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by

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Abstract

We study the geometric measure of quantum coherence recently proposed in [Phys. Rev. Lett. 115, 020403 (2015)]. Both lower and upper bounds of this measure are provided. These bounds are shown to be tight for a class of important coherent states – maximally coherent mixed states. The trade-off relation between quantum coherence and mixedness for this measure is also discussed.

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1 Introduction

Quantum coherence plays a vital role in quantum physics and quantum information processing. Being an important physical resource, it is tightly related to various research fields such as low-temperature thermodynamics [1, 2, 3, 4, 5], quantum biology [6, 7, 8, 9, 10, 11, 12], nanoscale physics [13, 14], ect. Formulating the theory of quantum coherence is a long-standing problem and considerable progress have been made in quantum optics [15, 16, 17]. Recently, a rigorous framework for the quantification of coherence was introduced [18]. Inspired by the quantitative theory of quantum entanglement, the authors in Ref. [18] provided a framework to quantify coherence by defining the so-called incoherent states and incoherent operations. These two concepts are analogous to the separable states and local operations and classical communication (LOCC) repectively in the quantum entanglement theory.

Fixing a basis $\{|i\rangle\}_{i=1}^d$ of a d-dimensional Hilbert space $\mathcal{H}$, the incoherent states are defined as [18]:

$$\sigma = \sum_{i=1}^d p_i |i\rangle\langle i|, \tag{1}$$

where $p_i \geq 0, \sum_{i=1}^d p_i = 1$. Quantum states that cannot be written in the above form are called coherent states. Let Λ be a completely positive trace preserving (CPTP) map,

$$\Lambda(\rho) = \sum_n K_n \rho K_n^\dagger, \tag{2}$$

where $\{K_n\}$ is a set of Kraus operators satisfying $\sum_n K_n^\dagger K_n = \mathbb{I}_d$. Let $\mathcal{I}$ be the set of incoherent states. If $K_n \mathcal{I} K_n^\dagger \subseteq \mathcal{I}$ for all n, then $\{K_n\}$ is called a set of incoherent Kraus operators, and the corresponding Λ is called an incoherent operation [18]. Obviously, $\Lambda(\mathcal{I}) \subseteq \mathcal{I}$. Similar to the quantification of quantum entanglement [19, 20, 21, 22], Baumgratz *et al.* proposed the following conditions to be satisfied as a measure of coherence $C(\rho)$ [18]:

(1) $C(\rho) \geq 0$, and $C(\rho) = 0$ if and only if $\rho \in \mathcal{I}$;

(2) $C(\Lambda(\rho)) \leq C(\rho)$ for any incoherent operation Λ;

(3) $\sum_n p_n C(\rho_n) \leq C(\rho)$, where $p_n = \text{Tr}(K_n \rho K_n^\dagger), \rho_n = K_n \rho K_n^\dagger / p_n, \{K_n\}$ is a set of incoherent Kraus operators;

(4) $C(\sum_i p_i \rho_i) \leq \sum_i p_i C(\rho_i)$ for any set of quantum states $\{\rho_i\}, p_i \geq 0, \sum_{i=1}^d p_i = 1$. It is obvious that conditions (3) and (4) imply condition (2). A quantity that satisfies conditions (1), (2), and (3) is called a coherence monotone. If it satisfies condition (4) in addition, then we call it a convex coherence monotone.

Distance-based measures are the best options for quantifying coherence, which are defined as $C_{\mathcal{D}}(\rho) = \min_{\sigma \in \mathcal{I}} \mathcal{D}(\rho, \sigma)$. If $\mathcal{D}$ is the l_1-norm [18], then $C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}|$, where $\rho_{ij} = \langle i|\rho|j \rangle$. If $\mathcal{D}$ is the quantum relative entropy [18], i.e., $\mathcal{D}(\rho, \sigma) = S(\rho\|\sigma) = \mathrm{Tr}(\rho \ln \rho) - \mathrm{Tr}(\rho \ln \sigma)$, then $C_{rel}(\rho) = S(\rho_d) - S(\rho)$, where $\rho_d = \sum_{i=1}^d \rho_{ii}|i\rangle\langle i|$, and $S(\rho) = -\mathrm{Tr}(\rho \ln \rho)$ is the von Neumann entropy. For other distance-based measures, many results have been obtained so far [23, 24, 25, 26].

In Ref. [26], the authors provided an operational link between coherence and entanglement. They showed that for any (convex) entanglement monotone E, one can define a corresponding (convex) coherence monotone C_E via an explicit formula. As an example, they proved that the geometric measure of coherence C_g, defined by the fidelity-based geometric measure of entanglement [27], is indeed a convex coherence monotone. The expression of C_g has been derived as $C_g(\rho) = 1 - \max_{\sigma \in \mathcal{I}} F(\rho, \sigma)$, where $F(\rho, \sigma) = (\mathrm{Tr}\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}})^2$ is the fidelity of two density operators ρ and σ, and analytical formula of $C_g(\rho)$ for any single-qubit state ρ is given [26]. However, for an arbitrary qudit state, computation of $C_g(\rho)$ is formidably difficult. Just like estimation of concurrence and entanglement of formation, it is also important to estimate the lower and upper bounds of $C_g(\rho)$.

Recently, Singh *et al.* studied quantum coherence and mixedness in any d-dimensional quantum system [28]. They have shown that for a fixed mixedness, the amount of coherence is restricted. A trade-off relation between coherence quantified by the l_1 norm and mixedness quantified by the normalized linear entropy is provided [28]. This result gives rise to the maximally coherent mixed states (MCMS), i.e., quantum states with maximal coherence and mixedness. They also discussed such trade-off relation and the form of MCMS for any qubit systems when the geometric coherence and geometric mixedness are considered. However, for any qudit system, the form of MCMS for geometric coherence is still unknown.

In this paper, we derive lower and upper bounds for the geometric measure of coherence for arbitrary dimension d, by using the concepts of sub-fidelity and super-fidelity introduced in Ref. [29, 30, 31, 32, 33]. These bounds are shown to be tight for a class

of maximally coherent mixed states. Furthermore, we also discuss the form of MCMS for the geometric measure of coherence for any qudit systems.

2 Estimation of geometric measure of coherence

Let $\{|i\rangle\}_{i=1}^d$ be a fixed basis of a d-dimensional Hilbert space. The incoherent states are represented as $\sigma = \sum_{i=1}^d x_i |i\rangle\langle i|$, where $x_i \geq 0, \sum_{i=1}^d x_i = 1$. Any density operator ρ is written as $\rho = \sum_{i,j=1}^d \rho_{ij} |i\rangle\langle j|$.

In Ref. [29], the authors introduced two quantities $E(\rho, \sigma)$ and $G(\rho, \sigma)$ called sub-fidelity and super-fidelity, respectively, as the lower and upper bounds of the fidelity $F(\rho, \sigma)$ for two quantum states ρ and σ. They are defined as

$$
\begin{aligned}
E(\rho, \sigma) &= \mathrm{Tr}(\rho\sigma) + \sqrt{2[(\mathrm{Tr}(\rho\sigma))^2 - \mathrm{Tr}(\rho\sigma\rho\sigma)]}, \\
G(\rho, \sigma) &= \mathrm{Tr}(\rho\sigma) + \sqrt{1 - \mathrm{Tr}(\rho^2)}\sqrt{1 - \mathrm{Tr}(\sigma^2)}.
\end{aligned}
\tag{3}
$$

It holds that $E(\rho, \sigma) \leq F(\rho, \sigma) \leq G(\rho, \sigma)$, and all three quantities are equal when $d = 2$, or at least one of ρ and σ is a pure state. Sub- and super-fidelity have many elegant properties, such as bounded, i.e., $0 \leq E(\rho, \sigma) \leq 1, 0 \leq G(\rho, \sigma) \leq 1$, symmetry, unitary invariance, concavity, ect. Here we use them to estimate the geometric measure of coherence $C_g(\rho) = 1 - \max_{\sigma \in \mathcal{I}} F(\rho, \sigma)$. It is obvious that $1 - \max_{\sigma \in \mathcal{I}} G(\rho, \sigma) \leq C_g(\rho) \leq 1 - \max_{\sigma \in \mathcal{I}} E(\rho, \sigma)$. We only need to find the maximal value of $E(\rho, \sigma)$ and $G(\rho, \sigma)$, when σ run over all the incoherent states. We have the following Theorem.

Theorem 1 *For any density operator ρ acting on a d-dimensional Hilbert space, the geometric measure of coherence $C_g(\rho)$ satisfies the following inequality:*

$$
1 - \frac{1}{d} - \frac{d-1}{d}\sqrt{1 - \frac{d}{d-1}\left(\mathrm{Tr}(\rho^2) - \sum_{i=1}^d \rho_{ii}^2\right)} \leq C_g(\rho) \leq 1 - \max_i\{\rho_{ii}\},
\tag{4}
$$

and $C_g(\rho) = 1 - \max_i\{\rho_{ii}\}$ when ρ is a pure state.

Proof. If ρ is a pure state, then $F(\rho, \sigma) = \mathrm{Tr}(\rho\sigma) = \sum_{i=1}^d \rho_{ii} x_i \leq \max_i\{\rho_{ii}\}$. Thus we have $C_g(\rho) = 1 - \max_i\{\rho_{ii}\}$. Next, we suppose that ρ is a mixed state.

4

We first estimate $\max_{\sigma \in \mathcal{I}} E(\rho, \sigma)$. Noting that $\mathrm{Tr}(\rho \sigma \rho \sigma) = \sum_{i,j=1}^{d} |\rho_{ij}|^2 x_i x_j$, we have

$$E(\rho, \sigma) = \sum_{i=1}^{d} \rho_{ii} x_i + \sqrt{2} \left[\sum_{i,j=1}^{d} \left(\rho_{ii} \rho_{jj} - |\rho_{ij}|^2 \right) x_i x_j \right]^{\frac{1}{2}} \tag{5}$$
$$:= f(x_1, \ldots, x_d).$$

To find the maximal value of $f(x_1, \ldots, x_d)$ over the closed domain $\overline{D} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i \geq 0, \sum_{i=1}^{d} x_i = 1\}$, we first assume that the maximal value is achieved in the open domain $D = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i > 0, \sum_{i=1}^{d} x_i = 1\}$, then the maximum point satisfies

$$\frac{\partial f}{\partial x_i} = \rho_{ii} + \frac{\sqrt{2} \sum_{j=1}^{d} \left(\rho_{ii} \rho_{jj} - |\rho_{ij}|^2 \right) x_j}{\sqrt{\sum_{i,j=1}^{d} \left(\rho_{ii} \rho_{jj} - |\rho_{ij}|^2 \right) x_i x_j}} = 0, \quad \forall i. \tag{6}$$

From $\sum_{i=1}^{d} \frac{\partial f}{\partial x_i} x_i = 0$, we get $\sum_{i=1}^{d} \rho_{ii} x_i + \sqrt{2} \left[\sum_{i,j=1}^{d} \left(\rho_{ii} \rho_{jj} - |\rho_{ij}|^2 \right) x_i x_j \right]^{\frac{1}{2}} = 0$. That is to say, the maximal value of f is equal to zero, which is impossible. Thus we can confirm that the maximum point of f must be on the boundary of $\overline{D}$, and $f_{\max}$ is no less than $f(\mathbf{x})$ when $\mathbf{x} \in D_1 \subseteq \overline{D}$, where $D_1 = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : \exists\, k, \text{ s.t. } x_k = 1, \text{ and } x_i = 0, \forall i \neq k\}$. Therefore we have $f_{\max} \geq \max_i \{\rho_{ii}\}$, and $1 - \max_{\sigma \in \mathcal{I}} E(\rho, \sigma) \leq 1 - \max_i \{\rho_{ii}\}$.

We now compute $\max_{\sigma \in \mathcal{I}} G(\rho, \sigma)$. Note that

$$G(\rho, \sigma) = \sum_{i=1}^{d} \rho_{ii} x_i + \sqrt{1 - \mathrm{Tr}(\rho^2)} \sqrt{1 - \sum_{i=1}^{d} x_i^2} \tag{7}$$
$$:= g(x_1, \ldots, x_d).$$

To find the maximal value of g over $\overline{D}$, we use the Lagrange multiplier method. Let $L(x_1, \ldots, x_d) = \sum_{i=1}^{d} \rho_{ii} x_i + \sqrt{1 - \mathrm{Tr}(\rho^2)} \sqrt{1 - \sum_{i=1}^{d} x_i^2} + \lambda(\sum_{i=1}^{d} x_i - 1)$. Then we have

$$\frac{\partial L}{\partial x_i} = \rho_{ii} - \sqrt{1 - \mathrm{Tr}(\rho^2)} \frac{x_i}{\sqrt{1 - \sum_{i=1}^{d} x_i^2}} + \lambda = 0, \quad \forall i. \tag{8}$$

This implies that

$$(\rho_{ii} + \lambda) \sqrt{1 - \sum_{i=1}^{d} x_i^2} = x_i \sqrt{1 - \mathrm{Tr}(\rho^2)}, \quad \forall i. \tag{9}$$

Summing over the above equations from 1 to d, we get

$$(1 + d\lambda) \sqrt{1 - \sum_{i=1}^{d} x_i^2} = \sqrt{1 - \mathrm{Tr}(\rho^2)}. \tag{10}$$

5

Thus

$$\rho_{ii} + \lambda = (1 + d\lambda)x_i, \quad \forall i. \tag{11}$$

From Eq. (10) and Eq. (11), we obtain

$$\sum_{i=1}^{d} x_i^2 = 1 - \frac{1 - \operatorname{Tr}(\rho^2)}{(1 + d\lambda)^2} = \sum_{i=1}^{d} \left(\frac{\rho_{ii} + \lambda}{1 + d\lambda} \right)^2, \tag{12}$$

which immediately yields that

$$\lambda = -\frac{1}{d} \pm \frac{1}{d} \sqrt{1 - \frac{d}{d-1} \left(\operatorname{Tr}(\rho^2) - \sum_{i=1}^{d} \rho_{ii}^2 \right)}. \tag{13}$$

We can also derive from Eq. (10) and Eq. (11) that

$$\begin{aligned}
g(x_1, \dots, x_d) &= \sum_{i=1}^{d} \left[(1 + d\lambda)x_i - \lambda \right] x_i + \frac{1 - \operatorname{Tr}(\rho^2)}{1 + d\lambda} \\
&= 1 + (d-1)\lambda.
\end{aligned} \tag{14}$$

Hence the maximal value of g over $\overline{D}$ is equal to $\frac{1}{d} + \frac{d-1}{d}\sqrt{1 - \frac{d}{d-1}\left(\operatorname{Tr}(\rho^2) - \sum_{i=1}^{d}\rho_{ii}^2\right)}$ by Eq. (13). Therefore we have $1 - \max_{\sigma \in \mathcal{I}} G(\rho, \sigma) = 1 - \frac{1}{d} - \frac{d-1}{d}\sqrt{1 - \frac{d}{d-1}\left(\operatorname{Tr}(\rho^2) - \sum_{i=1}^{d}\rho_{ii}^2\right)}$ for any mixed state. This completes the proof. $\square$

For $d = 2$, if ρ is a pure state, then $C_g(\rho) = 1 - \max\{\rho_{11}, \rho_{22}\}$. Taking into account that $\operatorname{Tr}(\rho^2) = 1$, we have $C_g(\rho) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4|\rho_{12}|^2}$. If ρ is a mixed state, then we have $C_g(\rho) = 1 - \max_{\sigma \in \mathcal{I}} G(\rho, \sigma) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4|\rho_{12}|^2}$. Therefore $C_g(\rho) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4|\rho_{12}|^2}$ holds for any qubit state ρ, which coincides with the result obtained in Ref. [26].

As an example of Theorem 1, let us consider a class of coherent states – maximally coherent mixed states (MCMS) [28], which are defined as

$$\rho_m = p|\psi_d\rangle\langle\psi_d| + \frac{1-p}{d}\mathbb{I}_d, \tag{15}$$

where $0 < p \le 1$, and $|\psi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle$ is the maximally coherent state. From (4), we get

$$1 - \frac{1}{d} - \frac{d-1}{d}\sqrt{1 - p^2} \le C_g(\rho_m) \le 1 - \frac{1}{d}. \tag{16}$$

We now compute $C_g(\rho_m)$. Note that

$$\sqrt{\sigma}\rho_m\sqrt{\sigma} = \frac{1}{d}\sum_i x_i|i\rangle\langle i| + \frac{p}{d}\sum_{i \ne j}\sqrt{x_i x_j}|i\rangle\langle j|, \tag{17}$$

and suppose that

$$\sqrt{\sqrt{\sigma}\rho_m\sqrt{\sigma}} = a\sum_i \sqrt{x_i}|i\rangle\langle i| + b\sum_{i,j}\sqrt{x_i x_j}|i\rangle\langle j|, \quad a \geq 0, b \geq 0. \tag{18}$$

Then we have

$$\frac{1}{d}\sum_i x_i|i\rangle\langle i| + \frac{p}{d}\sum_{i\neq j}\sqrt{x_i x_j}|i\rangle\langle j| = \sum_i \left[\left(a^2+b^2\right)x_i + 2abx_i\sqrt{x_i}\right]|i\rangle\langle i| \\ + \sum_{i\neq j}\left[b^2\sqrt{x_i x_j} + ab\left(x_i\sqrt{x_j} + x_j\sqrt{x_i}\right)\right]|i\rangle\langle j|. \tag{19}$$

Comparing both sides of the above equation, we obtain

$$\left(a^2+b^2\right) + 2ab\sqrt{x_i} = \frac{1}{d}, \quad \forall i, \\ b^2 + ab\left(\sqrt{x_j} + \sqrt{x_i}\right) = \frac{p}{d}, \quad i \neq j. \tag{20}$$

Summing over the above equations from 1 to d, respectively, we have

$$d\left(a^2+b^2\right) + 2ab\sum_i\sqrt{x_i} = 1, \\ db^2 + 2ab\sum_i\sqrt{x_i} = p, \tag{21}$$

which yields that $a = \sqrt{\frac{1-p}{d}}$. On the other hand, it holds that $\sum_{i=1}^d\left[\frac{1}{d} - \left(a^2+b^2\right)\right]^2 = 4a^2b^2\sum_{i=1}^d x_i = 4a^2b^2$ from Eq. (20), then we get $b = \frac{1}{d}\left(\sqrt{1-p+dp} - \sqrt{1-p}\right)$. Thus we have

$$\left(\text{Tr}\sqrt{\sqrt{\sigma}\rho_m\sqrt{\sigma}}\right)^2 = \left(a\sum_i\sqrt{x_i} + b\right)^2 \\ \leq \left(a\sqrt{d} + b\right)^2 \tag{22} \\ = \left[\sqrt{1-p} + \frac{1}{d}\left(\sqrt{1-p+dp} - \sqrt{1-p}\right)\right]^2$$

by use of Cauchy-Schwarz inequality, and the equality holds if and only if $x_i = \frac{1}{d}, \forall i$. Therefore we have

$$C_g(\rho_m) = 1 - \left[\sqrt{1-p} + \frac{1}{d}\left(\sqrt{1-p+dp} - \sqrt{1-p}\right)\right]^2. \tag{23}$$

For $d = 3$, the comparison between $C_g(\rho_m)$ and the lower and upper bounds of $C_g(\rho_m)$ in (16) is shown in FIG 1.

It seems that the upper bound we derived in Theorem 1 is large for mixed states. However, we have the following improved upper bound.
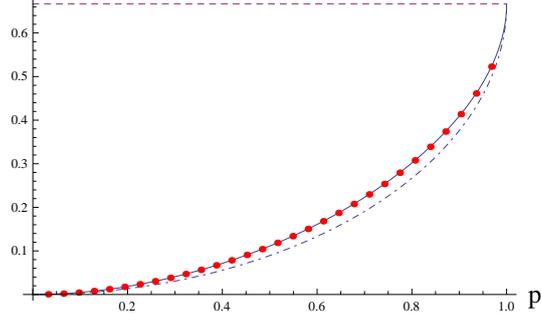
Figure 1: The solid line is the value of $C_g(\rho_m)$ for $0 < p \leq 1$. The dot-dashed line is the lower bound in (4). The dashed line is the upper bound in (4), and the dotted line is the upper bound in (24). It can be seen that the upper bound in (24) is achieved for ρ_m, and it is much tighter than the one in (4) in this case.

Theorem 2 *Let* $\sqrt{\rho} = \sum_{i,j} b_{ij} |i\rangle\langle j|$. *Then we have*

$$C_g(\rho) \leq 1 - \left(\sum_i b_{ii}^2 \right), \tag{24}$$

Proof. Since $\sqrt{F(\rho,\sigma)} = \max_U \mathrm{Tr}(U\sqrt{\rho}\sqrt{\sigma}) \geq \mathrm{Tr}(\sqrt{\rho}\sqrt{\sigma})$, we have $C_g(\rho) = 1 - \max_{\sigma \in \mathcal{I}} F(\rho,\sigma) \leq 1 - \max_{\sigma \in \mathcal{I}} \left[\mathrm{Tr}(\sqrt{\rho}\sqrt{\sigma}) \right]^2$. Note that

$$
\begin{aligned}
\mathrm{Tr}(\sqrt{\rho}\sqrt{\sigma}) &= \sum_i b_{ii}\sqrt{x_i} \\
&\leq \sqrt{\left(\sum_i b_{ii}^2 \right)\left(\sum_i x_i \right)} \\
&= \sqrt{\left(\sum_i b_{ii}^2 \right)}
\end{aligned}
\tag{25}
$$

by Cauchy-Schwarz inequality, and the equality holds if and only if $x_i = b_{ii}^2/(\sum_j b_{jj}^2), \forall i$. Then we complete the proof. $\square$

We consider again the state ρ_m given by Eq. (15). Setting $x_i = \frac{1}{d}, \forall i$, in Eq. (18), we get

$$\sqrt{\rho_m} = \frac{1}{\sqrt{d}}\left(\sqrt{1-p+dp} - \sqrt{1-p} \right)|\psi_d\rangle\langle\psi_d| + \sqrt{\frac{1-p}{d}}\mathbb{I}_d \tag{26}$$

directly. Hence the upper bound in (24) for ρ_m is equal to

$$1 - \left[\sqrt{1-p} + \frac{1}{d}\left(\sqrt{1-p+dp} - \sqrt{1-p} \right) \right]^2, \tag{27}$$

8

which is exactly the value of $C_g(\rho_m)$ by (23). Thus we can see from FIG 1 that the upper bound in (24) is tighter than the one in (4) for some cases.

From the above discussion, we have the following Theorem.

Theorem 3 *For any quantum state ρ, we have*

$$1 - \frac{1}{d} - \frac{d-1}{d}\sqrt{1 - \frac{d}{d-1}\left(\text{Tr}(\rho^2) - \sum_{i=1}^{d}\rho_{ii}^2\right)} \le C_g(\rho) \le \min\{l_1, l_2\}, \qquad (28)$$

where l_1 and l_2 denote the upper bounds in (4) and (24), respectively. When ρ is a pure state, $C_g(\rho) = l_1 \le l_2$.

Proof. We only need to prove that $l_1 \le l_2$ for pure states. Noting that $\sqrt{\rho} = \rho$ and $b_{ii} = \rho_{ii}$ for any pure state ρ, we have $\sum_i b_{ii}^2 = \sum_i \rho_{ii}^2 \le \max_i\{\rho_{ii}\}\sum_i \rho_{ii} = \max_i\{\rho_{ii}\}$. Thus $1 - \max_i\{\rho_{ii}\} \le 1 - (\sum_i b_{ii}^2)$. $\quad\square$

3   Discussion and Conclusion

We have investigated the geometric measure of coherence. Both lower and upper bounds of this measure have been derived. Our upper bound can be achieved for arbitrary pure states and a class of maximally coherent mixed states.

As a matter of fact, the amount of quantum coherence is closely related to the mixedness of a quantum state. There exits a kind of trade-off relation between the quantum coherence and the mixedness. Recently, Singh *et al.* [28] studied the trade-off relations between coherence and mixedness for an arbitrary d-dimensional quantum system. Employing the l_1-norm of coherence $C_{l_1}(\rho)$ and the normalized linear entropy [34], given by $M_l(\rho) = \frac{d}{d-1}(1 - \text{Tr}(\rho^2))$ as a measure of mixedness, they obtained the following inequality:

$$\frac{C_{l_1}^2(\rho)}{(d-1)^2} + M_l(\rho) \le 1. \qquad (29)$$

For a fixed mixedness M_l, quantum states with maximal coherence are called maximally coherent mixed states (MCMS). It is shown that, up to incoherent unitaries, ρ_m defined in (15) is the only form of MCMS with respect to the above inequality [28].

Besides linear entropy, the fidelity of a state ρ and the maximally mixed state $\frac{1}{d}\mathbb{I}$ is also a proper measure of mixedness. It is defined by $M_g(\rho) = F(\rho, \frac{1}{d}\mathbb{I}) = \frac{1}{d}(\text{Tr}\sqrt{\rho})^2$, $0 \le M_g(\rho) \le 1$, and called geometric measure of mixedness.

By the definition of geometric measure of coherence $C_g(\rho)$, one can easily get

$$C_g(\rho) + M_g(\rho) \leq 1. \tag{30}$$

It can be verified that the equality holds for ρ_m from (23) and (26). Whether there exist other form of states that satisfy the equality is still unknown, since there is no analytical formula for C_g in general. Thus ρ_m is a subset of MCMS with respect to the trade-off relation between coherence measured by C_g and mixedness measured by M_g.

For other quantifiers of coherence and mixedness, the trade-off relations remain to be investigated further. Our results may shed new light on the quantification of quantum coherence and present potential applications in quantum information theory.

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