Sigma$. If we denote it by $\theta_\nu(x)$, then

$$\frac{\epsilon_0^2}{2^m} \leq \theta_\nu(x) \leq C(m, p, \Omega, \Lambda_1, \Lambda, N) \delta_0^{2-m},$$

where $\delta_0 := \text{dist}(B_1(0), \partial\Omega)$.

Proof. For any $x \in \Omega$ and any two sequence $s_i \rightarrow 0$, $t_j \rightarrow 0$, by Corollary 2.5 and Sobolev's embedding $W^{1,p}(\Omega) \hookrightarrow L^{\frac{mp}{m-p}}(\Omega)$, we have

$$(4.7) \quad \frac{\mu_n(B_{s_i}(x))}{s_i^{m-2}} \leq \frac{\mu_n(B_{t_j}(x))}{t_j^{m-2}} + C(m, p, \Omega, \Lambda_1, \Lambda, N)(t_j)^{3-\frac{2m}{p}}$$

for $s_i \leq t_j$. Letting firstly $i \rightarrow \infty$ and secondly $j \rightarrow \infty$, we get

$$\limsup_{r \rightarrow 0} \frac{\mu(B_r(x))}{r^{m-2}} \leq \liminf_{r \rightarrow 0} \frac{\mu(B_r(x))}{r^{m-2}},$$

which implies that

$$\lim_{r \searrow 0} \frac{\mu(B_r(x))}{r^{m-2}}$$

exists for any $x \in \Omega$.

Noting that for H^{m-2} a.e. $x \in \Omega$,

$$(4.8) \quad \lim_{r \rightarrow 0} r^{2-m} \int_{B_r(x)} |\nabla \phi|^2 dx = 0,$$

therefore,

$$\lim_{r \rightarrow 0} \frac{\nu(B_r(x))}{r^{m-2}} = \lim_{r \rightarrow 0} \frac{\mu(B_r(x))}{r^{m-2}}.$$

Obviously, from (4.7), we can get

$$r^{2-m} \mu(B_r(x)) \leq C(\Lambda) \delta_0^{2-m} + C(m, p, \Omega, \Lambda_1, \Lambda, N) (\delta_0)^{3-\frac{2m}{p}} \leq C(m, p, \Omega, \Lambda_1, \Lambda, N) \delta_0^{2-m}.$$

This implies $\mu|_\Sigma$ is absolutely continuous with respect to $H^{m-2}|_\Sigma$ and the Radon-Nikodym theorem tells us that there exists a measurable function $\theta(x)$ such that

$$\mu|_\Sigma = \theta(x) H^{m-2}|_\Sigma.$$

Noting that for H^{m-2} a.e. $x \in \Sigma$,

$$2^{2-m} \leq \liminf_{r \rightarrow 0} \frac{H^{m-2}(\Sigma \cap B_r(x))}{r^{m-2}} \leq \limsup_{r \rightarrow 0} \frac{H^{m-2}(\Sigma \cap B_r(x))}{r^{m-2}} \leq 1$$

and (4.8), we have

$$\nu|_\Sigma = \theta(x) H^{m-2}|_\Sigma$$

and

$$\frac{\epsilon_0}{2^m} \leq \theta_\nu(x) = \theta(x) \leq C(m, p, \Omega, \Lambda_1, \Lambda, N) \delta_0^{2-m}.$$

□

By modifying Lin's method in [26] or applying Preiss's result [30], we have

Corollary 4.3. *The set of energy concentration points Σ is $(m-2)$ -rectifiable.*

For any $x \in \Sigma$ and $\lambda > 0$, we define a scaled Radon measure $\mu_{y,\lambda}$ by

$$\mu_{y,\lambda}(A) = \lambda^{2-m} \mu(y + \lambda A).$$

If there is a Radon measure μ_* such that

$$\mu_{y,\lambda} \rightarrow \mu_*$$

in the sense of Radon measure as $\lambda \searrow 0$, then we say that μ_* is the tangent measure of μ at y . (See [14, 38].)

Lemma 4.4. *Suppose $H^{m-2}(\Sigma) > 0$, then there exists a nonconstant harmonic sphere S^2 into N .*

Before we prove this lemma, let us state a basic proposition for the Dirac operator.

Proposition 4.5. *Suppose $\phi \in C^2(M, N)$, $\psi \in C^2(M, \Sigma M \otimes \phi^* TN)$. Let $\{e_\alpha\}_{\alpha=1}^m$ be a unit normal basis of TM and $e_\beta \in \Gamma(TM)$ a section satisfying*

$$[e_\beta, e_\alpha] = 0, \quad \alpha = 1, \dots, m,$$

then

$$(4.9) \quad \langle \psi, \widetilde{\nabla}_{e_\beta}(\not{D}\psi) \rangle = 2 \langle \text{Re}(P(\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi)), \phi_\beta \rangle + \langle \psi, \not{D}\psi_\beta \rangle,$$

where $[\cdot, \cdot]$ is the Lie bracket, $\phi_\beta = d\phi(e_\beta)$ and $\psi_\beta = \widetilde{\nabla}_{e_\beta} \psi$.

Proof. The proof is similar to the Proposition 2.2 in [22] (see also the computations of Proposition 3.3 in [9]), where the case of a two dimensional domain was considered. \square

Proof of Lemma 4.4. Since Σ is $(m-2)$ -rectifiable, we can find a point $x_0 \in \Sigma$, such that ν has a tangent measure at x_0 and

$$\nu_* = \theta(x_0)H^{m-2} \llcorner \Sigma_*$$

where $\Sigma_* \subset \mathbb{R}^m$ is a $(m-2)$ linear subspace which is usually called the tangent space of Σ at x_0 . Without loss of generality, we assume $x_0 = 0$ and $\Sigma_* = \mathbb{R}^{m-2} \times \{(0, 0)\}$.

In fact, by a diagonal argument, we can find a sequence $r_n \rightarrow 0$, such that,

$$|\nabla u_n|^2 dx \rightarrow \nu_*$$

in the sense of Radon measures (cf. [26]), where

$$(u_n(x), v_n(x)) := (\phi_n(x_0 + r_n x), \sqrt{r_n} \psi_n(x_0 + r_n x)).$$

It is easy to see that (u_n, v_n) is also a stationary Dirac-harmonic map with λ -curvature term. By Lemma 2.4, we have

$$\begin{aligned} & \int_{r_1}^{r_2} \int_{\partial B_r(x_0)} |x - x_0|^{2-m} (2|\frac{\partial u_n}{\partial r}|^2 + \operatorname{Re} \langle v_n, \partial_r \cdot \bar{\nabla}_{\partial_r} v_n \rangle) dH^{n-1} dr \\ &= r_2^{2-m} \int_{B_{r_2}(x_0)} (|\nabla u_n|^2 + \frac{\lambda}{6} R_{ikjl} \langle v_n^i, v_n^j \rangle \langle v_n^k, v_n^l \rangle) dx \\ & - r_1^{2-m} \int_{B_{r_1}(x_0)} (|\nabla u_n|^2 + \frac{\lambda}{6} R_{ikjl} \langle v_n^i, v_n^j \rangle \langle v_n^k, v_n^l \rangle) dx. \end{aligned} \quad (4.10)$$

By (2.9) and (2.10), we have

$$\begin{aligned} & \int_{B_r(x_0)} |x - x_0|^{2-m} |v_n| |\nabla v_n| dx \leq C(m) \|v_n\|_{L^{\frac{mp}{m-p}}(B_r(x_0))} \|\nabla v_n\|_{L^p(B_r(x_0))} r^{3-\frac{2m}{p}} \\ &= C(m) \|\psi_n\|_{L^{\frac{mp}{m-p}}(B_r(x_0))} \|\nabla \psi_n\|_{L^p(B_r(x_0))} (rr_n)^{3-\frac{2m}{p}} \\ &\leq C(m, p, \Lambda, \Omega) (rr_n)^{3-\frac{2m}{p}} \end{aligned} \quad (4.11)$$

and

$$r_1^{2-m} \int_{B_{r_1}(x_0)} |\psi|^4 dx + r_2^{2-m} \int_{B_{r_2}(x_0)} |\psi|^4 dx \leq C(m, p, \Lambda, \Omega) (r_2 r_n)^{6-\frac{4m}{p}}. \quad (4.12)$$

Since $r_2^{2-m} \nu_*(B_{r_2}(0)) = r_1^{2-m} \nu_*(B_{r_1}(0))$, letting $n \rightarrow \infty$ in (4.10), we get

$$\lim_{n \rightarrow \infty} \int_{B_2(0)} |\frac{\partial u_n}{\partial |x|}|^2 dx = 0. \quad (4.13)$$

Noting that $\nu_{*y,r} = \nu_*$ for $y \in \Sigma_*$, we also have

$$\lim_{n \rightarrow \infty} \int_{B_2(0)} |\frac{\partial u_n}{\partial |x-y|}|^2 dx = 0, \text{ for } y \in \Sigma_* \cap B_2. \quad (4.14)$$

These imply

$$(4.15) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{m-2} \int_{B_2(0)} \left| \frac{\partial u_n}{\partial x^k} \right|^2 dx = 0.$$

Let $x' = (x_1, \dots, x_{m-2})$, $x'' = (x_{m-1}, x_m)$, define $f_n : B_1^{m-2} \rightarrow \mathbb{R}$ by

$$f_n(x') := \sum_{k=1}^{m-2} \int_{B_1^2(0)} \left| \frac{\partial u_n}{\partial x_k} \right|^2(x', x'') dx''.$$

Then,

$$\lim_{n \rightarrow \infty} \|f_n(x')\|_{L^1(B_1^{m-2}(0))} = 0.$$

Let $M(f_n)(x')$ be the Hardy-Littlewood maximal function, i.e.

$$M(f_n)(x) = \sup_{0 < r < \frac{1}{2}} r^{2-m} \int_{B_r^{m-2}(x)} f_n(x') dx', \quad x \in B_{1/2}^{m-2}(0).$$

By the weak L^1 -estimate, for any $\rho > 0$, we have

$$|\{x \in B_{1/2}^{m-2}(0) | M(f_n) > \rho\}| \leq \frac{C(m)}{\rho} \|f_n\|_{L^1(B_{1/2}^{m-2}(0))},$$

which implies

$$|\{x \in B_{1/2}^{m-2}(0) | \limsup_{n \rightarrow \infty} M(f_n) > 0\}| = 0.$$

Combining this with Theorem 1.4, there exists a sequence of points $\{x'_n \in B_{1/2}^{m-2}(0)\}$, such that (u_n, v_n) is smooth near (x'_n, x'') for all $x'' \in B_1^2(0)$ and

$$(4.16) \quad \lim_{n \rightarrow \infty} M(f_n)(x'_n) = 0.$$

By the blow-up argument in [26], there exist sequences $\{\sigma_n\}$ and $\{x''_n\} \subset B_{1/2}^2(0)$ such that $\sigma_n \rightarrow 0$, $x''_n \rightarrow (0, 0)$ and

$$(4.17) \quad \max_{x'' \in B_{1/2}^2(0)} \sigma_n^{2-m} \int_{B_{\sigma_n}^{m-2}(x'_n) \times B_{\sigma_n}^2(x'')} |\nabla u_n|^2 dx = \frac{\epsilon_0^2}{C_1(m)},$$

where the maximum is achieved at the point x''_n and $C_1(m) \geq 4^m$ is a positive constant to be determined later.

In fact, denote

$$g_n(\sigma) := \max_{x'' \in B_{1/2}^2(0)} \sigma^{2-m} \int_{B_{\sigma}^{m-2}(x'_n) \times B_{\sigma}^2(x'')} |\nabla u_n|^2 dx.$$

On one hand, since (u_n, v_n) is smooth near $x'_n \times B_1^2(0)$, we have

$$\lim_{\sigma \rightarrow 0} g_n(\sigma) = 0.$$

On the other hand, for any $\sigma > 0$, when n is big enough, we must have

$$g_n(\sigma) \geq \frac{\epsilon_0^2}{2^m}.$$

Otherwise, by Theorem 1.2 (similar to deriving (3.6)), u_n will converge strongly in $W^{1,2}$ which is contradict to $|\nabla u_n|^2 dx \rightarrow \nu_*$. Thus, there exists σ_n , such that

$$g_n(\sigma_n) = \frac{\epsilon_0^2}{C_1(m)}$$

and we may assume the maximum is achieved at x_n'' . Next, we show $\sigma_n \rightarrow 0$ and $x_n'' \rightarrow (0, 0)$.

If $\sigma_n \geq \delta > 0$, by Corollary 2.5, we have

$$\frac{\epsilon_0^2}{C_1(m)} = \limsup_{n \rightarrow \infty} g_n(\sigma_n) \geq \limsup_{n \rightarrow \infty} \left(g_n(\delta) - C(m, p, \Lambda_1, \Lambda, \Omega, N)(r_n \sigma_n)^{3 - \frac{2m}{p}} \right) \geq \frac{\epsilon_0}{2^m},$$

which is a contradiction.

If $x_n'' \rightarrow x_0'' \in B_{1/2}^2(0)$ and $x_0'' \neq (0, 0)$, for any $0 < \sigma < \frac{|x_0''|}{2}$,

$$\frac{\epsilon_0}{2^m} \leq \limsup_{n \rightarrow \infty} g_n(\sigma) \leq \sigma^{2-m} \nu_*(B_1^{m-2}(0) \times B_{2\sigma}^2(x_0'')) = 0.$$

This is also a contradiction. □

Let $x_n = (x_n', x_n'')$ and

$$(\widetilde{u}_n(x), \widetilde{v}_n(x)) := (u_n(x_n + \sigma_n x), \sqrt{\sigma_n} v_n(x_n + \sigma_n x)).$$

Then $(\widetilde{u}_n(x), \widetilde{v}_n(x))$ is a stationary Dirac-harmonic map with λ -curvature term defined on $B_{R_n}^{m-2}(0) \times B_{R_n}^2(0)$, where $R_n = \frac{1}{4\sigma_n}$ which tends to infinity as $n \rightarrow \infty$.

By (4.16), we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sup_{0 < R < R_n} R^{2-m} \int_{B_R^{m-2}(0) \times B_{R_n}^2(0)} \sum_{k=1}^{m-2} \left| \frac{\partial \widetilde{u}_n}{\partial x_k} \right|^2 dx \\ &= \lim_{n \rightarrow \infty} \sup_{0 < R < R_n} (\sigma_n R)^{2-m} \int_{B_{\sigma_n R}^{m-2}(x_n') \times B_{\sigma_n R_n}^2(x_n'')} \sum_{k=1}^{m-2} \left| \frac{\partial u_n}{\partial x_k} \right|^2 dx \\ (4.18) \quad & \leq \lim_{n \rightarrow \infty} M(f_n)(x_n') = 0. \end{aligned}$$

By (4.17), we get

$$(4.19) \quad \frac{\epsilon_0^2}{C_1(m)} = \int_{B_1^{m-2}(0) \times B_1^2(0)} |\nabla \widetilde{u}_n|^2 dx = \max_{x'' \in B_{R_n-1}^2(0)} \int_{B_1^{m-2}(0) \times B_1^2(x'')} |\nabla \widetilde{u}_n|^2 dx.$$

By Corollary 2.5, for any $R > 0$, we obtain

$$\begin{aligned} & \int_{B_R^{m-2}(0) \times B_R^2(0)} |\nabla \widetilde{u}_n|^2 dx = (\sigma_n)^{2-m} \int_{B_{\sigma_n R}^{m-2}(x_n') \times B_{\sigma_n R}^2(x_n'')} |\nabla u_n|^2 dx \\ (4.20) \quad & \leq C(m, p, \delta_0, \Lambda_1, \Lambda, \Omega, N) R^{m-2}, \end{aligned}$$

when n is big enough.

Let $\zeta \in C_0^\infty(B_1^{m-2}(0))$ be a cut-off function such that $0 \leq \zeta \leq 1$ and $\zeta|_{B_{1/2}^{m-2}(0)} \equiv 1$. Let $\eta \in C_0^\infty(B_1^2(0))$ also be a cut-off function such that $0 \leq \zeta \leq 1$ and $\eta|_{B_{1/2}^2(0)} \equiv 1$. Similarly to [26], for any

$R > 0$, we define $F_n(a) : B_6^{m-2}(0) \times B_R^2(0) \rightarrow \mathbb{R}$ as follows:

$$F_n(a) = \int_{B_1^{m-2}(0) \times B_1^2(0)} |\nabla \widetilde{u}_n|^2(a+x) \zeta(x') \eta(x'') dx.$$

Computing directly, one has

$$\begin{aligned} \frac{\partial F_n(a)}{\partial a_k} &= \int_{B_1^{m-2}(0) \times B_1^2(0)} \frac{\partial}{\partial x_k} |\nabla \widetilde{u}_n|^2(a+x) \zeta(x') \eta(x'') dx \\ &= 2 \int_{B_1^{m-2}(0) \times B_1^2(0)} \left\langle \frac{\partial \widetilde{u}_n}{\partial x_l}, \frac{\partial^2 \widetilde{u}_n}{\partial x_l \partial x_k} \right\rangle (a+x) \zeta(x') \eta(x'') dx \\ &= -2 \int_{B_1^{m-2}(0) \times B_1^2(0)} \left\langle \Delta \widetilde{u}_n, \frac{\partial \widetilde{u}_n}{\partial x_k} \right\rangle (a+x) \zeta(x') \eta(x'') dx \\ &\quad - 2 \int_{B_1^{m-2}(0) \times B_1^2(0)} \left\langle \frac{\partial \widetilde{u}_n}{\partial x_l}, \frac{\partial \widetilde{u}_n}{\partial x_k} \right\rangle (a+x) \frac{\partial}{\partial x_l} (\zeta(x') \eta(x'')) dx. \end{aligned}$$

On the one hand, by Proposition 4.5,

$$(4.21) \quad \langle \widetilde{v}_n, \widetilde{\nabla}_{\frac{\partial}{\partial x^k}} (\mathcal{D} \widetilde{v}_n) \rangle = 2 \langle \text{Re} (P(\mathcal{A}(d\widetilde{u}_n(e_\alpha), e_\alpha \cdot \widetilde{v}_n); \widetilde{v}_n)), \nabla_{\frac{\partial}{\partial x^k}} \widetilde{u}_n \rangle + \langle \widetilde{v}_n, \mathcal{D} \widetilde{\nabla}_{\frac{\partial}{\partial x^k}} \widetilde{v}_n \rangle,$$

and (2.3), we have

$$\begin{aligned} &-2 \int_{B_1^{m-2}(0) \times B_1^2(0)} \left\langle \Delta \widetilde{u}_n, \frac{\partial \widetilde{u}_n}{\partial x_k} \right\rangle (a+x) \zeta(x') \eta(x'') dx \\ &= \int_{B_1^{m-2}(0) \times B_1^2(0)} \langle \widetilde{v}_n, \widetilde{\nabla}_{\frac{\partial}{\partial x^k}} (\mathcal{D} \widetilde{v}_n) - \mathcal{D} \widetilde{\nabla}_{\frac{\partial}{\partial x^k}} \widetilde{v}_n \rangle (a+x) \zeta(x') \eta(x'') dx \\ (4.22) \quad &- \int_{B_1^{m-2}(0) \times B_1^2(0)} \left\langle G(\widetilde{v}_n), \frac{\partial \widetilde{u}_n}{\partial x_k} \right\rangle (a+x) \zeta(x') \eta(x'') dx. \end{aligned}$$

Noting that

$$\left\langle G(\widetilde{v}_n), \frac{\partial \widetilde{u}_n}{\partial x_k} \right\rangle = \frac{1}{12} \frac{\partial (\lambda(\widetilde{u}_n) R_{ijkl}(\widetilde{u}_n))}{\partial x^k} \langle v_n^i, v_n^j \rangle \langle v_n^k, v_n^l \rangle,$$

integrating by parts and using Young's inequality, then the right hand side of (4.22) is controlled by

$$\begin{aligned} &C(\Lambda_1, N) \int_{B_1^{m-2}(0) \times B_1^2(0)} (|\nabla \widetilde{v}_n| |\widetilde{v}_n|^3 + |\widetilde{v}_n|^4 + |\nabla \widetilde{v}_n| |\widetilde{v}_n|) (a+x) dx \\ &\leq C(m, p, \Lambda_1, \Lambda, \Omega, N) (\sigma_n r_n)^{3-\frac{2m}{p}}. \end{aligned}$$

On the other hand, by Hölder's inequality, one has

$$\begin{aligned} &-2 \int_{B_1^{m-2}(0) \times B_1^2(0)} \left\langle \frac{\partial \widetilde{u}_n}{\partial x_l}, \frac{\partial \widetilde{u}_n}{\partial x_k} \right\rangle (a+x) \frac{\partial}{\partial x_l} (\zeta(x') \eta(x'')) dx \\ &\leq C \left(\int_{B_{R+1}^{m-2}(0) \times B_{R+1}^2(0)} |\nabla \widetilde{u}_n|^2 dx \right)^{1/2} \left(\int_{B_{R+1}^{m-2}(0) \times B_{R+1}^2(0)} \left| \frac{\partial \widetilde{u}_n}{\partial x_k} \right|^2 dx \right)^{1/2}. \end{aligned}$$

Combining these and letting $n \rightarrow \infty$, we obtain

$$\frac{\partial F_n(a)}{\partial a_k} \rightarrow 0, \quad k = 1, \dots, m-2,$$

uniformly in $B_6^{m-2}(0) \times B_R^2(0)$ for any fixed $R > 0$.

Thus, for any $a = (a', a'') = B_6^{m-2}(0) \times B_R^2(0)$,

$$\begin{aligned} \int_{B_{1/2}^{m-2}(a') \times B_{1/2}^2(a'')} |\nabla \tilde{u}_n|^2 dx &\leq F_n(a) \\ &\leq F_n((0, a'')) + C(m) \sum_{k=1}^{m-2} \left| \frac{\partial F_n(a)}{\partial a_k} \right| \\ &\leq \int_{B_1^{m-2}(0) \times B_1^2(a'')} |\nabla \tilde{u}_n|^2 dx + C(m) \sum_{k=1}^{m-2} \left| \frac{\partial F_n(a)}{\partial a_k} \right| \\ &\leq \frac{\epsilon_0^2}{C_1(m)} + C(m) \sum_{k=1}^{m-2} \left| \frac{\partial F_n(a)}{\partial a_k} \right|. \end{aligned}$$

Therefore, for any $R > 0$, when n is big enough, we have

$$(4.23) \quad 6^{2-m} \int_{B_6^{m-2}(0) \times B_6^2(0)} |\nabla \tilde{u}_n|^2(x', x'' + b) dx \leq \frac{C(m)\epsilon_0^2}{C_1(m)} \text{ for all } b \in B_{R-6}^2.$$

Taking $C_1(m) \geq 2^m C(m)$, similar to deriving (3.5), we have

$$\begin{aligned} &\sup_{x_0 \in B_3(0), 0 < r \leq 3} r^{2-m} \int_{B_r(x_0)} (|\nabla \tilde{u}_n|^2 + |\lambda|^2 |\tilde{v}_n|^4)(x', x'' + b) dx \\ &\leq \frac{\epsilon_0^2}{4} + C(m, p, \Lambda_1, \Lambda, \Omega, N)(r_n \sigma_n)^{3-\frac{2m}{p}} \leq \frac{\epsilon_0^2}{2}, \end{aligned}$$

whenever n is large enough.

By Theorem 1.2, we know $(\tilde{u}_n, \tilde{v}_n)$ sub-converges to a Dirac-harmonic map with λ -curvature term (u, v) in $C_{loc}^1(B_{3/2}^{m-2}(0) \times \mathbb{R}^2)$. Moreover, by (4.18)-(4.20), for any $R > 0$, we have

$$\int_{B_R(0)} \sum_{k=1}^{m-2} \left| \frac{\partial u}{\partial x_k} \right|^2 dx = 0,$$

and

$$\int_{B_1(0)} |\nabla u|^2 dx = \frac{\epsilon_0^2}{C_1(m)}, \quad \int_{B_R(0)} |\nabla u|^2 dx \leq C(m, p, \delta_0, \Lambda_1, \Lambda, \Omega, N) R^{m-2}.$$

Furthermore, since

$$\int_{B_R(0)} |v|^4 dx = \lim_{n \rightarrow \infty} \int_{B_R(0)} |\tilde{v}_n|^4 dx \leq \lim_{n \rightarrow \infty} C(m, p, \delta_0, \Lambda_1, \Lambda, \Omega, N)(r_n \sigma_n R)^{3-\frac{2m}{p}} = 0,$$

we know $v \equiv 0$ and $u : \mathbb{R}^2 \rightarrow N$ is a nonconstant harmonic map with finite energy. By the conformal invariance of harmonic maps in dimension two, u can be extended to a nonconstant harmonic sphere.

Proof of Theorem 1.5. The conclusion of Theorem 1.5 follows from Lemma 4.4 and the Federer dimension reduction argument which is similar to [35] for minimizing harmonic maps. We omit the details here. This completes the proof. \square

5. APPENDIX

In this section, for reader's convenience, we recall some known results which are used in this paper.

Lemma 5.1 (Lemma 6.1 in [24]). *Let $m \geq 2$ and $4 < p < \infty$. Let $\psi \in M^{4,2}(B_1, \mathbb{C}^L \otimes \mathbb{R}^K)$ be a weak solution of the nonlinear system*

$$(5.1) \quad \not{D}\psi^i = A_j^i \psi^j + B^i, \quad 1 \leq i \leq K,$$

where $A \in M^{2,2}(B_1, gl(L, \mathbb{C}) \otimes gl(K, \mathbb{R}))$ and $B \in M^{2,2}(B_1, \mathbb{C}^L \otimes \mathbb{R}^K)$. For any $U \subseteq B_1$, there exists $\epsilon_0 = \epsilon_0(m, p) > 0$ and $C = C(m, p, U) > 0$ such that if

$$\|A\|_{M^{2,2}(B_1)} \leq \epsilon_0,$$

then $\psi \in L^p(U)$ and the following estimate hold:

$$\|\psi\|_{L^p(U)} \leq C(m, p, U)(\|\psi\|_{M^{4,2}(B_1)} + \|B\|_{M^{2,2}(B_1)}).$$

Theorem 5.2 ([33], Theorem 1.2 in [36]). *Let $m \geq 2$ and $2 < p < \infty$. Let $u \in W^{1,2}(B_1, \mathbb{R}^d)$, $\Omega \in M^{2,2}(B_1, so(d) \otimes \wedge^1 \mathbb{R}^m)$ and $f \in L^p(B_1, \mathbb{R}^d)$ with $\frac{m}{2} < p < m$, satisfy weakly*

$$(5.2) \quad \Delta u = \Omega \cdot \nabla u + f \text{ in } B_1.$$

Then for any $U \subseteq B_1$, there exist $\epsilon = \epsilon(m, d, p) > 0$ and $C = C(m, d, p, U) > 0$ such that if $\|\Omega\|_{M^{2,2}(B_1)} \leq \epsilon$, then

$$\|\nabla^2 u\|_{M^{\frac{2p}{m}, 2}(U)} + \|\nabla u\|_{M^{\frac{2p}{m-p}, 2}(U)} \leq C(m, d, p, U)(\|u\|_{L^1(B_1)} + \|f\|_{L^p(B_1)}).$$

Lemma 5.3 ($W^{k,p}$ -estimates, c.f. [2]). *Let (M, g) be an m -dimensional spin Riemannian manifold. Suppose $\psi \in \Gamma(\Sigma M)$, $\psi \in L^4(B_r(x_0))$ is a weak solution of*

$$\not{D}\psi = f \text{ in } B_r(x_0)$$

where $B_r(x_0)$ is a geodesic ball of M and $f \in W^{k,p}(B_r(x_0))$ for some $1 < p < \infty$, $k \geq 1$. Then $\psi \in W^{k+1,p}(B_{\frac{r}{2}}(x_0))$ and

$$\|\psi\|_{W^{k+1,p}(B_{\frac{r}{2}}(x_0))} \leq C(p, k, r, M)(\|\psi\|_{L^4(B_r(x_0))} + \|f\|_{W^{k,p}(B_r(x_0))}).$$

Lemma 5.4 (Schauder estimates, c.f. [2]). *Let (M, g) be a m -dimensional spin Riemannian manifold. Suppose $\psi \in \Gamma(\Sigma M)$, $\psi \in L^4(B_r(x_0))$ is a weak solution of*

$$\not{D}\psi = f \text{ in } B_r(x_0)$$

where $f \in C^{k,\alpha}(B_r(x_0))$ for some $0 < \alpha < 1$ and $k \geq 1$. Then $\psi \in C^{k+1,\alpha}(B_{\frac{r}{2}}(x_0))$ and

$$\|\psi\|_{C^{k+1,\alpha}(B_{\frac{r}{2}}(x_0))} \leq C(\alpha, r, k, M)(\|\psi\|_{L^4(B_r(x_0))} + \|f\|_{C^{k,\alpha}(B_r(x_0))}).$$

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