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Random Spectrahedra

by

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ABSTRACT. Spectrahedral cones are linear sections of the cone of positive semidefinite symmetric matrices. We study statistical properties of random spectrahedral cones (intersected with the sphere)

$$\mathscr{S}_{\ell,n} = \left\{(x_0,\dots,x_\ell) \in S^\ell \mid x_0\mathbb{1} + x_1R_1 + \cdots + x_\ell R_\ell \succ 0\right\}$$

where $R_1,\dots,R_\ell$ are independent GOE(n)-distributed matrices rescaled by $(2n\ell)^{-1/2}$.

We relate the expectation of the volume of $\mathscr{S}_{\ell,n}$ with some statistics of the smallest eigenvalue of a GOE(n) matrix, by providing explicit formulas for this quantity. These formulas imply that as $\ell, n \to \infty$ on average $\mathscr{S}_{\ell,n}$ keeps a positive fraction of the volume of the sphere S^ℓ (the exact constant is $\Phi(-1) \approx 0.1587$, where Φ is the cumulative distribution function of a standard gaussian variable).

For $\ell = 2$ spectrahedra are generically smooth, but already when $\ell = 3$ singular points on their boundaries appear with positive probability. We relate the average number $\mathbb{E}\,\sigma_n$ of singular points on the boundary of a three-dimensional spectrahedron $\mathscr{S}_{3,n}$ to the volume of the set of symmetric matrices whose two smallest eigenvalues coincide. In the case of quartic spectrahedra ($n = 4$) we show that $\mathbb{E}\,\sigma_4 = 6 - \frac{4}{\sqrt{3}}$. Moreover, we prove that the average number $\mathbb{E}\,\rho_n$ of singular points on the random symmetroid surface

$$\Sigma_{3,n} = \left\{(x_0,x_1,x_2,x_3) \in S^3 \mid \det(x_0\mathbb{1} + x_1R_1 + x_2R_2 + x_3R_3) = 0\right\},$$

equals $n(n-1)$. This quantity is related to the volume of the set of symmetric matrices with repeated eigenvalues.

1. INTRODUCTION

A *spectrahedron* is an affine-linear section of the cone $\mathcal{P}_n \subset \mathrm{Sym}(n,\mathbb{R})$ of positive semidefinite symmetric matrices. On the space $\mathrm{Sym}(n,\mathbb{R})$ of $n \times n$ real symmetric matrices there is a partial order defined by $A \succ B$, if and only if $A - B \in \mathcal{P}_n$. Every spectrahedron can then be parametrized as the set of solutions of a linear matrix inequality:

$$M_0 + x_1M_1 + \cdots + x_\ell M_\ell \succ 0, \quad x = (x_1,\dots,x_\ell) \in \mathbb{R}^\ell,$$

for some symmetric matrices $M_0,\dots,M_\ell \in \mathrm{Sym}(n,\mathbb{R})$.

Optimization of a linear function over a spectrahedron is called *semidefinite programming* [MR95, Ali95]. This is a useful generalization of *linear programming*, i.e. optimization of a linear function over a polyhedron. Such problems as finding the smallest eigenvalue of a symmetric matrix or optimizing a polynomial function on the sphere can be approached using semidefinite programming. The presence of singularities on the boundary of a three-dimensional spectrahedron is relevant for optimization: with a positive probability a linear function constrained on a polyhedron attains its maximum in a vertex of the polyhedron, and, similarly, with a positive probability a linear function constrained on a spectrahedron attains its maximum in a singular point of the boundary of the spectrahedron. For example, consider the cubic spectrahedron shown on Figure 1.1:

$$\mathscr{S} = \left\{(x,z,y) \in \mathbb{R}^3 \mid \left(\begin{smallmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{smallmatrix}\right) \in \mathcal{P}_n\right\}.$$

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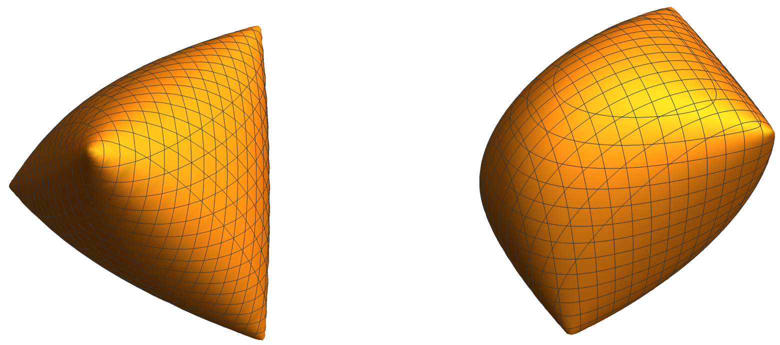

FIGURE 1.1. On the left is the cubic spectrahedron from the introduction. On the right is a quartic spectrahedron (in [GB12] it is called the “pillow”). The singularities on the boundaries of both spectrahedra are visible.

The linear function $\psi_w(x,y,z) = \langle w, (x,y,z)\rangle, w \in S^3$, constrained on $\mathscr{S}$ attains its maximum at a point $(x,y,z) \in \partial\mathscr{S}$ on the boundary of $\mathscr{S}$ at which the normal cone to $\partial\mathscr{S}$ contains w. At a singular point of the boundary the normal cone has positive dimension and hence the set of $w \in S^3$ for which the maximum of ψ_w is attained at a singular point of the boundary of the spectrahedron has positive volume in S^3.

Besides mentioned applications spectrahedra also appear in modern real algebraic geometry: in [JWH06] Helton and Vinnikov gave a beautiful characterization of two-dimensional spectrahedra and in [AD11] Degtyarev and Itenberg described all generic possibilities for the number of singular points on the boundary of a quartic three-dimensional spectrahedron (see Section 1.4 for more details). The reader can also look at [Vin14] for a survey article.

1.1. **Random spectrahedra.** In order to perform a probabilisitc study, it is more convenient to work instead with *spherical spectrahedra*, defined by:

$$x_0 M_0 + x_1 M_1 + \cdots + x_\ell M_\ell \succ 0, \quad x = (x_0, x_1, \ldots, x_\ell) \in S^\ell.$$

A generic spherical spectrahedron has nonempty interior and, after a change of coordinates in the space of symmetric matrices, it can be presented as:

$$\mathscr{S}_{\ell,n} = \left\{ x = (x_0, \ldots, x_\ell) \in S^\ell \mid x_0 \mathbb{1} + x_1 R_1 + \cdots + x_\ell R_\ell \succ 0 \right\}. \tag{1.1}$$

From now on, for simplicity, we abuse the terminology and use the term “spectrahedron” to refer to a spherical spectrahedron.

Our choice of *random* model for spectrahedra is as follows. In the representation (1.1) we sample the matrices $R_1, \ldots, R_\ell$ independently and identically distributed from the *Gaussian Orthogonal Ensemble* $\mathrm{GOE}(n)$ [Meh91, Tao12], rescaled by $(2n\ell)^{-1/2}$:

$$R_i = \frac{1}{\sqrt{2n\ell}} Q_i, \quad \text{where} \quad Q_i \sim \mathrm{GOE}(n),\ i = 1, \ldots, \ell.$$

By $Q \sim \mathrm{GOE}(n)$ we mean that the joint probability density of the entries of the symmetric matrix $Q \in \mathrm{Sym}(n, \mathbb{R})$ is $\varphi(Q) = \frac{1}{C_n} \exp(-\frac{1}{2}\mathrm{tr}(Q^2))$, where C_n is the normalization constant with $\int_{\mathrm{Sym}(n,\mathbb{R})} \varphi(Q)\,\mathrm{d}Q = 1$. In other words, the entries of Q are centered gaussian random variables, the diagonal entries having variance 1 and the off-diagonal entries having variance $\frac{1}{2}$. The scaling factor $(2n\ell)^{-1/2}$ serves to balance the order of magnitudes of eigenvalues of the two summands $x_0 \mathbb{1}$ and $x_1 R_1 + \cdots + x_\ell R_\ell$.

1.2. **Statistical properties of random spectrahedra.** In the following for a semialgebraic subset $X \subset S^\ell$ of dimension d by $|X|$ we denote the d-dimensional volume of the set of smooth points of X and by $|X|_{\mathrm{rel}} := \frac{|X|}{|S^d|}$ we denote the *relative volume* of X. The first statistic we will be interested in is the expected value of $|\mathscr{S}_{\ell,n}|_{\mathrm{rel}}$. Furthermore, we will be interested in the expected number of singular points on $\partial\mathscr{S}_{3,n}$. More specifically, the boundary $\partial\mathscr{S}_{3,n}$ of a 3-dimensional spectrahedron $\mathscr{S}_{3,n}$ is in general singular and, generically, has finitely many singular points that are all nodes. We denote their number by σ_n. Note that $\partial\mathscr{S}_{3,n}$ is a semialgebraic subset of

$$\Sigma_{3,n} = \{x \in S^3 \mid \det(x_0\mathbb{1} + x_1R_1 + x_2R_2 + x_3R_3) = 0\}, \tag{1.2}$$

called the *symmetroid surface* of $\mathscr{S}_{\ell,n}$. Hence σ_n is smaller than the number ρ_n of singular points on $\Sigma_{3,n}$. Summarizing, we will be interested in:

$$\mathbb{E}\,|\mathscr{S}_{\ell,n}|_{\mathrm{rel}}, \quad \mathbb{E}\,\sigma_n \quad \text{and} \quad \mathbb{E}\,\rho_n.$$

Our main results on those three quantities follow next.

1.3. **Main results.** To state our first main result, let $\lambda_{\min}(Q)$ denote the smallest eigenvalue of the matrix Q. For the scaled smallest eigenvalue we write

$$\tilde{\lambda}_{\min}(Q) := \frac{\lambda_{\min}(Q)}{\sqrt{2n}}. \tag{1.3}$$

The following is proved in Section 3 below.

Theorem 1.1 (Expected volume of the spectrahedron)**.** *Let F_ℓ denote the cumulative distribution function of the student's t-distribution with ℓ degrees of freedom* [NLJ95, Chapter 28] *and $\Phi(x)$ denote the cumulative distribution function of the normal distribution* [JS00, 40:14:2]. *Then:*

(1) $\mathbb{E}\,|\mathscr{S}_{\ell,n}|_{\mathrm{rel}} = \mathop{\mathbb{E}}_{Q\in \mathrm{GOE}(n)} F_\ell\big(\tilde{\lambda}_{\min}(Q)\big)$.

Moreover, we have

(2) $\mathbb{E}\,|\mathscr{S}_{\ell,n}|_{\mathrm{rel}} = F_\ell(-1) + \mathcal{O}(n^{-2/3})$ *uniformly in ℓ,*
(3) $\mathbb{E}\,|\mathscr{S}_{\ell,n}|_{\mathrm{rel}} = \Phi(-1) + \mathcal{O}(\ell^{-1}) + \mathcal{O}(n^{-2/3})$.

Note that $\Phi(-1) \approx 0.1587$. This means that asymptotically (in both n and ℓ) the average volume of a spectrahedron is at least 15% of the volume of the sphere.

For the average number of singular points σ_n on $\partial\mathscr{S}_{3,n}$ and number of singular points ρ_n on $\Sigma_{3,n}$ the result is more delicate to state. We denote the dimension of $\mathrm{Sym}(n,\mathbb{R})$ by $N := \frac{n(n+1)}{2}$ and the unit sphere there by $S^{N-1} := \{Q \in \mathrm{Sym}(n,\mathbb{R}) \mid \|Q\|^2 = \mathrm{tr}(Q^2) = 1\}$. Let $\Delta \subset S^{N-1}$ be the set of symmetric matrices of unit norm and with repeated eigenvalues and let $\Delta_1 \subset \Delta$ be its subset consisting of symmetric matrices whose two *smallest* eigenvalues coincide:

$$\begin{aligned}
\Delta &:= \{Q \in \mathrm{Sym}(n,\mathbb{R}) \cap S^{N-1} \mid \ \lambda_i(Q) = \lambda_j(Q) \text{ for some } i \neq j\},\\
\Delta_1 &:= \{Q \in \mathrm{Sym}(n,\mathbb{R}) \cap S^{N-1} \mid \lambda_1(Q) = \lambda_2(Q)\}.
\end{aligned}$$

Note that Δ and Δ_1 are both semialgebraic subsets of S^{N-1} of codimension two; Δ is actually algebraic. The following theorem relates $\mathbb{E}\,\sigma_n$ and $\mathbb{E}\,\rho_n$ to the volumes of Δ_1 and Δ, respectively.

Theorem 1.2 (The average number of singular points)**.** *The average number of singular points on the boundary of a random 3-dimensional spectrahedron $\mathscr{S}_{3,n} \subset S^3$ equals*

(1) $\mathbb{E}\,\sigma_n = 2\dfrac{|\Delta_1|}{|S^{N-3}|}$.

The average number of singular points on the symmetroid $\Sigma_{3,n} \subset S^3$ equals

(2) $\mathbb{E}\,\rho_n = 2\dfrac{|\Delta|}{|S^{N-3}|}$.

We prove Theorem 1.2 in Section 4. As a consequence of the proof of this theorem we will also derive the following interesting proposition.

Proposition 1.3. *For the generic choice of the matrices* $R_1, R_2, R_3 \in \mathrm{Sym}(n,\mathbb{R})$ *there is a one-to-one correspondence between* the number of singular points *of the projective symmetroid* $\mathbb{P}\Sigma_{3,n} = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{R}\mathrm{P}^3 \mid \det(x_0\mathbb{1} + x_1R_1 + x_2R_2 + x_3R_3) = 0\}$ *and the number of matrices with multiple eigenvalues in* $\mathbb{P}(\mathrm{span}\{R_1, R_2, R_3\})$.

The following theorem whose proof is given in Section 5 implies that the order of magnitude of $\mathbb{E}\,\rho_n$ is $\mathcal{O}(n^2)$. In particular, this gives an upper bound on $\mathbb{E}\,\sigma_n$.

Theorem 1.4 (The volume of the set of symmetric matrices with repeated eigenvalues)**.** *Let* $\Delta_1 \subset \Delta \subset S^{N-1}$ *be as above. Then*

(1) $$\frac{|\Delta_1|}{|S^{N-3}|} = \frac{2^{n-1}}{\sqrt{\pi}\,n!}\binom{n}{2}\int_{u\in\mathbb{R}} \underset{Q\sim\mathrm{GOE}(n-2)}{\mathbb{E}}\left[\det(Q-u\mathbb{1})^2\mathbf{1}_{\{Q-u\mathbb{1}\succ 0\}}\right]e^{-u^2}\,\mathrm{d}u.$$

(2) $$\frac{|\Delta|}{|S^{N-3}|} = \frac{2^{n-1}}{\sqrt{\pi}\,n!}\binom{n}{2}\int_{u\in\mathbb{R}} \underset{Q\sim\mathrm{GOE}(n-2)}{\mathbb{E}}\left[\det(Q-u\mathbb{1})^2\right]e^{-u^2}\,\mathrm{d}u.$$

The last quantity we compute explicitly:

(3) $$\frac{|\Delta|}{|S^{N-3}|} = \binom{n}{2}.$$

The theorem is of independent interest: as we explain at the beginning of Section 5 it gives some information on the geometry of the set of matrices with repeated eigenvalues. In particular, from Theorem 1.4 we can extract the following interesting corollary.

Corollary 1.5. *Let* $Q \in S^{N-1}$ *be a random symmetric matrix uniformly distributed in the sphere* S^{N-1}. *Let* $\mathrm{dist}_{S^{N-1}}(Q,\Delta)$ *denote the distance in* S^{N-1} *between* Q *and* Δ, *then*

$$\mathrm{Prob}\,\{\mathrm{dist}_{S^{N-1}}(Q,\Delta) < \epsilon\} \le 2\pi(1-\cos\epsilon)\tbinom{n}{2}|S^{N-3}|.$$

The proof of Corollary 1.5 is by integrating the volume of a spherical disc of radius ϵ over Δ.

1.4. **Quartic spectrahedra.** Quartic spectrahedra are a special case of our study, corresponding to $n = 4$. In this case the symmetroid surface

$$\Sigma_{3,4} = \{x \in S^3 \mid \det(x_0\mathbb{1} + x_1R_1 + x_2R_2 + x_3R_3) = 0\}$$

has degree four, since $\mathbb{1}, R_1, R_2, R_3 \in \mathrm{Sym}(4,\mathbb{R})$. In [AD11] Degtyarev and Itenberg proved that all possibilities for σ_4 and ρ_4 are realized by some generic spectrahedra $\mathscr{S}_{3,4}$ and their symmetroids $\Sigma_{3,4}$ under the following constraints:

(1.4) $\quad \sigma_4$ is even and $2 \le \sigma_4 \le 10$; $\quad \rho_4$ is a multiple of 4 and $4 \le \rho_4 \le 20$.

(Degtyarev and Itenberg proved this for the spectrahedron and its symmetroid in projective space, that is why in our condition (1.4) above we have to double their estimates.) An "average picture" of this result is given in the following proposition.

Proposition 1.6 (The average number of nodes on the boundary of a quartic spectrahedron)**.** *We have*

$$\mathbb{E}\,\sigma_4 = 6 - \frac{4}{\sqrt{3}} \approx 3.69 \quad \textit{and} \quad \mathbb{E}\,\rho_4 = 12.$$

It would be interesting to understand the distribution of the random variables σ_4, ρ_4 and compare it with the "deterministic" picture (1.4). We believe that the study of metric properties of spectrahedra such as volume is very promising for future research. This lines up with our discussion presented at the beginning of Section 5.

In the general case we conjecture that $\lim_{n\to\infty} \frac{\mathbb{E}\,\sigma_n}{\mathbb{E}\,\rho_n} = 0$, but it is difficult to predict how small is $\mathbb{E}\,\sigma_n$ compared to $\mathbb{E}\,\rho_n$. The main challenge is handling the characteristic function in the integral from Theorem 1.4 (1).

1.5. **Another possible random model.** Another natural model of random spectrahedra is by defining them as linear sections of $\mathcal{P}_n \cap S^{N-1}$ by a uniformly distributed $(\ell+1)$-dimensional plane V in $\mathrm{Sym}(n,\mathbb{R})$:

$$\mathcal{S}_{\ell,n}(V) := \mathcal{P}_n \cap S^{N-1} \cap V, \tag{1.5}$$

Before proceeding we argue in favor of the model (1.1) over the random linear section model (1.5). The main reason for this is that the expected volume of $\mathcal{S}_{\ell,n}(V)$ decays to zero for fixed ℓ and $n \to \infty$, which we prove in Proposition 1.7 below. In fact, typically a spherical spectrahedron of the form $\mathcal{S}_{\ell,n}(V)$ is empty (this is essentially due to the fact that the volume of the positive semidefinite cone $\mathcal{P}_n$ decays exponentially fast as $n \to \infty$), and this model is inaccessible for probabilistic studies. For the model introduced in (1.1) this appears differently: for large n and ℓ the spectrahedron $\mathscr{S}_{\ell,n}$ keeps a fraction of about 15% of the volume of the sphere S^ℓ; cf. Theorem 1.1. In fact, the spectrahedron $\mathscr{S}_{\ell,n}$ is never empty, as it contains an open neighborhood of $(1,0,\ldots,0) \in S^\ell$.

Proposition 1.7 (Decay of the random linear section model)**.** *Let V be uniformly distributed in the Grassmannian of $(\ell+1)$-dimensional subspaces of Sym$(n,\mathbb{R})$. Then for every $c > 0$ we have $\mathbb{P}\{\mathcal{S}_{\ell,n}(V) \neq \emptyset\} \leq O(n^{-c})$.*

Proof. Let us denote by $\mu(V)$ the maximum number of positive eigenvalues that a matrix in V has. For $n \geq 4$, we have the simple bound $\mathbb{P}\{\mathcal{S}_{\ell,n}(V) \neq \emptyset\} = \mathbb{P}\{\mu(V) = n\} \leq \mathbb{P}\{\mu(V) \geq \frac{n}{2} + \sqrt{n}\}$. By [AL16, Lemma 4] the last quantity is smaller than $O(n^{-c})$ for every $c > 0$. $\square$

1.6. **Notation.** Throughout the article some symbols are repeatedly used for the same purposes: $\mathrm{Sym}(n,\mathbb{R})$ stands for the space of $n \times n$ real symmetric matrices. By the symbols $\mathcal{Q} = (Q_1,\ldots,Q_\ell) \in \mathrm{Sym}(n,\mathbb{R})^\ell$ and $\mathcal{R} = (R_1,\ldots,R_\ell) \in \mathrm{Sym}(n,\mathbb{R})^\ell$ we denote a collection of ℓ symmetric matrices and its rescaled version respectively, i.e. $R_i = \frac{1}{\sqrt{2n\ell}} Q_i$. The k-dimensional sphere endowed with the standard metric is denoted S^k. The symbol $\mathbb{1}$ stands for the unit matrix (of any dimension). For $x = (x_0, x_1, \ldots, x_\ell) \in S^\ell$ we denote the matrices $Q(x) = x_1 Q_1 + \cdots + x_\ell Q_\ell$ and $A(x) = x_0 \mathbb{1} + Q(x)$. By $\mathscr{S}_{\ell,n}$, $\partial\mathscr{S}_{\ell,n}$ and $\Sigma_{\ell,n}$ we denote a (random) spectrahedron, its boundary and a symmetroid hypersurface respectively. Letters α, λ and μ are used to denote eigenvalues and $\tilde{\lambda} = \frac{1}{\sqrt{2n}} \lambda$ stands for the rescaled eigenvalue λ.

1.7. **Organization of the article.** The organization of the paper is as follows. In the next section we recall some known deviation inequalities for the smallest eigenvalue of a GOE(n)-matrix. In Sections 3–5 we prove our main theorems, Section 6 deals with the case of quartic spectrahedra. In the Appendix we prove a technical result, based on the computation of the expectation of the square of the characteristic polynomial of a GOE matrix.

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2. Deviation inequalities for the smallest eigenvalue

In this section we want to summarize known inequalities for the deviation of $\lambda_{\min}(Q)$ from its expected value in the GOE(n) random matrix model. The results that we present are due to [ML10]. Note that in that reference, however, the inequalities are given for the *largest* eigenvalue $\lambda_{\max}(Q)$. Since the GOE(n)-distribution is symmetric around 0, we have $\lambda_{\max}(Q) \sim -\lambda_{\min}(Q)$. Using this we translate the deviation inequalities for $\lambda_{\max}(Q)$ from [ML10] into deviation inequalities for $\lambda_{\min}(Q)$. Furthermore, note that in [ML10, (1.2)] the variance for the GOE(n)-ensemble is defined differently than it is here: eigenvalues of a random matrix in [ML10] are $\sqrt{2}$ times eigenvalues in our definition.

We express the deviation inequalities in terms of the scaled eigenvalue $\tilde{\lambda}_{\min}(Q)$, cf. (1.3). The following Proposition is [ML10, Theorem 1]. We will not need this result in the rest of the paper directly, but we decided to recall it here because it gives an idea of the behavior of the smallest eigenvalue of a random GOE(n) matrix, in terms of which our theorem on the volume of random spectrahedra is stated.

Proposition 2.1. *For some constant $C > 0$, all $n \geq 1$ and $0 < \epsilon < 1$, we have*

$$\operatorname*{Prob}_{Q\in \mathrm{GOE}(n)} \left\{\tilde{\lambda}_{\min}(Q) \leq -(1+\epsilon)\right\} \leq Ce^{-C^{-1}n\epsilon^{\frac{3}{2}}}$$

and

$$\operatorname*{Prob}_{Q\in \mathrm{GOE}(n)} \left\{\tilde{\lambda}_{\min}(Q) \geq -(1-\epsilon)\right\} \leq Ce^{-C^{-1}n^2\epsilon^{3}}.$$

Proposition 2.1 shows that for large n the mass of $\tilde{\lambda}_{\min}(Q)$ concentrates exponentially around -1. Thus $\mathbb{E}\,\tilde{\lambda}_{\min}(Q)$ converges to -1 as the following proposition shows.

Proposition 2.2. *For some constant $C > 0$ and all $n \geq 1$ we have*

$$|\mathbb{E}\,\tilde{\lambda}_{\min}(Q) + 1| \leq \mathbb{E}|\tilde{\lambda}_{\min}(Q) + 1| \leq Cn^{\frac{-2}{3}}.$$

Proof. By [ML10, Equation after Corollary 3] we have

$$\limsup_{n\to\infty} 2^p n^{\frac{2p}{3}}\, \mathbb{E}|\tilde{\lambda}_{\min}(Q) + 1|^p < \infty. \tag{2.1}$$

The assertion follows from monotonicity of the integral: $|\mathbb{E}\,\tilde{\lambda}_{\min}(Q) + 1| \leq \mathbb{E}|\tilde{\lambda}_{\min}(Q) + 1|$ and (2.1) with $p = 1$. $\square$

Remark. The distribution of the scaled largest eigenvalue of a GOE(n) matrix for $n \to \infty$ is known as the *Tracy-Widom distribution* [CT96]. Suprisingly, this distribution appears in branches of probability that at first sight seem unrelated. For instance, the length of the longest increasing subsequence in a permutation that is chosen uniformly at random in the limit follows the Tracy-Widom distribution [JB99]. In the survey article [CAT02] Tracy and Widom give an overview of appearances of the distribution in growth processes, random tilings, statistics, queuing theory and superconductors. The present article adds spectrehedra to that list.

3. Expected volume of the spectrahedron

3.1. **Proof of Theorem 1.1 (1).** Note that due to the rotational invariance of the standard Gaussian distribution $N(0,1)$ the volume of a spectrahedron $\mathscr{S}_{\ell,n}$ can be computed as follows:

$$|\mathscr{S}_{\ell,n}|_{\mathrm{rel}} = \frac{|\mathscr{S}_{\ell,n}|}{|S^\ell|} = \operatorname*{Prob}_{\xi_0,\ldots,\xi_\ell \overset{\mathrm{iid}}{\sim} N(0,1)} \left\{ \xi_0\,\mathbb{1} + \frac{1}{\sqrt{2n\ell}}\sum_{i=1}^{\ell} \xi_i Q_i \succ 0 \right\} \tag{3.1}$$

Using this and the following shorthand notation

$$Q(x) = \sum_{i=1}^{\ell} x_i Q_i, \quad A(x) = x_0\,\mathbb{1} + \tfrac{1}{\sqrt{2n\ell}} Q(x). \tag{3.2}$$

we now write the expectation $\mathbb{E}\,|\mathscr{S}_{\ell,n}|_{\mathrm{rel}}$ of the relative volume of the random spectrahedron as:

$$\mathbb{E}|\mathscr{S}_{\ell,n}|_{\mathrm{rel}} = \operatorname*{\mathbb{E}}_{\mathcal{Q}\in \mathrm{GOE}(n)^\ell} \operatorname*{Prob}_{\xi_0,\ldots,\xi_\ell \overset{\mathrm{iid}}{\sim} N(0,1)} \left\{ \xi_0\,\mathbb{1} + \frac{1}{\sqrt{2n\ell}}\sum_{i=1}^{\ell} \xi_i Q_i \succ 0 \right\} = \operatorname*{\mathbb{E}}_{\mathcal{Q}} \operatorname*{\mathbb{E}}_{\xi} \mathbf{1}_{\{A(\xi)\succ 0\}} =: (\star)$$

where $\mathbf{1}_Y$ denotes the characteristic function of the set Y. Using Tonelli's theorem the two integrations can be exchanged:

$$(\star) = \operatorname*{\mathbb{E}}_{\xi} \operatorname*{\mathbb{E}}_{\mathcal{Q}} \mathbf{1}_{\{A(\xi)\succ 0\}} = \operatorname*{\mathbb{E}}_{\xi} \operatorname*{Prob}_{\mathcal{Q}\in \mathrm{GOE}(n)^\ell} \left\{ \xi_0\,\mathbb{1} + \frac{1}{\sqrt{2n\ell}} Q(\xi) \succ 0 \right\}. \tag{3.3}$$

For a unit vector $x = (x_1, \dots, x_\ell) \in S^{\ell-1}$ by the orthogonal invariance of the GOE-ensemble we have $Q(x) \sim \mathrm{GOE}(n)$ which leads to

$$(\star) = \mathop{\mathbb{E}}_{\xi} \mathop{\mathrm{Prob}}_{Q\in\mathrm{GOE}(n)} \left\{ \frac{\xi_0}{(\xi_1^2+\cdots+\xi_\ell^2)^{\frac{1}{2}}} \mathbb{1} + \frac{1}{\sqrt{2n\ell}} Q \succ 0 \right\}$$
$$= \mathop{\mathbb{E}}_{Q\in\mathrm{GOE}(n)} \mathop{\mathrm{Prob}}_{\xi} \left\{ \frac{\xi_0\sqrt{\ell}}{(\xi_1^2+\cdots+\xi_\ell^2)^{\frac{1}{2}}} \mathbb{1} + \frac{1}{\sqrt{2n}} Q \succ 0 \right\},$$

where in the second equality we again used Tonelli's theorem. Let us put $t_\ell := \frac{\xi_0\sqrt{\ell}}{(\xi_1^2+\cdots+\xi_\ell^2)^{\frac{1}{2}}}$. Note that by [NLJ95, (28.1)] the random variable t_ℓ follows the Student's t-distribution with ℓ degrees of freedom. Since this distribution is symmetric around the origin and $t_\ell \mathbb{1} + \frac{1}{\sqrt{2n}} Q \succ 0$ is equivalent to $-t_\ell \le \frac{1}{\sqrt{2n}} \lambda_{\min}(Q)$, we have

$$(\star) = \mathop{\mathbb{E}}_{Q\in\mathrm{GOE}(n)} \mathop{\mathrm{Prob}}_{t_\ell} \left\{ t_\ell \le \frac{1}{\sqrt{2n}} \lambda_{\min}(Q) \right\} = \mathop{\mathbb{E}}_{Q\in\mathrm{GOE}(n)} F_\ell\big(\tilde{\lambda}_{\min}(Q)\big),$$

where F_ℓ is the cumulative distribution function of the random variable t_ℓ. This proves Theorem 1.1 (1) since $(\star) = \mathbb{E}\,|\mathscr{S}_{\ell,n}|_{\mathrm{rel}}$. □

3.2. **Proof of Theorem 1.1 (2).** The random variable t_ℓ is absolutely continuous. Moreover, its density F_ℓ' is continuous and bounded uniformly in ℓ [NLJ95, (28.2)]. This combined with the following lemma proves Theorem 1.1 (2):

Lemma 3.1. *Let $f_\ell : \mathbb{R} \to \mathbb{R}$ be a sequence of smooth functions such that there exists a constant $c > 0$ with $\|f_\ell'\|_\infty \le c$, $\ell \ge 1$. Then $\mathbb{E}_{Q\in\mathrm{GOE}(n)}\, f_\ell\big(\tilde{\lambda}_{\min}(Q)\big) = f_\ell(-1) + \mathcal{O}(n^{-2/3})$ uniformly in ℓ.*

Proof. Write $f_\ell(\tilde{\lambda}_{\min})$ as $f_\ell(\tilde{\lambda}_{\min}) = f_\ell(-1) + f_\ell'(x)(\tilde{\lambda}_{\min}+1)$ for some $x = x(\tilde{\lambda}_{\min}) \in \mathbb{R}$. Taking expectation we obtain

$$\mathbb{E}\, f_\ell(\tilde{\lambda}_{\min}) = f_\ell(-1) + \mathbb{E}(f_\ell'(x)(\tilde{\lambda}_{\min}+1)) \le f_\ell(-1) + c\,\mathbb{E}\,|\tilde{\lambda}_{\min}+1| = f_\ell(-1) + \mathcal{O}(n^{-3/2})$$

where the last inequality follows from Proposition 2.2. □

3.3. **Proof of Theorem 1.1 (3).** In the preceding subsection we have shown the (uniform in ℓ) asymptotic $\mathbb{E}|\Sigma_{\ell,n}|_{\mathrm{rel}} = F_\ell(-1) + \mathcal{O}(n^{-\frac{2}{3}})$, where $F_\ell(x) = \mathrm{Prob}\,\{t_\ell \le x\}$ and the random variable t_ℓ follows the student's t-distribution with ℓ degrees of freedom. By [NLJ95, (28.15)] for fixed x we have $F_\ell(x) = \Phi(x)(1 + \mathcal{O}(\ell^{-1}))$. where Φ is the cumulative distribution function of the standard normal distribution. Plugging in $x = -1$ settles Theorem 1.1 (3). □

4. The average number of singular points

For the study of the average number of singular points on the boundary of a random spectrahedron and on a symmetroid surface we will rely on the following proposition, which implies that this number is finite.

Proposition 4.1. *Let $\mathscr{S}_{\ell,n}^{(k)}$ be the set of matrices of corank k in the spectrahedron $\mathscr{S}_{\ell,n}$ and $\Sigma_{\ell,n}^{(k)}$ the set of matrices of corank k in the symmetroid hypersurface $\Sigma_{\ell,n}$. For a generic choice of $\mathcal{R} = (R_1, \dots, R_\ell) \in \mathrm{Sym}(n, \mathbb{R})^\ell$ the sets $\mathscr{S}_{\ell,n}^{(k)}, \Sigma_{\ell,n}^{(k)} \subset S^\ell$ are semialgebraic of codimension $\binom{k+1}{2}$.*

Proof. In the space $\mathrm{Sym}(n, \mathbb{R})$ consider the semialgebraic stratification given by the corank: $\mathrm{Sym}(n, \mathbb{R}) = \coprod_{k=0}^n \mathcal{Z}^{(k)}$, where $\mathcal{Z}^{(k)}$ denotes the set matrices of corank k, and the induced stratification on the cone $\mathcal{P}_n$ of positive semidefinite matrices $\mathcal{P}_n = \coprod_{k=0}^n (\mathcal{Z}^{(k)} \cap \mathcal{P}_n)$. These are Nash stratifications [AA12, Proposition 9] and the codimensions of both $\mathcal{Z}^{(k)}$ and $\mathcal{Z}^{(k)} \cap \mathcal{P}_n$ are equal to $\binom{k+1}{2}$.

Consider now the semialgebraic map

$$F : S^\ell \times (\mathrm{Sym}(n,\mathbb{R}))^\ell \to \mathrm{Sym}(n,\mathbb{R}),\ (x,\mathcal{R}) \mapsto x_0\mathbb{1} + x_1R_1 + \cdots + x_\ell R_\ell. \tag{4.1}$$

Then $\Sigma_{\ell,n}^{(k)} = \{x \in S^\ell \,|\, F(\mathcal{R},x) \in \mathcal{Z}^{(k)}\}$ and $\mathscr{S}_{\ell,n}^{(k)} = \{x \in S^\ell \,|\, F(\mathcal{R},x) \in \mathcal{Z}^{(k)} \cap \mathcal{P}_n\}$ and consequently they are semialgebraic.

We now prove that F is transversal to all the strata of these stratifications. Then the parametric transversality theorem [Hir94, Chapter 3, Theorem 2.7] will imply that for a generic choice of $\mathcal{R}$ the set $\mathscr{S}_{\ell,n}$ is stratified by the $\mathscr{S}_{\ell,n}^{(k)}$ and the same for the set $\Sigma_{\ell,n}$. To see that F is transversal to all the strata of the stratifications we compute its differential. At points $(x,\mathcal{R})$ with $x \neq e_0 = (1,0,\ldots,0)$ we have $D_{(x,\mathcal{R})}F(0,\dot{\mathcal{R}}) = x_1\dot{R}_1 + \cdots + x_\ell \dot{R}_\ell$ and the equation $D_{(x,\mathcal{R})}F(\dot{x},\dot{\mathcal{Q}}) = P$ can be solved by taking $\dot{x} = 0$ and $\dot{\mathcal{R}} = (0,\ldots,0,x_i^{-1}P,0,\ldots,0)$ where $x_i^{-1}P$ is in the i-th entry and i is such that $x_i \neq 0$ (in other words, already variations in $\mathcal{R}$ ensure surjectivity of $D_{(x,\mathcal{R})}F$). All points of the form $(e_0,\mathcal{R})$ are mapped by F to the identity matrix $\mathbb{1}$ which belongs to the open stratum $\mathcal{Z}^{(0)}$, on which transversality is automatic (because this stratum has full dimension). This concludes the proof. □

Proposition 4.2. *For generic $\mathcal{R} \in \mathrm{Sym}(n,\mathbb{R})^3$ the number of singular points ρ_n on the symmetroid $\Sigma_{3,n}$ and hence the number of singular points σ_n on $\partial\mathscr{S}_{3,n}$ is finite and satisfies*

$$\sigma_n \le \rho_n \le \frac{n(n+1)(n-1)}{3}.$$

Moreover, for any $n \ge 1$ there exists a generic symmetroid $\Sigma_{3,n}$ with $\rho_n = \frac{n(n+1)(n-1)}{3}$ singular points on it.

Proof. The fact that $\sigma_n \le \rho_n$ are generically finite follows from Proposition 4.1 with $k = 2$, as remarked before. Observe that ρ_n is bounded by twice (since $\Sigma_{3,n}$ is a subset of S^3) the number $\#\mathrm{Sing}(\Sigma_{3,n}^{\mathbb{C}})$ of singular points on the complex symmetroid projective surface

$$\Sigma_{3,n}^{\mathbb{C}} = \{x \in \mathbb{C}\mathrm{P}^3 |\, \det(x_0\mathbb{1} + x_1R_1 + x_2R_2 + x_3R_3)) = 0\}$$

Since $\mathrm{Sing}(\Sigma_{3,n}^{\mathbb{C}})$ is obtained as a linear section of the set $\mathcal{Z}_{\mathbb{C}}^{(2)}$ of $n \times n$ *complex* symmetric matrices of corank two (using similar transversality arguments as in Proposition 4.1) we have that generically $\#\mathrm{Sing}(\Sigma_{3,n}^{\mathbb{C}}) = \deg(\mathcal{Z}_{\mathbb{C}}^{(2)})$. The latter is equal to $\frac{n(n+1)(n-1)}{6}$; see [JH84].

Now comes the proof of the second claim, we are thankful to Bernd Sturmfels and Simone Naldi for helping us with this. For a generic collection of $n+1$ linear forms $L_1,\ldots,L_{n+1}$ in $\ell+1$ variables we denote by $p(x) := L_1(x)\cdots L_{n+1}(x)$ their product and by $P = \{x \in \mathbb{R}^{\ell+1} | L_i(x) > 0, i = 1,\ldots,n+1\}$ the polyhedral cone. Let $e \in \mathrm{int}(P)$ be any interior point of P. Then [San13, Thm 1.1] implies that the derivative $\langle \nabla p, e\rangle$ of p along the constant vector field $e \in \mathbb{R}^{\ell+1}$ is a hyperbolic polynomial in direction e and that the closure of the connected component of $\mathbb{R}^{\ell+1} \setminus \{\langle \nabla p, e\rangle = 0\}$ containing e is a spectrahedral cone. Let's consider the intersection of this spectrahedral cone with the generic linear 4-space $V \subset \mathbb{R}^{\ell+1}$ and denote by $\mathscr{S}_{3,n}, \Sigma_{3,n}$ the corresponding spectrahedron and its symmetroid surface respectively. It is straightforward to check that the triple intersections of the hyperplanes $L_1,\ldots,L_{n+1}$ when intersected with V produce $2\binom{n+1}{3} = \frac{(n+1)n(n-1)}{3}$ singular points on $\Sigma_{3,n}$. This completes the proof since the above number coincides with the complex bound. □

Now we prove Theorem 1.2 (1).

4.1. **Proof of Theorem 1.2 (1).** Recall that $\Delta_1 \subset S^{N-1} \subset \mathrm{Sym}(n,\mathbb{R})$ denotes the set of unit symmetric matrices such that $\lambda_1 = \lambda_2$. The ordered eigenvalues $\alpha_1(x) \le \cdots \le \alpha_n(x)$ of $A(x) = x_0\mathbb{1} + \frac{1}{\sqrt{2n\ell}}\sum_{i=1}^3 x_iQ_i$ satisfy

$$\alpha_i(x) = x_0 + \frac{1}{\sqrt{6n}}\mu_i(x),$$

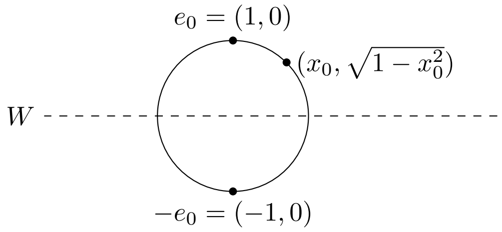

FIGURE 4.1. Two-dimensional depiction of $\mathbb{R}^4$: The north pole e_0 is identified with the unit matrix $\mathbb{1}$, while the south pole $-e_0$ is identified with $-\mathbb{1}$. By continuity there must be a point $(x_0, \sqrt{1-x_0^2})$ on the arc joining e_0 and $-e_0$ that has 0 as eigenvalue.

where $\mu_1(x) \le \ldots \le \mu_n(x)$ are the ordered eigenvalues of $Q(x) = \sum_{i=1}^3 x_i Q_i$. Then

$$\begin{aligned}\mathrm{Sing}(\partial\mathscr{S}_{3,n}) &= \{x \in S^3 \mid \alpha_1(x) = \alpha_2(x) = 0\}\\ &= \{x = (x_0, x_1, x_2, x_3) \in S^3 \mid \mu_1(x) = \mu_2(x) \text{ and } \mu_1(x) = -\sqrt{6n}x_0\}.\end{aligned}$$

Consider now the vector space:

$$W = \mathrm{span}\{Q_1, Q_2, Q_3\} \subset \mathrm{Sym}(n, \mathbb{R}).$$

Because $\mathcal{Q} = (Q_1, Q_2, Q_3) \sim \mathrm{GOE}(n)^3$, with probability one W is three-dimensional. In the sequel, for fixed $\mathcal{Q}$ we will naturally identify $\mathbb{R}^4$ with $V := \mathrm{span}\{\mathbb{1}, Q_1, Q_2, Q_3\}$ by using the isomorphism $A : \mathbb{R}^4 \to V, x \mapsto A(x)$.

By Proposition 4.1 (for $k = 2$) the set of symmetric matrices with repeated eigenvalues has codimension two in $\mathrm{Sym}(n, \mathbb{R})$. Hence, with probability one there are finitely many half lines $L_1, \ldots, L_s$ where $\mu_1(x) = \mu_2(x)$. Because on $\mathrm{span}\{\mathbb{1}, L_i\}$ we have $\alpha_1 = \alpha_2$, every singular point in $\partial\mathscr{S}_{3,n}$ maps into $\mathrm{span}\{\mathbb{1}, L_i\}$ for some $i \in \{1, \ldots, s\}$.

We examine now the condition for one of the half lines $L_1, \ldots, L_s$ actually contributing to a singular point. The construction we make for this is depicted in Figure 4.1. Let $V_0 := A(S^3)$ (an ellipsoid), and let $A(y)$ be the point of intersection of some fixed $L \in \{L_1, \ldots, L_s\}$ with V_0. Note that, because $L \subset W$, we have $A(y) \in W$, i.e., the first coordinate of y is zero. Moreover, by orthogonal invariance of the $\mathrm{GOE}(n)$ distribution we can assume that $y = (0, 1, 0, 0)$. The arc on the sphere S^3 through y joining the north pole $e_0 = (1, 0, 0, 0) \in S^3$ and the south pole $-e_0 = (-1, 0, 0, 0) \in S^3$ is then parametrized by $[-1, 1] \to S^3, x_0 \mapsto (x_0, \sqrt{1-x_0^2}, 0, 0)$. For the smallest eigenvalue we get

$$x_0 \mapsto \alpha_1(x_0) := x_0 + \frac{\sqrt{1-x_0^2}}{\sqrt{6n}}\,\lambda_1,$$

where λ_1 is the smallest eigenvalue of the first matrix Q_1. This function is strictly monotone with $\alpha_1(-1) = -1$ and $\alpha_1(1) = 1$. Hence it has exactly one zero, which correspond to a point on this arc where $\alpha_1 = \alpha_2 = 0$, i.e. a singular points.

The above considerations show that the number of singular points on $\partial\mathscr{S}_{3,n} = \partial\mathscr{S}_{3,n}(\mathcal{Q})$ equals the number s of half-lines on $W = \mathrm{span}\{Q_1, Q_2, Q_3\}$ that intersect Δ_1. Let us write $s(\mathcal{Q}) := s$ to put emphasis on its dependence on $\mathcal{Q}$. Observe that $s(\mathcal{Q})$ is equal to the number of lines (not half-lines) that intersect Δ_1, unless there exists a matrix in W, where both the two smallest and the two largest eigenvalues coincide. Since the latter is a non-generic property, $s(\mathcal{Q}) = \#(W \cap \Delta_1)$ with probability one. By the kinematic formula from [How93] we have:

$$\mathbb{E}\,\sigma_n = \int_{\mathrm{GOE}(n)^3} s(\mathcal{Q})\,\mathrm{d}\mathcal{Q} = \mathop{\mathbb{E}}_{W \in G(3, \mathrm{Sym}(n,\mathbb{R}))} \#(W \cap \Delta_1) = 2\frac{|\Delta_1|}{|S^{\frac{n(n+1)}{2}-3}|}$$

since the $\mathrm{GOE}(n)$ distribution of the Q_i induces the uniform distribution on W. This shows the assertion. $\square$

4.2. **Proof of Theorem 1.2 (2) and Proposition 1.3.** Recall that $\Delta \subset S^{N-1} \subset \mathrm{Sym}(n,\mathbb{R})$ denotes the set of unit symmetric matrices with repeated eigenvalues.

By the same argument as in the proof of Theorem 1.2 (1) singular points of the symmetroid $\Sigma_{3,n} = \Sigma_{3,n}(\mathcal{Q})$ are in one-to-one correspondence with matrices in $\mathrm{span}\{Q_1, Q_2, Q_3\} \cap \Delta$. Invoking again the kinematic formula from [How93] we obtain:

$$\mathbb{E}\,\rho_n = \underset{W\in G(3,\mathrm{Sym}(n,\mathbb{R}))}{\mathbb{E}} \#(W\cap\Delta) = 2\frac{|\Delta|}{|S^{\frac{n(n+1)}{2}-3}|}. \tag{4.2}$$

This finishes the proof of Theorem 1.2 and at the same time proves Proposition 1.3. □

5. The volume of the set of symmetric matrices with repeated eigenvalues

In this section we give a proof of Theorem 1.4. That is, we compute the volume of the set of symmetric matrices with repeated eigenvalues and of its subset consisting of matrices whose two smallest eigenvalues coincide. Before, however, we want to discuss some of it's consequences.

5.1. **The degree of the set of complex symmetric matrices with repeated eigenvalues.** Let us consider the complex hypersurface $\Delta^{\mathbb{C}} \subset \mathbb{P}\,\mathrm{Sym}(n,\mathbb{C})$ consisting of complex symmetric matrices with repeated eigenvalues. Its ideal is generated by the *discriminant* D of the characteristic polynomial of a matrix: $D(Q) = \prod_{i<j}(\lambda_i(Q)-\lambda_j(Q))^2$. The discriminant is a homogeneous polynomial of degree $n(n-1)$ in the entries of Q and it's a sum of squares of real polynomials [Par02, Chapter 2]. By definition of the degree for a generic line $L \subset \mathbb{P}\,\mathrm{Sym}(n,\mathbb{C})$ we have

$$\#\left(L\cap\mathbb{P}\Delta^{\mathbb{C}}\right) = \deg(\Delta^{\mathbb{C}}) = n(n-1).$$

On the other hand, from the integral geometry formula [How93] and Theorem 1.4 (3) it follows that

$$\mathbb{E}\,\#\left(W\cap\mathbb{P}\Delta\right) = \frac{n(n-1)}{2} = \frac{1}{2}\deg(\Delta^{\mathbb{C}})$$

where W is a uniformly distributed projective 2-plane in $\mathbb{P}\,\mathrm{Sym}(n,\mathbb{R})$.

Note, however, that since the real zero locus $\mathbb{P}\Delta \subset \mathbb{P}\,\mathrm{Sym}(n,\mathbb{R})$ of the discriminant hypersurface $\Delta^{\mathbb{C}} \subset \mathbb{P}\,\mathrm{Sym}(n,\mathbb{C})$ is of codimension two the degree $\deg(\Delta^{\mathbb{C}})$ does not give an upper bound on the number $\#(W\cap\mathbb{P}\Delta)$ of real symmetric matrices with repeated eigenvalues in a generic projective 2-plane W. Indeed, by Proposition 1.3 for generic $R_1, R_2, R_3 \in \mathrm{Sym}(n,\mathbb{R})$ matrices in $\mathbb{P}\,\mathrm{span}(R_1,R_2,R_3)) \cap \mathbb{P}\Delta$ are in one-to-one correspondence with singular points on the projective symmetroid $\mathbb{P}\Sigma_{3,n} = \{x \in \mathbb{R}\mathrm{P}^3 \mid \det(x_0\mathbb{1} + x_1R_1 + x_2R_2 + x_3R_3) = 0\}$. This together with Proposition 4.2 implies that the number $\#(W\cap\mathbb{P}\Delta)$ of matrices with repeated eigenvalues in the generic projective 2-plane W is bounded by $\binom{n+1}{3}$. Moreover, this bound is sharp: for some generic W we have $\#(W\cap\mathbb{P}\Delta) = \binom{n+1}{3}$ which is bigger than $\deg(\Delta^{\mathbb{C}}) = n(n-1)$ for $n > 5$.

The number of symmetric matrices with repeated eigenvalues in a generic projective 2-plane can be estimated using general Milnor-type bounds, i.e. estimates for the sum of Betti numbers of a real algebraic variety in terms of the degree of defining it polynomial. Indeed, for any 3-dimensional space $\tilde{W} \subset \mathrm{Sym}(n,\mathbb{R})$ the set $\tilde{W}\cap\Delta \subset S^3 = \tilde{W}\cap S^{N-1}$ is given by a homogeneous equation of degree $n(n-1)$. Therefore by [Ler16, Proposition 14] we have $b(\tilde{W}\cap\Delta) \leq 4n^4 + \mathcal{O}(n^3)$, where $b(\cdot)$ denotes the sum of Betti numbers with $\mathbb{Z}_2$ coefficients. In particular, when $\tilde{W}\cap\Delta$ is finite we have $\#(\tilde{W}\cap\Delta) = b(\tilde{W}\cap\Delta) \leq 4n^4 + \mathcal{O}(n^3)$. Hence, for generic projective 2-plane W general techniques give the fourth order bound $\#(W\cap\mathbb{P}\Delta) \leq 2n^4 + \mathcal{O}(n^3)$ whereas the above discussion shows that $\binom{n+1}{3}$ is the sharp cubic bound for $\#(W\cap\mathbb{P}\Delta)$.

5.2. **Proof of Theorem 1.4.** As before let us denote by $\lambda_1(Q) \le \cdots \le \lambda_n(Q)$ the ordered eigenvalues of a symmetric matrix $Q \in \mathrm{Sym}(n, \mathbb{R})$, by $N := \dim(\mathrm{Sym}(n,\mathbb{R})) = \frac{n(n+1)}{2}$ the dimension of the set of $n \times n$ real symmetric matrices and by S^{N-1} the unit sphere in the space $\mathrm{Sym}(n,\mathbb{R})$. Let Δ_j, $j = 1, \ldots, n-1$ denote the set of $n \times n$ real symmetric matrices of unit norm, whose j-th and $(j+1)$-th eigenvalues are equal:

$$\Delta_j := \{Q \in S^{N-1} \mid \lambda_j(Q) = \lambda_{j+1}(Q)\}, \quad j = 1, \ldots, n-1 \tag{5.1}$$

It is easily seen that Δ_j is a semialgebraic subset of S^N , and from Section 4 one can deduce that it is of codimension two. The smooth locus $(\Delta_j)_{sm}$ of Δ_j consists of matrices of unit norm whose j-th and $(j+1)$-th eigenvalues are equal and all other eigenvalues are of multiplicity one:

$$(\Delta_j)_{\mathrm{sm}} = \{Q \in S^{N-1} \mid \lambda_1(Q) < \cdots < \lambda_j(Q) = \lambda_{j+1}(Q) < \cdots < \lambda_n(Q)\} \tag{5.2}$$

Recall that Δ denotes the algebraic set of $n \times n$ real symmetric matrices of unit norm that have at least one repeated eigenvalue. Observe that Δ is a union of the sets Δ_j, $j = 1, \ldots, n-1$ and its smooth locus Δ_{sm} is a disjoint union of $(\Delta_j)_{sm}$, $j = 1, \ldots, n-1$. Furthermore, denote by Z_n the normalization constant for the density of eigenvalues of the GOE(n)-ensemble:

$$Z_n := \int_{\mathbb{R}^n} e^{-\frac{\|\lambda\|^2}{2}} |\Delta(\lambda)| \, \mathrm{d}\lambda$$

where $\Delta(\lambda) = \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)$ is the Vandermonde determinant. It is equal to

$$Z_n = \sqrt{2\pi}^{\,n} \prod_{i=1}^{n} \frac{\Gamma(1 + \frac{i}{2})}{\Gamma(\frac{3}{2})}, \tag{5.3}$$

see, e.g., [Meh91, (17.5.9) for $\gamma = \frac{1}{2}$]. The proof of Theorem 1.4 is based on the following proposition, which we prove in the subsequent subsection.

Proposition 5.1. *Let $1 \le j < n$. Then*

$$\frac{|\Delta_j|}{|S^{N-3}|} = \frac{4}{Z_n} \binom{n}{2} \binom{n-2}{j-1} \int_{\substack{\mu_1, \ldots, \mu_{j-1} < u \\ u < \mu_j, \ldots, \mu_{n-2}}} \prod_{i=1}^{n-2} (\mu_i - u)^2 \, e^{-\frac{\|\mu\|^2}{2} - u^2} |\Delta(\mu)| \, \mathrm{d}(\mu, u),$$

Before proving the proposition we first use it to finish the proof of Theorem 1.4:

Proof of Theorem 1.4. According to Proposition 5.1 we have

$$\frac{|\Delta_1|}{|S^{N-3}|} = \frac{4}{Z_n} \binom{n}{2} \int_{u < \mu_1, \ldots, \mu_{n-2}} \prod_{i=1}^{n-2} (\mu_i - u)^2 \, e^{-\frac{\|\mu\|^2}{2} - u^2} |\Delta(\mu)| \, \mathrm{d}(\mu, u).$$

Interpreting $\mu_1, \ldots, \mu_{n-2}$ as the eigenvalues of a GOE$(n-2)$ matrix we can rewrite this as follows:

$$\frac{|\Delta_1|}{|S^{N-3}|} = \frac{4 Z_{n-2}}{Z_n} \binom{n}{2} \int_{u \in \mathbb{R}} \mathop{\mathbb{E}}_{Q \sim \mathrm{GOE}(n-2)} \left[\mathbf{1}_{\{Q - u\mathbb{1} \succ 0\}} \det(Q - u\mathbb{1})^2 \right] e^{-u^2} \, \mathrm{d}u.$$

From (5.3) it's easy to see that $Z_n = 8\,\Gamma(\frac{n+1}{2})\Gamma(\frac{n+2}{2})\, Z_{n-2}$ or, using the duplication formula for Gamma function, $Z_n = 2^{-n+3}\sqrt{\pi}\, n!\, Z_{n-2}$. From this we get

$$\frac{|\Delta_1|}{|S^{N-3}|} = \frac{2^{n-1}}{\sqrt{\pi}\, n!} \binom{n}{2} \int_{u \in \mathbb{R}} \mathop{\mathbb{E}}_{Q \sim \mathrm{GOE}(n-2)} \left[\mathbf{1}_{\{Q - u\mathbb{1} \succ 0\}} \det(Q - u\mathbb{1})^2 \right] e^{-u^2} \, \mathrm{d}u. \tag{5.4}$$

which proves Theorem 1.4 (1).

For Theorem 1.4 (2) note that since $\Delta_{sm} = \cup_{j=1}^{n-1} (\Delta_j)_{sm}$ is a disjoint union we have that $|\Delta| = \sum_{j=1}^{n-1} |\Delta_j|$ and hence, by Proposition 5.1,

$$\frac{|\Delta|}{|S^{N-3}|} = \frac{4}{Z_n} \binom{n}{2} \sum_{j=1}^{n-1} \binom{n-2}{j-1} \int_{\substack{\mu_1, \ldots, \mu_{j-1} < u \\ u < \mu_j, \ldots, \mu_{n-2}}} \prod_{i=1}^{n-2} (\mu_i - u)^2 \, e^{-\frac{\|\mu\|^2}{2} - u^2} |\Delta(\mu)| \, \mathrm{d}(\mu, u),$$

This, together with the summation lemma [Bre17, Lemma E.3.5], gives

$$\frac{|\Delta|}{|S^{N-3}|} = \frac{4}{Z_n}\binom{n}{2}\int_{u\in\mathbb{R}}\int_{\mu\in\mathbb{R}^{n-2}}\prod_{i=1}^{n-2}(\mu_i-u)^2\, e^{-\frac{\|\mu\|^2}{2}-u^2}|\Delta(\mu)|\,\mathrm{d}\mu\mathrm{d}u,$$

Again, treating $\mu_1,\ldots,\mu_{n-2}$ as the eigenvalues of a GOE$(n-2)$ matrix and then proceeding as we did to get (5.4) we obtain

$$\frac{|\Delta|}{|S^{N-3}|} = \frac{2^{n-1}}{\sqrt{\pi}\,n!}\binom{n}{2}\int_{u\in\mathbb{R}}\mathop{\mathbb{E}}_{Q\sim\mathrm{GOE}(n-2)}\left[\det(Q-u\mathbb{1})^2\right]e^{-u^2}\,\mathrm{d}u. \tag{5.5}$$

This proves Theorem 1.4 (2).

Finally, Theorem A.1 entails that the integral on the right-hand-side of (5.5) is equal to $\frac{\sqrt{\pi}\,n!}{2^{n-1}}$, which implies that $\frac{|\Delta|}{|S^{N-3}|} = \binom{n}{2}$. This finishes the proof. □

5.3. **Proof of Proposition 5.1.** Recall that

$$(\Delta_j)_{\mathrm{sm}} = \{Q\in S^{N-1} \mid \lambda_1(Q)<\cdots<\lambda_j(Q)=\lambda_{j+1}(Q)<\cdots<\lambda_n(Q)\}.$$

In the following, we denote for brevity $\lambda_i := \lambda_i(Q)$. In order to compute $|\Delta_j| = |(\Delta_j)_{\mathrm{sm}}|$ define for $\delta>0$

$$X_j(\delta) := \{Q\in(\Delta_j)_{\mathrm{sm}} \mid \lambda_j-\lambda_{j-1}>\delta,\ \lambda_{j+2}-\lambda_{j+1}>\delta\} \tag{5.6}$$

Then $(\Delta_j)_{\mathrm{sm}} = \bigcup_{\delta>0}X_j(\delta)$ and by continuity of the Lebesgue measure

$$|\Delta_j| = \lim_{\delta\to 0}|X_j(\delta)|. \tag{5.7}$$

For a fixed $\delta>0$ and for any $\varepsilon>0$ let $T^{\perp}(X_j(\delta),\varepsilon)\subset S^{N-1}$ denote the ε-tube around $X_j(\delta)\subset S^{N-1}$. Weyl's formula [Wey39] gives the expansion of the volume of the ε-tube around a submanifold of the sphere. Here it is enough to have it in the following simplified form.

Theorem 5.2 (Weyl's tube formula for $X_j(\delta)$)**.** *For any $\varepsilon>0$, such that the fibres of $T^{\perp}(X_j(\delta),\varepsilon)$ do not intersect, the volume of the ε-tube around $X_j(\delta)$ is $|T^{\perp}(X_j(\delta),\varepsilon)| = \pi\varepsilon^2|X_j(\delta)| + \mathcal{O}(\varepsilon^3)$.*

In the lemma below we describe the ε-tube around $X_j(\delta)$ and show that for a sufficiently small $\varepsilon>0$ its fibers do not intersect. We postpone its proof to the next subsection.

Lemma 5.3. *For $0<\varepsilon<\arctan(\sqrt{2}\delta)$ we have*

$$T^{\perp}(X_j(\delta),\varepsilon) = \left\{ Q\in S^{N-1} \;\middle|\; \begin{array}{l} \lambda_1<\cdots<\lambda_j\le\lambda_{j+1}<\cdots<\lambda_n, \\ \lambda_{j+1}-\lambda_j<\sqrt{2}\sin\varepsilon, \\ \lambda_{j+2}-\dfrac{\lambda_j+\lambda_{j+1}}{2}>\delta\cos\varepsilon,\ \dfrac{\lambda_j+\lambda_{j+1}}{2}-\lambda_{j-1}>\delta\cos\varepsilon \end{array}\right\}$$

and the fibers of $T^{\perp}(X_j(\delta),\varepsilon)$ do not intersect.

Combining this lemma with Weyl's formula we are allowed to compute the volume of $X_j(\delta)$ as $|X_j(\delta)| = \lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}|T^{\perp}(X_j(\delta),\varepsilon)|$ and, consequently, by (5.7):

$$|\Delta_j| = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\,|T^{\perp}(X_j(\delta),\varepsilon)|. \tag{5.8}$$

To actually compute this limit, we will rewrite the volume of $T^{\perp}(X_j(\delta),\varepsilon)$ in terms of a GOE(n) random variable. The following formula gives the relation between the volume of a measurable set $E\subset S^{N-1}$ and the GOE(n)-measure of the homogeneous cone $C(E)=\{A\in \mathrm{Sym}(n,\mathbb{R}) \mid A/\|A\|\in E\}$ over E

$$\frac{|E|}{|S^{N-1}|} = \mathop{\mathrm{Prob}}_{Q\sim\mathrm{GOE}(n)} C(E).$$

Applying this to the measurable set $T^{\perp}(X_j(\delta),\varepsilon)$ we get

$$\frac{|T^{\perp}_{S^{N-1}}(X_j(\delta),\varepsilon)|}{|S^{N-1}|} = \operatorname*{Prob}_{Q\sim\mathrm{GOE}(n)}\left\{\begin{array}{l}\lambda_1<\cdots<\lambda_j\leq\lambda_{j+1}<\cdots<\lambda_n,\\ \lambda_{j+1}-\lambda_j<\sqrt{2}\|Q\|\sin\varepsilon,\\ \lambda_{j+2}-\dfrac{\lambda_j+\lambda_{j+1}}{2}>\delta\|Q\|\cos\varepsilon\\ \dfrac{\lambda_j+\lambda_{j+1}}{2}-\lambda_{j-1}>\delta\|Q\|\cos\varepsilon\end{array}\right\} = (\star) \tag{5.9}$$

In the following we denote the event

$$E(\lambda) := \left\{\begin{array}{l}\lambda_1<\cdots<\lambda_j\leq\lambda_{j+1}<\cdots<\lambda_n,\\ \lambda_{j+1}-\lambda_j<\sqrt{2}\|Q\|\sin\varepsilon,\\ \lambda_{j+2}-\dfrac{\lambda_j+\lambda_{j+1}}{2}>\delta\|Q\|\cos\varepsilon\\ \dfrac{\lambda_j+\lambda_{j+1}}{2}-\lambda_{j-1}>\delta\|Q\|\cos\varepsilon\end{array}\right\}.$$

Writing (5.9) in terms of the density of eigenvalues of the GOE(n) ensemble it becomes

$$(\star) = \frac{n!}{Z_n}\int_{\mathbb{R}^n}\mathbf{1}_{E(\lambda)}\, e^{-\frac{\|\lambda\|^2}{2}}|\Delta(\lambda_1,\ldots,\lambda_n)|\,\mathrm{d}\lambda,$$

see [Mui82, Theorem 3.2.17]. Here, $\Delta(\lambda_1,\ldots,\lambda_n)=\prod_{1\leq i<j\leq n}(\lambda_j-\lambda_i)$ is the Vandermonde determinant, $\mathbf{1}_{E(\lambda)}$ is the characteristic function of the set $E(\lambda)$, and Z_n is the normalization constant from (5.3). We express now the integral in terms of the following event:

$$\tilde{E}(\lambda) := \left\{\begin{array}{l}\lambda_1,\ldots,\lambda_{j-1}<\lambda_j,\lambda_{j+1}<\lambda_{j+1},\ldots,\lambda_n,\\ |\lambda_{j+1}-\lambda_j|<\sqrt{2}\|\lambda\|\sin\varepsilon,\\ \lambda_i-\dfrac{\lambda_j+\lambda_{j+1}}{2}>\delta\|\lambda\|\cos\varepsilon\ \text{ for } i\geq j+2,\\ \dfrac{\lambda_j+\lambda_{j+1}}{2}-\lambda_i>\delta\|\lambda\|\cos\varepsilon\ \text{ for } i\leq j-1\end{array}\right\}.$$

There are $(j-1)!$ possibilities to arrange the first $j-1$ eigenvalues, 2 possibilities to arrange λ_j and λ_{j+1} and $(n-(j+1))!$ possibilities to arrange the last $n-(j+1)$ eigenvalues. Hence,

$$\begin{aligned}(\star) &= \frac{n!}{Z_n}\,\frac{1}{2(j-1)!(n-(j+1))!}\int_{\mathbb{R}^n}\mathbf{1}_{\tilde{E}(\lambda)}\, e^{-\frac{\|\lambda\|^2}{2}}|\Delta(\lambda_1,\ldots,\lambda_n)|\,\mathrm{d}\lambda\\ &= \frac{1}{Z_n}\binom{n}{2}\binom{n-2}{j-1}\int_{\mathbb{R}^n}\mathbf{1}_{\tilde{E}(\lambda)}\, e^{-\frac{\|\lambda\|^2}{2}}|\Delta(\lambda_1,\ldots,\lambda_n)|\,\mathrm{d}\lambda\end{aligned}$$

Next, we perform the following orthogonal change of variables

$$\begin{aligned}&\mu_1:=\lambda_1,\ \ldots,\ \mu_{j-1}:=\lambda_{j-1},\ \mu_j:=\lambda_{j+2},\ \ldots,\ \mu_{n-2}:=\lambda_n \text{ and}\\ &x=\frac{\lambda_j+\lambda_{j+1}}{\sqrt{2}},\ y=\frac{\lambda_{j+1}-\lambda_j}{\sqrt{2}}\end{aligned}$$

($\mu_1,\ldots,\mu_{n-2}$ now become the eigenvalues of a new GOE($n-2$) matrix and we treat the variables x,y separately). We get

$$(\star) = \frac{1}{Z_n}\binom{n}{2}\binom{n-2}{j-1}\int_{(\mu,x,y)\in\mathbb{R}^{n-2}\times\mathbb{R}\times\mathbb{R}}\mathbf{1}_{\widehat{E}(\mu,x,y)}\,g(\mu,x,y)e^{-\frac{\|\mu\|^2+y^2+x^2}{2}}|\Delta(\mu)|\,\mathrm{d}(\mu,x,y) \tag{5.10}$$

where

$$g(\mu,x,y)=\sqrt{2}|y|\prod_{i=1}^{n-2}\left(\left(\mu_i-\frac{x}{\sqrt{2}}\right)^2-\frac{y^2}{2}\right) \tag{5.11}$$

and

$$\widehat{E}(\mu,x,y) := \left\{ \begin{matrix} \mu_1,\dots,\mu_{j-1} < \frac{1}{\sqrt{2}}(x-y),\, \frac{1}{\sqrt{2}}(x+y) < \mu_j,\dots,\mu_{n-2}, \\ |y| < \|(\mu,x,y)\| \sin\varepsilon, \\ \mu_i - \dfrac{x}{\sqrt{2}} > \delta\|(\mu,x,y)\|\cos\varepsilon \ \text{ for } i \ge j, \\ \dfrac{x}{\sqrt{2}} - \mu_i > \delta\|(\mu,x,y)\|\cos\varepsilon \ \text{ for } i \le j-1 \end{matrix} \right\}.$$

We perform another change of varables:

$$\begin{aligned} t &= \tfrac{y}{\sin\varepsilon\,\|(\mu,x,y)\|} \quad \mathrm{d}y = \frac{\sin\varepsilon\|(\mu,x)\|}{(1-(\sin\varepsilon)^2t^2)^{3/2}} \\ x,\mu_1,&\dots,\mu_{n-2} \quad \text{are as before} \end{aligned} \tag{5.12}$$

Note that after this change a factor of $(\sin\varepsilon)^2$ appears and the function $y(t,x,\mu,\varepsilon)\to 0$ in the limits $\varepsilon\to 0$. We multiply the integral in (5.10) by $\frac{1}{\pi\varepsilon^2}$ and, thereafter, invoke the dominated convergence theorem that allows us to pass to the limit $\varepsilon\to 0$ under the integral:

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\int_{(\mu,x,y)\in\mathbb{R}^{n-2}\times\mathbb{R}\times\mathbb{R}} \mathbf{1}_{\widehat{E}(\mu,x,y)} g(\mu,x,y) e^{-\frac{\|\mu\|^2+y^2+x^2}{2}} |\Delta(\mu)|\,\mathrm{d}(\mu,x,y) \\ =&\frac{\sqrt{2}}{\pi}\int_{(\mu,x)\in\mathbb{R}^{n-2}\times\mathbb{R}}\int_{t=-1}^{1}\mathbf{1}_{\bar{E}(\mu,x)}|t|\|\mu,x\|^2\prod_{i=1}^{n-2}\left(\mu_i-\frac{x}{\sqrt{2}}\right)^2 e^{-\frac{\|\mu\|^2+x^2}{2}}|\Delta(\mu)|\,\mathrm{d}t\,\mathrm{d}(\mu,x), \end{aligned}$$

where

$$\bar{E}(\mu,x) := \left\{ \begin{matrix} \mu_1,\dots,\mu_{j-1} < \frac{x}{\sqrt{2}} < \mu_j,\dots,\mu_{n-2}, \\ \mu_i - \dfrac{x}{\sqrt{2}} > \delta\|(\mu,x)\| \ \text{ for } i\ge j, \\ \dfrac{x}{\sqrt{2}} - \mu_i > \delta\|(\mu,x)\| \ \text{ for } i \le j-1 \end{matrix} \right\}.$$

Using that $\int_{t=-1}^{1}|t|\,\mathrm{d}t = 1$ we have

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\,|T^\perp(X_j(\delta),\varepsilon)| \\ =&\frac{\sqrt{2}|S^{N-1}|}{\pi Z_n}\binom{n}{2}\binom{n-2}{j-1}\int_{\mathbb{R}^{n-2}\times\mathbb{R}}\mathbf{1}_{\bar{E}(\mu,x)}\|\mu,x\|^2\prod_{i=1}^{n-2}\left(\mu_i-\frac{x}{\sqrt{2}}\right)^2 e^{-\frac{\|\mu\|^2+x^2}{2}}|\Delta(\mu)|\,\mathrm{d}(\mu,x), \end{aligned}$$

Plugging this into (5.8) and again using the dominated convergence theorem we get

$$\begin{aligned} |\Delta_j| &= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\,|T^\perp(X_j(\delta),\varepsilon)| \\ &= \frac{\sqrt{2}|S^{N-1}|}{\pi Z_n}\binom{n}{2}\binom{n-2}{j-1}\int_D\|\mu,x\|^2\prod_{i=1}^{n-2}\left(\mu_i-\frac{x}{\sqrt{2}}\right)^2 e^{-\frac{\|\mu\|^2+x^2}{2}}|\Delta(\mu)|\,\mathrm{d}(\mu,x), \end{aligned}$$

where the region of integration is $D=\{\mu_1,\dots,\mu_{j-1}<\frac{x}{\sqrt{2}}<\mu_j,\dots,\mu_{n-2}\}$. Next, we use the fact that for a homogeneous function f of degree d in m variables one has

$$\int_{x\in\mathbb{R}^m}\|x\|^2 f(x)e^{-\frac{\|x\|^2}{2}}\,\mathrm{d}x = (d+m)\int_{x\in\mathbb{R}^m} f(x)e^{-\frac{\|x\|^2}{2}}\,\mathrm{d}x$$

(the proof is straightforward using polar coordinates). In our case, we have $m=n-1$ and $d=2(n-2)+\frac{(n-2)(n-3)}{2}=\frac{(n-2)(n+1)}{2}$. Thus $d+m=\frac{n^2+n-4}{2}$ and

$$|\Delta_j| = \frac{(n^2+n-4)|S^{N-1}|}{\sqrt{2}\,\pi Z_n}\binom{n}{2}\binom{n-2}{j-1}\int_D\prod_{i=1}^{n-2}\left(\mu_i-\frac{x}{\sqrt{2}}\right)^2 e^{-\frac{\|\mu\|^2+x^2}{2}}|\Delta(\mu)|\,\mathrm{d}(\mu,x).$$

Finally, we make a change of variables $u := \frac{x}{\sqrt{2}}$ and use $(n^2+n-4)|S^{N-1}| = 4\pi|S^{N-3}|$ to conclude that

$$\frac{|\Delta_j|}{|S^{N-3}|} = \frac{4}{Z_n}\binom{n}{2}\binom{n-2}{j-1}\int_{\substack{\mu_1,\ldots,\mu_{j-1}<u\\ u<\mu_j,\ldots,\mu_{n-2}}}\prod_{i=1}^{n-2}(\mu_i-u)^2\, e^{-\frac{\|\mu\|^2}{2}-u^2}|\Delta(\mu)|\,\mathrm{d}(\mu,u).$$

This completes the proof of Proposition 5.1. $\square$

5.4. **Proof of Lemma 5.3.** We can assume without loss of generality that $Q \in X_j(\delta)$ is diagonal: $Q = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\lambda_1 < \cdots < \lambda_j = \lambda_{j+1} < \cdots < \lambda_n$. Then, the fiber $N_Q \subset T_Q S^{N-1}$ of the normal bundle to $X_j(\delta) \subset S^{N-1}$ at Q is described as follows. For $a, b \in \mathbb{R}$ let $V_{a,b} = (v_{i,j}) \in \mathrm{Sym}(n, \mathbb{R})$ be the matrix that has zeros everywhere except for the following block on the diagonal: $\left(\begin{smallmatrix} v_{j,j} & v_{j,j+1} \\ v_{j+1,j} & v_{j+1,j+1}\end{smallmatrix}\right) = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} a & b \\ b & -a\end{smallmatrix}\right)$. Note that $V_{a,b} \in S^{N-1}$ if and only if $a^2 + b^2 = 1$. We claim that

$$N_Q = \{V_{a,b} \mid a, b \in \mathbb{R}\}. \tag{5.13}$$

It is easy to see that $V_{a,b}$ is orthogonal to Q, i.e., $V_{a,b} \in T_Q S^{N-1}$, and that the tangent space $T_Q X_j(\delta) \subset T_Q S^{N-1}$ to $X_j(\delta)$ at Q is spanned by the following $\binom{n+1}{2} - 3$ vectors:

$$\begin{aligned}
&\mathrm{diag}(\lambda_2 e_1 - \lambda_1 e_2),\\
&\quad\vdots\\
&\mathrm{diag}(\lambda_{j-1} e_{j-2} - \lambda_{j-2} e_{j-1})\\
&\mathrm{diag}(\lambda_{j+2} e_{j-1} - \lambda_{j-1} e_{j+2}).\\
&\mathrm{diag}(\lambda_{j+3} e_{j+2} - \lambda_{j+2} e_{j+3}).\\
&\quad\vdots\\
&\mathrm{diag}(\lambda_n e_{n-1} - \lambda_{n-1} e_n),
\end{aligned}$$

and

$$\mathrm{diag}\big(-2\lambda_j \sum_{i\neq j,j+1} \lambda_i e_i + \sum_{i\neq j,j+1} \lambda_i^2 (e_j + e_{j+1})\big),$$

and

$$\begin{array}{c|c}
 & \begin{array}{ccccc} & r & & s & \end{array} \\
\begin{array}{c} \\ r \\ \\ s \\ \\ \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \ldots & 0 & \ldots & \lambda_s - \lambda_r & \ldots \\ & \vdots & & \vdots & \\ \ldots & \lambda_s - \lambda_r & \ldots & 0 & \ldots \\ & \vdots & & \vdots & \end{pmatrix}
\end{array}, \; r, s = 1, \ldots n, r \neq s, \{r, s\} \neq \{j, j+1\}.$$

It is immediate to see that these vectors are all orthogonal to $V_{a,b}$. Thus, $N_Q = \{V_{a,b} \,|\, a, b \in \mathbb{R}\}$.

Now we prove that $T^\perp(X_j(\delta), \varepsilon)$ has the asserted form and that the fibers of the normal ε-tube $T^\perp(X_j(\delta), \varepsilon)$ do not intersect provided that $\varepsilon < \arctan(\sqrt{2}\delta)$. The fibers are swept out by geodesics of length less than ε starting at Q in the direction of some $V_{a,b} \in S^{N-1}$, in formulas:

$\{\cos t\, Q + \sin t\, V_{a,b} \mid 0 \le t < \varepsilon\}$. We write explicitly the matrix $\cos t\, Q + \sin t\, V_{a,b}$:

$$\begin{pmatrix} \lambda_1 \cos t &&&&&&& \\ & \ddots &&&&&& \\ && \lambda_{j-1}\cos t &&&&& \\ &&& \lambda_j \cos t + \frac{a}{\sqrt{2}}\sin t & \frac{b}{\sqrt{2}}\sin t &&& \\ &&& \frac{b}{\sqrt{2}}\sin t & \lambda_{j+1}\cos t - \frac{a}{\sqrt{2}}\sin t &&& \\ &&&&& \lambda_{j+2}\cos t && \\ &&&&&& \ddots & \\ &&&&&&& \lambda_n \cos t \end{pmatrix}$$

Provided that $\varepsilon < \arctan(\sqrt{2}\delta)$ the eigenvalues of this matrix are

$$\lambda_1 \cos t < \cdots < \lambda_{j-1}\cos t < \lambda_j \cos t \pm \frac{\sin t}{\sqrt{2}} < \lambda_{j+2}\cos t < \cdots < \lambda_n \cos t \tag{5.14}$$

since $Q \in X_j(\delta)$ (see (5.6)). Moreover, for $0 \le t < \varepsilon$ these eigenvalues satisfy the inequalities

$$\begin{aligned} &\frac{(\lambda_j \cos t + \frac{\sin t}{\sqrt{2}}) + (\lambda_j \cos t - \frac{\sin t}{\sqrt{2}})}{2} - \lambda_{j-1}\cos t = (\lambda_j - \lambda_{j-1})\cos t > \delta\cos\varepsilon, \\ &(\lambda_j \cos t + \frac{\sin t}{\sqrt{2}}) - (\lambda_j \cos t - \frac{\sin t}{\sqrt{2}}) = \sqrt{2}\sin t < \sqrt{2}\sin\varepsilon, \\ &\lambda_{j+2}\cos t - \frac{(\lambda_{j+1}\cos t + \frac{\sin t}{\sqrt{2}}) + (\lambda_{j+1}\cos t - \frac{\sin t}{\sqrt{2}})}{2} = (\lambda_{j+2} - \lambda_{j+1})\cos t > \delta\cos\varepsilon \end{aligned}$$

This shows that $T^\perp(X_j(\delta), \varepsilon)$ is contained in the set we claim it to be. To show the other inclusion let $A \in S^{N-1}$ be a matrix whose eigenvalues $\alpha_1 < \cdots < \alpha_{j-1} < \alpha_j \le \alpha_{j+1} < \alpha_{j+2} < \cdots < \alpha_n$ satisfy

$$\begin{aligned} \frac{\alpha_j + \alpha_{j+1}}{2} - \alpha_{j-1} &> \delta\cos\varepsilon, \\ \alpha_{j+1} - \alpha_j &< \sqrt{2}\sin\varepsilon, \\ \text{and} \quad \alpha_{j+2} - \frac{\alpha_j + \alpha_{j+1}}{2} &> \delta\cos\varepsilon. \end{aligned}$$

We can assume again that $A = \operatorname{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$ is diagonal. Let $0 \le t < \varepsilon$ be such that $\alpha_{j+1} - \alpha_j = \sqrt{2}\sin t$. One can easily verify that $A = \cos t\, Q + \sin t\, V_{-1,0}$ for

$$Q = \frac{1}{\cos t}\operatorname{diag}(\alpha_1, \dots, \alpha_{j-1}, \tfrac{1}{2}(\alpha_j + \alpha_{j+1}), \tfrac{1}{2}(\alpha_j + \alpha_{j+1}), \alpha_{j+2}, \dots, \alpha_n) \in X_j(\delta)$$

This implies that $A \in T^\perp(X_j(\delta), \varepsilon)$ and $T^\perp(X_j(\delta), \varepsilon)$ has the claimed form.

It remains to show that the fibers of the normal ε-tube $T^\perp(X_j(\delta), \varepsilon)$ do not intersect when $\varepsilon < \arctan(\sqrt{2}\delta)$. For this assume there is another representation $A = \cos\tilde{t}\, Q_0 + \sin\tilde{t}\, V$ of the matrix $A = \operatorname{diag}(\alpha_1, \dots, \alpha_n) \in T^\perp(X_j(\delta), \varepsilon)$, where $Q_0 \in X_j(\delta)$, $V \in N_{Q_0}$ and $0 \le \tilde{t} < \varepsilon$. We will prove that actually $Q_0 = Q, V = V_{-1,0}$ and $\tilde{t} = t$. To show this, we consider the diagonalization of Q_0; that is, $Q_0 = C_1^T Q_1 C_1$, where $Q_1 = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is diagonal and C_1 is orthogonal. We may assume $\lambda_1 < \cdots < \lambda_{j-1} < \lambda_j = \lambda_{j+1} < \lambda_{j+2} < \cdots < \lambda_n$. Note that the normal bundle N_{Q_0} to $X_j(\delta)$ at $Q_0 = C_1^T Q_1 C_1$ is given by $N_{C_1^T Q_1 C_1} = C_1^T N_{Q_1} C_1 = \{C_1^T V_{a,b} C_1 \mid a, b \in \mathbb{R}\}$. It follows that $V = C_1^T V_{a,b} C_1$ for some $a, b \in \mathbb{R}$ and we can write $A = C_1^T(\cos\tilde{t}\, Q_1 + \sin\tilde{t}\, V_{a,b})C_1$. Note that the eigenvalues of the inner matrix are given as in (5.14). Therefore, we can write

$A = C_1^T C_2^T Q_2 C_2 C_1$, where the orthogonal matrix C_2 commutes with Q_1 and

$$\begin{aligned} Q_2 &= \mathrm{diag}\left(\lambda_1 \cos\tilde{t}, \dots, \lambda_{j-1}\cos\tilde{t}, \lambda_j \cos\tilde{t} - \frac{\sin\tilde{t}}{\sqrt{2}}, \lambda_j \cos\tilde{t} + \frac{\sin\tilde{t}}{\sqrt{2}}, \lambda_{j+2}\cos\tilde{t}, \dots, \lambda_n \cos\tilde{t}\right) \\ &= \cos\tilde{t}\, Q_1 + \sin\tilde{t}\, V_{-1,0} \end{aligned}$$

The condition $\varepsilon < \arctan(\sqrt{2}\delta)$ together with $Q_1 \in X_j(\delta)$ ensures $\lambda_{j-1}\cos\tilde{t} < \lambda_j \cos\tilde{t} - \frac{\sin\tilde{t}}{\sqrt{2}}$ and $\lambda_j \cos\tilde{t} + \frac{\sin\tilde{t}}{\sqrt{2}} < \lambda_{j+2}\cos\tilde{t}$. Now since the diagonal matrices A and Q_2 both have ordered entries it follows that $C_2 C_1$ can be taken to be the identity matrix. Therefore $\alpha_i = \lambda_i \cos\tilde{t}$ for $i = 1, \dots, j-1, j+2, \dots, n$, and $\alpha_j = \lambda_j \cos\tilde{t} - \frac{\sin\tilde{t}}{\sqrt{2}}$ and $\alpha_{j+1} = \lambda_j \cos\tilde{t} + \frac{\sin\tilde{t}}{\sqrt{2}}$. It is straightforward now to see that $\tilde{t} = t$, $Q_0 = Q$ and $V = V_{-1,0}$. □

6. Proof of Proposition 1.6

The identity $\mathbb{E}\,\rho_4 = 12$ follows immediately from Theorem 1.2 and Theorem 1.4 for $n = 4$. For the other identity we apply again Theorem 1.2 and Theorem 1.4 and write:

$$\begin{aligned} \mathbb{E}\,\sigma_4 &= 2\frac{|\Delta_1|}{S^{N-3}} \\ &= \frac{2^4}{\sqrt{\pi}}\frac{1}{4!}\binom{4}{2}\int_{u\in\mathbb{R}} \mathop{\mathbb{E}}_{Q\sim\mathrm{GOE}(2)}\left[\mathbf{1}_{\{Q-u\mathbb{1}\succ 0\}}\det(Q-u\mathbb{1})^2\right]e^{-u^2}\,\mathrm{d}u \\ &= \frac{4}{\sqrt{\pi}}\int_{\mathbb{R}}\left(\frac{1}{Z_2}\int_{\mathbb{R}^2}(\lambda_1-u)^2(\lambda_2-u)^2|\lambda_1-\lambda_2|e^{-\frac{\lambda_1^2+\lambda_2^2}{2}}\mathbf{1}_{\{\lambda_1-u>0,\lambda_2-u>0\}}d\lambda_1 d\lambda_2\right)e^{-u^2}\,\mathrm{d}u \\ &= (\star). \end{aligned}$$

We apply now the change of variable $\alpha_1 = \lambda_1 - u$ and $\alpha_2 = \lambda_2 - u$ in the innermost integral, obtaining:

$$\begin{aligned} (\star) &= \frac{4}{\sqrt{\pi}}\int_{\mathbb{R}}\left(\frac{1}{Z_2}\int_{\mathbb{R}^2_+}(\alpha_1\alpha_2)^2|\alpha_1-\alpha_2|e^{-\frac{\alpha_1^2+\alpha_2^2}{2}}e^{-u^2-u(\alpha_1+\alpha_2)}\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\right)e^{-u^2}\,\mathrm{d}u \\ &= \frac{4}{\sqrt{\pi}}\frac{1}{Z_2}\int_{\mathbb{R}^2_+}(\alpha_1\alpha_2)^2|\alpha_1-\alpha_2|e^{-\frac{\alpha_1^2+\alpha_2^2}{2}}\left(\int_{\mathbb{R}}e^{-2u^2-u(\alpha_1+\alpha_2)}du\right)\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \\ &= \frac{4}{\sqrt{\pi}}\frac{\Gamma\left(\frac{3}{2}\right)}{2\pi}\int_{\mathbb{R}^2_+}(\alpha_1\alpha_2)^2|\alpha_1-\alpha_2|e^{-\frac{\alpha_1^2+\alpha_2^2}{2}}\left(\sqrt{\frac{\pi}{2}}e^{\frac{(\alpha_1+\alpha_2)^2}{8}}\right)\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^2_+}(\alpha_1\alpha_2)^2|\alpha_1-\alpha_2|e^{-\frac{\alpha_1^2+\alpha_2^2}{2}+\frac{(\alpha_1+\alpha_2)^2}{8}}\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \\ &= \frac{2}{\sqrt{2\pi}}\int_{\mathbb{R}^2_+\cap\{\alpha_1<\alpha_2\}}(\alpha_1\alpha_2)^2|\alpha_1-\alpha_2|e^{-\frac{\alpha_1^2+\alpha_2^2}{2}+\frac{(\alpha_1+\alpha_2)^2}{8}}\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2. \end{aligned}$$

In the last equality we have used the fact that the integrand is invariant under the symmetry $(\alpha_1, \alpha_2) \mapsto (\alpha_2, \alpha_1)$. Consider now the map $F : \mathbb{R}^2_+ \cap \{\alpha_1 < \alpha_2\} \to \mathbb{R}[x] \simeq \mathbb{R}^2$ given by

$$F(\alpha_1, \alpha_2) = (x-\alpha_1)(x-\alpha_2) = x^2 - (\alpha_1+\alpha_2)x + \alpha_1\alpha_2.$$

Essentially, F maps the pair (α_1, α_2) to a monic polynomial of degree two whose ordered roots are (α_1, α_2). Observe that F is injective on the region $\mathbb{R}^2_+ \cap \{\alpha_1 < \alpha_2\}$ with never-vanishing Jacobian $|JF(\alpha_1, \alpha_2)| = |\alpha_1 - \alpha_2|$. What is the image of F in the space of polynomials $\mathbb{R}[x]$? Denoting by a_1, a_2 the coefficients of a monic polynomial $p(x) = x^2 - a_1 x + a_2 \in \mathbb{R}[x]$, we see first that the conditions $\alpha_1, \alpha_2 > 0$ imply $a_1, a_2 > 0$. Moreover the polynomial $p(x) = F(\alpha_1, \alpha_2)$ has by construction real roots, hence its discriminant $a_1^2 - 4a_2$ must be positive. Viceversa, given (a_1, a_2) such that $a_1, a_2 > 0$ and $a_1^2 - 4a_2 > 0$, the roots of $p(x) = x^2 - a_1 x + a_2$ are real and positive. Hence, $F(\mathbb{R}^2_+ \cap \{\alpha_1 < \alpha_2\}) = \{(a_1,a_2) \in \mathbb{R}^2 \,|\, a_1,a_2 > 0, a_1^2 - 4a_2 > 0\}$. Thus we can write the above integral as

$$(\star) = \frac{2}{\sqrt{2\pi}} \int_0^\infty \int_0^{\frac{a_1^2}{4}} a_2^2 e^{-\frac{a_1^2-2a_2}{2}+\frac{a_1^2}{8}} \,\mathrm{d}a_2 \mathrm{d}a_1 = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{3a_1^2}{8}} \left(\int_0^{\frac{a_1^2}{4}} a_2^2 e^{a_2} da_2 \right) \mathrm{d}a_1$$

and performing elementary integration we obtain $(\star) = \mathbb{E}\,\sigma_4 = 6 - \frac{4}{\sqrt{3}}$. □

Appendix.

In this section we will give a proof for the following formula, that is needed in the proof of Theorem 1.4.

Theorem A.1. *For a fixed positive integer k we have*

$$\int_{u\in\mathbb{R}} \mathop{\mathbb{E}}_{Q\sim \mathrm{GOE}(k)} \det(Q - u\mathbb{1})^2 \, e^{-u^2} \,\mathrm{d}u = \sqrt{\pi}\, \frac{(k+2)!}{2^{k+1}}.$$

Before proving Theorem A.1 below we give definition of Hermite polynomials, recall their properties that we will need and then prove an auxiliary proposition.

A.1. **Hermite polynomials.** The *(physicist's) Hermite polynomials* $H_i(x)$, $i = 0,1,2,\dots$ form a family of orthogonal polynomials on the real line with respect to the measure $e^{-x^2}dx$. They are defined by

$$H_i(x) = (-1)^i e^{x^2} \frac{\mathrm{d}^i}{\mathrm{d}x^i} e^{-x^2}, \quad i \geq 0 \tag{A.1}$$

and satisfy

$$\int_{u\in\mathbb{R}} H_i(u) H_j(u) e^{-u^2} \,\mathrm{d}u = \begin{cases} 2^i i! \sqrt{\pi}, & \text{if } i = j \\ 0, & \text{else.} \end{cases} \tag{A.2}$$

A Hermite polynomial is either odd (if the degree is odd) or even (if the degree is even) function:

$$H_i(-x) = (-1)^i H_i(x); \tag{A.3}$$

and its derivative satisfies

$$H_i'(x) = 2i H_{i-1}(x) \tag{A.4}$$

(see [JS00, (24:5:1)], [IG15, (8.952.1)] for these properties).

A.2. **The expected value of the square of the characteristic polynomial of a GOE-matrix.** The following proposition is crucial for the proof of Theorem A.1.

Proposition A.2 (Expected value of the square of the characteristic polynomial)**.** *For a fixed positive integer k and a fixed $u \in \mathbb{R}$ the following holds.*

(1) If $k = 2m$ is even, then

$$\mathop{\mathbb{E}}_{Q\sim \mathrm{GOE}(k)} \det(Q - u\mathbb{1})^2 = \frac{(2m)!}{2^{2m}} \sum_{j=0}^{m} \frac{2^{-2j-1}}{(2j)!} \det X_j(u),$$

where

$$X_j(u) = \begin{pmatrix} H_{2j}(u) & H_{2j}'(u) \\ H_{2j+1}(u) - H_{2j}'(u) & H_{2j+1}'(u) - H_{2j}''(u) \end{pmatrix}.$$

(2) If $k = 2m+1$ is odd, then

$$\underset{Q\sim\mathrm{GOE}(k)}{\mathbb{E}} \det(Q - u\mathbb{1})^2 = \frac{\sqrt{\pi}(2m+1)!}{2^{4m+2}\,\Gamma(m+\frac{3}{2})} \sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!} \det Y_j(u),$$

where

$$Y_j(u) = \begin{pmatrix} \frac{(2j)!}{j!} & H_{2j}(u) & H'_{2j}(u) \\ 0 & H_{2j+1}(u) - H'_{2j}(u) & H'_{2j+1}(u) - H''_{2j}(u) \\ \frac{(2m+2)!}{(m+1)!} & H_{2m+2}(u) & H'_{2m+2}(u) \end{pmatrix}.$$

For the proof of Proposition A.2 we need the following lemma.

Lemma A.3. *Let $P_m = 2^{1-m^2}\sqrt{\pi}^m \prod_{i=0}^m (2i)!$ and let Z_{2m} denote the normalization constant from (5.3). Then $P_m = 2^{1-2m}\, Z_{2m}$.*

Proof. By (5.3) we have $Z_{2m} = (2\pi)^m \prod_{i=0}^{2m} \frac{\Gamma(\frac{i}{2}+1)}{\Gamma(\frac{3}{2})}$. Use the duplication formula for Gamma functions to transform this into

$$Z_{2m} = (2\pi)^m \prod_{i=0}^{m} \frac{\Gamma(\frac{2i-1}{2}+1)\Gamma(\frac{2i}{2}+1)}{\Gamma(\frac{3}{2})^2}.$$

Now we use the formula $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\Gamma(2z)\sqrt{\pi}$ [JS00, 43:5:7] with $z = \frac{2i-1}{2}$ to get

$$Z_{2m} = (2\pi)^m \prod_{i=0}^{m} \frac{2^{1-(2i-1)-2}\sqrt{\pi}\,\Gamma(2i-1+2)}{\Gamma(\frac{3}{2})} = (2\pi)^m \prod_{i=0}^{m} \frac{2^{-2i}\sqrt{\pi}\,\Gamma(2i+1)}{\Gamma(\frac{3}{2})}.$$

Using furthermore $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$ we get $Z_{2m} = 2^{2m-m^2}\,\sqrt{\pi}^m\,\prod_{i=0}^m (2i)!$. Therefore,

$$\frac{P_m}{Z_{2m}} = \frac{2^{1-m^2}\sqrt{\pi}^m \prod_{i=0}^m (2i)!}{2^{2m-m^2}\,\sqrt{\pi}^m\,\prod_{i=0}^m (2i)!} = 2^{1-2m}$$

as claimed. ☐

Now we prove Proposition A.2.

Proof of Proposition A.2. In [Meh91, Section 2.2] one finds two different formulas for the cases $k = 2m$ even and $k = 2m+1$ odd.

If $k = 2m$, we have by [Meh91, (2.2.38)] that

$$\mathbb{E}\det(Q - u\mathbb{1})^2 = \frac{(2m)!\,P_m}{Z_{2m}} \sum_{j=0}^{m} \frac{2^{2j-1}}{(2j)!} \det\begin{pmatrix} R_{2j}(u) & R'_{2j}(u) \\ R_{2j+1}(u) & R'_{2j+1}(u) \end{pmatrix},$$

where $P_m = 2^{1-m^2}\sqrt{\pi}^m \prod_{i=0}^m (2i)!$ is as in Lemma A.3, Z_{2m} is the normalization constant (5.3) and where $R_{2j}(u) = 2^{-2j}H_{2j}(u)$ and $R_{2j+1}(u) = 2^{-(2j+1)}(H_{2j+1}(u) - H'_{2j}(u))$. Using the multilinearity of the determinant we get

$$\mathbb{E}\det(Q - u\mathbb{1})^2 = \frac{(2m)!\,P_m}{Z_{2m}} \sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!} \det X_j(u).$$

By Lemma A.3 we have $\frac{P_m}{Z_{2m}} = 2^{1-2m}$. Putting everything together yields the first claim.

In the case $k = 2m+1$ we get from [Meh91, (2.2.38)] that

$$\mathbb{E}\det(Q - u\mathbb{1})^2 = \frac{(2m+1)!\,P_m}{Z_{2m+1}} \sum_{j=0}^{m} \frac{2^{2j-1}}{(2j)!} \det\begin{pmatrix} g_{2j} & R_{2j}(u) & R'_{2j}(u) \\ g_{2j+1} & R_{2j+1}(u) & R'_{2j+1}(u) \\ g_{2m+2} & R_{2m+2}(u) & R'_{2m+2}(u) \end{pmatrix},$$

where P_m, $R_{2j}(u), R_{2j+1}(u)$ are as above and

$$g_i = \int_{u\in\mathbb{R}} R_i(u)\exp(-\tfrac{u^2}{2})\, du.$$

Note that by (A.3) $H_{2j+1}(u)$ is an odd function. Hence, we have $g_{2j+1} = 0$. For even indices we use [IG15, (7.373.2)] to get $g_{2j} = 2^{-2j}\sqrt{2\pi}\frac{(2j)!}{j!}$. By the multilinearity of the determinant:

$$\mathbb{E}\det(Q - u\mathbb{1})^2 = \frac{\sqrt{2\pi}(2m+1)!\, P_m}{2^{2m+2} Z_{2m+1}} \sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!} \det Y_j(u). \tag{A.5}$$

By (5.3) we have $Z_{2m+1} = 2\sqrt{2}\,\Gamma(m+\frac{3}{2})\, Z_{2m}$. Using also Lemma A.3 we see that

$$\frac{P_m}{Z_{2m+1}} = \frac{2^{-2m}}{\sqrt{2}\,\Gamma(m+\frac{3}{2})}.$$

Plugging this into (A.5) we get

$$\mathbb{E}\det(Q - u\mathbb{1})^2 = \frac{\sqrt{\pi}(2m+1)!}{2^{4m+2}\,\Gamma(m+\frac{3}{2})} \sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!} \det Y_j(u).$$

This finishes the proof. □

A.3. **Proof of Theorem A.1.** Due to the nature of Proposition A.2 we also have to make a distinction in the proof of Theorem A.1.

In the case $k = 2m$ we use the formula from Proposition A.2 (1) to write

$$\int_{u\in\mathbb{R}} \mathbb{E}\det(Q - u\mathbb{1})^2 e^{-u^2}\, du = \frac{(2m)!}{2^{2m}} \sum_{j=0}^{m} \frac{2^{-2j-1}}{(2j)!} \int_{u\in\mathbb{R}} \det X_j(u)\, du.$$

By (A.4) we have $H_i'(u) = 2iH_{i-1}(u)$. Hence, $X_j(u)$ can be written as

$$\begin{pmatrix} H_{2j}(u) & 4jH_{2j-1}(u) \\ H_{2j+1}(u) - 4jH_{2j-1}(u) & 2(2j+1)H_{2j}(u) - 8j(2j-1)H_{2j-2}(u) \end{pmatrix}.$$

From (A.2) we can deduce that

$$\begin{aligned}\int_{u\in\mathbb{R}} \det X_j(u)\, du &= 2(2j+1)2^{2j}(2j)!\sqrt{\pi} + 16j^2 2^{2j-1}(2j-1)!\sqrt{\pi} \\ &= 2^{2j+1}(2j)!\sqrt{\pi}(4j+1).\end{aligned}$$

From this we see that

$$\sum_{j=0}^{m} \frac{2^{-2j-1}}{(2j)!} \int_{u\in\mathbb{R}} \det X_j(u)\, du = \sqrt{\pi}\sum_{j=0}^{m}(4j+1) = \sqrt{\pi}\,(m+1)(2m+1). \tag{A.6}$$

and hence,

$$\int_{u\in\mathbb{R}} \mathbb{E}\det(Q - u\mathbb{1})^2 e^{-u^2}\, du = \frac{(2m)!}{2^{2m}}\sqrt{\pi}\,(m+1)(2m+1) = \frac{(2m+2)!}{2^{2m+1}}\sqrt{\pi}.$$

Plugging back in $m = \frac{k}{2}$ finishes the proof of the case $k = 2m$.

In the case $k = 2m+1$ we use the formula from Proposition A.2 (2) to see that

$$\int_u \mathbb{E}\det(Q - u\mathbb{1})^2\, e^{-u^2}\, du = \frac{\sqrt{\pi}(2m+1)!}{2^{4m+2}\,\Gamma(m+\frac{3}{2})} \sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!} \int_u \det Y_j(u)\, e^{-u^2}\, du.$$

Note that the top right 2×2-submatrix of $Y_j(u)$ is $X_j(u)$, so that $\det Y_j(u)$ is equal to

$$\frac{(2m+2)!}{(m+1)!}\det X_j(u) + \frac{(2j)!}{j!}\det\begin{pmatrix} H_{2j+1}(u) - H_{2j}'(u) & H_{2j+1}'(u) - H_{2j}''(u) \\ H_{2m+2}(u) & H_{2m+2}'(u) \end{pmatrix}. \tag{A.7}$$

Because taking derivatives of Hermite polynomials decreases the index by one (A.4) and because the integral over a product of two Hermite polynomials is only non-vanishing, if their indices agree, the integral of the determinant in (A.7) is only non-vanishing for $j = m$, in which case it is equal to

$$\int_{u\in\mathbb{R}} H_{2m+1}(u)H'_{2m+2}(u)\, e^{-u^2}\, \mathrm{d}u = 2(2m+2)2^{2m+1}(2m+1)!\,\sqrt{\pi},$$

by (A.2) and (A.4). Hence,

$$\begin{aligned}&\int_{u\in\mathbb{R}} \det Y_j(u)\, e^{-u^2}\, \mathrm{d}u\\ =&\begin{cases} \frac{(2m+2)!}{(m+1)!}\int_{u\in\mathbb{R}} \det X_m(u)\, e^{-u^2}\, \mathrm{d}u + \frac{(2m)!}{m!}\, 2^{2m+2}(2m+2)!\sqrt{\pi}, & \text{if } j = m,\\ \frac{(2m+2)!}{(m+1)!}\int_{u\in\mathbb{R}} \det X_j(u)\, e^{-u^2}\, \mathrm{d}u, & \text{else.}\end{cases}\end{aligned}$$

We find that

$$\begin{aligned}&\sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!}\int_u \det Y_j(u)\, e^{-u^2}\, \mathrm{d}u\\ =&\frac{(2m+2)!}{m!}\sqrt{\pi} + \frac{(2m+2)!}{(m+1)!}\sum_{j=0}^{m}\frac{2^{-2j-2}}{(2j)!}\int_u \det X_j(u)\, e^{-u^2}\, \mathrm{d}u\\ =&\frac{(2m+2)!}{m!}\sqrt{\pi} + \frac{(2m+2)!}{(m+1)!}\,\frac{\sqrt{\pi}}{2}\,(m+1)(2m+1)\\ =&\frac{\sqrt{\pi}}{2}\,\frac{(2m+3)!}{m!};\end{aligned}$$

the second-to-last line by (A.6). It follows that

$$\begin{aligned}\int_{u\in\mathbb{R}} \mathbb{E}\det(Q-u\mathbb{1})^2\, e^{-u^2}\, \mathrm{d}u &= \frac{\sqrt{\pi}(2m+1)!}{2^{4m+2}\,\Gamma(m+\frac{3}{2})}\,\frac{\sqrt{\pi}}{2}\,\frac{(2m+3)!}{m!}\\ &= \frac{\pi(2m+1)!(2m+3)!}{2^{4m+3}\,\Gamma(m+\frac{3}{2})\, m!}.\end{aligned}$$

Using the duplication formula for Gamma function, it is not difficult to verify that the last term is $2^{-2m-2}\sqrt{\pi}\,(2m+3)!$. Substituting $2m+1 = k$ shows the assertion in this case.

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