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Existence of solutions of a mixed elliptic-parabolic boundary value problem coupling a harmonic-like map with a nonlinear spinor

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EXISTENCE AND ASYMPTOTIC ANALYSIS FOR SOLUTIONS OF A MIXED ELLIPTIC-PARABOLIC BOUNDARY VALUE PROBLEM COUPLING A HARMONIC-LIKE MAP WITH A NONLINEAR SPINOR. I

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ABSTRACT. In this paper, we solve a new elliptic-parabolic system arising in geometric analysis that is motivated by the nonlinear supersymmetric sigma model of quantum field theory. The corresponding action functional involves two fields, a map from a Riemann surface into a Riemannian manifold and a spinor coupled to the map. The first field has to satisfy a second order elliptic system, which we turn into a parabolic system so as to apply heat flow techniques. The spinor, however, satisfies a first order Dirac type equation. We carry that equation as a nonlinear constraint along the flow.

With this novel scheme, in more technical terms, we can show the existence of Dirac-harmonic maps from a compact spin Riemann surface with smooth boundary to a general compact Riemannian manifold via a heat flow method when a Dirichlet boundary condition is imposed on the map and a chiral boundary condition on the spinor.

1. INTRODUCTION

In this paper, we explore a new scheme in geometric analysis. We show the existence of solutions of an elliptic system consisting of second and first order equations. Such systems frequently arise in geometric analysis, and here we look at the system for Dirac-harmonic maps. Another such system would be the minimal surface system in Riemannian manifolds where a harmonic map, satisfying a semilinear second order elliptic system, is further constrained by the conformality condition, a non-linear first order system. We carry the analysis out here for Dirac-harmonic maps, but want to emphasize the potential of the method for other problems of this type.

Such systems typically represent borderline cases for the Palais-Smale condition, and therefore cannot be solved by standard tools. One needs to understand the potential formation of singularities in such systems, usually called bubbles, and one then needs suitable conditions to prevent them. Here, we deal with a system that, while of variational origin, cannot be solved by minimizing a suitable integral, because in our case, the corresponding integral is not bounded from below. We therefore study a parabolic system. Here, a difficulty arises from the first order elliptic part of the system. The second order elliptic part can be easily converted into a parabolic system, by letting the solution depend on time and equating the elliptic part with its time derivative. Since this does not work for the first order part, we simply carry it along as an elliptic side condition. Our contribution then consists in showing the long time existence of the resulting parabolic-elliptic system and the convergence of its solutions to solutions of the original elliptic system as time goes to infinity. The scheme developed in the present paper should lead to applications to a broad range

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of variational problems arising in geometry and physics for which the existence of critical points can neither be obtained by a direct minimization procedure nor by a minimax scheme. The Dirac-harmonic system thus seems to be a good new model problem in geometric analysis because on one hand, it shares the important classical features, like conformal invariance, but on the other hand does not succumb to the classical techniques. It therefore forces the development of new methods that are of wider interest. In particular, the interplay between second and first order elliptic equations is typical for many problems in geometric analysis, like minimal surfaces, as explained above, or more generally, constant mean curvature surfaces, where the first order equation is the conformality condition. In symplectic geometry, solutions of such first order equations give rise to pseudoholomorphic curves, leading to the famous Gromov-Witten invariants. It would be of great interest to also couple them with a spinorial field, with the hope to produce refined invariants.

Let us now describe the geometric origin and the motivation for the system that we shall solve in this paper. Harmonic maps from Riemann surfaces are important both in mathematics and in physics. In mathematics, they have been intensively investigated since the seminal work of Sacks-Uhlenbeck [29], and the phenomena discovered and the techniques developed have been fundamental for the subject of geometric analysis, see for instance [17] and the references given there. In physics, they arise from the nonlinear sigma model of quantum field theory. From that perspective, however, they only contain part of the story. In fact, in the supersymmetric sigma model, they have a partner, a spinor field, see e.g. [11, 18]. In order to also incorporate that field into the mathematical theory, [5, 6] introduced the concept of Dirac-harmonic maps that couple a harmonic map type field with a spinor field.

This then naturally leads to the question to what extent the mathematical results obtained for harmonic maps carry over to Dirac-harmonic maps. Foremost here is of course the existence question. Already for harmonic maps, the existence question becomes subtle because of the phenomenon of bubbling, that is the concentration of the energy at single points. After rescaling, this leads to the emergence of harmonic spheres. Thus, while this seems to be an obstruction to an existence scheme, after all, such a harmonic sphere is a nontrivial harmonic map, and so, in any case, we find nontrivial solutions. When bubbling occurs, these solutions may, however, lie in different homotopy classes than the maps that one started with. When, however, the target manifold contains no nontrivial minimal 2-spheres, no bubbling can occur, and one can then show the existence of a solution in any given homotopy class.

For Dirac-harmonic maps, the existence question is more difficult. We are seeking solutions that are nontrivial in a more constrained sense. That is, not only should the map be nonconstant, like a harmonic sphere, but also the spinor part should be nontrivial. But the schemes developed for harmonic maps are blind to the nontriviality of the spinor part. This makes the existence question much harder.

Here, we address the question of the existence of Dirac-harmonic maps in the case of boundary value problems. We impose a Dirichlet type boundary condition for the map and a chiral boundary condition for the spinor, see [8]. The reason is that a nontrivial boundary condition for the spinor should also give the existence of a nontrivial spinor.

The harmonic map and the Dirac-harmonic map problem are limit cases for the Palais-Smale condition, because they are conformally invariant, and the two-dimensional conformal group is noncompact. In that situation, Sacks-Uhlenbeck [29] approximated the underlying variational integral, which contains the square of the derivative of the map, by variational integrals that contain higher powers and therefore satisfy the Palais-Smale condition. Their main technical achievement then consists in controlling the limit when that power goes to 2. In our situation, however, as explained, we cannot use a variational method and need to rely on a parabolic technique instead. Such parabolic methods have been developed in the context of harmonic maps in [33, 4]. While in outline, the scheme may look similar to the variational approach, already for harmonic maps, that is, in a situation where the variational method can be applied, the parabolic approach encounters different problems and needs to develop new estimates, as can be seen in [33, 4]. The same is the case here. While we transfer the approximation scheme of Sacks-Uhlenbeck [29] to the parabolic case, the detailed estimates required are rather different, and in particular, we cannot directly take over their reasoning. In fact, extending the Sacks-Uhlenbeck scheme to the parabolic case may be of interest in itself, but this is not our main point. For us, this scheme just provides the necessary background, and we need to develop it here because this has not yet been done in the literature. In fact, in our situation, that scheme still cannot fully solve the problem. This is due to the spinor part, as the reader will amply see in the main body of this paper. We thus have to consider a new type of flow that can also handle the first-order component of our system. For Dirac-harmonic maps, the relevant flow has been introduced in [9] and further studied in [19, 20]. Its novel feature consists in carrying the Dirac equation for the spinor, which is a first order elliptic equation, as a constraint along the parabolic equation for the map. Thus, we need to deal with an elliptic-parabolic system. And this is where the difficult part of our estimates is required.

We shall now describe our results in more precise terms. This will need some technical preparation.

Let M be a compact Riemann surface, equipped with a Riemannian metric g and with a fixed spin structure, ΣM be the spinor bundle over M and $\langle\cdot,\cdot\rangle_{\Sigma M}$ be the natural Hermitian inner product on ΣM. Choosing a local orthonormal basis $e_\gamma, \gamma = 1, 2$ on M, the usual Dirac operator is defined as $\partial\!\!\!/ := e_\gamma \cdot \nabla_{e_\gamma}$, where ∇ is the spin connection on ΣM and $\cdot$ is the Clifford multiplication. This multiplication is skew-adjoint:

$$\langle X\cdot\psi, \varphi\rangle_{\Sigma M} = -\langle\psi, X\cdot\varphi\rangle_{\Sigma M}$$

for any $X \in \Gamma(TM)$, $\psi,\ \varphi \in \Gamma(\Sigma M)$. For more details on spin geometry and Dirac operators, one can refer to [22].

Let ϕ be a smooth map from M to another compact Riemannian manifold (N, h) with dimension $n \geq 2$. Denote ϕ^*TN the pull-back bundle of TN by ϕ and then we get the twisted bundle $\Sigma M \otimes \phi^*TN$. Naturally, there is a metric $\langle\cdot,\cdot\rangle_{\Sigma M\otimes\phi^*TN}$ on $\Sigma M \otimes \phi^*TN$ which is induced from the metrics on ΣM and ϕ^*TN. Also we have a natural connection $\widetilde{\nabla}$ on $\Sigma M \otimes \phi^*TN$ which is induced from the connections on ΣM and ϕ^*TN. Let ψ be a section of the bundle $\Sigma M \otimes \phi^*TN$. In local coordinates, it can be written as

$$\psi = \psi^i \otimes \partial_{y^i}(\phi),$$

where each ψ^i is a usual spinor on M and ∂_{y^i} is the nature local basis on N. Then $\widetilde{\nabla}$ becomes

$$\widetilde{\nabla}\psi = \nabla\psi^i \otimes \partial_{y^i}(\phi) + (\Gamma^i_{jk}\nabla\phi^j)\psi^k \otimes \partial_{y^i}(\phi), \tag{1.1}$$

where Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection of N. The Dirac operator along the map ϕ is defined by

$$D\!\!\!\!/\,\psi := e_\alpha \cdot \widetilde{\nabla}_{e_\alpha}\psi.$$

We consider the following functional

$$L(\phi,\psi) = \frac{1}{2}\int_M \Big(|d\phi|^2 + \langle\psi, D\!\!\!/\psi\rangle_{\Sigma M\otimes\phi^\star TN}\Big)dM.$$

The functional $L(\phi,\psi)$ is conformally invariant, see [6]. That is, for any conformal diffeomorphism $f : M \to M$, setting

$$\widetilde{\phi} = \phi\circ f \qquad \text{and} \qquad \widetilde{\psi} = \lambda^{-1/2}\psi\circ f.$$

where λ is the conformal factor of the conformal map f, *i.e.* $f^*g = \lambda^2 g$. Then there holds

$$L(\widetilde{\phi},\widetilde{\psi}) = L(\phi,\psi).$$

Critical points (ϕ,ψ) are called *Dirac-harmonic maps* from M to N.

The Euler-Lagrange equations of the functional L are

$$\Big(\Delta_g\phi^i + \Gamma^i_{jk}g^{\alpha\beta}\phi^j_\alpha\phi^k_\beta\Big)\frac{\partial}{\partial y^i}(\phi(x)) = R(\phi,\psi), \tag{1.2}$$

$$D\!\!\!/\psi = 0, \tag{1.3}$$

where $\Delta_g := \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\beta}(\sqrt{g}g^{\beta\gamma}\frac{\partial}{\partial x^\gamma})$ is the Laplacian operator with respect to the Riemannian metric g, $R(\phi,\psi)$ is defined by

$$R(\phi,\psi) = \frac{1}{2}R^m_{lij}(\phi(x))\langle\psi^i, \nabla\phi^l\cdot\psi^j\rangle\frac{\partial}{\partial y^m}(\phi(x)).$$

Here R^m_{lij} stands for the Riemann curvature tensor of the target manifold (N,h).

By Nash's embedding theorem, we embed N isometrically into some $\mathbb{R}^K$. Then, critical points (ϕ,ψ) of the functional L satisfy the Euler-Lagrange equations

$$\Delta\phi = A(\phi)(d\phi,d\phi) + Re(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi);\psi)), \tag{1.4}$$

$$\partial\!\!\!/\psi = \mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi), \tag{1.5}$$

where $\partial\!\!\!/$ is the usual Dirac operator, A is the second fundamental form of N in $\mathbb{R}^K$, and

$$\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi) := (\nabla\phi^i\cdot\psi^j)\otimes A(\partial_{y^i},\partial_{y^j}),$$

$$Re(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi);\psi)) := P(A(\partial_{y^l},\partial_{y^j});\partial_{y^i})Re(\langle\psi^i, d\phi^l\cdot\psi^j\rangle).$$

Here $P(\xi;\cdot)$ denotes the shape operator, defined by $\langle P(\xi;X),Y\rangle = \langle A(X,Y),\xi\rangle$ for $X,Y\in\Gamma(TN)$, and $Re(z)$ denotes the real part of $z\in\mathbb{C}$.

For $p > 1$, we denote

$$W^{1,p}(M,N) := \Big\{\phi\in W^{1,p}(M,\mathbb{R}^K) \mid \phi(x)\in N,\ a.e.\ x\in M\Big\},$$

$$W^{1,p}(\Sigma M\otimes\phi^*TN) := \Big\{\psi\in W^{1,p}(\Sigma M\otimes\phi^*\mathbb{R}^K) \mid \psi(x) \text{ is along the map } \phi,\ a.e.\ x\in M\Big\}.$$

Here $\psi\in\Gamma(\Sigma M\otimes\phi^*TN)$ is along the map ϕ should be understood as a K-tuple of spinors $(\psi^1,...,\psi^K)$ satisfying

$$\sum_{i=1}^K \nu_i\psi^i(x) = 0$$

for any normal vector $\nu = (\nu_1,...,\nu_K)\in\mathbb{R}^K$ at $\phi(x)$. For more details on the set up of Dirac-harmonic maps, we refer to [5, 6, 37, 8, 31].

The blow-up theory for sequences of Dirac-harmonic maps including the energy identity and the no neck property, i.e., bubble tree convergence, was systematically explored in [5, 36, 28]. For the existence results of Dirac-harmonic maps, however, since the functional $L(\phi, \psi)$ does not have a lower bound due to the fact that the second term in L does not have a fixed sign, classical variational methods developed for harmonic maps cannot be applied directly and hence the problem becomes very difficult. Up to now, there are only few results in this regard. See [7] for some attempt via the maximum principle, where some partial existence results were obtained. See [3] for a regularized heat flow approach for regularized Dirac-harmonic maps, which is different from ours to be introduced in a moment. See [1, 10] for some existence results of uncoupled Dirac-harmonic maps (here uncoupled means that the map part is harmonic) based on index theory and the Riemann-Roch theorem.

In order to study the general existence problem, a heat flow approach for Dirac-harmonic maps from spin Riemannian manifolds with boundary was introduced in [9], and the short time existence of a solution was shown. (Recently, Wittmann [35] could show short time existence also in the case of a closed domain under certain conditions on the initial data.) Furthermore, the existence of a global weak solution to this flow in dimension two was obtained in [19]. By studying the limit behaviour as time approaches infinity, they proved the existence results of Dirac-harmonic maps with Dirichlet-chiral boundary condition in a given homotopy class under a certain smallness assumption on the boundary-initial value in [19]. A technical difficulty stems from the fact that along the Dirac-harmonic map flow considered in [19], one only have that the energy of the map ϕ is uniformly bounded, i.e.,

$$E(\phi(\cdot, t)) = \frac{1}{2}\int_M |\nabla\phi(\cdot, t)|^2 dM \leq C < +\infty.$$

However, the Dirac type equation (1.5) for the spinor ψ does not control the energy of the spinor field

$$E(\psi(\cdot, t)) = \int_M |\psi(\cdot, t)|^4 dM,$$

as time t approaches the first singular time $T_1 > 0$, even for the L^1-norm. This is the main difficulty and why we need to impose an additional boundary-initial constraint in [19] in order to obtain a global weak solution to the Dirac-harmonic map flow and show some existence results by letting t goes to infinity. For the qualitative blow-up behavior of this flow, one can refer to [20]. The general question, however, is

Question: Does there exist a Dirac-harmonic map from a compact Riemann surface with boundary to a compact Riemannian manifold with general given Dirichlet-chiral boundary data?

In this paper, we will give an affirmative answer to this question. To achieve this, we shall introduce a new parabolic-elliptic system.

In our new approach, one crucial observation is the following *key estimate* for the Dirac operator $\not{D}$ along a given map (see Lemma 2.4):

Key estimate: Let $\phi \in W^{1,q}(M,N)$ for some $q > 2$ and $\psi \in W^{1,p}(M, \Sigma M \otimes \phi^*TN)$ for some $1 < p < 2$, then there holds

$$\|\psi\|_{W^{1,p}(M)} \leq C(p, M, N, \|\nabla\phi\|_{L^q(M)})(\|\not{D}\psi\|_{L^p(M)} + \|\mathbf{B}\psi\|_{W^{1-1/p,p}(\partial M)}). \tag{1.6}$$

Here $\mathbf{B}$ is an extension to spinors along a map of the chiral boundary operator for usual spinors introduced by Gibbons-Hawking-Horowitz-Perry [13] to study positive mass theorems for black holes via Witten's approach through the spinor equation. See (1.13) for more details on this boundary operator. There are two key properties of the above estimate. The first one is that the constant $C(p, M, N, \|\nabla\phi\|_{L^q(M)}) > 0$ depends on the norm $\|\nabla\phi\|_{L^q(M)}$ with $q > 2$, which was already observed in [9]. The second one is that the two numbers $q > 2$ and $1 < p < 2$ are independent of each other. This fact was not exploited in [9] while here, as we will see later, it plays an important role. In fact, such kind of key estimate holds true for Dirac type systems of more general type, see Lemma 2.3.

Since the key estimate for the Dirac operator $\not{D}$ along a map in (1.6) requires that the map ϕ lies in $W^{1,q}(M,N)$ for some $q > 2$, inspired by this fact, we introduce the following functional

$$L_\alpha(\phi,\psi) = \frac{1}{2}\int_M \left\{(1 + |d\phi|^2)^\alpha + \langle\psi, \not{D}\psi\rangle\right\} dM, \tag{1.7}$$

where $\alpha > 1$ is a constant. Critical points $(\phi_\alpha, \psi_\alpha)$ of L_α are called *α-Dirac-harmonic maps* from M to N. When the spinor field is vanishing, the above functional reduces to Sacks-Uhlenbeck's approximation for harmonic maps in [29].

By a direct computation, critical points $(\phi_\alpha, \psi_\alpha)$ of the functional L_α satisfy the following Euler-Lagrange equations (see Lemma 2.2)

$$\Delta_g\phi = -(\alpha - 1)\frac{\nabla_g|\nabla_g\phi|^2\nabla_g\phi}{1 + |\nabla_g\phi|^2} + A(d\phi, d\phi) + \frac{Re\left(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi); \psi)\right)}{\alpha(1 + |\nabla_g\phi|^2)^{\alpha-1}}, \tag{1.8}$$

$$\not{\partial}\psi = \mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi). \tag{1.9}$$

One crucial step in our scheme is to get the existence result of Dirac-harmonic maps through studying the limit behaviour of a sequence of α-Dirac-harmonic maps as $\alpha \searrow 1$ [1]. If there exists a sequence of α-Dirac-harmonic maps $(\phi_\alpha, \psi_\alpha)$ with

$$E_\alpha(\phi_\alpha) := \frac{1}{2}\int_M (1 + |d\phi_\alpha|^2)^\alpha dM \leq \Lambda < \infty,$$

then the key estimate (1.6) implies the following uniform control of the spinors:

$$\|\psi_\alpha\|_{W^{1,p}(M)} \text{ with } 1 < p < 2, \text{ is uniformly bounded as } \alpha \searrow 1.$$

Thus, we can do the blow-up analysis and we will show that the weak limit is just the desired Dirac-harmonic map. This is better than for the Dirac-harmonic map flow [9, 19], and so, here lies the advantage of considering α-Dirac-harmonic maps which is the basic idea in this paper.

The remaining task is to show the existence of such an α-Dirac-harmonic map sequence. This is in fact one key step in our new scheme. Since the second term of the functional L_α is not bounded from below, the classical Ljusternik-Schnirelman theory may not be applied here to obtain critical points. Therefore, we need to develop a new method to proceed with our scheme.

[1]Here and in the sequel, for simplicity of notations, when talking about a sequence of $(\phi_\alpha, \psi_\alpha)$ for $\alpha \searrow 1$, we mean the sequence of $(\phi_{\alpha_k}, \psi_{\alpha_k})$ for a given sequence of $\alpha_k \searrow 1$.

In the present work, we shall consider the following new parabolic-elliptic system:

$$\partial_t\phi = \Delta_g\phi + (\alpha-1)\frac{\nabla_g|\nabla_g\phi|^2\nabla_g\phi}{1+|\nabla_g\phi|^2} - A(d\phi,d\phi) - \frac{Re\left(P(\mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi);\psi)\right)}{\alpha(1+|\nabla_g\phi|^2)^{\alpha-1}}, \tag{1.10}$$

$$\partial\!\!\!/\psi = \mathcal{A}(d\phi(e_\gamma),e_\gamma\cdot\psi), \tag{1.11}$$

with the following boundary-initial data:

$$\begin{cases} \phi(x,t) = \varphi(x), & on \quad \partial M\times[0,T]; \\ \phi(x,0) = \phi_0(x), & in \quad M; \\ \mathbf{B}\psi(x,t) = \mathbf{B}\psi_0(x), & on \quad \partial M\times[0,T]; \\ \phi_0(x) = \varphi(x), & on \quad \partial M. \end{cases} \tag{1.12}$$

where $\mathbf{B}=\mathbf{B}^\pm$ is the chiral boundary operator defined as follows:

$$\begin{aligned} \mathbf{B}^\pm : L^2(\partial M,\Sigma M\otimes\phi^*TN|_{\partial M}) &\to L^2(\partial M,\Sigma M\otimes\phi^*TN|_{\partial M}) \\ \psi &\mapsto \frac{1}{2}\left(Id\pm\overrightarrow{n}\cdot G\right)\cdot\psi, \end{aligned} \tag{1.13}$$

where $\overrightarrow{n}$ is the outward unit normal vector field on ∂M, $G=ie_1\cdot e_2$ is the chiral operator defined using a local orthonormal frame $\{e_\gamma\}_{\gamma=1}^2$ on M and satisfying:

$$G^2 = Id, \quad G^* = G, \quad \nabla G = 0, \quad G\cdot X = -X\cdot G, \tag{1.14}$$

for any $X\in\Gamma(TM)$. One can also take $\mathbf{B}$ to be the MIT bag boundary operator $\mathbf{B}^\pm_{MIT}$ as considered in [9]. See e.g. [15, 2] for more detailed discussions on these boundary operators. For convenience, in the sequel, we shall only consider the case of chiral boundary conditions and omit the other case of boundary conditions, as the arguments for them are the same. We call (1.10)-(1.11) the *α-Dirac-harmonic map flow*.

Now, we state our first main result about the global existence of the α-Dirac-harmonic map flow with a Dirichlet-chiral boundary condition.

Theorem 1.1. *Let M be a compact spin Riemann surface with smooth boundary ∂M and let $N\subset\mathbb{R}^K$ be a compact Riemannian manifold. Suppose*

$$1<\alpha<1+\min\{\epsilon_1,\epsilon_2\}$$

where ϵ_1 and ϵ_2 are the positive constants in Theorem 3.1 and Lemma 3.4 depending only on M, N. Then for any $\phi_0\in C^{2+\mu}(M,N)$, $\varphi\in C^{2+\mu}(\partial M,N)$, $\psi_0\in C^{1+\mu}(\partial M,\Sigma M\otimes\varphi^*TN)$ where $0<\mu<1$ is a constant, there exists a unique global solution

$$\phi\in C^{2+\mu,1+\frac{\mu}{2}}_{loc}(M\times[0,\infty),N)$$

and

$$\psi\in C^{\mu,\frac{\mu}{2}}_{loc}(M\times[0,\infty),\Sigma M\otimes\phi^*TN)\cap L^\infty([0,\infty),\|\psi(\cdot,t)\|_{C^{1+\mu}(M)})$$

to the problem (1.10)-(1.11) *with boundary-initial data* (1.12)*, satisfying*

$$E_\alpha(\phi(t))\le E_\alpha(\phi_0)+\sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}$$

and

$$\|\psi(\cdot,t)\|_{W^{1,p}(M)} \le C(p, M, N, E_\alpha(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}),$$

where $1 < p < 2$.

Moreover, there exist a time sequence $t_i \to \infty$ *and an* α*-Dirac-harmonic map*

$$(\phi_\alpha, \psi_\alpha) \in C^{2+\mu}(M, N) \times C^{1+\mu}(M, \Sigma M \otimes \phi_\alpha^* TN)$$

with the boundary data

$$(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0),$$

such that $(\phi(\cdot, t_i), \psi(\cdot, t_i))$ *converges to* $(\phi_\alpha, \psi_\alpha)$ *in* $C^2(M) \times C^1(M)$.

We remark that the harmonic map flow from a closed Riemann surface has been solved in [33], and from a compact Riemann surface with smooth boundary in [14, 4]. When the spinor field is vanishing and the domain is a closed surface, our flow reduces to the one in [16].

By Theorem 1.1, for any $\alpha > 1$ sufficiently close to 1, there exists an α-Dirac-harmonic map $(\phi_\alpha, \psi_\alpha) \in C^{2+\mu}(M, N) \times C^{1+\mu}(M, \Sigma M \otimes \phi_\alpha^* TN)$ with the Dirichlet-chiral boundary condition $(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$ and with the properties

$$E_\alpha(\phi_\alpha) \le E_\alpha(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)} \tag{1.15}$$

and

$$\|\psi_\alpha\|_{W^{1,p}(M)} \le C(p, M, N, E_\alpha(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}), \tag{1.16}$$

for any $1 < p < 2$. With this in hand, we can prove the existence of Dirac-harmonic maps by using the blow-up analysis.

Generally, we have the following existence and concentration compactness theorem of Dirac-harmonic maps corresponding to the previous **Question**.

Theorem 1.2. *Let* $(\phi_\alpha, \psi_\alpha) : M \to N$ *be a sequence of* α*-Dirac-harmonic maps with Dirichlet-chiral boundary condition* $(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$ *and with uniformly bounded energy*

$$E_\alpha(\phi_\alpha) + \|\psi_\alpha\|_{L^4(M)} \le \Lambda.$$

Denoting $E(\phi_\alpha; \Omega) := \frac{1}{2}\int_\Omega |\nabla\phi_\alpha|^2 dvol_g$, $\Omega \subset M$ *and the energy concentration set*

$$\mathcal{S} := \left\{ x \in M | \liminf_{\alpha \to 1} E(\phi_\alpha; B_r^M(x)) \ge \frac{\epsilon_0}{2} \; for \; all \; r > 0 \right\},$$

where ϵ_0 *is the positive constant in Lemma 4.1 and Lemma 4.2,* $B_r^M(x)$ *is the geodesic ball in* M *with center point* x *and radius* r*, then* $\mathcal{S}$ *is a finite set. Moreover, after selection of a subsequence of* $(\phi_\alpha, \psi_\alpha)$ *(without changing notation), there exists a Dirac-harmonic map*

$$(\phi, \psi) \in C^{2+\mu}(M, N) \times C^{1+\mu}(M, \Sigma M \otimes \phi^* TN)$$

with Dirichlet-chiral boundary data $(\phi, \mathbf{B}\psi)|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$*, such that*

$$(\phi_\alpha, \psi_\alpha) \to (\phi, \psi) \; in \; C^2_{loc}(M \setminus \mathcal{S}) \times C^1_{loc}(M \setminus \mathcal{S}).$$

Remark 1.3. Since we can impose nontrivial boundary conditions for both the map and the spinor, we shall obtain Dirac-harmonic maps with nontrivial map part and nontrivial spinor part.

Moreover, we show that at each singular point x_0, that is, when the energy of the map concentrates, after suitable rescaling, a bubble, namely, a nontrivial Dirac-harmonic sphere splits off. Here, however, we cannot employ the usual bubbling argument for a blow-up sequence of Dirac-harmonic maps which are conformally invariant [5], since α-Dirac-harmonic maps are not conformally invariant. We need to develop a different type of rescaling argument by adding a new rescaling factor $r_\alpha^{\alpha-1}$, with r_α being the blow-up radii, to the spinor part. Therefore, the blow-up analysis for α-Dirac-harmonic maps is more difficult and complicated than the case of Dirac-harmonic maps. To achieve this, we shall introduce the notation of *general α-Dirac-harmonic maps* and develop the appropriate analytical background, see Section 4.

Theorem 1.4. *Under the same assumption as in Theorem 1.2, suppose $x_0 \in \mathcal{S}$ is an energy concentration point, i.e.,*

$$\liminf_{\alpha\to 1} E(\phi_\alpha; B_r^M(x_0)) \geq \frac{\epsilon_0}{2} \ for\ all\ r > 0. \tag{1.17}$$

Then,

(1) *if $x_0 \in M \setminus \partial M$, there exist a subsequence of $(\phi_\alpha, \psi_\alpha)$ (still denoted by $(\phi_\alpha, \psi_\alpha)$) and sequences $x_\alpha \to x_0$, $r_\alpha \to 0$ and a nontrivial Dirac-harmonic map $(\sigma, \xi) : \mathbb{R}^2 \to N$, such that as $\alpha \to 1$,*[2]

$$\left(\phi_\alpha(x_\alpha + r_\alpha x), r_\alpha^{\alpha-1}\sqrt{r_\alpha}\psi_\alpha(x_\alpha + r_\alpha x)\right) \to (\sigma(x), \xi(x)) \ in\ C^1_{loc}(\mathbb{R}^2) \times C^0_{loc}(\mathbb{R}^2).$$

(σ, ξ) has finite energy and conformally extends to a smooth Dirac-harmonic sphere.

(2) *if $x_0 \in \partial M$, then $\frac{dist(x_\alpha, \partial M)}{r_\alpha} \to \infty$ and the same bubbling statement as in (1) holds.*

So far, we have answered the **Question** about the existence of Dirac-harmonic maps with given Dirichlet-chiral boundary data. It is natural to ask whether the map component ϕ of the limit Dirac-harmonic map stays in the same homotopy class as ϕ_0.

Here we give a positive answer under the same condition as in the harmonic map case. To see this, we recall that we can actually choose a sequence of α-Dirac-harmonic maps satisfying the properties (1.15)-(1.16), for any $1 < p < 2$. Therefore we are in a better situation than the case of $p = \frac{4}{3}$ considered in Theorem 1.2. In fact, if we take $\frac{4}{3} < p < 2$, then we can show that the bubbles in Theorem 1.4 are just nontrivial harmonic spheres, i.e., harmonic maps from S^2 to N. Thus, we have the following stronger version of the existence result:

Theorem 1.5. *Let M be a compact spin Riemann surface with smooth boundary ∂M and let $N \subset \mathbb{R}^K$ be a compact Riemannian manifold. For any $\phi_0 \in C^{2+\mu}(M, N)$, $\varphi \in C^{2+\mu}(\partial M, N)$, $\psi_0 \in C^{1+\mu}(\partial M, \Sigma M \otimes \varphi^* TN)$ where $\phi_0|_{\partial M} = \varphi$ and $0 < \mu < 1$ is a constant, if (N, h) dose not admit any nontrivial harmonic sphere, then there exists a Dirac-harmonic map*

$$(\phi, \psi) \in C^{2+\mu}(M, N) \times C^{1+\mu}(M, \Sigma M \otimes \phi^* TN)$$

with Dirichlet-chiral boundary data $(\phi, \mathbf{B}\psi)|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$ such that the map component ϕ is in the same homotopy class as ϕ_0.

[2]Compared to the usual rescaling, i.e. $\left(\phi_n(x_n + r_n x), \sqrt{r_n}\psi_n(x_n + r_n x)\right)$, for a blow-up sequence of Dirac-harmonic maps given in [5], here the additional factor $r_\alpha^{\alpha-1}$ comes from the fact that α-Dirac-harmonic maps are not conformally invariant, see Section 4.

At the end of this part, we want to state some refined analytical properties corresponding to the blow-up sequence in Theorem 1.2. We leave the detailed proofs of these theorems to the sequel to this paper [21].

Consider a sequence of α-Dirac-harmonic maps $\{(\phi_\alpha, \psi_\alpha)\} : M \to N$ with uniformly bounded energy

$$E_\alpha(\phi_\alpha) + E(\psi_\alpha) \leq \Lambda.$$

From Theorem 1.2, we know that, by passing to a subsequence, $(\phi_\alpha, \psi_\alpha)$ converges smoothly to some limit Dirac-harmonic map $(\phi, \psi) : M \to N$ away from at most finitely many blow-up points $\mathcal{S} = \{x_i\}_{i=1}^I$ as $\alpha \searrow 1$. For a fixed point x_i, $1 \leq i \leq I$, we may assume there are k_i bubbles occurring at this point, i.e. there are a sequence of points $\{x_\alpha^{ij}\}$, $j = 1, ..., k_i$, and a sequence of positive numbers $\{r_\alpha^{ij}\}$ with $x_\alpha^{ij} \to x_i$, $r_\alpha^{ij} \to 0$ as $\alpha \searrow 1$ and one of the following two alternatives holds true: if $1 \leq j_1, j_2 \leq k_i$ and $j_1 \neq j_2$,

(A1) for any fixed $R > 0$, $D^M_{Rr_\alpha^{ij_1}}(x_\alpha^{ij_1}) \cap D^M_{Rr_\alpha^{ij_2}}(x_\alpha^{ij_2}) = \emptyset$, whenever α is sufficiently close to 1.

(A2) $\frac{r_\alpha^{ij_1}}{r_\alpha^{ij_2}} + \frac{r_\alpha^{ij_2}}{r_\alpha^{ij_1}} = \infty$, as $\alpha \searrow 1$.

Moreover, the rescaled fields

$$\sigma_\alpha^{ij} := \phi_\alpha(x_\alpha^{ij} + r_\alpha^{ij} x), \quad \xi_\alpha^{ij} := \sqrt{r_\alpha^{ij}}\psi_\alpha(x_\alpha^{ij} + r_\alpha^{ij} x)$$

converge in $C^k_{loc}(\mathbb{R}^2 \setminus \{p_1^{ij}, ..., p_{s_j}^{ij}\})$ to a nontrivial Dirac-harmonic map (σ^{ij}, ξ^{ij}). Define two types of quantities:

$$\mu_{ij} = \liminf_{\alpha \searrow 1} (r_\alpha^{ij})^{2-2\alpha}, \quad \nu_{ij} = \liminf_{\alpha \searrow 1} (r_\alpha^{ij})^{-\sqrt{\alpha - 1}}. \tag{1.18}$$

We have the following generalized energy identities

Theorem 1.6. *Let M be a smooth closed Riemann surface, N be a n-dimensional smooth compact Riemannian manifold. Let $(\phi_{\alpha_k}, \psi_{\alpha_k}) \in C^\infty(M, N)$, $\alpha_k \searrow 1$ be a sequence of α_k-Dirac-harmonic maps with uniformly bounded energy, i.e. $E_{\alpha_k}(\phi_{\alpha_k}) + E(\psi_{\alpha_k}) \leq \Lambda$. Then there exists a finite set $\mathbf{S} = \{p_1, ..., p_I\}$ such that, passing to a subsequence, there exists $(\phi, \psi) : M \to N$ which is a smooth Dirac-harmonic map and there are finitely many bubbles: a finite set of Dirac-harmonic spheres $(\sigma_i^l, \xi_i^l) : S^2 \to N$, $l = 1, ...l_i$, where $l_i \geq 1$, $i = 1, ..., I$, such that $(\phi_{\alpha_k}, \psi_{\alpha_k}) \to (\phi, \psi)$ weakly in $W^{1,2}(M, N)$ and strongly in $C^\infty_{loc}(M \setminus \mathbf{S}, N)$. Moreover, the following generalized energy identities hold:*

$$\lim_{k\to\infty} E_{\alpha_k}(\phi_{\alpha_k}) = E(\phi) + |M| + \sum_{i=1}^{I} \sum_{l=1}^{l_i} \mu_{il}^2 E(\sigma_i^l),$$

$$\lim_{k\to\infty} E(\psi_{\alpha_k}) = E(\psi) + \sum_{i=1}^{I} \sum_{l=1}^{l_i} \mu_{il}^2 E(\xi_i^l),$$

where the quantities μ_{il} are defined as in (1.18).

Furthermore, we shall show that the Dirac-harmonic necks appearing during the blow-up process are converging to geodesics in the target manifold N. Precisely, we have

Theorem 1.7. *Under the same assumptions as in Theorem 1.6, assume* $\mathbf{S} = \{x_1\}$ *and there is only one bubble in* $D_r^M(x_1) \subset M$ *for some* $r > 0$*, for the sequence* $\{(\phi_{\alpha_k}, \psi_{\alpha_k})\}$*, denoted by* (σ^1, ξ^1)*, which is a Dirac-harmonic sphere. Let*

$$\nu^1 = \liminf_{\alpha \searrow 1} (r_\alpha^1)^{-\sqrt{\alpha-1}}. \tag{1.19}$$

Then the Dirac-harmonic neck appearing during the blow-up precess is converging to a geodesic in the target manifold N*. Moreover, we have the following alternatives:*

(1) *when* $\nu^1 = 1$*, the set* $\phi(D_r^M(x_1)) \cup \sigma^1(S^2)$ *is a connected set in* N*;*

(2) *when* $\nu^1 \in (1, \infty)$*, then the set* $\phi(D_r^M(x_1))$ *and* $\sigma(S^2)$ *are connected by a geodesic with length*

$$L = \sqrt{\frac{E(\sigma^1)}{\pi}} \log \nu^1;$$

(3) *when* $\nu^1 = \infty$*, the map part of the limit of the Dirac-harmonic neck contains at least an infinite length curve which is a geodesic in* N*;*

Although Theorem 1.7 is stated for the case of a single bubble, it is not hard to extend the results to the case of multiple bubbles and the corresponding length formulas will be more complicated. When the spinor field is vanishing, namely, in the case of α-harmonic maps, the refined analytical properties for a blow-up sequence were given in [25]. To handle the more complicated case of the coupled system of α-Dirac-harmonic maps, we need to develop new methods in order to show Theorem 1.6 and Theorem 1.7 and we leave the detailed proofs to the forthcoming paper [21].

From the perspective of differential geometry, it is natural and interesting to find some geometric and topological conditions on the target manifold such that the energy identities hold. In particular, a natural question is whether or not we can exploit some geometric and topological conditions to ensure that the limiting necks are some geodesics of finite length and hence the energy identity follows immediately.

In view of the works on minimal hypersurfaces (see e.g. [32]), it seems reasonable to impose the assumptions that the Ricci curvature of the target has a positive lower bound and the Dirac-harmonic sequence has bounded Morse index.

Let $(\phi, \psi) : M \to N$ be a α-Dirac-harmonic map. $\phi^*(TN)$ is the pull-back bundle over M. Let V be a section of $\phi^*(TN)$. We use V to vary (ϕ, ψ) via

$$\phi_\tau(x) = exp_{\phi(x)}(\tau V), \ \psi_\tau(x) = \psi^i(x) \otimes \frac{\partial}{\partial y^i}(\phi_\tau(x)). \tag{1.20}$$

Definition 1.8. Let $\Gamma(\phi^*TN)$ denote the linear space of the smooth sections of ϕ^*TN. The index of (ϕ, ψ) is defined as the maximal dimension of a linear subspace Ξ of $\Gamma(\phi^*TN)$ on which the second variation of L_α with respect to the variations (1.20) is negative, i.e., for any $V \in \Xi \subset \Gamma(\phi^*TN)$, there holds

$$\delta^2 L_\alpha(\phi, \psi)(V, V) < 0,$$

where

$$\begin{aligned}
&\delta^2 L_\alpha(\phi,\psi)(V,V)\\
&= \frac{d^2}{d\tau^2}\bigg|_{\tau=0} L_\alpha(\phi_\tau,\psi_\tau)\\
&= 2\alpha\int_M (1+|d\phi|^2)^{\alpha-1}\left(\langle\nabla V,\nabla V\rangle - R(V,\nabla\phi,\nabla\phi,V)\right) + 4\alpha(\alpha-1)\int_M (1+|d\phi|^2)^{\alpha-2}\langle d\phi,\nabla V\rangle^2\\
&\quad + 2\int_M \left\langle \psi^j\otimes\nabla_V\frac{\partial}{\partial y^j}, e_\alpha\cdot\widetilde{\nabla}_{e_\alpha}\left(\psi^i\otimes\nabla_V\frac{\partial}{\partial y^i}\right) + e_\alpha\cdot\psi^i\otimes R(V,e_\alpha)\frac{\partial}{\partial y^i}\right\rangle\\
&\quad + \int_M \left\langle \psi, e_\alpha\cdot\psi^i\otimes\left(R^j_{ikl;p}V^pV^k d\phi^l(e_\alpha)\frac{\partial}{\partial y^j} + R(V,e_\alpha)\nabla_V\frac{\partial}{\partial y^i} + R(V,\nabla_{e_\alpha}V)\frac{\partial}{\partial y^i}\right)\right\rangle.
\end{aligned}$$

Theorem 1.9. *Under the assumption of Theorem 1.6, suppose the Ricci curvature of the target manifold (N,g) has a positive lower bound, i.e. there exists a positive constant $\lambda_0>0$ such that $Ric_N\geq\lambda_0>0$ and assume the sequence $(\phi_\alpha,\psi_\alpha)$ has bounded index. Then the limit of the necks consist of geodesics of finite length. Moreover, the energy identities hold, i.e.*

$$\lim_{k\to\infty} E_{\alpha_k}(\phi_{\alpha_k}) = E(\phi) + |M| + \sum_{i=1}^{I}\sum_{l=1}^{l_i} E(\sigma_i^l),$$

$$\lim_{k\to\infty} E(\psi_{\alpha_k}) = E(\psi) + \sum_{i=1}^{I}\sum_{l=1}^{l_i} E(\xi_i^l).$$

The rest of this paper is organized as follows. In Section 2, we derive the Euler-Lagrange equations for α-Dirac-harmonic maps and prove the estimate (1.6). In Section 3, we establish some properties of α-Dirac-harmonic maps flow and obtain the global existence Theorem 1.1. In Section 4, we study the blow-up behaviour for a sequence of α-Dirac-harmonic maps. Theorem 1.2, Theorem 1.4 and Theorem 1.5 are proved in this section.

2. Euler-Lagrange equations

In this section, we derive the Euler-Lagrange equations for α-Dirac-harmonic maps and prove the key estimate (1.6) for the Dirac type operator $\not{D}$.

Lemma 2.1. *The Euler-Lagrange equations for L_α are*

$$\tau_\alpha = \frac{1}{\alpha}\mathrm{R}(\phi,\psi) \tag{2.1}$$

$$\not{D}\psi = 0 \tag{2.2}$$

where $\tau_\alpha=(\tau_\alpha^1,...,\tau_\alpha^n)$ and $\mathrm{R}(\phi,\psi)$ are defined respectively by

$$\tau_\alpha^i(\phi) := \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\beta}\left((1+|d\phi|^2)^{\alpha-1}\sqrt{g}g^{\beta\gamma}\frac{\partial\phi^i}{\partial x^\gamma}\right) - (1+|d\phi|^2)^{\alpha-1}g^{\beta\gamma}\Gamma^i_{jk}\frac{\partial\phi^j}{\partial x^\beta}\frac{\partial\phi^k}{\partial x^\gamma} \tag{2.3}$$

and

$$\mathrm{R}(\phi,\psi)(x) := \frac{1}{2}\langle\psi^i, \nabla\phi^l\cdot\psi^j\rangle R^m_{lij}(\phi(x))\frac{\partial}{\partial y^m}(\phi(x)). \tag{2.4}$$

Proof. Let ψ_t be a variation of ψ with $\frac{d\psi_t}{dt}|_{t=0} = \eta$ and fix ϕ. By Proposition 2.1 in [6], we know

$$\frac{dL_\alpha(\psi_t)}{dt}\bigg|_{t=0} = \int_M \langle\eta, \not{D}\psi\rangle.$$

Then (2.2) follows immediately.

For the equation of ϕ, let ϕ_t be a variation of ϕ such that $\frac{d\phi_t}{dt}|_{t=0} = \xi$ and ψ^i $(i = 1, ..., n)$ in $\psi(x) = \psi^i(x)\otimes\frac{\partial}{\partial y^i}(\phi(x))$ are independent of t. Also, by Proposition 2.1 in [6], we get

$$\frac{d}{dt}\bigg|_{t=0}\frac{1}{2}\int_M\langle\psi_t, \not{D}\psi_t\rangle = \frac{1}{2}\int_M\langle\psi^i, \nabla\phi^l\cdot\psi^j R_{mlij}\xi^m\rangle.$$

Finally, it is easy to check that

$$\begin{aligned}
&\frac{d}{dt}\bigg|_{t=0}\frac{1}{2}\int_M(1+|d\phi_t|^2)^\alpha \\
&= \alpha\int_M\left\{-\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\beta}\left((1+|d\phi|^2)^{\alpha-1}\sqrt{g}g^{\beta\gamma}\frac{\partial\phi^i}{\partial x^\gamma}\right) + (1+|d\phi|^2)^{\alpha-1}g^{\beta\gamma}\Gamma^i_{jk}\frac{\partial\phi^j}{\partial x^\beta}\frac{\partial\phi^k}{\partial x^\gamma}\right\}h_{im}\xi^m \\
&:= \alpha\int_M -\tau^i_\alpha h_{im}\xi^m.
\end{aligned}$$

Thus, we obtain

$$\frac{dL_\alpha(\phi_t)}{dt}|_{t=0} = \int_M\left\{-\alpha\tau^i_\alpha h_{im} + \frac{1}{2}\langle\psi^i, \nabla\phi^l\cdot\psi^j\rangle R_{mlij}\right\}\xi^m,$$

which implies the equation (2.1). □

By Nash's embedding theorem, we embed N isometrically into some $\mathbb{R}^K$, denoted by $f : N \to \mathbb{R}^K$. Set

$$\phi' = f\circ\phi \quad \textit{and} \quad \psi' = f_*\psi.$$

If we identify ϕ with ϕ' and ψ with ψ', we can get the following extrinsic form of the Euler-Lagrange equations:

Lemma 2.2. *Let $(\phi,\psi) : M \to N$ be an α-Dirac-harmonic map. Then, (ϕ,ψ) satisfies*

$$\Delta_g\phi = -(\alpha-1)\frac{\nabla_g|\nabla_g\phi|^2\nabla_g\phi}{1+|\nabla_g\phi|^2} + A(d\phi, d\phi) + \frac{Re\,(P(\mathcal{A}(d\phi(e_\alpha), e_\alpha\cdot\psi);\psi))}{\alpha(1+|\nabla_g\phi|^2)^{\alpha-1}}, \tag{2.5}$$

$$\partial\!\!\!/\psi = \mathcal{A}(d\phi(e_\alpha), e_\alpha\cdot\psi). \tag{2.6}$$

Proof. Firstly, it is easy to see that $\tau'_\alpha(\phi')$ and $\tau_\alpha(\phi)$ satisfy

$$\tau'_\alpha(\phi') = (1+|d\phi|^2)^{\alpha-1}A(d\phi(e_\beta), d\phi(e_\beta)) + df(\tau_\alpha(\phi)). \tag{2.7}$$

Secondly, by similar arguments as in [5, 6, 8], we know

$$\not{D}'\psi' = f_*(\not{D}\psi) + \mathcal{A}(d\phi(e_\beta), e_\beta\cdot\psi)$$

and

$$df(\tau_\alpha(\phi)) = \frac{1}{\alpha} Re\left(P(\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi)\right).$$

Then the conclusion of the lemma follows from the fact that $\not{D}' = \not{\partial}$ (here, $\not{D}'$ is the Dirac operator along the map ϕ') and

$$\tau'_\alpha(\phi) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\beta}\left((1+|d\phi|^2)^{\alpha-1}\sqrt{g} g^{\beta\gamma}\frac{\partial \phi}{\partial x^\gamma}\right).$$

□

In the end of this section, we shall prove the key estimate (1.6). The idea is to use a contradiction argument, where a crucial ingredient is the uniqueness of the Dirac equation, i.e.

$$\begin{cases} \not{D}\psi &= 0, \ M, \\ \mathbf{B}\psi &= 0, \ \partial M \end{cases}$$

has only the trivial solution $\psi \equiv 0$ when $\phi \in W^{1,p}(M,N)$ for some $p > 2$, see Theorem 1.2 and Theorem 4.1 in [9].

Lemma 2.3. *Let M be a compact spin Riemann surface with smooth boundary ∂M and $\psi = (\psi^1, ..., \psi^K)$, $\psi^A \in \Sigma M$, $A = 1, ..., K$. Let $\Omega \in \Gamma(\Lambda^1 T^*M \otimes so(K))$, i.e. $\Omega^A_B = -\Omega^B_A$ and $\Omega \in L^{2p'}(M)$, $d\Omega \in L^{p'}(M)$ for some $p' > 1$. Suppose $\psi \in W^{1,p}(M, \mathbb{R}^K)$ and $\eta = (\eta^1, ..., \eta^K) \in L^p(M, \mathbb{R}^K)$, $1 < p < 2$ satisfy*

$$\not{\partial}\psi^A + \Omega^A_B \cdot \psi^B = \eta^A$$

then there exists a positive constant $C = C(p, M, K, \|\Omega\|_{L^{2p'}(M)} + \|d\Omega\|_{L^{p'}(M)})$ such that

$$\|\psi\|_{W^{1,p}(M)} \leq C\left(\|\not{\partial}\psi + \Omega\cdot\psi\|_{L^p(M)} + \|\mathbf{B}\psi\|_{W^{1-1/p,p}(\partial M)}\right). \tag{2.8}$$

Our proof will follow the scheme of Remark 3.3, Theorem 3.11 and Remark 3.7 in [9]. The main difference is that, on a two dimensional domain considered in our lemma, the two real numbers $p' > 1$ and $1 < p < 2$ can be arbitrary and be independent of each other, while Theorem 3.11 in [9] requires that $1 < p < p'$, which is too strong and hence can not be applied to our blow-up analysis of a sequence of α-Dirac-harmonic map as $\alpha \searrow 1$. This is a new and crucial observation in the present paper.

Proof. First, by Theorem 3.3 in [9], we have

$$\|\psi\|_{W^{1,p}(M)} \leq C\left(\|\not{\partial}\psi + \Omega\cdot\psi\|_{L^p(M)} + \|\mathbf{B}\psi\|_{W^{1-1/p,p}(\partial M)} + \|\psi\|_{L^p(M)}\right), \tag{2.9}$$

where $C = C(p, M, K, \Omega)$ is a positive constant.

Next, we claim:

$$\|\psi\|_{W^{1,p}(M)} \leq C\left(\|\not{\partial}\psi + \Omega\cdot\psi\|_{L^p(M)} + \|\mathbf{B}\psi\|_{W^{1-1/p,p}(\partial M)}\right), \tag{2.10}$$

where $C = C(p, M, K, \Omega)$ is a positive constant.

In fact, if (2.10) does not hold, then there exists $\psi_i \in W^{1,p}(M, \mathbb{R}^K)$, such that

$$\|\psi_i\|_{W^{1,p}(M)} \geq i\left(\|\not{\partial}\psi_i + \Omega\cdot\psi_i\|_{L^p(M)} + \|\mathbf{B}\psi_i\|_{W^{1-1/p,p}(\partial M)}\right). \tag{2.11}$$

Without loss of generality, we may assume $\|\psi_i\|_{L^p} = 1$. Then by (2.9) and (2.11), we have

$$\|\partial\!\!\!/\psi_i + \Omega\cdot\psi_i\|_{L^p(M)} + \|\mathbf{B}\psi_i\|_{W^{1-1/p,p}(\partial M)} \le \frac{C}{i} \tag{2.12}$$

and

$$\|\psi_i\|_{W^{1,p}(M)} \le C. \tag{2.13}$$

Then there exists a subsequence of $\{\psi_i\}$ (also denoted by $\{\psi_i\}$) with $\psi \in W^{1,p}(M,\mathbb{R}^K)$, such that,

$$\psi_i \rightharpoonup \psi \textit{ weakly in } W^{1,p}(M) \textit{ and } \psi_i \to \psi \textit{ strongly in } L^p(M).$$

Moreover, it is easy to see that ψ is a weak solution of

$$\partial\!\!\!/\psi + \Omega\cdot\psi = 0$$

with boundary condition

$$\mathbf{B}\psi = 0.$$

Since $p' > 1$, by Theorem 4.1 in [9], there must hold $\psi \equiv 0$. However, the fact that $\|\psi_i\|_{L^p(M)} = 1$ tells us $\|\psi\|_{L^p(M)} = 1$. This is a contradiction and hence (2.10) holds.

For (2.8), we can also prove it by a contradiction argument. In fact, if it does not hold, then we can find a sequence $\Omega_i \in \Gamma(\Lambda^1T^*M\otimes so(K))$ and $\psi_i \in W^{1,p}(M,\mathbb{R}^K)$, such that

$$1 = \|\psi_i\|_{W^{1,p}(M)} \ge i\left(\|\partial\!\!\!/\psi_i + \Omega_i\cdot\psi_i\|_{L^p(M)} + \|\mathbf{B}\psi_i\|_{W^{1-1/p,p}(\partial M)}\right), \tag{2.14}$$

and

$$\|\Omega_i\|_{L^{2p'}(M)} + \|d\Omega_i\|_{L^{p'}(M)} \le C.$$

By the weak compactness and compact embedding, there exists a subsequence of (Ω_i,ψ_i) (without changing notation) and $\psi \in W^{1,p}(M,\mathbb{R}^K)$, $\Omega \in W^{1,p'}(M)\cap L^{2p'}(M)$, such that

$$\Omega_i \rightharpoonup \Omega \textit{ weakly in } L^{2p'}(M) \textit{ and } d\Omega_i \rightharpoonup d\Omega \textit{ weakly in } L^{p'}(M)$$

and

$$\psi_i \rightharpoonup \psi \textit{ weakly in } W^{1,p}(M) \textit{ and } \psi_i \to \psi \textit{ strongly in } L^{p^*}(M),$$

for any p^* satisfying $\frac{1}{p^*} > \frac{1}{p} - \frac{1}{2}$.

Then it is easy to see that ψ is a weak solution of $\partial\!\!\!/\psi + \Omega\cdot\psi = 0$ with boundary condition $\mathbf{B}\psi = 0$ which implies $\psi \equiv 0$ by Theorem 4.1 in [9], since $p' > 1$. Thus

$$\lim_{i\to\infty}\|\psi_i\|_{L^{p^*}(M)} = 0.$$

Therefore, we have

$$\begin{aligned}\|\partial\!\!\!/\psi_i + \Omega\cdot\psi_i\|_{L^p(M)} &\le \|\partial\!\!\!/\psi_i + \Omega_i\cdot\psi_i\|_{L^p(M)} + \|(\Omega-\Omega_i)\psi_i\|_{L^p(M)}\\ &\le \|\partial\!\!\!/\psi_i + \Omega_i\cdot\psi_i\|_{L^p(M)} + \|\Omega-\Omega_i\|_{L^{2p'}(M)}\|\psi_i\|_{L^{p^*}(M)}\\ &\le \frac{1}{i} + C(N)\|\left(|\Omega|+|\Omega_i|\right)\|_{L^{2p'}(M)}\|\psi_i\|_{L^{p^*}(M)} \to 0\end{aligned}$$

as $i\to\infty$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{2p'} > \frac{1}{p} - \frac{1}{2}$.

But, (2.10) tells us

$$1 = \|\psi_i\|_{W^{1,p}(M)} \le C(p,M,K,\Omega)\left(\|\partial\!\!\!/\psi_i + \Omega\cdot\psi_i\|_{L^p(M)} + \|\mathbf{B}\psi_i\|_{W^{1-1/p,p}(\partial M)}\right) \to 0, \tag{2.15}$$

as $i\to\infty$, which is a contradiction. We proved this lemma.

□

As a direct application of Lemma 2.3, we have

Lemma 2.4. *Let M be a compact spin Riemann surface with boundary ∂M, N be a compact Riemannian manifold. Let $\phi \in W^{1,2\alpha}(M,N)$ for some $\alpha > 1$ and $\psi \in W^{1,p}(M, \Sigma M \otimes \phi^*TN)$, $1 < p < 2$, then there exists a positive constant $C = C(p, M, N, \|\nabla\phi\|_{L^{2\alpha}(M)})$, such that*

$$\|\psi\|_{W^{1,p}(M)} \leq C\left(\|\not{D}\psi\|_{L^p(M)} + \|\mathbf{B}\psi\|_{W^{1-1/p,p}(\partial M)}\right). \tag{2.16}$$

Proof. Noting that $\mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi) = -\Omega \cdot \psi$ where

$$\Omega = \sum_{i=n+1}^{K} [\nu^i(\phi), d\nu^i(\phi)] = \sum_{i=n+1}^{K} \left((\nu^i)^A(\nabla_{e_\gamma}\nu^i)^B e_\gamma - (\nu^i)^B(\nabla_{e_\gamma}\nu^i)^A e_\gamma\right)$$

and $\{\nu^i\}_{i=n+1}^K$ is an orthonormal basis of the normal bundle $T^\perp N$ and $\nu^i = ((\nu^i)^1, ..., (\nu^i)^K)$ (see Remark 2.1 in [9]), thus

$$\not{D}\psi = \not{\partial}\psi - \mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi) = \not{\partial}\psi + \Omega(\phi) \cdot \psi.$$

Then the conclusion of the lemma follows immediately from Lemma 2.3 and the fact that $d\Omega = [d\nu(\phi), d\nu(\phi)]$. □

3. Global existence of α-Dirac-harmonic map flow

In this section, we will prove the global existence result for the α-Dirac-harmonic map flow and show that the limit map at infinity time is an α-Dirac-harmonic map.

The equations (1.10)-(1.11) have the equivalent intrinsic form

$$\begin{aligned}
&\partial_t\phi = \frac{1}{(1+|d\phi|^2)^{\alpha-1}}\left(\tau_\alpha(\phi) - \frac{1}{\alpha}R(\phi,\psi)\right),\\
&\not{D}\psi = 0,
\end{aligned}$$

where we regard ϕ as a map into N and ψ as a section of $\Sigma M \otimes \phi^*TN$. This leads us to consider another isometrical embedding. In fact, [14] (Page 108) says that (N, h) can be embedded isometrically into some $\mathbb{R}^L$ with some non-flat metric denoted by h_L. Moreover, this isometric embedding is totally geodesic and there exist a tubular neighborhood $\mathcal{N}$ of N and an isometric involution $i : \mathcal{N} \to \mathcal{N}$ which has precisely N for its fixed point set. Since N is a totally geodesic submanifold, then $\tau_\alpha(\phi) = \tau_\alpha^{\mathbb{R}^L}$ and it suffices to study [3]

$$\partial_t\phi = \Delta_g\phi + (\alpha-1)\frac{\nabla_g|\nabla_g\phi|^2\nabla_g\phi}{1+|\nabla_g\phi|^2} + \Gamma(\phi)\#\nabla\phi\#\nabla\phi + R(\phi)\#\nabla\phi\#\psi\#\psi, \tag{3.1}$$

$$\not{D}\psi = 0, \tag{3.2}$$

where Γ is the Levi-Civita connection of $(\mathbb{R}^L, h_L)$, R is the curvature of $(\mathbb{R}^L, h_L)$ and

$$\not{D}\psi = \not{\partial}\psi + \Gamma(\phi)\#\nabla\phi\#\psi.$$

[3]Here and in the sequel, # denotes a multi-linear map with smooth coefficients.

Next, in order to emphasize the Dirac operator $\not{D}$ depends on the map ϕ, we sometimes use the notation

$$\not{D}_\phi := \not{D}.$$

Noting that

$$|\nabla\phi|^2 = (h_L)_{ij}(\phi)\nabla\phi^i\nabla\phi^j,$$

if we expand $\nabla|\nabla\phi|^2$, there is an additional term like $(h_L)_{ij,k}\nabla\phi^i\nabla\phi^j\nabla\phi^k$. This term and $\Gamma(\phi)\#\nabla\phi\#\nabla\phi$ will be put together into the term $\Gamma(\phi)\#\nabla\phi\#\nabla\phi\#\nabla\phi$. Therefore, the equations can be rewritten as

$$\begin{aligned} \partial_t\phi &= \Delta_g\phi + 2(\alpha-1)\frac{\nabla^2_{\beta\gamma}\phi^i\nabla_\beta\phi^i\nabla_\gamma\phi}{1+|\nabla_g\phi|^2} + \Gamma(\phi)\#\nabla\phi\#\nabla\phi\#\nabla\phi \\ &\quad + R(\phi)\#\nabla\phi\#\psi\#\psi, \end{aligned} \tag{3.3}$$

$$\not{D}_\phi\psi = 0. \tag{3.4}$$

Firstly, we have the following short-time existence result for the α-Dirac-harmonic map flow with Dirichlet-chiral boundary condition. We use some ideas from harmonic map flows [12, 14, 24]. The argument here is not quite the same as in the case of Dirac-harmonic maps in [9]. For the case of α-harmonic map flow from a closed Riemann surface, see the appendix in [16].

Theorem 3.1. *Let (M, g) be a compact spin Riemann surface with a smooth boundary ∂M and (N, h) be another compact Riemannian manifold. Then there exists a positive constant ϵ_1 depending only on M, N, such that, for any $1 < \alpha < 1 + \epsilon_1$ and any*

$$\phi_0 \in C^{2+\mu}(M, N),\ \varphi \in C^{2+\mu}(\partial M, N),\ \psi_0 \in C^{1+\mu}(\partial M, \Sigma M \otimes \varphi^* TN),$$

where $0 < \mu < 1$, the problem (1.10)-(1.11) and (1.12) admits a unique solution

$$\phi \in C^{2+\mu, 1+\mu/2}(M \times [0, T], N),$$

and

$$\psi \in C^{\mu,\mu/2}(M \times [0, T], \Sigma M \otimes \phi^* TN),\ \psi \in L^\infty([0, T]; C^{1+\mu}(M)),$$

for some time $T > 0$.

Proof. **Step 1:** Short-time existence of (3.3)-(3.4).

For every $T > 0$, we define

$$\mathcal{U} := \left\{u, du \in C^{\mu,\mu/2}(M \times [0, T])\Big|\ \|u\|_{C^{\mu,\mu/2}(M\times[0,T])} + \|du\|_{C^{\mu,\mu/2}(M\times[0,T])} \le 1, u|_{M\times\{0\}\cup\partial M\times[0,T]} = 0\right\}.$$

Consider the following linear parabolic-elliptic system:

$$\begin{aligned} \partial_t\phi &= \Delta_g\phi + 2(\alpha-1)\frac{\nabla^2_{\beta\gamma}\phi^i\nabla_\beta u^i\nabla_\gamma u}{1+|\nabla_g u|^2} + \Gamma(u)\#\nabla u\#\nabla u\#\nabla u \\ &\quad + R(u)\#\nabla u\#\psi\#\psi + \Delta_g\phi_0 + 2(\alpha-1)\frac{\nabla^2_{\beta\gamma}\phi_0^i\nabla_\beta u^i\nabla_\gamma u}{1+|\nabla_g u|^2}, \end{aligned} \tag{3.5}$$

$$\not{D}_u\psi = 0. \tag{3.6}$$

Now, let us begin a routine iteration argument as in [24] to show the local existence. For every $u \in \mathcal{U}$, on the one hand, by Theorem 4.6 in [9], there exists a unique solution $v_1 \in C^{1+\mu}(M, \Sigma M \otimes u^*\mathbb{R}^L)$ to the problem (3.6) with boundary condition $\mathbf{B}\psi = \mathbf{B}\psi_0$, satisfying

$$\|v_1\|_{C^{1+\mu}(M)} \leq C(\mu, M, N, \|u\|_{C^{1+\mu}(M)})\|\mathbf{B}\psi_0\|_{C^{1+\mu}(\partial M)}.$$

Moreover, for any $0 < t, s < T$, it is easy to see that $v_1(\cdot, t) - v_1(\cdot, s)$ satisfy the following equation

$$\begin{aligned}\partial\!\!\!/(v_1(\cdot, t) - v_1(\cdot, s)) = & -\Gamma(u(t))\#\nabla u(t)\#(v_1(\cdot, t) - v_1(\cdot, s)) \\ & - \Gamma(u(t))\#\nabla(u(t) - u(s))\#v_1(\cdot, s) \\ & - (\Gamma(u(t)) - \Gamma(u(s)))\#\nabla u(s)\#v_1(\cdot, s) \text{ in } M,\end{aligned}$$

i.e.

$$\begin{aligned}D\!\!\!/_{u(t)}(v_1(\cdot, t) - v_1(\cdot, s)) = & -\Gamma(u(t))\#\nabla(u(t) - u(s))\#v_1(\cdot, s) \\ & - (\Gamma(u(t)) - \Gamma(u(s)))\#\nabla u(s)\#v_1(\cdot, s) \text{ in } M,\end{aligned} \tag{3.7}$$

with boundary data

$$\mathbf{B}(v_1(\cdot, t) - v_1(\cdot, s)) = 0 \text{ on } \partial M.$$

By Theorem 1.2 in [9] and Sobolev embedding, for any $\delta \in (0, 1)$, we have

$$\begin{aligned}&\|v_1(\cdot, t) - v_1(\cdot, s)\|_{C^\delta(M)} \\ &\leq C(\delta, M, N, \|u\|_{C^1(M)})\|v_1\|_{L^\infty(M)}(\|u(\cdot, t) - u(\cdot, s)\|_{L^\infty(M)} + \|du(\cdot, t) - du(\cdot, s)\|_{L^\infty(M)}) \\ &\leq C(\delta, M, N, \|u\|_{C^1(M)})\|\mathbf{B}\psi_0\|_{C^{1+\mu}(\partial M)}|t - s|^{\frac{\mu}{2}}.\end{aligned}$$

Therefore,

$$\|v_1\|_{C^{\mu, \frac{\mu}{2}}(M\times[0,T])} \leq C(\mu, M, N, \|u\|_{C^{1+\mu}(M)})\|\mathbf{B}\psi_0\|_{C^{1+\mu}(\partial M)}.$$

On the other hand, when $\alpha - 1$ is sufficiently small, by the standard theory of linear parabolic systems, for above (u, v_1), there exists a unique solution $u_1 \in C^{2+\mu, 1+\frac{\mu}{2}}(M\times[0, T], \mathbb{R}^L)$ to the problem (3.5) with the initial-boundary data $\phi|_{M\times\{0\}\cup\partial M\times[0,T]} = 0$, such that

$$\|u_1\|_{C^{2+\mu, 1+\frac{\mu}{2}}(M\times[0,T])} \leq C(\mu, M, N)(\|u_1\|_{C^0(M\times[0,T])} + \|\phi_0\|_{C^{2+\mu}(M)} + \|\mathbf{B}\psi_0\|_{C^{1+\mu}(\partial M)} + 1).$$

Noting that $u_1(\cdot, 0) = 0$, we have

$$\|u_1\|_{C^0(M\times[0,T])} \leq C(\mu, M, N)T(\|u_1\|_{C^0(M\times[0,T])} + \|\phi_0\|_{C^{2+\mu}(M)} + \|\mathbf{B}\psi_0\|_{C^{1+\mu}(\partial M)} + 1).$$

Taking T small enough, we obtain

$$\|u_1\|_{C^0(M\times[0,T])} \leq CT(\|\phi_0\|_{C^{2+\mu}(M)} + \|\mathbf{B}\psi_0\|_{C^{1+\mu}(\partial M)} + 1).$$

By the interpolation inequality for Hölder spaces (see Proposition 4.2 in [26]), we have

$$\|u_1\|_{C^{\mu, \frac{\mu}{2}}(M\times[0,T])} + \|\nabla u_1\|_{C^{\mu, \frac{\mu}{2}}(M\times[0,T])} \leq C\|u_1\|^{\frac{1-\mu}{2}}_{C^0(M\times[0,T])}\|u_1\|^{\frac{1+\mu}{2}}_{C^{2,1}(M\times[0,T])}.$$

Thus, if we choose T sufficiently small, then $u_1 \in \mathcal{U}$. This is the first step. Similarly, we can get (u_2, v_2) by using the above argument and substituting u with $u_1 + \phi_0$. After a standard induction procedure, we will get a solution (u_{k+1}, v_{k+1}) of (3.5) and (3.6) with $u = u_k + \phi_0$, satisfying

$$\|v_{k+1}\|_{C^{\mu, \frac{\mu}{2}}(M\times[0,T])} \leq C(\mu, M, N, \|\phi_0\|_{C^{1+\mu}(M)})\|\mathbf{B}\psi_0\|_{C^{1+\mu}(\partial M)}$$

and

$$\|u_{k+1}\|_{C^{2+\mu,1+\frac{\mu}{2}}(M\times[0,T])} \leq C(\mu, M, N)(\|\phi_0\|_{C^{2+\mu}(M)} + \|\mathbf{B}\psi_0\|_{C^{1+\mu}(\partial M)} + 1).$$

After passing to a subsequence, we know u_k converges to some ϕ in $C^{2,1}(M \times [0,T])$ and v_k converges to some ψ in $C^0(M \times [0,T])$. Then $(\phi + \phi_0, \psi)$ is a solution of equation (3.3)-(3.4) with boundary-initial data (1.12). Since $\phi \in C^{2,1}(M \times [0,T], \mathbb{R}^L)$ and $\psi \in C^0(M \times [0,T], \Sigma M \otimes (\phi + \phi_0)^* T\mathbb{R}^L)$, by the standard theory of Dirac-harmonic maps (see Lemma 3.6 in [19] or Lemma 3.4 below), it is easy to see that $\phi \in C^{2+\mu,1+\frac{\mu}{2}}(M \times [0,T], \mathbb{R}^L)$ and $\psi \in C^{\mu,\frac{\mu}{2}}(M \times [0,T], \Sigma M \otimes (\phi + \phi_0)^* T\mathbb{R}^L) \cap L^\infty([0,T]; C^{1+\mu}(M))$.

Step 2: Uniqueness.

If there are two solutions (u_1, v_1) and (u_2, v_2) to equation (3.3)-(3.4) with boundary-initial data (1.12), subtracting the equations of u_1 and u_2, then multiplying by $u_1 - u_2$ and integrating over M, we have

$$\begin{aligned}
&\int_M \partial_t(u_1 - u_2)(u_1 - u_2) \\
&\leq \int_M \Delta_g(u_1 - u_2)(u_1 - u_2) + 2(\alpha - 1)\int_M \frac{\nabla^2_{\beta\gamma}(u_1^i - u_2^i)\nabla_\beta u_1^i \nabla_\gamma u_1}{1 + |\nabla_g u_1|^2}(u_1 - u_2) \\
&\quad + C\int_M |u_1 - u_2|^2 + C\int_M |\nabla u_1 - \nabla u_2||u_1 - u_2| + C\int_M |v_1 - v_2||u_1 - u_2|.
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_M |u_1 - u_2|^2 \\
&\leq \int_M -|\nabla u_1 - \nabla u_2|^2 - 2(\alpha - 1)\int_M \frac{\nabla_\beta(u_1^i - u_2^i)\nabla_\beta u_1^i \nabla_\gamma u_1}{1 + |\nabla_g u_1|^2}\nabla_\gamma(u_1 - u_2) \\
&\quad + C\int_M |u_1 - u_2|^2 + C\int_M |\nabla u_1 - \nabla u_2||u_1 - u_2| + C\int_M |v_1 - v_2||u_1 - u_2|.
\end{aligned}$$

By Young's inequality and noting that the second term on the right hand side of the above inequality is nonpositive, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_M |u_1 - u_2|^2 \leq &-\frac{1}{2}\int_M |\nabla u_1 - \nabla u_2|^2 + C\int_M |u_1 - u_2|^2 \\
&+ C\int_M |v_1 - v_2||u_1 - u_2|.
\end{aligned} \tag{3.8}$$

Similar to deriving (3.7), we know $v_1 - v_2$ satisfies the following equation

$$\begin{aligned}
\not{D}_{u_1}(v_1 - v_2) = &-\Gamma(u_1)\#\nabla(u_1 - u_2)\#v_2 \\
&- (\Gamma(u_1) - \Gamma(u_2))\#\nabla u_2\#v_2 \textit{ in } M,
\end{aligned} \tag{3.9}$$

with the boundary data

$$\mathbf{B}(v_1 - v_2) = 0 \textit{ on } \partial M.$$

By Theorem 1.2 in [9], we have

$$\|v_1 - v_2\|_{W^{1,2}(M)} \leq C(\|u_1 - u_2\|_{L^2(M)} + \|\nabla u_1 - \nabla u_2\|_{L^2(M)}). \tag{3.10}$$

Therefore, by (3.8) and Young's inequality, we have

$$\frac{d}{dt}\int_M |u_1 - u_2|^2 \leq C\int_M |u_1 - u_2|^2, \tag{3.11}$$

which implies $u_1 \equiv u_2$ on $M \times [0, T]$ if $u_1 = u_2$ for $t = 0$. Then $v_1 \equiv v_2$ follows immediately from (3.10).

Step 3: $\phi(x, t) \in N$ for all $(x, t) \in M \times [0, T]$.

Since $i : \mathcal{N} \to \mathcal{N}$ is an isometric involution and $\phi_0 \in N$, $\varphi \in N$, then $(i \circ \phi, i_*\psi)$ is also a solution to (3.3)-(3.4) with the same boundary-initial data (1.12). By the uniqueness, $i \circ \phi = \phi$ which implies $\phi(x, t) \in N$. We finished the proof of this theorem. □

Next, we shall control the α-energy of the map part, i.e. $E_\alpha(\phi)$, along the α-Dirac-harmonic maps flow. Precisely, we have

Lemma 3.2. *Suppose* (ϕ, ψ) *is a solution of* (1.10)-(1.11) *with the boundary-initial data* (1.12)*, then there holds*

$$E_\alpha(\phi(t)) + \alpha\int_{M^t}(1 + |\nabla_g\phi|^2)^{\alpha-1}|\partial_t\phi|^2 dMdt \leq E_\alpha(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}.$$

Moreover, $E_\alpha(\phi(t)) + \frac{1}{2}\int_{\partial M}\langle\overrightarrow{n}\cdot\mathbf{B}\psi_0, \psi(t)\rangle$ *is absolutely continuous on* $[0, T]$ *and non-increasing.*

Proof. Firstly, it is easy to see that the equation (1.10) can be written as follows:

$$\begin{aligned}(1 + |\nabla_g\phi|^2)^{\alpha-1}\partial_t\phi &= div\left((1 + |\nabla_g\phi|^2)^{\alpha-1}\nabla_g\phi\right) - (1 + |\nabla_g\phi|^2)^{\alpha-1}A(d\phi, d\phi)\\ &\quad - \frac{1}{\alpha}Re\left(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi); \psi)\right).\end{aligned} \tag{3.12}$$

Multiplying the above equation by $\alpha\partial_t\phi$ and using the Lemma 3.1 in [19] that,

$$\int_{M_s^t}\langle P(\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi); \psi), \frac{\partial\phi}{\partial t}\rangle dMdt = -\frac{1}{2}\int_s^t \frac{d}{dt}\int_{\partial M}\langle\mathbf{B}\psi_0, \overrightarrow{n}\cdot\psi\rangle(t)dt, \tag{3.13}$$

we have

$$\begin{aligned}&\alpha\int_{M_s^t}(1 + |\nabla_g\phi|^2)^{\alpha-1}|\partial_t\phi|^2 dMdt - \alpha\int_{M_s^t} div\left((1 + |\nabla_g\phi|^2)^{\alpha-1}\nabla_g\phi\right)\partial_t\phi dMdt\\ &= -\int_{M_s^t}\langle P(\mathcal{A}(d\phi(e_\gamma), e_\gamma\cdot\psi); \psi), \partial_t\phi\rangle dMdt = \frac{1}{2}\int_s^t\frac{d}{dt}\int_{\partial M}\langle\mathbf{B}\psi_0, \overrightarrow{n}\cdot\psi\rangle dt,\end{aligned}$$

for any $0 \leq s \leq t \leq T$. Integrating by parts, we get

$$\begin{aligned}&\frac{1}{2}\int_s^t\frac{d}{dt}\int_M(1 + |\nabla_g\phi|^2)^\alpha dMdt + \alpha\int_{M_s^t}(1 + |\nabla_g\phi|^2)^{\alpha-1}|\partial_t\phi|^2 dMdt\\ &= \frac{1}{2}\int_s^t\frac{d}{dt}\int_{\partial M}\langle\mathbf{B}\psi_0, \overrightarrow{n}\cdot\Psi\rangle dt.\end{aligned} \tag{3.14}$$

So, we have

$$
\begin{aligned}
&E_\alpha(\phi(t)) + \alpha \int_{M^t} (1+|\nabla_g \phi|^2)^{\alpha-1} |\partial_t \phi|^2 dM dt \\
&\le E_\alpha(\phi_0) + \frac{1}{2}\Big|\int_{\{0\}\times\partial M} \langle \mathbf{B}\psi_0, \overrightarrow{n}\cdot\psi\rangle\Big| + \frac{1}{2}\Big|\int_{\{t\}\times\partial M} \langle \mathbf{B}\psi_0, \overrightarrow{n}\cdot\psi\rangle\Big| \\
&\le E_\alpha(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)},
\end{aligned}
$$

where the last inequality follows from Proposition 2.5 in [19] that

$$
\|\psi\|_{L^2(\partial M)} = \sqrt{2}\|\mathbf{B}\psi\|_{L^2(\partial M)} = \sqrt{2}\|\mathbf{B}\psi_0\|_{L^2(\partial M)},
$$

since $\not{D}\psi \equiv 0$. Also, we have

$$
\begin{aligned}
&\int_s^t \frac{d}{dt}\Big(\frac{1}{2}\int_M (1+|\nabla_g\phi|^2)^\alpha dM + \frac{1}{2}\int_{\partial M} \langle \overrightarrow{n}\cdot \mathbf{B}\psi_0, \Psi\rangle\Big) dt \\
&= -\alpha \int_{M_s^t} (1+|\nabla_g\phi|^2)^{\alpha-1}|\partial_t\phi|^2 dM dt,
\end{aligned}
\tag{3.15}
$$

and the claims follow. □

Consequently, using the key estimate for the Dirac operator along a map in Lemma 2.4, we are able to control the spinor part along the flow. For the Dirac-harmonic map flow studied in [9, 19], however, there is in general no such a nice property.

Lemma 3.3. *Suppose (ϕ, ψ) is a solution of* (1.10)-(1.11) *with the boundary-initial data* (1.12)*, then for any $1 < p < 2$, there holds*

$$
\|\psi(\cdot, t)\|_{W^{1,p}(M)} \le C\|\mathbf{B}\psi_0\|_{C^1(\partial M)}, \ \forall\, 0 \le t \le T,
\tag{3.16}
$$

where C is a positive constant depending only on p, M, N, $E_\alpha(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}$.

Proof. According to Lemma 3.2, for any $0 \le t \le T$, we get

$$
E_\alpha(\phi(\cdot, t)) \le E_\alpha(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}.
$$

Then by Lemma 2.4, we have

$$
\begin{aligned}
\|\psi(\cdot,t)\|_{W^{1,p}(M)} &\le C(p, M, N, \|\nabla\phi\|_{L^{2\alpha}(M)})\|\mathbf{B}\psi_0\|_{C^1(\partial M)} \\
&\le C(p, M, N, E_\alpha(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)})\|\mathbf{B}\psi_0\|_{C^1(\partial M)}, \ \forall\, 0 \le t \le T.
\end{aligned}
$$

□

Next, we derive a small energy regularity theory for the α-Dirac-harmonic map flow.

Lemma 3.4. *Suppose that $\phi_0 \in C^{2+\mu}(M, N)$, $\varphi \in C^{2+\mu}(\partial M, N)$ and $\psi_0 \in C^{1+\mu}(\partial M, \Sigma M \otimes \varphi^* TN)$, where $0 < \mu < 1$ is a positive constant. Let (ϕ, ψ) be a solution of* (1.10)-(1.11) *in $M \times [0, T]$ with boundary-initial data* (1.12)*. Given $z_0 = (x_0, t_0) \in M \times (0, T]$, denote $P^M_R(z_0) := B^M_R(x_0) \times$*

$[t_0 - R^2, t_0]$. *Then there exist three positive constants* $\epsilon_2 = \epsilon_2(M, N)$, $\epsilon_3 = \epsilon_3(M, N, \phi_0, \varphi, \psi_0)$ *and* $C = C(\mu, R, M, N, E_\alpha(\phi_0), \|\phi_0\|_{C^{2+\mu}(M)}, \|\mathbf{B}\psi_0\|_{C^{1+\mu}(\partial M)})$ *such that if*

$$1 < \alpha < 1 + \epsilon_2 \text{ and } \sup_{[t_0-4R^2, t_0]} E(\phi(t); B^M_{2R}(x_0)) \le \epsilon_3,$$

then

$$\sqrt{R}\|\psi\|_{L^\infty(P^M_R(z_0))} + R\|\nabla\phi\|_{L^\infty(P^M_R(z_0))} \le C \tag{3.17}$$

and for any $0 < \beta < 1$,

$$\sup_{t_0 - \frac{R^2}{4} \le t \le t_0} \|\psi(t)\|_{C^{1+\mu}(B^M_{R/2}(z_0))} + \|\nabla\phi\|_{C^{\beta, \frac{\beta}{2}}(P^M_{R/2}(z_0))} \le C(\beta), \tag{3.18}$$

Moreover, if

$$\sup_{x_0 \in M} \sup_{[t_0-4R^2, t_0]} E(\phi(t); B^M_{2R}(x_0)) \le \epsilon_3,$$

then

$$\|\phi\|_{C^{2+\mu, 1+\frac{\mu}{2}}(M\times[t_0 - \frac{R^2}{8}, t_0])} + \|\psi\|_{C^{\mu, \frac{\mu}{2}}(M\times[t_0 - \frac{R^2}{8}, t_0])} + \sup_{t_0 - \frac{R^2}{8} \le t \le t_0} \|\psi\|_{C^{1+\mu}(M)} \le C. \tag{3.19}$$

Proof. For simplicity of notation and a better expression of the idea of proof, we assume $M \subset \mathbb{R}^2$ is a bounded closed domain with the standard Euclidean metric.

Step 1: We derive (3.18) and (3.19) from (3.17).

Take a cut-off function $\eta \in C^\infty_0(P^M_R(z_0))$ such that $0 \le \eta \le 1$, $\eta|_{P^M_{3R/4}(z_0)} \equiv 1$, $|\nabla^j\eta| \le \frac{C}{R^j}$, $j = 1, 2$ and $|\partial_t\eta| \le \frac{C}{R^2}$. Set $U = \eta\phi$, then

$$\begin{cases} U_t - a_{\beta\gamma}\frac{\partial^2 U}{\partial x^\beta \partial x^\gamma} = f, & in \quad P^M_R(z_0); \\ U(x, t) = 0, & on \quad B^M_R(z_0) \times \{t = t_0 - R^2\}; \\ U(x, t) = \eta\varphi, & on \quad \partial M \times (t_0 - R^2, t_0), \end{cases}$$

where $f = f(\nabla\phi, \phi, \psi, \partial_t\eta, \nabla^2\eta, \nabla\eta, \eta)$ and

$$a_{\beta\gamma} = \delta_{\beta\gamma} + 2(\alpha - 1)\frac{\nabla_\beta\phi\nabla_\gamma\phi}{1 + |\nabla\phi|^2}.$$

Under the assumption (3.17), we know $f \in L^\infty$. Noting that

$$\partial_t - a_{\beta\gamma}\frac{\partial^2}{\partial x^\beta \partial x^\gamma}$$

is a parabolic operator when $\alpha - 1$ is sufficiently small, by standard parabolic theory, for any $1 < p < \infty$, we have

$$\|U\|_{W^{2,1}_p(P^M_R(z_0))} \le C(\|f\|_{L^p(P^M_R(z_0))} + \|\eta\varphi\|_{W^{2,1}_p(\partial P^M_R(z_0))}) \le C(1 + \|\phi_0\|_{C^2(M)})$$

Then for any $0 < \beta = 1 - 4/p < 1$, Sobolev embedding tells us,

$$\begin{aligned} \|\nabla\phi\|_{C^{\beta,\beta/2}(P^M_{3R/4}(z_0))} &\le \|\nabla U\|_{C^{\beta,\beta/2}(P^M_R(z_0))} \\ &\le C\|U\|_{W^{2,1}_p(P^M_R(z_0))} \le C(\beta)(1 + \|\phi_0\|_{C^2(M)}). \end{aligned} \tag{3.20}$$

Choose a cut-off function $\chi \in C_0^\infty(B_R^M(x_0))$ satisfying $0 \le \chi \le 1, \chi|_{B^M_{3R/4}(x_0)} \equiv 1$ and $|\nabla^j \chi| \le \frac{C}{R^j}$, $j = 1, 2$. Set $V = \chi\psi$, then we have

$$\begin{cases} \partial\!\!\!/ V = h, & in \quad B_R^M(x_0); \\ \mathbf{B}V(x) = \chi \mathbf{B}\psi_0, & on \quad \partial B_R^M(x_0), \end{cases}$$

where

$$h = \chi \partial\!\!\!/\psi + \nabla\chi \cdot \psi = \chi \mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi) + \nabla\chi \cdot \psi \in L^\infty,$$

since the assumption (3.17) holds. By the standard theory of the usual Dirac operator and Sobolev embedding, we have

$$\begin{aligned} \|\psi\|_{C^{1-2/p}(B^M_{3R/4}(x_0))} &\le C\|V\|_{W^{1,p}(B^M_R(x_0))} \\ &\le C(\|h\|_{L^p(B^M_R(x_0))} + \|\mathbf{B}V\|_{W^{1-1/p,p}(\partial B^M_R(x_0))}) \\ &\le C(1 + \|\mathbf{B}\psi_0\|_{C^1(\partial M)}) \end{aligned} \tag{3.21}$$

for any $2 < p < \infty$. Then (3.20) and (3.21) tell us $\partial\!\!\!/\psi \in C^\mu(B^M_{3R/4}(z_0))$. By the Schauder estimates Theorem 4.6 in [9] and taking some suitable cut-off function as before, we have

$$\|\psi(t)\|_{C^{1+\mu}(B^M_{R/2}(x_0))} \le C\big(1 + \|\mathbf{B}\psi_0\|_{C^{1+\mu}(\partial M)}\big)\big(1 + \|\phi_0\|_{C^2(M)}\big) \tag{3.22}$$

for any $t_0 - \frac{R^2}{4} \le t \le t_0$. Then the inequality (3.18) follows from (3.20), (3.22) immediately.

For the estimate (3.19), we first rewrite the equation $\partial\!\!\!/\psi = \mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi)$ as

$$\partial\!\!\!/\psi + \Omega \cdot \psi = 0$$

where

$$\Omega = \sum_{i=n+1}^{K} [\nu^i(\phi), d\nu^i(\phi)] = \sum_{i=n+1}^{K} \Big((\nu^i)^A (\nabla_{e_\gamma}\nu^i)^B e_\gamma - (\nu^i)^B (\nabla_{e_\gamma}\nu^i)^A e_\gamma\Big)$$

and $\{\nu^i\}_{i=n+1}^K$ is an orthonormal basis of the normal bundle $T^\perp N$ and $\nu^i = ((\nu^i)^1, ..., (\nu^i)^K)$ (see Remark 2.1 in [9]), then for any $t_0 - \frac{R^2}{4} < t, s < t_0$, we have

$$\begin{cases} \partial\!\!\!/(\psi(\cdot,t) - \psi(\cdot,s)) = -\Omega(\cdot,t)\big(\psi(\cdot,t) - \psi(\cdot,s)\big) + (\Omega(\cdot,s) - \Omega(\cdot,t))\psi(\cdot,s) & in \quad M; \\ \mathbf{B}(\psi(\cdot,t) - \psi(\cdot,s)) = 0 & on \quad \partial M. \end{cases}$$

Since $d\Omega = [d\nu(\phi), d\nu(\phi)]$, with (3.20) and (3.22), according to Theorem 4.1 in [9], for any $0 < \beta < 1$, by Sobolev embedding, we have

$$\|\psi(\cdot,t) - \psi(\cdot,s)\|_{C^\beta(M)} \le C\big(\|\Omega(\cdot,t) - \Omega(\cdot,s)\|_{L^\infty(M)}\big) \le C|s-t|^\beta.$$

So, we get $\|\psi\|_{C^{\beta,\frac{\beta}{2}}(M\times[t_0 - \frac{R^2}{4}, t_0])} \le C$ and

$$\begin{cases} \partial_t\phi - a_{\beta\gamma}\frac{\partial^2\phi}{\partial x^\beta \partial x^\gamma} & \in C^{\beta,\beta/2}(M \times [t_0 - \frac{R^2}{4}, t_0]) \quad for \quad any \quad 0 < \beta < 1; \\ \phi|_{\partial M} = \varphi & \in C^{2+\mu}(\partial M). \end{cases}$$

Taking some suitable cut-off function and by standard Schauder estimates for second order parabolic equations, when $\alpha - 1$ is sufficiently small, we have $\phi \in C^{2+\mu,1+\frac{\mu}{2}}(M \times [t_0 - \frac{R^2}{8}, t_0])$ and

$$\begin{aligned}&\|\phi\|_{C^{2+\mu,1+\frac{\mu}{2}}(M\times[t_0-\frac{R^2}{8},t_0])}\\ &\leq C(\|\partial_t\phi - a_{\beta\gamma}\frac{\partial^2\phi}{\partial x^\beta \partial x^\gamma}\|_{C^{\mu,\mu/2}(M\times[t_0-\frac{R^2}{4},t_0])} + \|\phi\|_{C^0(M\times[t_0-\frac{R^2}{4},t_0])} + \|\varphi\|_{C^{2+\mu}(\partial M)}) \leq C.\end{aligned}$$

So we have proved (3.19).

Step 2: We prove (3.17).

We follow a similar idea as in [30, 34, 27]. Without loss of generality, we may assume $R = \frac{1}{2}$. Take $0 \leq \rho < 1$ such that

$$(1-\rho)^2 \sup_{P^M_\rho(z_0)} |\nabla\phi|^2 = \max_{0\leq\sigma\leq 1}\{(1-\sigma)^2 \sup_{P^M_\sigma(z_0)} |\nabla\phi|^2\}$$

and then choose $z_1 = (x_1, t_1) \in P^M_\rho(z_0)$ such that

$$|\nabla\phi|^2(z_1) = \sup_{P^M_\rho(z_0)} |\nabla\phi|^2 := e.$$

We claim:

$$(1-\rho)^2 e \leq 4.$$

We proceed by contradiction. If $(1-\rho)^2 e > 4$, we set

$$u(x,t) := \phi(x_1 + e^{-\frac{1}{2}}x, t_1 + e^{-1}t) \quad and \quad v(x) := e^{-\frac{1}{4}}\psi(x_1 + e^{-\frac{1}{2}}x, t_1 + e^{-1}t).$$

Denoting $P_r(0) = D_r(0) \times [-r^2, 0] \subset \mathbb{R}^2 \times \mathbb{R}$ and

$$S_r := P_r(0) \cap \{(x,t)|(x_1 + e^{-\frac{1}{2}}x, t_1 + e^{-1}t) \in P^M_1(0)\},$$

then (u, v) satisfy

$$\begin{aligned}&(3.23) \qquad \partial_t u = \Delta u + (\alpha-1)\frac{\nabla|\nabla u|^2\nabla u}{e^{-1} + |\nabla u|^2} - A(\nabla u, \nabla u) - \frac{Re\left(P(\mathcal{A}(du(e_\gamma), e_\gamma \cdot v); v)\right)}{\alpha(1 + e|\nabla u|^2)^{\alpha-1}},\\ &(3.24) \qquad \partial\!\!\!/ v = \mathcal{A}(du(e_\gamma), e_\gamma \cdot v)\end{aligned}$$

with the boundary data

$$(3.25) \qquad (u(x,t), \mathbf{B}v(x,t)) = (\varphi(x_1 + e^{-\frac{1}{2}}x), e^{-\frac{1}{4}}\mathbf{B}\psi_0(x_1 + e^{-\frac{1}{2}}x)), \quad if \quad x_1 + e^{-\frac{1}{2}}x \in \partial M.$$

Moreover,

$$\sup_{S_1}|\nabla u|^2 = e^{-1} \sup_{P^M_{e^{-1/2}}(z_1)} |\nabla\phi|^2 \leq e^{-1} \sup_{P^M_{\rho+e^{-1/2}}(z_0)} |\nabla\phi|^2 \leq e^{-1} \sup_{P^M_{\frac{1+\rho}{2}}(z_0)} |\nabla\phi|^2 \leq 4$$

and

$$|\nabla u|^2(0) = e^{-1}|\nabla\phi|^2(z_1) = 1.$$

Noting that v satisfies the equation $\partial\!\!\!/ v = \mathcal{A}(du(e_\gamma), e_\gamma \cdot v)$ and the facts

$$|du| \leq 2, \qquad \sup_{-1\leq t\leq 0} \|v\|_{L^4(D_1)} \leq \sup_t \|\psi(\cdot, t)\|_{L^4(M)} \leq C,$$

where in the last step we have used Lemma 3.3 by taking $p = \frac{4}{3}$. By elliptic estimates of the usual Dirac operator and Sobolev embedding, we have

$$\sup_{-1\le t\le 0} \|v\|_{L^\infty(D_{3/4})} \le C \sup_{-1\le t\le 0} \|v\|_{W^{1,4}(D_{3/4})} \le C(1 + \|\mathbf{B}\psi_0\|_{C^1(\partial M)}).$$

Next, we want to show that there exists a constant $C > 0$ such that

$$1 \le C \int_{S_{3/4}} |\nabla u|^2 dxdt. \tag{3.26}$$

In fact, if such a C does not exist, then there exists a sequence $\{(u_i, v_i)\}$ satisfying (3.23)-(3.24) with the boundary data (3.25) and

$$\sup_{S_{3/4}} (|\nabla u_i| + |v_i|) \le C, \tag{3.27}$$

$$|\nabla u_i|^2(0) = 1, \tag{3.28}$$

$$\int_{S_{3/4}} |\nabla u_i|^2 dxdt \le \frac{1}{i}. \tag{3.29}$$

Similar to the argument in **Step 1** (since (u_i, v_i) satisfy (3.23)-(3.24), (3.25) and (3.27)), we obtain

$$\|\nabla u_i\|_{C^{\beta,\beta/2}(S_{1/2})} \le C(\beta)$$

for any $0 < \beta < 1$.

Therefore, there exists a subsequence of $\{u_i\}$ (we still denote it by $\{u_i\}$) and a function $\overline{u} \in C^{\delta,\frac{\delta}{2}}(S_{1/2})$ such that

$$\nabla u_i \to \nabla \overline{u} \quad in \quad C^{\delta,\delta/2}(S_{1/2})$$

where $0 < \delta < \beta$. By (3.29), we know

$$\int_{S_{1/2}} |\nabla \overline{u}|^2 dxdt = 0 \tag{3.30}$$

which implies $\nabla \overline{u} \equiv 0$ in $S_{1/2}$. But, (3.28) tells us $|\nabla \overline{u}|(0) = 1$. This is a contradiction and then (3.26) must be true. Thus, we have

$$\begin{aligned}
1 &\le C \int_{S_{3/4}} |\nabla u|^2 dxdt \\
&\le C \sup_{-1<t<0} \int_{B^M_{e^{\frac{1}{2}}}(x_1)} |\nabla \phi|^2 (t_1 + e^{-1} t) dx \\
&\le C \sup_{-1<t<0} \int_{B^M_1(z_0)} |\nabla \phi|^2 (t) dx \le C\epsilon_3.
\end{aligned}$$

Choosing $\epsilon_3 > 0$ sufficiently small leads to a contradiction, so we must have $(1-\rho)^2 e \le 4$ and then

$$(1 - 3/4)^2 \sup_{P^M_{3/4}(z_0)} |\nabla \phi|^2 \le (1-\rho)^2 e \le 4.$$

Since ψ satisfies the equation $\partial\!\!\!/\psi = \mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi)$ and $\|d\phi\|_{L^\infty(P^M_{3/4}(z_0))} \le 8$, $\|\psi\|_{L^4(M)} \le C$, by the elliptic theory for the Dirac operator and Sobolev embedding again, we shall easily obtain

$$\|\psi\|_{L^\infty(P^M_{1/2}(z_0))} \le C.$$

Thus we get the inequality (3.17). This finishes the proof of the lemma. □

In the end of this section, we prove our first main Theorem 1.1.

Proof of Theorem 1.1. By the short-time existence result Theorem 3.1, there is a unique solution

$$\phi \in C^{2+\mu,1+\frac{\mu}{2}}_{loc}(M \times [0,T), N)$$

and

$$\psi \in \cap_{0<S<T} L^\infty([0,S], \|\psi(\cdot,t)\|_{C^{1+\mu}(M)}) \cap C^{\mu,\frac{\mu}{2}}_{loc}(M \times [0,T), \Sigma M \otimes \phi^* TN)$$

to the problem (1.10)-(1.11) with boundary data (1.12) for some $T > 0$.

Next, we will show that the solution (ϕ,ψ) can be extended to the time T. In fact, by Lemma 3.2, we have

$$\frac{1}{2}\int_M (1 + |\nabla_g \phi|^2)^\alpha(\cdot,t) dM \le E_\alpha(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}.$$

Then it is easy to see that, for any $0 < \epsilon < \epsilon_3$, there exists a positive constant r_0, depending only on $\epsilon,\ \alpha,\ M,\ E_\alpha(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}$, such that for all $x \in M$ and $0 \le t < T$, there holds

$$\frac{1}{2}\int_{B^M_{r_0}(x)} |\nabla_g \phi|^2(\cdot,t) dM \le C E_\alpha(\phi)^{1/\alpha} r_0^{1-\frac{1}{\alpha}} \le \epsilon.$$

By Lemma 3.4, we can extend the solution $(\phi(\cdot,t),\psi(\cdot,t))$ to the time T with $(\phi(\cdot,T),\psi(\cdot,T)) \in C^{2+\mu}(M,N) \times C^{1+\mu}(M, \Sigma M \otimes \phi(\cdot,T)^* TN)$. Then the short-time existence result implies $T = \infty$.

For the limit behaviour as $t \to \infty$, by Lemma 3.2, we get

$$\int_0^\infty \int_M |\partial_t \phi|^2 dM dt \le C,$$

which implies that there exists a time sequence $t_i \to \infty$, such that

$$\int_M |\partial_t \phi|^2(\cdot,t_i) dM \to 0.$$

By Lemma 3.4, we have

$$\|\phi(t_i)\|_{C^{2+\mu}(M)} + \|\psi(t_i)\|_{C^{1+\mu}(M)} \le C.$$

Thus, there exists a subsequence of $\{t_i\}$ (still denoted by $\{t_i\}$) and an α-Dirac-harmonic map $(\phi_\alpha,\psi_\alpha)$ with boundary data

$$(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0),$$

such that $(\phi(\cdot,t_i),\psi(\cdot,t_i))$ converges to $(\phi_\alpha,\psi_\alpha)$ in $C^2(M) \times C^1(M)$. Since $(\varphi,\psi_0) \in C^{2+\mu}(\partial M, N) \times C^{1+\mu}(\partial M, \varphi^* TN)$, it is standard to obtain

$$(\phi_\alpha,\psi_\alpha) \in C^{2+\mu}(M,N) \times C^{1+\mu}(M, \Sigma M \otimes \phi^*_\alpha TN)$$

from the Schauder theory for second order elliptic operators and Dirac operators. This completes the proof of theorem. □

4. Blow-up analysis for α-Dirac-harmonic map sequences and existence of Dirac-harmonic maps

In the previous section, it is shown that there exists a sequence of α-Dirac-harmonic maps $\{(\phi_\alpha, \psi_\alpha)\}$ as $\alpha \searrow 1$ with Dirichlet-chiral boundary condition $(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$, such that

$$E_\alpha(\phi_\alpha) \leq E_\alpha(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)} \tag{4.1}$$

and

$$\|\psi_\alpha\|_{W^{1,p}(M)} \leq C(p, M, N, E_\alpha(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}), \tag{4.2}$$

for any $1 < p < 2$. In this section, we will study the limit behaviour of the sequence as $\alpha \searrow 1$ and show that the limit is just the Dirac-harmonic map we want to find.

First of all, we consider the blow-up sequence under the following more general assumption that

$$E_\alpha(\phi_\alpha) + \|\psi_\alpha\|_{L^4(M)} \leq \Lambda < \infty.$$

Note that the functional L_α and the equations of α-Dirac-harmonic maps are not conformally invariant in dimension two. For example, on an isothermal coordinate system around a point $p \in M$, if the metric is given by

$$g = e^\rho((dx^1)^2 + (dx^2)^2)$$

with $\rho(p) = 0$, setting

$$(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x)) := \Big(\phi_\alpha(p + r_\alpha x),\ \sqrt{r_\alpha}\psi_\alpha(p + r_\alpha x)\Big)$$

for some small positive number $r_\alpha > 0$. By the conformal invariance of the spinor equation, it is easy to check that $(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x))$ satisfies the following system

$$\begin{cases} \Delta_{g_\alpha}\widetilde{u}_\alpha &= -(\alpha-1)\frac{\nabla_{g_\alpha}|\nabla_{g_\alpha}\widetilde{u}_\alpha|^2\nabla_{g_\alpha}\widetilde{u}_\alpha}{\sigma_\alpha + |\nabla_{g_\alpha}\widetilde{u}_\alpha|^2} + A(d\widetilde{u}_\alpha, d\widetilde{u}_\alpha) + \frac{Re\big(P(\mathcal{A}(d\widetilde{u}_\alpha(e_\gamma), e_\gamma\cdot\widetilde{v}_\alpha); \widetilde{v}_\alpha)\big)}{\alpha(1+\sigma_\alpha^{-1}|\nabla_{g_\alpha}\widetilde{u}_\alpha|^2)^{\alpha-1}}, \\ \partial\!\!\!/\widetilde{v}_\alpha &= \mathcal{A}(d\widetilde{u}_\alpha(e_\gamma), e_\gamma\cdot\widetilde{v}_\alpha), \end{cases} \tag{4.3}$$

where $g_\alpha = e^{\rho(p+r_\alpha x)}((dx^1)^2 + (dx^2)^2)$ and $\sigma_\alpha = r_\alpha^2 > 0$.

Since α-Dirac-harmonic maps are not conformally invariant, in order to get unified bubbling equations, we need to add another factor $r_\alpha^{\alpha-1}$ in the rescaling. Setting

$$(u_\alpha(x), v_\alpha(x)) := \Big(\widetilde{u}_\alpha(x), r_\alpha^{\alpha-1}\widetilde{v}_\alpha(x)\Big) = \Big(\phi_\alpha(p + r_\alpha x), r_\alpha^{\alpha-1}\sqrt{r_\alpha}\psi_\alpha(p + r_\alpha x)\Big)$$

and noting that the equation for the spinor part is also invariant by multiplying a constant to the spinor, then one can verify that $(u_\alpha(x), v_\alpha(x))$ satisfies the following system:

$$\begin{cases} \Delta_{g_\alpha}u_\alpha &= -(\alpha-1)\frac{\nabla_{g_\alpha}|\nabla_{g_\alpha}u_\alpha|^2\nabla_{g_\alpha}u_\alpha}{\sigma_\alpha + |\nabla_{g_\alpha}u_\alpha|^2} + A(du_\alpha, du_\alpha) + \frac{Re\big(P(\mathcal{A}(du_\alpha(e_\gamma), e_\gamma\cdot v_\alpha); v_\alpha)\big)}{\alpha(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}}, \\ \partial\!\!\!/v_\alpha &= \mathcal{A}(du_\alpha(e_\gamma), e_\gamma\cdot v_\alpha), \end{cases} \tag{4.4}$$

For a general Riemannian metric $g_\alpha = e^{\rho_\alpha}((dx^1)^2 + (dx^2)^2)$, $\rho_\alpha \in C^\infty(D_1(0))$, $\rho_\alpha(0) = 0$, and positive constant $0 < \sigma_\alpha \leq 1$, we call (u_α, v_α) a *general α-Dirac-harmonic map* if it satisfies the system (4.3) or it satisfies the system (4.4) with $0 < \beta_0 \leq \liminf_{\alpha\searrow 1}\sigma_\alpha^{\alpha-1} \leq 1$ for some positive constant $\beta_0 > 0$.

Since the spinor equation is conformally invariant, it is easy to check that the system (4.3) is equivalent to

$$\begin{cases}\Delta u_\alpha &= -(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}u_\alpha|^2\nabla u_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2}+A(u_\alpha)(du_\alpha,du_\alpha)+\frac{Re\left(P(\mathcal{A}(du_\alpha(e_\gamma),e_\gamma\cdot v_\alpha);v_\alpha)\right)}{\alpha(1+\sigma_\alpha^{-1}|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}}=0,\\ \partial\!\!\!/ v_\alpha &= \mathcal{A}(du_\alpha(e_\gamma),e_\gamma\cdot v_\alpha),\end{cases}\tag{4.5}$$

and the system (4.4) is equivalent to

$$\begin{cases}\Delta u_\alpha &= -(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}u_\alpha|^2\nabla u_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2}+A(u_\alpha)(du_\alpha,du_\alpha)+\frac{Re\left(P(\mathcal{A}(du_\alpha(e_\gamma),e_\gamma\cdot v_\alpha);v_\alpha)\right)}{\alpha(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}}=0,\\ \partial\!\!\!/ v_\alpha &= \mathcal{A}(du_\alpha(e_\gamma),e_\gamma\cdot v_\alpha),\end{cases}\tag{4.6}$$

where $\Delta=\frac{\partial^2}{(\partial x^1)^2}+\frac{\partial^2}{(\partial x^2)^2}$, the derivative ∇ and the Dirac operator $\partial\!\!\!/$ are taken with respect to the standard Euclidean metric. The $\{e_\gamma\}$ in (4.5) and (4.6) is a local orthonormal basis with respect to the standard Euclidean metric and hence it is different from the one in (4.3) and (4.4), however, for simplicity, we shall use the same notation. More precisely, the above equivalences of the systems mean that $(u_\alpha\circ Id, e^{\frac{\rho_\alpha}{2}}v_\alpha\circ Id)$ satisfies (4.5)-(4.6), where

$$Id:(D_1(0),(dx^1)^2+(dx^2)^2)\to(D_1(0),g_\alpha)$$

is a conformal map defined by $Id(x)=x$. In the sequel, for simplicity of notation, we shall identify $(u_\alpha\circ Id, e^{\frac{\rho_\alpha}{2}}v_\alpha\circ Id)$ with (u_α,v_α) and use the appropriate forms of the systems.

Let $D_1(0)\subset\mathbb{R}^2$ be the unit ball centered at 0. Denote

$$D_1^+(0):=\{(x^1,x^2)\in D_1(0)|x^2\geq 0\},\ \partial^0D_1^+(0):=\{(x^1,x^2)\in D_1(0)|x^2=0\}.$$

Next, we show a small energy regularity lemma for α-Dirac-harmonic maps. This kind of regularity theorem was introduced by Sacks and Uhlenbeck for the critical points of functional E_α in [29]. For the interior case, we have

Lemma 4.1. *For any $1<p<\infty$, there exist two positive constants ϵ_0 and $\alpha_0>1$ depending only on g,N, such that if $(\phi_\alpha,\psi_\alpha):(D_1(0),g_\alpha)\to N$ is a general α-Dirac-harmonic map with $E_\alpha(\phi_\alpha)+\|\psi_\alpha\|_{L^4(D_1(0))}\leq\Lambda$ and*

$$E(\phi_\alpha)\leq\epsilon_0,\ 1\leq\alpha\leq\alpha_0,$$

where $g_\alpha=e^{\rho_\alpha}((dx^1)^2+(dx^2)^2)$ and $\rho_\alpha(0)=0$, $\rho_\alpha\to\rho$ in $C^\infty(D_1(0))$ as $\alpha\to 1$, then there holds

$$\|\nabla\phi_\alpha\|_{W^{1,p}(D_{1/2}(0))}\leq C(p,g,\Lambda,N)\|\nabla\phi_\alpha\|_{L^2(D_1(0))},\quad \|\psi_\alpha\|_{W^{1,p}(D_{1/2}(0))}\leq C(p,g,\Lambda,N)\|\psi_\alpha\|_{L^4(D_1(0))}.$$

Near the boundary, we have

Lemma 4.2. *For any $1<p<\infty$, there exist two positive constants ϵ_0 and $\alpha_0>1$ depending only on g,N, such that if $(\phi_\alpha,\psi_\alpha):(D_1^+(0),g_\alpha)\to N$ is a general α-Dirac-harmonic map with Dirichlet-chiral boundary condition*

$$(\phi_\alpha,\mathbf{B}\psi_\alpha)|_{\partial^0D_1^+(0)}=(\varphi,\mathbf{B}\psi_0),$$

satisfying $E_\alpha(\phi_\alpha)+\|\psi_\alpha\|_{L^4(D_1(0))}\leq\Lambda$ and

$$E(\phi_\alpha)\leq\epsilon_0,\ 1\leq\alpha\leq\alpha_0,$$

where $g_\alpha = e^{\rho_\alpha}((dx^1)^2 + (dx^2)^2)$ and $\rho_\alpha(0) = 0$, $\rho_\alpha \to \rho$ in $C^\infty(D_1^+(0))$ as $\alpha \to 1$, then there holds

$$\|\nabla\phi_\alpha\|_{W^{1,p}(D^+_{1/2}(0))} \le C(\|\nabla\phi_\alpha\|_{L^2(D^+)} + \|\nabla\varphi\|_{C^1(\partial^0 D^+)},$$
$$\|\psi_\alpha\|_{W^{1,p}(D^+_{1/2}(0))} \le C(\|\psi_\alpha\|_{L^4(D^+)} + \|\mathbf{B}\psi_0\|_{C^1(\partial^0 D^+)}),$$

where C is a positive constant depending on $p, g, \Lambda, N, \|\varphi\|_{C^2}, \|\mathbf{B}\psi_0\|_{C^1}$.

Since the proof for the interior case is similar to, but simpler than that of the boundary case, we only prove Lemma 4.2 here and omit the interior case.

Proof of Lemma 4.2. We prove the lemma for the case that $(\phi_\alpha, \psi_\alpha)$ satisfies (4.3). For the other case, i.e. $(\phi_\alpha, \psi_\alpha)$ satisfies (4.4) with $0 < \beta_0 \le \liminf_{\alpha\searrow 1} \sigma_\alpha^{\alpha-1} \le 1$ for some positive constant $\beta_0 > 0$, the proof is almost the same.

Without loss of generality, we assume $\int_{\partial^0 D^+_{\frac{1}{2}}} \varphi = 0$.

Choose a cut-off function $\eta \in C_0^\infty(D^+)$ satisfying $0 \le \eta \le 1, \eta|_{D^+_{3/4}} \equiv 1, |\nabla\eta| + |\nabla^2\eta| \le C$. Noting that the ψ_α-equation is conformally invariant in dimension two, by standard theory of first order elliptic equations, for any $1 < q < 2$, we have

$$\begin{aligned}
\|\eta\psi_\alpha\|_{W^{1,q}(D^+)} &\le C(\|\partial\!\!\!/(\eta\psi_\alpha)\|_{L^q(D^+)} + \|\mathbf{B}\psi_\alpha\|_{W^{1-1/q,q}(\partial^0 D^+)}) \\
&\le C(\|\nabla\eta \cdot \psi_\alpha + \eta\partial\!\!\!/\psi_\alpha\|_{L^q(D^+)} + \|\mathbf{B}\psi_\alpha\|_{W^{1-1/q,q}(\partial^0 D^+)}) \\
&\le C\left(\|\psi_\alpha\|_{L^q(D^+)} + \||d\phi_\alpha||\eta\psi_\alpha|\|_{L^q(D^+)} + \|\mathbf{B}\psi_0\|_{W^{1-1/q,q}(\partial^0 D^+)}\right) \\
&\le C\|d\phi_\alpha\|_{L^2(D^+)}\|\eta\psi_\alpha\|_{L^{\frac{2q}{2-q}}(D^+)} + C(\|\psi_\alpha\|_{L^q(D^+)} + \|\mathbf{B}\psi_0\|_{W^{1-1/q,q}(\partial^0 D^+)}) \\
&\le C\epsilon_0\|\eta\psi_\alpha\|_{L^{\frac{2q}{2-q}}(D^+)} + C(\|\psi_\alpha\|_{L^q(D^+)} + \|\mathbf{B}\psi_0\|_{W^{1-1/q,q}(\partial^0 D^+)}).
\end{aligned}$$

Taking $\epsilon_0 > 0$ sufficiently small, by Sobolev embedding, we get

$$\|\eta\psi_\alpha\|_{L^{\frac{2q}{2-q}}(D^+)} \le \|\eta\psi_\alpha\|_{W^{1,q}(D^+)} \le C(\|\psi_\alpha\|_{L^q(D^+)} + \|\mathbf{B}\psi_\alpha\|_{W^{1-1/q,q}(\partial^0 D^+)}). \tag{4.7}$$

In particular, taking $q = \frac{8}{5}$, we get

$$\|\eta\psi_\alpha\|_{L^8(D^+)} \le \|\eta\psi_\alpha\|_{W^{1,\frac{8}{5}}(D^+)} \le C(\|\psi_\alpha\|_{L^4(D^+)} + \|\mathbf{B}\psi_\alpha\|_{C^{3/8,8/5}(\partial^0 D^+)}). \tag{4.8}$$

Noting that

$$\Delta\phi_\alpha = -(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}\phi_\alpha|^2\nabla\phi_\alpha}{\sigma_\alpha + |\nabla_{g_\alpha}\phi_\alpha|^2} + A(d\phi_\alpha, d\phi_\alpha) + \frac{Re\left(P(\mathcal{A}(d\phi_\alpha(e_\gamma), e_\gamma\cdot\psi_\alpha);\psi_\alpha)\right)}{\alpha(1+\sigma_\alpha^{-1}|\nabla_{g_\alpha}\phi_\alpha|^2)^{\alpha-1}},$$

where $\Delta = (\frac{\partial}{\partial x^1})^2 + (\frac{\partial}{\partial x^2})^2$ is the Laplace operator of the standard Euclidean metric, computing directly, we obtain

$$\begin{aligned}
|\Delta(\eta\phi_\alpha)| &= |\eta\Delta\phi_\alpha + 2\nabla\eta\nabla\phi_\alpha + \phi_\alpha\Delta\eta| \\
&\le C\left(|\phi_\alpha| + |d\phi_\alpha| + (\alpha-1)|\eta\nabla^2\phi_\alpha| + |d\phi_\alpha||\eta d\phi_\alpha| + |\psi_\alpha|^2|\eta d\phi_\alpha|\right) \\
&\le C(|d\phi_\alpha||d(\eta\phi_\alpha)| + (\alpha-1)|\nabla^2(\eta\phi_\alpha)|) + C\left(|\phi_\alpha| + |d\phi_\alpha| + |\psi_\alpha|^2|\eta d\phi_\alpha|\right). \qquad (4.9)
\end{aligned}$$

By standard elliptic estimates and the Poincaré inequality, for any $1 < p < 2$, we have

$$\begin{aligned}
\|\eta\phi_\alpha\|_{W^{2,p}(D^+)} &\le C(\||d\phi_\alpha||d(\eta\phi_\alpha)|\|_{L^p(D^+)} + (\alpha-1)\|\nabla^2(\eta\phi_\alpha)\|_{L^p(D^+)}) \\
&\quad + C(\|d\phi_\alpha\|_{L^p(D^+)} + \|\varphi\|_{W^{2-1/p,p}(\partial^0 D^+)} + \||\psi_\alpha|^2|\eta d\phi_\alpha|\|_{L^p(D^+)}) \\
&\le C\|d(\eta\phi_\alpha)\|_{L^{\frac{2p}{2-p}}(D^+)}\|d\phi_\alpha\|_{L^2(D^+)} + C(\alpha-1)\|\nabla^2(\eta\psi_\alpha)\|_{L^p(D^+)} \\
&\quad + C(\|d\phi_\alpha\|_{L^p(D^+)} + \|||\psi_\alpha|^2|\eta d\phi_\alpha|\|\|_{L^p(D^+)} + \|\varphi\|_{W^{2,p}(\partial^0 D^+)}) \\
&\le C(\epsilon_0 + \alpha - 1)\|d(\eta\phi_\alpha)\|_{W^{1,p}(D^+)} + C(\|d\phi_\alpha\|_{L^p(D^+)} \\
&\quad + \||\psi_\alpha|^2|\eta d\phi_\alpha|\|_{L^p(D^+)} + \|\nabla\varphi\|_{W^{1,p}(\partial^0 D^+)}).
\end{aligned}$$

Choosing $\epsilon_0 > 0$ and $\alpha_0 - 1$ sufficiently small, we have

$$\|\nabla(\eta\phi_\alpha)\|_{W^{1,p}(D^+)} \le C(\|d\phi_\alpha\|_{L^p(D^+)} + \|||\psi_\alpha|^2|\eta d\phi_\alpha|\|\|_{L^p(D^+)} + \|\nabla\varphi\|_{C^1(\partial^0 D^+)}). \tag{4.10}$$

In particular, we take $p = \frac{4}{3}$, then

$$\begin{aligned}
\|\nabla\phi_\alpha\|_{L^4(D^+_{\frac{5}{8}})} &\le C\|\nabla\phi_\alpha\|_{W^{1,\frac{4}{3}}(D^+_{\frac{5}{8}})} \\
&\le C(\|d\phi_\alpha\|_{L^2(D^+)} + \|\psi_\alpha\|^2_{L^8(D^+_{3/4})}\|d\phi_\alpha\|_{L^2(D^+)} + \|\nabla\varphi\|_{C^1(\partial^0 D^+)}) \\
&\le C(\|d\phi_\alpha\|_{L^2(D^+)} + \|\nabla\varphi\|_{C^1(\partial^0 D^+)}), \qquad (4.11)
\end{aligned}$$

where the last inequality follows from (4.8).

Applying the $W^{1,2}$-estimate for the usual Dirac operator, we have

$$\begin{aligned}
\|\psi_\alpha\|_{W^{1,2}(D^+_{\frac{9}{16}})} &\le C(\|\partial\!\!\!/\psi_\alpha\|_{L^2(D^+_{\frac{5}{8}})} + \|\psi_\alpha\|_{L^2(D^+_{\frac{5}{8}})} + \|\mathbf{B}\psi_\alpha\|_{W^{1/2,2}(\partial^0 D^+)}) \\
&\le C(\|d\phi_\alpha\|_{L^4(D^+_{\frac{5}{8}})}\|\psi_\alpha\|_{L^4(D^+_{\frac{5}{8}})} + \|\psi_\alpha\|_{L^2(D^+_{\frac{5}{8}})} + \|\mathbf{B}\psi_\alpha\|_{W^{1/2,2}(\partial^0 D^+)}) \\
&\le C(\|\psi_\alpha\|_{L^4(D^+)} + \|\mathbf{B}\psi_\alpha\|_{C^1(\partial^0 D^+)}).
\end{aligned}$$

By (4.9), we get

$$|\Delta(\eta\phi_\alpha)| \le C(\alpha-1)|\nabla^2(\eta\phi_\alpha)| + C\left(|\phi_\alpha| + |d\phi_\alpha| + |d\phi_\alpha|^2 + |\psi_\alpha|^2|d\phi_\alpha|\right). \tag{4.12}$$

Applying the $W^{2,2}$-estimate for the Laplace operator and choosing $\alpha_0 - 1$ small enough, by (4.8) and (4.11), we obtain

$$\|\nabla\phi_\alpha\|_{W^{1,2}(D^+_{\frac{9}{16}})} \le C(\|d\phi_\alpha\|_{L^2(D^+)} + \|\nabla\varphi\|_{C^1(D^+)}). \tag{4.13}$$

By the Sobolev embedding theorem, we know $\nabla\phi_\alpha \in L^p(D^+_{9/16})$ and $\psi_\alpha \in L^p(D^+_{9/16})$ for any $1 < p < \infty$. Then the conclusions of the lemma follow from the standard L^p-estimate for the Dirac operator and the $W^{2,p}$-estimate for the following elliptic operator immediately

$$\Delta + (\alpha-1)\frac{2\nabla_\beta\phi_\alpha\nabla_\gamma\phi_\alpha}{\sigma_\alpha + |\nabla\phi_\alpha|^2}\nabla^2_{\beta\gamma}.$$

□

Applying the above small energy regularity for general α-Dirac-harmonic maps, we can now show Theorem 1.2 and Theorem 1.4.

Proof of Theorem 1.2: Without loss of generality, let $\{x_1, ..., x_I\} \subset \mathcal{S}$ be any subset with finite points. Choosing $r > 0$ sufficiently small such that $B^M_r(x_i) \cap B^M_r(x_j) = \emptyset,\ i \neq j$, then

$$\Lambda \geq \liminf_{\alpha\to 1} E(\phi_\alpha; M) \geq \sum_{i=1}^{I} \liminf_{\alpha\to 1} E(\phi_\alpha; B^M_r(x_i)) \geq \frac{\epsilon_0}{2} I,$$

which implies $I \leq \frac{2\Lambda}{\epsilon_0}$. Therefore, $\mathcal{S}$ is a set with at most finitely many points..

For any $x_0 \in M \setminus \mathcal{S}$, there exist $r_0 > 0$ and a subsequence of $\alpha \searrow 1$, such that

$$E(\phi_\alpha; B^M_{r_0}(x_0)) < \frac{\epsilon_0}{2}.$$

If $x_0 \in M \setminus \partial M$, without loss of generality, we may assume $B^M_{r_0}(x_0) \cap \partial M = \emptyset$. By Lemma 4.1, we have

$$r_0\|\nabla\phi_\alpha\|_{L^\infty(B^M_{r_0/2}(x_0))} + \sqrt{r_0}\|\psi_\alpha\|_{L^\infty(B^M_{r_0/2}(x_0))} \leq C(\Lambda, M, N).$$

If $x_0 \in \partial M$, by Lemma 4.2, we have

$$r_0\|\nabla\phi_\alpha\|_{L^\infty(B^M_{r_0/2}(x_0))} + \sqrt{r_0}\|\psi_\alpha\|_{L^\infty(B^M_{r_0/2}(x_0))} \leq C(\Lambda, M, N, \|\varphi\|_{C^2}, \|\mathbf{B}\psi_0\|_{C^1}).$$

According to the standard theory of Dirac and second order elliptic operators, we can obtain

$$\|\phi_\alpha\|_{C^k(B^M_{r_0/4}(x_0))} + \|\psi_\alpha\|_{C^k(B^M_{r_0/4}(x_0))} \leq C(k, r_0, \Lambda, M, N) \tag{4.14}$$

for $x_0 \in M \setminus \partial M$ and

$$\|\phi_\alpha\|_{C^{2+\mu}(B^M_{r_0/4}(x_0))} + \|\psi_\alpha\|_{C^{1+\mu}(B^M_{r_0/4}(x_0))} \leq C(r_0, \mu, \Lambda, M, N, \|\varphi\|_{C^{2+\mu}}, \|\mathbf{B}\psi_0\|_{C^{1+\mu}}) \tag{4.15}$$

for $x_0 \in \partial M$.

Suppose (ϕ, ψ) is the weak limit of $(\phi_\alpha, \psi_\alpha)$ in $W^{1,2}(M) \times L^4(M)$, then by (4.14) and (4.15), we know there exists a subsequence of $(\phi_\alpha, \psi_\alpha)$ (not changing notation) such that

$$(\phi_\alpha, \psi_\alpha) \to (\phi, \psi) \ in\ C^2_{loc}(M \setminus \mathcal{S}) \times C^1_{loc}(M \setminus \mathcal{S}),$$

where

$$(\phi, \mathbf{B}\psi)|_{\partial M} = (\varphi, \mathbf{B}\psi_0).$$

By the removable singularity theory of Dirac-harmonic maps (see Theorem 4.6 in [6] for the interior singularity case and the proof of Theorem 1.4 and Theorem 1.5 in [19] for the boundary singularity case), we have $(\phi, \psi) \in C^2(M) \times C^1(M)$. Then, $(\phi, \psi) \in C^{2+\mu}(M) \times C^{1+\mu}(M)$ follows from the standard Schauder theory. □

Proof of Theorem 1.4: Take $r_0 > 0$ such that $x_0 \in \mathcal{S}$ is the only energy concentration point in $B^M_{r_0}(x_0)$. By the standard blow-up analysis argument for harmonic map type problems, we can assume that, for the sequence $\alpha \searrow 1$, there exist sequences $x_\alpha \to x_0$ and $r_\alpha \to 0$ such that

$$E(\phi_\alpha; B^M_{r_\alpha}(x_\alpha)) = \sup_{\substack{x\in B^M_{r_0}(x_0), r\leq r_\alpha \\ B^M_r(x)\subset B^M_{r_0}(x_0)}} E(\phi_\alpha; B^M_r(x)) = \frac{\epsilon_0}{4}, \tag{4.16}$$

where $\epsilon_0 > 0$ is the constant in Lemma 4.1 and Lemma 4.2.

Step 1: Let $x_0 \in \partial M$ and we prove the statement (2) under the assumption that

$$\limsup_{\alpha \to 1} \frac{dist(x_\alpha, \partial M)}{r_\alpha} = \infty. \tag{4.17}$$

Without loss of generality, we may assume $x_0 = 0 \in D_1^+(0) \subset \mathbb{R}^2$ is the only energy concentration point in $D_1^+(0)$ and

$$g(x) = e^{\rho(x)}((dx^1)^2 + (dx^2)^2),$$

where ρ is a smooth function satisfying $\rho(0) = 0$.

Setting

$$(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x)) = \left(\phi_\alpha(x_\alpha + r_\alpha x),\ \sqrt{r_\alpha}\psi_\alpha(x_\alpha + r_\alpha x)\right), \tag{4.18}$$

by (4.17) and (4.5), it is easy to see that, for any $R > 0$, $(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x))$ lives in $D_R(0) \subset \mathbb{R}^2$ for α close to 1 and satisfies

$$\begin{cases} \Delta \widetilde{u}_\alpha &= -(\alpha - 1)\frac{\nabla|\nabla_{g_\alpha}\widetilde{u}_\alpha|^2\nabla\widetilde{u}_\alpha}{r_\alpha^2 + |\nabla_{g_\alpha}\widetilde{u}_\alpha|^2} + A(d\widetilde{u}_\alpha, d\widetilde{u}_\alpha) + \frac{Re\left(P(\mathcal{A}(d\widetilde{u}_\alpha(e_\gamma), e_\gamma \cdot \widetilde{v}_\alpha); \widetilde{v}_\alpha)\right)}{\alpha(1 + r_\alpha^{-2}|\nabla_{g_\alpha}\widetilde{u}_\alpha|^2)^{\alpha-1}}, \\ \partial\!\!\!/ \widetilde{v}_\alpha &= \mathcal{A}(d\widetilde{u}_\alpha(e_\gamma), e_\gamma \cdot \widetilde{v}_\alpha), \end{cases} \tag{4.19}$$

where $g_\alpha(x) = e^{\rho(x_\alpha + r_\alpha x)}((dx^1)^2 + (dx^2)^2)$ and we used the fact that the second equation, i.e. the equation for the spinor part, is conformally invariant.

Since $(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x))$ is a general α-Dirac-harmonic map, by (4.16) and the small energy regularity result Lemma 4.1, we know there exists a subsequence of $\{\alpha\}$ (still denoted by the same symbols) and $(\widetilde{\sigma}, \widetilde{\xi}) \in W^{2,2}_{loc}(\mathbb{R}^2) \times W^{1,2}_{loc}(\mathbb{R}^2)$, such that $E(\widetilde{\sigma}; D_1(0)) = \frac{\epsilon_0}{4}$ and

$$(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x)) \to (\widetilde{\sigma}, \widetilde{\xi}) \ in\ C^1_{loc}(\mathbb{R}^2) \times C^0_{loc}(\mathbb{R}^2). \tag{4.20}$$

Next, we make the following

Claim 1:

$$1 \le \liminf_{\alpha \searrow 1} r_\alpha^{2(1-\alpha)} \le \limsup_{\alpha \searrow 1} r_\alpha^{2(1-\alpha)} \le \mu_{max} < \infty. \tag{4.21}$$

To show this claim, we just need to prove that

$$\limsup_{\alpha \searrow 1} r_\alpha^{2(1-\alpha)} < \infty.$$

In fact, if it does not hold, then there exists a subsequence $\alpha_j \to 1$ such that

$$\lim_{j \to \infty} r_{\alpha_j}^{2(1-\alpha_j)} := \mu_1 = \infty.$$

By (4.19) and (4.20), it is easy to see that $\widetilde{\sigma} : \mathbb{R}^2 \to N$ is a harmonic map such that $\widetilde{u}_{\alpha_j} \to \widetilde{\sigma}$ in $C^1_{loc}(\mathbb{R}^2)$ as $j \to \infty$. Then we have

$$\begin{aligned} 2\Lambda \ge \lim_{R\to\infty}\lim_{j\to\infty}\int_{D_{r_{\alpha_j}R}(x_{\alpha_j})} |\nabla_{g_{\alpha_j}}\phi_{\alpha_j}|^{2\alpha_j} dvol_{g_{\alpha_j}} &= \lim_{R\to\infty}\lim_{j\to\infty}(r_{\alpha_j})^{2-2\alpha_j}\int_{D_R(0)} |\nabla_{g_{\alpha_j}}\widetilde{u}_{\alpha_j}|^{2\alpha_j} dvol_{g_{\alpha_j}(x_{\alpha_j}+r_{\alpha_j}x)} \\ &= \lim_{R\to\infty}\mu_1\int_{D_R(0)} |\nabla\widetilde{\sigma}|^2 dx = 2\mu_1 E(\widetilde{\sigma}). \end{aligned}$$

which is a contradiction to the fact that $E(\widetilde{\sigma}) \geq \overline{\epsilon} > 0$ which follows from the well known energy gap theorem for harmonic spheres, since $\widetilde{\sigma} : \mathbb{R}^2 \to N$ is a nontrivial harmonic map with finite energy and hence it can be conformally extended to a harmonic sphere. Thus, the **Claim 1** holds true.

Now setting

$$(u_\alpha(x), v_\alpha(x)) := \left(\widetilde{u}_\alpha(x), r_\alpha^{\alpha-1}\widetilde{v}_\alpha(x)\right) = \left(\phi_\alpha(x_\alpha + r_\alpha x), r_\alpha^{\alpha-1}\sqrt{r_\alpha}\psi_\alpha(x_\alpha + r_\alpha x)\right), \tag{4.22}$$

since the equation for the spinor part is also invariant by multiplying a constant to the spinor, it is easy to see that (u_α, v_α) satisfies

$$\begin{cases} \Delta u_\alpha &= -(\alpha-1)\frac{\nabla|\nabla_{g_\alpha} u_\alpha|^2 \nabla u_\alpha}{r_\alpha^2 + |\nabla_{g_\alpha} u_\alpha|^2} + A(du_\alpha, du_\alpha) + r_\alpha^{2(1-\alpha)}\frac{Re\left(P(\mathcal{A}(du_\alpha(e_\gamma), e_\gamma\cdot v_\alpha); v_\alpha)\right)}{\alpha(1+r_\alpha^{-2}|\nabla_{g_\alpha} u_\alpha|^2)^{\alpha-1}}, \\ \partial\!\!\!/ v_\alpha &= \mathcal{A}(du_\alpha(e_\gamma), e_\gamma \cdot v_\alpha). \end{cases} \tag{4.23}$$

From (4.21), we know that (u_α, v_α) is a general α-Dirac-harmonic map with $\sigma_\alpha = r_\alpha^2 > 0$. By (4.16), (4.23), the small energy regularity result Lemma 4.1 and the fact that $r_\alpha^{\alpha-1} \leq 1$, we know there exists a subsequence of $\{\alpha\}$ (still denoted by the same symbols) and a nontrivial Dirac-harmonic map $(\sigma, \xi) : \mathbb{R}^2 \to N$, such that

$$(u_\alpha(x), v_\alpha(x)) \to (\sigma, \xi) \ in \ C^1_{loc}(\mathbb{R}^2) \times C^0_{loc}(\mathbb{R}^2).$$

Next, we will show that (σ, ξ) has finite energy, i.e.

$$\|\nabla\sigma\|_{L^2(\mathbb{R}^2)} + \|\xi\|_{L^4(\mathbb{R}^2)} \leq C < \infty.$$

In fact, for any $R > 0$,

$$\begin{aligned} \|\nabla\sigma\|_{L^2(D_R(0))} + \|\xi\|_{L^4(D_R(0))} &= \lim_{\alpha\searrow 1}(\|\nabla u_\alpha\|_{L^2(D_R(0))} + \|v_\alpha\|_{L^4(D_R(0))}) \\ &= \lim_{\alpha\searrow 1}(\|\nabla\phi_\alpha\|_{L^2(D_{r_\alpha R}(x_\alpha))} + r_\alpha^{\alpha-1}\|\psi_\alpha\|_{L^4(D_{r_\alpha R}(x_\alpha))}) \\ &\leq \lim_{\alpha\searrow 1}(\|\nabla\phi_\alpha\|_{L^2(D_{r_\alpha R}(x_\alpha))} + \|\psi_\alpha\|_{L^4(D_{r_\alpha R}(x_\alpha))}) \leq C(\Lambda) < \infty. \end{aligned}$$

Step 2: Let $x_0 \in \partial M$, then

$$\limsup_{\alpha\to 1}\frac{dist(x_\alpha, \partial M)}{r_\alpha} = \infty. \tag{4.24}$$

If not, then there exists a converging subsequence of $\frac{dist(x_\alpha,\partial M)}{r_\alpha}$. Without loss of generality, we may assume

$$\lim_{\alpha\to 1}\frac{dist(x_\alpha, \partial M)}{r_\alpha} = a$$

where $a \geq 0$ is a constant.

Denoting

$$B_\alpha := \left\{x \in \mathbb{R}^2 |\ x_\alpha + r_\alpha x \in D_1^+(0)\right\},$$

then

$$B_\alpha \to \mathbb{R}^2_a := \left\{(x^1, x^2) | x^2 \geq -a\right\}.$$

Noting that $(\widetilde{u}_\alpha(x), \widetilde{v}_\alpha(x))$ (see (4.18)) lives in B_α and satisfies (4.19) with the boundary data

$$(\widetilde{u}_\alpha(x), \mathbf{B}\widetilde{v}_\alpha(x)) = (\varphi(x_\alpha + r_\alpha x),\ \sqrt{r_\alpha}\mathbf{B}\psi_0(x_\alpha + r_\alpha x)), \quad if\ x_\alpha + r_\alpha x \in \partial^0 D_1^+(0),$$

by (4.16), Lemma 4.1 and Lemma 4.2, we have

$$\|\widetilde{u}_\alpha\|_{W^{2,p}(D_{4R}(0)\cap B_\alpha(0))} + \|\widetilde{v}_\alpha\|_{W^{1,p}(D_{4R}(0)\cap B_\alpha(0))} \leq C(p,R,g,\Lambda,N,\|\varphi\|_{C^{2+\mu}},\|\mathbf{B}\psi_0\|_{C^{1+\mu}}) \tag{4.25}$$

for any $D_R(0)\subset\mathbb{R}^2$ and $p>1$, which implies

$$\|\widetilde{u}_\alpha(x-(0,\frac{d_\alpha}{r_\alpha}))\|_{W^{2,2}(D^+_{3R}(0))} + \|\widetilde{v}_\alpha(x-(0,\frac{d_\alpha}{r_\alpha}))\|_{W^{1,2}(D^+_{3R}(0))} \leq C$$

when $\frac{1}{\alpha-1}$, R are large, where $d_\alpha := dist(x_\alpha,\partial^0 D^+)$.

Then there exist a subsequence of $(\widetilde{u}_\alpha,\widetilde{v}_\alpha)$ (also denoted by $(\widetilde{u}_\alpha,\widetilde{v}_\alpha)$) and

$$(\widetilde{u},\widetilde{v})\in W^{2,2}_{loc}(\mathbb{R}^{2+}_a)\times W^{1,2}_{loc}(\mathbb{R}^{2+}_a)$$

with the boundary data $(\widetilde{u},\mathbf{B}\widetilde{v})|_{\partial\mathbb{R}^{2+}_a} = (\varphi(x_0),0)$ where $\mathbb{R}^{2+}_a := \left\{(x^1,x^2)|x^2>-a\right\}$, such that for any $R>0$,

$$\lim_{\alpha\to1}\|\widetilde{u}_\alpha(x-(0,\frac{d_\alpha}{r_\alpha}))-\widetilde{u}(x)\|_{W^{1,2}(D^+_{3R}(0))}=0,\ \lim_{\alpha\to1}\|\widetilde{v}_\alpha(x-(0,\frac{d_\alpha}{r_\alpha}))-\widetilde{v}(x)\|_{L^4(D^+_{3R}(0))}=0.$$

We set $\widetilde{\sigma}(x):=\widetilde{u}(x+(0,a))$ and $\widetilde{\xi}(x):=\widetilde{v}(x+(0,a))$ and then conclude that, for any $R>0$,

$$\lim_{\alpha\to1}\|\widetilde{u}_\alpha(x)-\widetilde{\sigma}(x)\|_{W^{1,2}(D_{2R}(0)\cap B_\alpha\cap\mathbb{R}^2_a)}=0,\ \lim_{\alpha\to1}\|\widetilde{v}_\alpha(x)-\widetilde{\xi}(x)\|_{L^4(D_{2R}(0)\cap B_\alpha\cap\mathbb{R}^2_a)}=0.$$

Combining this with (4.25) and noting that the measures of $D_{2R}(0)\cap B_\alpha\setminus\mathbb{R}^2_a$ and $D_{2R}(0)\cap\mathbb{R}^2_a\setminus B_\alpha$ go to zero, we have

$$\lim_{\alpha\to1}\|\nabla\widetilde{u}_\alpha(x)\|_{L^2(D_R(0)\cap B_\alpha)}=\|\nabla\widetilde{\sigma}(x)\|_{L^2(D_R(0)\cap\mathbb{R}^2_a)},\ \lim_{\alpha\to1}\|\widetilde{v}_\alpha(x)\|_{L^4(D_R(0)\cap B_\alpha)}=\|\widetilde{\xi}(x)\|_{L^4(D_R(0)\cap\mathbb{R}^2_a)}. \tag{4.26}$$

By (4.16), we have $E(\widetilde{\sigma};D_1(0)\cap\mathbb{R}^2_a)=\frac{\epsilon_0}{4}$.

Next, similarly to **Claim 1** in **Step 1**, we make the following

Claim 2:

$$1\leq\liminf_{\alpha\searrow1}r_\alpha^{2(1-\alpha)}\leq\limsup_{\alpha\searrow1}r_\alpha^{2(1-\alpha)}\leq\mu_{max}<\infty. \tag{4.27}$$

In fact, if it is not true, then there exists a subsequence $\alpha_j\to1$ such that

$$\lim_{j\to\infty}r_{\alpha_j}^{2(1-\alpha_j)}\to\infty.$$

In view of the equation (4.5), it follows from the above fact that $(\widetilde{u}_{\alpha_j},\widetilde{v}_{\alpha_j})\rightharpoonup(\widetilde{\sigma},\widetilde{\xi})$ weakly in $W^{2,2}_{loc}(\mathbb{R}^{2+}_a)\times W^{1,2}_{loc}(\mathbb{R}^{2+}_a)$ as $j\to\infty$ and $\widetilde{\sigma}:\mathbb{R}^{2+}_a\to N$ is a harmonic map with boundary data $\widetilde{\sigma}|_{\partial\mathbb{R}^{2+}_a}=\varphi(x_0)$. By a well known result of Lemaire [23], we have that $\widetilde{\sigma}$ is a constant map, which is a contradiction to the fact that $E(\widetilde{\sigma};D_1(0)\cap\mathbb{R}^2_a)=\frac{\epsilon_0}{4}$. Thus, **Claim 2** holds.

Then we know (u_α,v_α) (see (4.22)) is a general α-Dirac-harmonic map. By Lemma 4.2 and above arguments, there exist a subsequence of $\{\alpha\}$ (still denoted by itself) and a Dirac-harmonic map $(\sigma,\xi):\mathbb{R}^{2+}_a\to N$ with the boundary data $(\sigma,\mathbf{B}\xi)|_{\partial\mathbb{R}^{2+}_a}=(\varphi(x_0),0)$, such that

$$\lim_{\alpha\to1}\|\nabla u_\alpha(x)\|_{L^2(D_R(0)\cap B_\alpha)}=\|\nabla\sigma(x)\|_{L^2(D_R(0)\cap\mathbb{R}^2_a)},\ \lim_{\alpha\to1}\|v_\alpha(x)\|_{L^4(D_R(0)\cap B_\alpha)}=\|\xi(x)\|_{L^4(D_R(0)\cap\mathbb{R}^2_a)}$$

for any $R > 0$, which implies $E(\sigma; D_1(0) \cap \mathbb{R}^2_a) = \frac{\epsilon_0}{4}$ according to (4.16). However, by Theorem 1.4 in [19], we know σ is a constant map and $\xi \equiv 0$. This is a contradiction and hence the statement (2) holds.

Step 3: For the first statement (1), i.e., the case of $x_0 \in M \setminus \partial M$, the argument is almost the same as in **Step 1**, so we omit it. The proof of the theorem is finished. □

Finally, we show Theorem 1.5.

Proof of Theorem 1.5. By Theorem 1.1, we know there exists a sequence of α-Dirac-harmonic maps $(\phi_\alpha, \psi_\alpha) \in C^{2+\mu}(M, N) \times C^{1+\mu}(M, \Sigma M \otimes \phi_\alpha^* TN)$ for $\alpha \searrow 1$ with the Dirichlet-chiral boundary condition

$$(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0),$$

satisfying

$$E_\alpha(\phi_\alpha) \leq E_\alpha(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)} \tag{4.28}$$

and

$$\|\psi_\alpha\|_{W^{1,p}(M)} \leq C(p, M, N, E_\alpha(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}), \tag{4.29}$$

for any $1 < p < 2$. All ϕ_α are in the homotopy class of ϕ_0.

Now, we claim that if the target manifold N does not admit any harmonic sphere, then the energy concentration set $\mathcal{S}$ defined in Theorem 1.2 is empty.

In fact, if not, taking a point $x_0 \in \mathcal{S}$, then by Theorem 1.4, there exist sequences $x_\alpha \to x_0$, $r_\alpha \to 0$ and a nontrivial Dirac-harmonic map $(\sigma, \xi) : \mathbb{R}^2 \to N$, such that

$$\left(\phi_\alpha(x_\alpha + r_\alpha x), r_\alpha^{\alpha-1}\sqrt{r_\alpha}\psi_\alpha(x_\alpha + r_\alpha x)\right) \to (\sigma, \xi) \textit{ in } C^2_{loc}(\mathbb{R}^2),$$

as $\alpha \to 1$. For any $4 < q < \infty$, taking $p = \frac{2q}{2+q} \in (\frac{4}{3}, 2)$ in (4.29), we have

$$\|\psi_\alpha\|_{L^q(M)} \leq C(q, M, N, E_\alpha(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}), \tag{4.30}$$

and for any $R > 0$,

$$\|\xi\|_{L^4(D_R(0))} = \lim_{\alpha\to 1} r_\alpha^{\alpha-1}\|\psi_\alpha\|_{L^4(D_{Rr_\alpha}(x_\alpha))} \leq \lim_{\alpha\to 1} C\|\psi_\alpha\|_{L^q(M)}(Rr_\alpha)^{2(\frac{1}{4}-\frac{1}{q})} = 0.$$

Thus, $\xi \equiv 0$ and the Dirac-harmonic map $(\sigma, \xi) : \mathbb{R}^2 \to N$ is just a nontrivial harmonic map $\sigma : \mathbb{R}^2 \to N$ with finite energy, which can be extended to a nontrivial smooth harmonic sphere. This is a contradiction and hence $\mathcal{S}$ must be empty.

By Theorem 1.2, we have

$$(\phi_\alpha, \psi_\alpha) \to (\phi, \psi) \textit{ in } C^2(M) \times C^1(M), \quad \textit{as } \alpha \to 1,$$

where $(\phi, \psi) \in C^{2+\mu}(M, N) \times C^{1+\mu}(M, \Sigma M \otimes \phi^* TN)$ is a Dirac-harmonic map with Dirichlet-chiral boundary data

$$(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0).$$

Moreover, it is easy to see that (ϕ, ψ) is in the same homotopy class as ϕ_0. We have finished the proof. □

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