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Real Space Sextics and their Tritangents

by

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ABSTRACT

The intersection of a quadric and a cubic surface in 3-space is a canonical curve of genus 4. It has 120 complex tritangent planes. We present algorithms for computing real tritangents, and we study the associated discriminants. We focus on space sextics that arise from del Pezzo surfaces of degree one. Their numbers of planes that are tangent at three real points vary widely; both 0 and 120 are attained. This solves a problem suggested by Arnold Emch in 1928.

1 INTRODUCTION

We present a computational study of canonical curves of genus 4 over the field $\mathbb{R}$ of real numbers. Such a curve C, provided it is smooth and non-hyperelliptic, is the complete intersection in $\mathbb{P}^3$ of a unique surface Q of degree two and a (non-unique) surface K of degree three. Conversely, any smooth complete intersection of a quadric and a cubic in $\mathbb{P}^3$ is a genus 4 curve. The degree of $C = Q \cap K$ is six: any plane in $\mathbb{P}^3$ meets C in six complex points, counting multiplicity. We refer to such a curve C as a *space sextic.*

Any space sextic C has at least 120 complex tritangent planes, one for each odd theta characteristic of C. If the quadric Q is smooth, then these 120 planes are exactly the tritangents [9, Theorem 2.2]. However, if Q is singular, then the curve C has infinitely many tritangents. We can see this as follows. Any plane H tangent to Q contains the singular point of Q, and it is tangent to Q at every point in the line $H \cap Q$. Since the intersection of H and C is contained in Q, the plane H is tangent to C at every point in $C \cap H$.

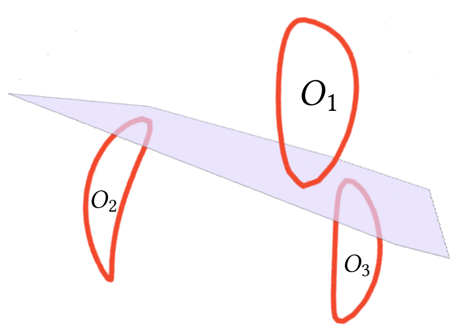

Figure 1: Totally real tritangent of a curve with three ovals. The plane touches O_1 on one side and O_2, O_3 on the other.

In what follows we focus on the case when the quadric surface Q containing the space sextic C is singular. We adopt the convention that a *tritangent* of C is one of the 120 complex planes corresponding to the odd theta characteristics of C. A tritangent is *real* if it is defined by a linear form with real coefficients. A real tritangent is *totally real* if it touches the curve C at three distinct real points.

A space sextic C has at most five ovals [9, §3], since the maximum number of ovals is the genus of C plus one. By [9, Proposition 3.1], all 120 tritangents of C are real if and only if the number of ovals of C attains this upper bound. A heuristic argument suggests that at least $80 = \binom{5}{3}\times 8$ of the 120 real tritangents are totally real, since eight planes can touch three ovals as in Figure 1. The analogous fact for genus three curves is true: a plane quartic with four ovals has 28 real bitangents, of which at least $24 = \binom{4}{2}\times 4$ are totally real. The situation is more complicated in genus 4, as seen in Figure 2.

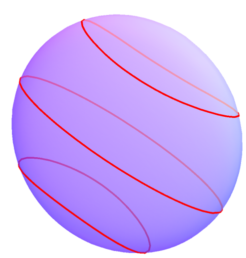

Figure 2: No tritangent touches all three ovals of this curve.

In 1928, Emch [6, §49] asked whether there exists a space sextic with all of its 120 tritangent planes totally real. He exhibited a curve suspected to attain the bound 120. However, ninety years later, Harris and Len [9, Theorem 3.2] showed that only 108 of the tritangents of Emch's curve are totally real. In [9, Question 3.3] they reiterated the question whether 120 totally real tritangents are possible. Our Example 2.2 answers that question affirmatively.

Theorem 1.1. *The number of totally real tritangents of a space sextic with five ovals can be any integer between* 84 *and* 120*. Each of these numbers is realized by an open semialgebraic set of such curves.*

This article is organized as follows. In Section 2 we construct space sextics associated with del Pezzo surfaces of degree one. These curves lie on a singular quadric Q and are obtained by blowing up eight points in the plane. This construction has the advantage of producing 120 rational tritangents when the points are rational. In Section 2 we also prove Theorem 1.1. In Section 3 we extend this construction to real curves obtained from complex configurations in $\mathbb{P}^2$ that are invariant under complex conjugation. Theorem 3.1 summarizes what we know about these special space sextics. In Section 4 we turn to arbitrary space sextics, where Q is now generally smooth, and we show how to compute the 120 tritangents of $C = Q \cap K$ directly from the equations defining Q and K. Section 5 offers a study of the discriminants associated with our polynomial system, and Section 6 sketches some directions for future research. Finally, the scripts used throughout this article are available at [12].

2 EIGHT POINTS IN THE PLANE

We shall employ the classical construction of space sextics from del Pezzo surfaces of degree one. We describe this construction below and direct the reader to [5, §8] or [10, §2] for further details. Any space sextic C that is obtained from this construction is special: the quadric Q that contains C is singular. See also [11], where these curves C are referred to as *uniquely trigonal genus* 4 *curves*.

Fix a configuration $\mathcal{P} = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\}$ of eight points in $\mathbb{P}^2_{\mathbb{R}}$. We may assume that $\mathcal{P}$ is sufficiently generic to allow for the choices to be made below. Additionally, genericity of $\mathcal{P}$ ensures that the resulting space sextic C is a smooth curve in $\mathbb{P}^3$. For practical computations we always choose points P_i whose coordinates are in the field $\mathbb{Q}$ of rational numbers. This ensures that each object arising in our computations is defined over $\mathbb{Q}$.

The space of ternary cubics that vanish on $\mathcal{P}$ is two-dimensional. We compute a basis $\{u, v\}$ for that space. The space of ternary sextics that vanish doubly on $\mathcal{P}$ is four-dimensional, and it contains the three-dimensional subspace spanned by $\{u^2, uv, v^2\}$. We augment this to a basis by another sextic w that vanishes to order two on $\mathcal{P}$.

The blow-up of $\mathbb{P}^2$ at the eight points in $\mathcal{P}$ is a *del Pezzo surface* $X_{\mathcal{P}}$ of degree one. Our basis $\{u^2, uv, v^2, w\}$ specifies a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ that is regular outside $\mathcal{P}$ and hence lifts to $X_{\mathcal{P}}$. This map is 2-to-1 and its image is the singular quadric $V(x_0x_2 - x_1^2)$. The ramification locus consists of two connected components, the isolated point $(0:0:0:1)$ and the intersection of the quadric $V(x_0x_2 - x_1^2)$ with a cubic C that is unique modulo $\langle x_0x_2 - x_1^2\rangle$.

Following [10, Example 2.5], we parametrize the singular quadric Q as $\{(1:t:t^2:W)\}$. This represents C by a polynomial in two unknowns (t, W) that has Newton polygon $\operatorname{conv}\{(0,0),(6,0),(0,3)\}$:

$$C: t^6+c_1t^5+c_2t^4W+c_3t^4+c_4t^3W+c_5t^2W^2+c_6t^3+c_7t^2W+ \\ c_8tW^2+c_9W^3+c_{10}t^2+c_{11}tW+c_{12}W^2+c_{13}t+c_{14}W+c_{15}. \tag{1}$$

We derive 120 tritangents of our curve C in $\mathbb{P}^3$ from the 240 exceptional curves on the del Pezzo surface $X_{\mathcal{P}}$ (cf. Lemma 2.1). There is an order two automorphism ι of $X_{\mathcal{P}}$, called the *Bertini involution*. The image of an exceptional curve C_1 under the Bertini involution ι is another exceptional curve $C_2 = \iota(C_1)$. If $\varphi\colon X_{\mathcal{P}} \to V(x_0x_2 - x_1^2)$ is the 2-to-1 covering branched along C, then $\varphi \circ \iota = \varphi$. In particular, $\varphi(C_1) = \varphi(C_2)$. The intersection $C_1 \cap C_2$ consists of three points on $X_{\mathcal{P}}$. Their image under φ is the triple of points at which the tritangent corresponding to $\{C_1, C_2\}$ touches C. We can thus decide whether a tritangent is totally real by checking whether the intersection $C_1 \cap C_2$ in $X_{\mathcal{P}}$ contains one or three real points. This intersection can be carried out in $\mathbb{P}^2$, as we shall explain next.

Recall that $X_{\mathcal{P}}$ is the blow-up of $\mathbb{P}^2$ at $\mathcal{P}$. By blowing down, we may view the eight exceptional fibers of the blow-up as the eight points of $\mathcal{P}$, and we may view the remaining 112 exceptional curves of $X_{\mathcal{P}}$ as (possibly singular) curves in $\mathbb{P}^2$. We can determine the images of the exceptional curves in $\mathbb{P}^2$ from [17, Table 1], as well as how they are matched into pairs $\{C_1, C_2\}$ via the Bertini involution:

- 8: The exceptional fiber at one point P_i matches the sextic vanishing triply at P_i and doubly at the other seven points. The three components of the tangent cone of this sextic determine the three desired points on the branch curve C.
- 28: The line through P_i and P_j matches the quintic vanishing at all eight points and doubly at the six points in $\mathcal{P}\backslash\{P_i, P_j\}$. Their intersection in $\mathbb{P}^2 \setminus \mathcal{P}$ consists of three complex points. Either one or three of them are real (see Figure 3).

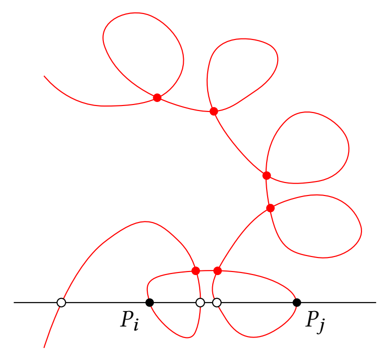

Figure 3: $\mathcal{P}$ determines 28 lines meeting a rational quintic

- 56: The conic through $P_{i_1}, \ldots, P_{i_5}$ matches the quartic vanishing at $\mathcal{P}$ and doubly at the three other points. Their intersection in $\mathbb{P}^2 \setminus \mathcal{P}$ consists of three complex points (see Figure 4).

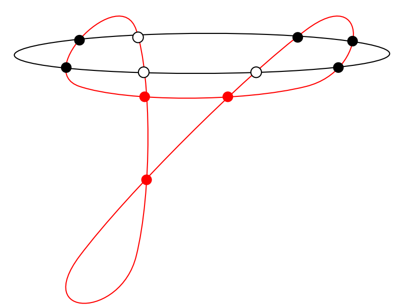

Figure 4: $\mathcal{P}$ determines 56 conics meeting a rational quartic

- 56/2: For two points P_i and P_j, the cubic vanishing doubly at P_i, non-vanishing at P_j, and vanishing singly at $\mathcal{P}\backslash\{P_i, P_j\}$ matches the cubic vanishing doubly at P_j, non-vanishing at P_i, and vanishing singly at $\mathcal{P}\backslash\{P_i, P_j\}$. Their intersection in $\mathbb{P}^2\backslash\mathcal{P}$ consists of three points in $\mathbb{P}^2$ (see Figure 5).

The following lemma summarizes the reality issues on the del Pezzo surface $X_{\mathcal{P}}$ that arises from the constructions in $\mathbb{P}^2$ described above.

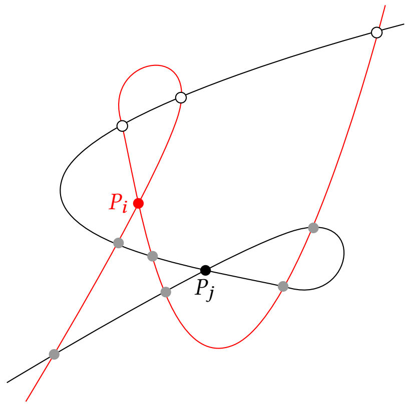

Figure 5: $\mathcal{P}$ determines $56/2$ **pairs of rational cubics**

Lemma 2.1. *Let $\{C_1, C_2\}$ be a pair of exceptional curves of type 8, 28, 56 or 56/2 contained in the del Pezzo surface $X_{\mathcal{P}}$ Then $\varphi(C_1 \cap C_2)$ spans a tritangent plane of the space sextic C in $\mathbb{P}^3$. That tritangent is totally real if and only if the intersection $C_1 \cap C_2$ is real on $X_{\mathcal{P}}$.*

Proof. Let $-K$ be the anticanonical divisor class of $X_{\mathcal{P}}$. Then $-K$ and $-2K$ are ample but not very ample. The class $-3K$ is very ample, and its linear system embeds $X_{\mathcal{P}}$ into $\mathbb{P}^6$. Consider the sequence of maps $\mathbb{P}^2 \dashrightarrow X_{\mathcal{P}} \longrightarrow V(x_0x_2 - x_1^2) \subset \mathbb{P}^3$. The first map is the blow-up, which is birational. The second map is the 2-1 morphism φ given by the linear system $|-2K|$. The second map takes the 240 exceptional curves in pairs $\{C_1, C_2\}$ onto the 120 hyperplane sections of $V(x_0x_2 - x_1^2)$ defined by the tritangent planes of C.

The pairs are as indicated above, since their classes add up to $-2K$ by [17, Table 1]. Intersection points of the pairs of curves on $X_{\mathcal{P}}$ become singular points of the intersection curves on $V(x_0x_2 - x_1^2)$, so the planes are tangent at those points. The tritangent being totally real means that these three points have real coordinates. $\square$

In our computations, the del Pezzo surface $X_{\mathcal{P}}$ is represented by $(\mathbb{P}^2, \mathcal{P})$. For each of the triples of points described above, we can compute their images in $V(x_0x_2 - x_1^2) \subset \mathbb{P}^3$ using Gröbner-based elimination. These triples are the contact points of the corresponding tritangent plane of C. We may choose an affine open subset of $V(x_0x_2 - x_1^2)$, isomorphic to $\mathbb{A}^2$, containing these three points. The intersection of a plane in $\mathbb{P}^3$ with the singular quadric Q is represented on this open subset by a plane curve with Newton polygon $\operatorname{conv}\{(0,0),(2,0),(0,1)\}$. We normalize this as follows:

$$\text{tritangent planes:} \qquad t^2 + e_1 t + e_2 + e_3 W. \tag{2}$$

The upper bound in Theorem 1.1 is attained with Example 2.2.

Example 2.2. Consider the following configuration of eight points:

$$\mathcal{P} = \Big\{ (1{:}0{:}0),\ (0{:}1{:}0),\ (0{:}0{:}1),\ (1{:}1{:}1),\ (10{:}11{:}1),\\ (27{:}2{:}17),\ (-19{:}11{:}-12),\ (-15{:}-19{:}20) \Big\} \subset \mathbb{P}^2_{\mathbb{R}}.$$

The resulting space sextic C in $V(x_0x_2 - x_1^2)$ has 120 totally real tritangents. We prove this by computing the pairs of special curves in $\mathbb{P}^2$ and by computing their triples of intersection points as described above. For each of the $112 = 28 + 56 + 56/2$ pairs of curves as above, we found that all three intersection points are real. We verified that the remaining eight tritangents of C are also totally real by computing the tangent cones of the special sextics in item 8.

We now convert the curve C to the format in (1). From that we can recover the pair (Q, K) defining the canonical model of C, for the independent verification in Example 4.1. We start by computing the cubics u, v. They are minimal generators of the ideal $I := \bigcap_{i=1}^{8} \mathfrak{m}_{P_i}$, where $\mathfrak{m}_{P_i}$ denotes the maximal ideal corresponding to the point P_i:

$$\begin{aligned} u \ =\ & 7151648400xy^2 - 434820164119x^2z + 354394201544xyz \\ & - 38806821565y^2z + 692107405715xz^2 - 580026269975yz^2, \end{aligned}$$

$$\begin{aligned} v \ =\ & 14303296800x^2y - 782195108453x^2z + 613370275528xyz - \\ & 49450554755y^2z + 1245021817105xz^2 - 1041049726225yz^2. \end{aligned}$$

Next, we compute the sextic w. It is the element of lowest degree in $I^{(2)} \setminus I^2$, where $I^{(2)}$ is the *symbolic square* of the ideal I. We find

$$\begin{aligned} w \ =\ & 175674063641748261863073581969689280x^4yz \\ & + 111155154295545647506864393467014440x^3y^2z \\ & - 445819563363162103552629662552521920x^2y^3z \\ & + 264167833624792096768707005238371200xy^4z \\ & - 200369626564548183654878856379681107x^4z^2 \\ & - 294913066878605444782558855953184976x^3yz^2 \\ & - 440622710904767923701179945218196420x^2y^2z^2 \\ & + 755657199632193956412295956477085200xy^3z^2 \\ & - 416363969347671237983809688854251675y^4z^2 \\ & + 329055128149267102548173318886152300x^3z^3 \\ & + 289931566371655705099858080895789300x^2yz^3 \\ & + 408084518267021777532269243486778900xy^2z^3 \\ & - 786825285955642438281852193533136500y^3z^3 \\ & - 174528373018867309329004510048947500x^2z^4 \\ & - 523785002916549858130362906690985000xyz^4 \\ & - 246023791579452575541006631825987500y^2z^4. \end{aligned}$$

The curve C is defined by the generator of the principal ideal

$$\left(\left(\langle \det J(u,v,w) \rangle + \mathrm{Minors}_{2\times 2} \begin{pmatrix} u^2 & uv & v^2 & w \\ 1 & t & t^2 & W \end{pmatrix} \right) : \langle u, v \rangle^2 \right) \cap \mathbb{Q}[t, W],$$

where $J(u,v,w)$ is the Jacobian matrix of the map $(x,y,z) \mapsto (u,v,w)$. The determinant of $J(u,v,w)$ gives the singular model of the branch curve in $\mathbb{P}^2$ and the 2×2 minors determine its image in the singular quadric in $\mathbb{P}^3$. In our case, the generator of the principal ideal is in the form of (1), and explicitly is given by

$$\begin{aligned} C:\ & 22070179871476654215734436981460373192064947078797748209t^6 \\ & + 5585831392725719195345163470516310362705889042844010328t^5 \\ & + 1417556981272444739350023378987784853149126 5t^4W \\ & - 4477180786035007172164248960407378691578283216077040398 64t^4 \\ & - 8656765538657190122323659315169836296202744 0t^3W \\ & + 57114529769698357624742306475t^2W^2 \\ & + 474302309016648096934423520799618219755274954155075926592t^3 \\ & + 192856342071229007723481356183461213738057680t^2W \\ & - 1943027060436044532587529594 00tW^2 - 26371599148125W^3 \\ & + 234139781685386481761784798116294507058448352826151077518 4t^2 \\ & - 1835288562819411262638933768610093443263299 20tW \\ & + 16496924410592194938861213540 0W^2 \\ & - 539025869397077269511781194383341975448880792033814574656 0t \\ & + 6155049906970017347872406308938765481230840 0W \\ & + 3193966974265623365398753846860968247266969720956505401600. \end{aligned}$$

We next compute each of the 120 tritangent planes explicitly, in the format (2). For instance, the tritangent that arises from the line spanned by the points $(10:11:1)$ and $(27:2:17)$ in $\mathcal{P}$ is found to be

$$345059077005W - 15320817327762671698417994 9t^2$$
$$+ 277165925195542929496239488t - 261340014239142448236734 0.$$

We now have a list of 120 such polynomials. Each of these intersects the curve C in three complex points with multiplicity two in the (t, W)-plane. All of these complex points are found to be real.

Example 2.3. A similar computation verifies that the following configuration of eight points gives 84 totally real tritangents:

$$\mathcal{P} = \{ (-12:9:11), (7:-5:-7), (1:3:3), (2:2:-1), (-2:2:1), (1:3:1), (3:3:2), (8:-8:-7) \} \subset \mathbb{P}^2_{\mathbb{R}}.$$

Proof of Theorem 1.1. The 120 tritangent planes arising from the construction above correspond to the odd theta characteristics of C. They are tritangent to C but they do not pass through the singular point $(0:0:0:1)$ of the quadric $V(x_0x_2 - x_1^2)$ in $\mathbb{P}^3$. Each such tritangent is an isolated regular solution to the polynomial equations that define the tritangents of C. These equations are described explicitly as the tritangent ideal in Section 4. We may perturb the equation $x_0x_2 - x_1^2$ to obtain a new curve C'. By the Implicit Function Theorem, for each tritangent H of C there is a nearby tritangent plane H' of C'. Moreover, if the perturbation is sufficiently small and the three points of $C \cap H$ are real and distinct, then $C' \cap H'$ also consists of three distinct real points. Conversely, if two points of $C \cap H$ are distinct and complex conjugate, then two points of $C' \cap H'$ will also be distinct and complex conjugate.

Hence, if our blow-up construction gives m totally real tritangents for some $m \leq 120$ then that same number of real solutions persists throughout some open semialgebraic subset in the space $\mathbb{P}^9_{\mathbb{R}} \times \mathbb{P}^{19}_{\mathbb{R}}$ of pairs (Q, K) of a real quadric and a real cubic in $\mathbb{P}^3$.

Examples 2.3 and 2.2 exhibit configurations with $m = 84$ and $m = 120$. Every integer m between these two values can be realized as well. We verified that assertion computationally, by constructing a configuration $\mathcal{P}$ in $\mathbb{P}^2_{\mathbb{Q}}$ for every integer between 84 and 120. □

Remark 2.4. It may be possible to prove by hand that every integer m between 84 and 120 is realizable. The idea is to connect the two extreme configurations with a general semialgebraic path in $\mathbb{P}^9_{\mathbb{R}} \times \mathbb{P}^{19}_{\mathbb{R}}$. That path crosses the *tritangent discriminant* Δ_2 (cf. Section 5) transversally. At such a crossing point, precisely one of the 120 configurations marked 8, 28, 56 or 56/2 fails to have its three intersection points distinct. This means that the number of real triples changes by exactly one. So, the number of totally real tritangents of the associated space sextic changes by exactly one. This is not yet a proof because the path might cross the discriminant Δ_1.

3 SPACE SEXTICS WITH FEWER OVALS

In Section 2 we started with eight points in the real projective plane $\mathbb{P}^2_{\mathbb{R}}$. Here we generalize by taking a configuration $\mathcal{P}$ in the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ that is invariant under complex conjugation. This also defines a real curve C in $V(x_0x_2 - x_1^2) \subset \mathbb{P}^3_{\mathbb{R}}$. To be precise, for $s \in \{1, 2, 3, 4, 5\}$, let $\mathcal{P}$ consist of $2s - 2$ real points and $5 - s$ complex conjugate pairs. Such a configuration of eight points defines a real del Pezzo surface $X_{\mathcal{P}}$. Additionally, the map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ and its branch curve C are defined over $\mathbb{R}$. The space sextic C has s ovals and it is not of dividing type when $s \leq 4$. By, [9, Proposition 3.1], the number of real tritangents of C equals 2^{s+2}. For curves which come from the construction in Section 2, we can derive this number by examining how complex conjugation acts on the special curves in $\mathbb{P}^2_{\mathbb{C}}$ we had associated with the point configuration $\mathcal{P}$:

- 8: The exceptional fiber over a point P_i defines a real tritangent if and only if the point P_i itself is real.
- 28: This tritangent is real if and only if the pair $\{P_i, P_j\}$ is real, i.e. either P_i and P_j are both real, or P_j is the conjugate of P_i. Among the 28 pairs, the number of real pairs is thus $\mathbf{4} = 0+4$, $\mathbf{4} = \binom{2}{2} + 3$, $\mathbf{8} = \binom{4}{2} + 2$ and $\mathbf{16} = \binom{6}{2} + 1$ for $s = 1, 2, 3, 4$.
- 56: This tritangent is real if and only if the triple of singular points in the quartic is real. This happens if either the three points are real, or there is one real point and a conjugate pair. Among the 56 triples, the number of real triples is thus $\mathbf{0}$, $\mathbf{6} = 0 + 2 \cdot 3$, $\mathbf{12} = \binom{4}{3} + 4 \cdot 2$, $\mathbf{26} = \binom{6}{3} + 6 \cdot 1$ for $s = 1, 2, 3, 4$.
- 56/2: In this case, the tritangent is real if and only if the two cubics are conjugate, and this happens if and only if the pair $\{P_i, P_j\}$ is real. Hence the count is $\mathbf{4}, \mathbf{4}, \mathbf{8}, \mathbf{16}$, as in the case 28.

For each value of $s \in \{1, 2, 3, 4\}$, if we add up the respective four numbers then we obtain 2^{s+2}. For instance, for $s = 3$, the analysis above shows that $4 + 8 + 12 + 8 = 32$ of the 120 tritangents are real.

We wish to know how many of these 2^{s+2} real tritangents can be totally real, as $\mathcal{P}$ ranges over the various types of real configurations. Our investigations led to the findings summarized in Theorem 3.1.

Theorem 3.1. *The third row in Table 1 lists the ranges of currently known values for the number of totally real tritangents of real space sextics C that are constructed by blowing up eight points in $\mathbb{P}^2$:*

s ovals	1	2	3	4	5
real	8	16	32	64	120
totally real	[0, 8]	[1, 15]	[10, 32]	[35, 63]	[84, 120]

Table 1: Real and totally real tritangents of a space sextic C on a singular quadric Q, according to number of ovals of C.

The following examples exhibit some lower and upper bounds.

Example 3.2 ($s = 1$). Let $\mathcal{P}$ be the following configuration in $\mathbb{P}^2_{\mathbb{C}}$:

$$\begin{array}{lcl} P_1 = & (i : 1 - i : 0) & P_2 = \overline{P_1} \\ P_3 = & (2 - i : -3 - i : 3 + i) & P_4 = \overline{P_3} \\ P_5 = & (2 - i : 1 - i : -2 - i) & P_6 = \overline{P_5} \\ P_7 = & (4i : -i : 4) & P_8 = \overline{P_7} \end{array}$$

The curve C consists of only one oval in $\mathbb{P}^3_{\mathbb{R}}$. One checks that **none** of the eight real tritangents of C is totally real, i.e. no plane is tangent to C at three real points. On the other hand, for the following configuration, **all** eight real tritangents are totally real:

$$\begin{array}{lcl} P_1 = & (i : 0 : 1) & P_2 = \overline{P_1} \\ P_3 = & (1 - 3i : -3 + 2i : 1) & P_4 = \overline{P_3} \\ P_5 = & (0 : 2 + 3i : -3 - 2i) & P_6 = \overline{P_5} \\ P_7 = & (4i : -3 + 4i : 1 + i) & P_8 = \overline{P_7} \end{array}$$

Example 3.3 ($s = 2$). We fix the following configuration $\mathcal{P}$ of two real points and three pairs of complex conjugate points in $\mathbb{P}^2$.

$$\begin{array}{rcrl} P_1 = & (1 : -2i : 2i) & P_2 = & \overline{P_1} \\ P_3 = & (1 : 3 + 2i : -3i) & P_4 = & \overline{P_3} \\ P_5 = & (1 + 2i : 4 + 2i : -4 + i) & P_6 = & \overline{P_5} \\ P_7 = & (1 : 0 : -1) & P_8 = & (0 : 4 : 1) \end{array}$$

The associated curve C has two ovals. Of its 16 real tritangents, **exactly one** is totally real. By a random search, we found examples where up to 15 of the real tritangents of the curve C are totally real. At present, we have not found any $\mathcal{P}$ where the associated curve has either 0 or 16 totally real tritangents.

Figure 6 shows the empirical distribution we observed for $s = 3$ (left) and $s = 4$ (right). The respective ranges are $[10, 32]$ and $[35, 63]$.

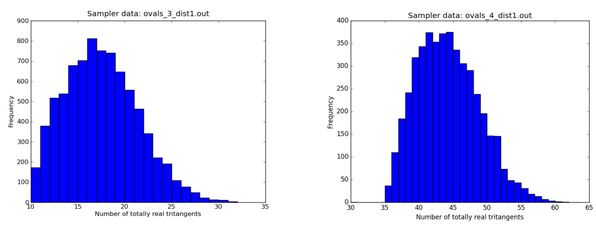

Figure 6: Count of totally real tritangents for $s = 3$ and $s = 4$.

Example 3.4 ($s = 3$). The following configuration $\mathcal{P}$ gives a space sextic C with three ovals that has 32 totally real tritangents:

$P_1 = (-204813760 - 55982740i : 452442430 + 319792532i : 1)$, $P_2 = \overline{P_1}$

$P_3 = (252002303 - 508295920i : 418802957 + 255990940i : 1)$, $P_4 = \overline{P_2}$

$P_5 = (420794066 : 346448315 : 1)$, $P_6 = (64527687 : 183049780 : 1)$,

$P_7 = (410335352 : 364471450 : -1)$, $P_8 = (-210806629 : -146613813 : 1)$.

4 SOLVING THE TRITANGENT EQUATIONS

In Sections 2 and 3 we studied space sextics C lying on a singular quadric surface Q. By perturbing these, we obtained generic space sextics with many different numbers of totally real tritangents. However, not all numbers between 0 and 120 were attained by this method. To remedy this, we considered arbitrary space sextics $C = Q \cap K$, defined by a random quadric Q and a random cubic K.

However, we found the problem of computing the tritangents directly from (Q, K) to be quite challenging. We conjecture that all integers between 0 and 120 can be realized by the totally real tritangents of some space sextic. But, at present, the gaps from Table 1 persist. We do not yet have realizations for $33, 34$ or $64, 65, \ldots, 83$.

In what follows we describe our algorithm – and its implementation – for computing the 120 tritangents directly from the homogeneous polynomials of degree two resp. three in x_0, x_1, x_2, x_3 that define the quadric Q resp. the cubic K. We introduce four unknowns u_0, u_1, u_2, u_3 that serve as coordinates on the space $(\mathbb{P}^3)^\vee$ of planes:

$$H \;:\; u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3 \;=\; 0. \tag{3}$$

For generic real values of the u_i, the intersection $Q \cap K \cap H = C \cap H$ consists of six distinct complex points in $\mathbb{P}^3$. We are interested in the special cases when these six points become three double points. We seek to find the *tritangent ideal* $\mathcal{I}_C$, consisting of polynomials in u_0, u_1, u_2, u_3 that vanish at those H that are tritangent planes of C.

We fix the projective space $\mathbb{P}^6$ whose points are the binary sextics

$$f \;=\; a_0t_0^6 + a_1t_0^5t_1 + a_2t_0^4t_1^2 + a_3t_0^3t_1^3 + a_4t_0^2t_1^4 + a_5t_0t_1^5 + a_6t_1^6.$$

Inside that $\mathbb{P}^6$ we consider the threefold of squares of binary cubics:

$$f \;=\; \left(b_0t_0^3 + b_1t_0^2t_1 + b_2t_0t_1^2 + b_3t_1^3\right)^2. \tag{4}$$

The defining prime ideal of that threefold is minimally generated by 45 quartics in $a_0, a_1, a_2, a_3, a_4, a_5, a_6$. This is revealed by the row labeled $\lambda = (2, 2, 2)$ in [13, Table 1]. Computing these 45 quartics is a task of elimination, which we carried out in a preprocessing step.

Consider now a specific instance (Q, K), defining $C = Q \cap K$. We then transform the above 45 quartics in $a_0, \ldots, a_6$ into higher degree equations in $u_0, \ldots, u_3$. This is done by projecting $C \cap H$ onto a line. This gives a univariate polynomial of degree six whose seven coefficients are polynomials of degree 12 in u_0, u_1, u_2, u_3. We replace $a_0, \ldots, a_6$ by these polynomials. Theoretically, it suffices to project onto a single generic line. Practically, we had more success with multiple (possibly degenerate) projections onto the coordinate axes, and gathering the resulting systems of 45 equations each.

To be more precise, fix one of the 12 ordered pairs (x_i, x_j). First, solve the equation (3) for x_i, substitute into the equations of Q and K, and clear denominators. Next, eliminate x_j from the resulting ternary quadric and cubic. The result is a binary sextic f in the two unknowns $\{x_0, x_1, x_2, x_3\} \backslash \{x_i, x_j\}$ whose coefficients $a_0, \ldots, a_6$ are expressions of degree 12 in $u_0, \ldots, u_3$. We substitute these expressions into the 45 quartics precomputed above. This results in 45 polynomials of degree 48 in $u_0, \ldots, u_3$ that lie in the tritangent ideal $\mathcal{I}_C$. Repeating this elimination process for the other 11 pairs (x_i, x_j), we obtain additional polynomials in $\mathcal{I}_C$. Altogether, we have now enough polynomials of degree 48 to generate $\mathcal{I}_C$ on any desired affine open subset in the dual $(\mathbb{P}^3)^\vee$ of planes in $\mathbb{P}^3$. The homogeneous ideal $\mathcal{I}_C$ is radical and it has 120 zeros in $(\mathbb{P}^3)^\vee$.

To compute these zeros, we restrict ourselves to an open chart, say $U = \{u_3 \neq 0\} \simeq \mathbb{C}^3$. The resulting system (with $u_3 = 1$) is grossly over-constrained, with up to 12×45 equations in the three unknowns u_0, u_1, u_2. We compute a lexicographic Gröbner bases, using `fglm` [8], as our ideal is zero-dimensional. For generic instances (Q, K), the lexicographic Gröbner basis has the shape

$$\left\{\, u_1 - p_1(u_3),\; u_2 - p_2(u_3),\; p_3(u_3) \,\right\}, \tag{5}$$

where $\deg(p_3) = 120$ and $\deg(p_1) = \deg(p_2) = 119$. For degenerate (Q, K) we proceed with a triangular decomposition.

We implemented this method in MAGMA [1]. The Gröbner basis computation was very hard to carry out. It took several days to finish for Example 4.2. The output had coefficients of size $\sim 10^{680}$.

We applied our implementation to several curves C, some from configurations $\mathcal{P} \subset \mathbb{P}^2_{\mathbb{Q}}$, and some from general instances (Q, K).

The first case is used as a tool for independent verification, e.g. for Example 2.2. Here, p_3 decomposes into linear factors over $\mathbb{Q}$. Each factor yields a rational tritangent, for which we compute the three (double) points in $H \cap C$ symbolically. To check whether one or three are real, we again project onto a line. This yields a univariate rational polynomial of degree 6. We can test whether it is the square of a cubic with positive discriminant. More generally, any nonlinear factor with only real roots also allows us to continue our computations symbolically over an algebraic field extension.

In the second case, the univariate polynomial p_3 is typically irreducible over $\mathbb{Q}$, and we solve (5) numerically. We compute all real tritangents H and their intersections $H \cap C$. Based on the resulting numerical data, we decide which H are totally real. Complex zeroes are also counted, to attest that there are indeed 120 solutions. This certifies that the chosen open chart U was indeed generic.

Example 4.1. The polynomial $C(t, W)$ in Example 2.2 translates into a cubic $K(x_0, x_1, x_2, x_3)$ which is unique modulo the quadric $Q = x_0x_2 - x_1^2$. We apply the algorithm above to the instance (Q, K) with $U = \{u_3 \neq 0\}$. The result verifies that all 120 tritangents are rational and totally real. Interestingly, two of the 120 tritangents have a coordinate that is zero. These two special planes are

$$\begin{aligned} 0 \quad = \quad & 666727858907928630542805134887161895157u_0 \\ & -371406861222752391050720128495402169926u_1 \\ & -13148859997292971155483015u_3 \end{aligned}$$

and

$$\begin{aligned} 0 \quad = \quad & 79848789064366287163873087455437884472u_1 \\ & -444610889957505571930558230563361607 1u_2 \\ & +106897050552377064523 95u_3 . \end{aligned}$$

Example 4.2. The curve $C = Q \cap K$ in [9, §3] is given by

$$\begin{aligned} Q \quad &= \quad x_0^2 + x_1^2 + x_2^2 - 25x_3^2 \\ K \quad &= \quad \left(x_0 + \sqrt{3}x_3\right)\left(x_0 - \sqrt{3}x_1 - 3x_3\right)\left(x_0 + \sqrt{3}x_1 - 3x_3\right) - 2x_3^3. \end{aligned}$$

It has five ovals, so all tritangents are real. Our computation shows that there are only 108 distinct tritangents. Twelve are solutions of multiplicity two in the ideal I_C, and none of the tritangents are rational. This verifies [9, Theorem 3.2]. Figure 7 shows three tritangents, meeting 3, 2 and 1 ovals of the red curve respectively.

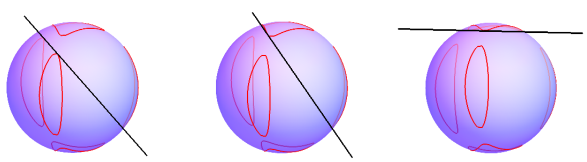

Figure 7: The curve in Example 4.2 has 108 real tritangents

In [9, Question 3.3], Harris and Len asked whether this example can be replaced by one with 120 distinct totally teal tritangents. Our computations in Examples 2.2 and 4.1 establish the affirmative answer. However, we do not yet know whether all integers between 0 and 120 are possible for the number of totally real tritangents.

5 DISCRIMINANTS

In this paper we considered two parameter spaces for space sextics. First, there is the space $\mathbb{P}^9_{\mathbb{R}} \times \mathbb{P}^{19}_{\mathbb{R}}$ of pairs (Q, K) consisting of a real quadric and a real cubic in $\mathbb{P}^3$. The regions for which the number of real tritangents remains constant partitions $\mathbb{P}^9_{\mathbb{R}} \times \mathbb{P}^{19}_{\mathbb{R}}$ into open strata. This stratification is refined by regions for which the number of totally real tritangents remains constant. We are interested in the discriminantal hypersurfaces that separate these strata.

Second, there is the space $(\mathbb{P}^2_{\mathbb{R}})^8$ of configurations $\mathcal{P}$ of eight labeled points in the plane. This space works for any fixed value of s in $\{1, 2, 3, 4, 5\}$, representing configurations of $2s - 2$ real points and $5 - s$ complex conjugate pairs. For simplicity of exposition we focus on the fully real case $s = 5$. In any case, the number of real tritangents is fixed, and we care about the open strata in $(\mathbb{P}^2_{\mathbb{R}})^8$ in which the number of totally real tritangents is constant. Again, we seek to describe the discriminantal hypersurface, but now in $(\mathbb{P}^2)^8$.

For $\mathcal{P} \in (\mathbb{P}^2)^8$, denote the associated space sextic by $C_{\mathcal{P}}$. Let Σ denote the locus of configurations in $(\mathbb{P}^2)^8$ which are not in general position. We define the *tritangent discriminant locus* by

$$Y = \overline{\left\{ \mathcal{P} \in (\mathbb{P}^2)^8 \backslash \Sigma : \begin{array}{l} C_{\mathcal{P}} \text{ has a tritangent with contact} \\ \text{order at least 4 at some point} \end{array} \right\}},$$

where the over-line denotes the Zariski closure.

Lemma 5.1. *Every irreducible component of Y is a hypersurface.*

Proof. Let $\mathcal{P}_0 \in Y \backslash \Sigma$, and fix local coordinates $\bar{p} = (p_1, \ldots, p_{16})$ for a neighborhood $\mathcal{U}$ of $\mathcal{P}_0 = \bar{p}_0$ in $(\mathbb{P}^2)^8$. The bivariate equation (1) that represents C_0 is the specialization at $\bar{p}_0$ of a general equation

$$C: \quad c(t, W) \;=\; t^6 + d_1(\bar{p})t^5 + d_2(\bar{p})t^4W + \cdots + d_{15}(\bar{p}), \tag{6}$$

where the coefficients $d_i(\bar{p})$ are rational functions regular at p_0. Let H_0 be a tritangent plane to C_0 with a contact point of order at least 4. Then H_0 is either the tritangent associated to a point in $\mathcal{P}_0$ or associated to one of the patterns in Figure 3, 4 or 5. Either way, we see that H_0 is obtained by specializing an equation of the form

$$H: \quad h(t, W) \;=\; t^2 \;+\; e_1(\bar{p})t \;+\; e_2(\bar{p}) \;+\; e_3(\bar{p})W, \tag{7}$$

where the coefficients $e_i(\bar{p})$ are rational functions regular at $\bar{p}_0$.

The resultant of $c(t, W)$ and $h(t, W)$ with respect to W is a polynomial $f(t)$ of degree 6 whose coefficients are rational functions in $\bar{p}$. Note that H is a tritangent plane to C, so $f = g^2$ as in (4). The roots of the cubic g correspond to the contact points of H with C. In particular, $H_{\bar{p}}$ has a point of contact with $C_{\bar{p}}$ of order at least 4 precisely when the discriminant of g is zero. Since the coefficients of g are rational functions in $\bar{p}$, regular at $\bar{p}_0$, this means that a neighborhood of $\mathcal{P}_0$ in Y has codimension 1 in $\mathcal{U}$. This implies that every irreducible component of Y has codimension 1. □

The following theorem describes these irreducible components:

Theorem 5.2. *The tritangent discriminant locus Y is the union of* $120 = 8+28+56+56/2$ *irreducible hypersurfaces in* $(\mathbb{P}^2)^8$*, one for each point in $\mathcal{P}$ and each pattern in Figures 3, 4 and 5. The components of type 8 have total degree* 306*, namely* 54 *in the point corresponding to the exceptional curve and* 36 *in the other seven points. The components of type 28 have total degree* 216*, namely* 18 *in each of the two points on the line and* 30 *for the six on the quintic. The components of type 56 have total degree* 162*, namely* 18 *in each of the five points on the conic and* 24 *for the three on the quartic. The components of type 56/2 have total degree* 144*, namely* 18 *in each of the eight points.*

We prove Theorem 5.2 computationally. In order to do so, it is convenient to make the following observation. Let $Y = V(f)$ with f a $\mathbb{Z}^8$-homogeneous polynomial of $\mathbb{Z}^8$-degree $(d_1, \ldots, d_8)$. We scale

f so that its coefficients are relatively prime integers. For a prime p, let f_p denote the reduction of f modulo p. If p is large, then

$$Y_p = V(f_p) \subset (\mathbb{P}^2_{\mathbb{F}_p})^8$$

has the same $\mathbb{Z}^8$-degree as Y. We can thus calculate $(d_1, \ldots, d_8)$ by using Gröbner bases over a large finite field $\mathbb{F}_p$.

Let $k = \mathbb{F}_p$ be the field with $p = 10^6 + 3$ elements and k^{al} its algebraic closure. Let $S = \mathbb{P}^1_k$ and let $R = k[a, b]$ be the coordinate ring of S. Let $\mathbb{P}^2_R := \operatorname{Proj} R[x, y, z]$ be the projective plane over R. If X is some family, its specialization to $(a : b) \in S$ is denoted $X_{(a:b)}$.

We use the following configuration of eight points in $\mathbb{P}^2_R$:

$$\mathcal{P} = \big\{ (24 : -23 : 57), (11 : 25 : -27), (-30 : 29 : 79), (14 : -23 : 26), \\ (43 : 92 : 61), (-34 : 81 : 7), (88 : 29 : 69), (a : b : 0) \big\} \subset \mathbb{P}^2_{\mathbb{R}}.$$

Then $\mathcal{P}$ is in general position for generic a, b. Let $\mathcal{U}$ be the open subset of S parameterizing specializations in general position. The following result concerns generic specializations. We omit the proof.

Proposition 5.3. *There exists a pair of ternary cubics $u, v \in R[x, y, z]$, a ternary sextic $w \in R[x, y, z]$, bivariate polynomials $c, h \in R[t, W]$ as in (6) and (7), and an explicitly computable finite set $X \subset S(k^{\mathrm{al}})$ such that, whenever $(a : b) \in \mathcal{U} \backslash X$, the following hold:*

(a) The specializations $u_{(a:b)}, v_{(a:b)}$ span the space of cubics passing through all eight points in $\mathcal{P}_{(a:b)}$.
(b) The specializations $u^2_{(a:b)}, uv_{(a:b)}, v^2_{(a:b)}, w_{(a:b)}$ span the space of sextics vanishing doubly at each point in $\mathcal{P}_{(a:b)}$.
(c) The specialization $\{c_{(a:b)}(t, W) = 0\}$ is a smooth genus 4 curve $C_{a:b}$ lying on a singular quadric surface.
(d) The specialization $\{h_{(a:b)}(t, W) = 0\}$ is a tritangent plane to $C_{(a:b)}$ where the coefficient of W is nonzero.
(e) For any $(a : b) \in X$, the curve $C_{(a:b)}$ is smooth, genus 4, and none of the tritangent planes have a point of contact order larger than 4.

We now derive Theorem 5.2 from Proposition 5.3. The degree d_8 of Y in the last point P_8 is computed by restricting to the slice

$$\{(24 : -23 : 57)\} \times \{(11 : 25 : -27)\} \times \ldots \times \{(88 : 29 : 69)\} \times \mathbb{P}^2.$$

This restriction of Y is a curve of degree d_8 in $\mathbb{P}^2$. We compute this degree as the number of points in the intersection with the line

$$S \;=\; \{(a : b : c) \in \mathbb{P}^2 : c = 0\}.$$

The same argument works also for each irreducible component of Y. These components correspond to the various tritangent patterns, marked 8, 28, 56 and 56/2. We perform this computation for each pattern over $\mathbb{F}_p$, and we obtain the numbers stated in Theorem 5.2.

We now turn to the canonical representation of arbitrary space sextics $C = Q \cap K$, namely by pairs (Q, K) in $\mathbb{P}^9 \times \mathbb{P}^{19}$. We shall identify three irreducible hypersurfaces in $\mathbb{P}^9 \times \mathbb{P}^{19}$ that serve as discriminants for different scenarios of how C can degenerate. For each hypersurface, we shall determine its *bidegree* (α, β). Here α is the degree of its defining polynomial in the coefficients of Q, and β is the degree of its defining polynomial in the coefficients of K.

First, there is the classical discriminant Δ_1, which parametrizes all pairs (Q, K) such the curve $C = Q \cap K$ is singular. This is an irreducible hypersurface in $\mathbb{P}^9 \times \mathbb{P}^{19}$, revisited recently in [2]. The general points of Δ_1 are irreducible curves C of arithmetic genus 4 that have one simple node, so the geometric genus of C is 3. The discriminant Δ_1 specifies the wall to be crossed when the number of real tritangents changes as (Q, K) moves throughout $\mathbb{P}^9_{\mathbb{R}} \times \mathbb{P}^{19}_{\mathbb{R}}$.

Second, there is the wall to be crossed when the number of totally real tritangents changes. The discriminant Δ_2 comprises space sextics with a tritangent H that is degenerate, in the sense that H is tangent at one point and doubly tangent at another point of C. For real pairs (C, H), such a point of double tangency deforms into two contact points of a tritangent H_ϵ at a nearby curve C_ϵ, and this pair is either real or complex conjugate. On the hypersurface in $\mathbb{P}^9 \times \mathbb{P}^{19}$ where Q is singular, the locus Δ_2 is the image of the discriminant with 120 components in Theorem 5.3 under the map that takes a configuration $\mathcal{P} \in (\mathbb{P}^2)^8$ to its associated curve $C_{\mathcal{P}}$.

Our third discriminant Δ_3 parametrizes pairs (Q, K) such that the curve $C = Q \cap K$ has two distinct tritangents that share a common contact point on C. In other words, the curve C has a point whose tangent line is contained in two tritangent planes. The discriminant Δ_3 furnishes an embedded realization of the *common contact locus* that was studied in the dissertation of Emre Sertöz [16, §2.4].

The following theorem was found with the help of Gavril Farkas and Emre Sertöz. The numbers are derived from results in [7, 16].

Theorem 5.4. *The discriminantal loci Δ_1, Δ_2 and Δ_3 are irreducible and reduced hypersurfaces in $\mathbb{P}^9 \times \mathbb{P}^{19}$. Their bidegrees are*

$$\begin{aligned} \operatorname{bidegree}(\Delta_1) &= (33, 34), \\ \operatorname{bidegree}(\Delta_2) &= (744, 592), \\ \operatorname{bidegree}(\Delta_3) &= (8862, 5236). \end{aligned}$$

Proof. Consider the discriminant Δ_1 for curves in $\mathbb{P}^3$ that are intersections of two surfaces of degree d and e. It has bidegree

$$\big(e(3d^2{+}2de{+}e^2{-}8d{-}4e{+}6) \, , \, d(3e^2{+}2de{+}d^2{-}8e{-}4d{+}6) \big).$$

This can be found in many sources, including [2, Proposition 3]. For $d = 2$ and $e = 3$ we obtain $\operatorname{bidegree}(\Delta_1) = (33, 34)$, as desired.

To determine the other two bidegrees, we employ known facts from the enumerative geometry of $\overline{\mathcal{M}}_4$, the moduli space of stable curves of genus 4. The Picard group $\operatorname{Pic}(\overline{\mathcal{M}}_4)$ is generated by four classes $\lambda, \delta_0, \delta_1, \delta_2$. Here λ is the *Hodge class*, and the δ_i are classes of irreducible divisors in the boundary $\overline{\mathcal{M}}_4 \backslash \mathcal{M}_4$. They represent:

δ_0: a genus 3 curve that self-intersects at one point;
δ_1: a genus 1 curve intersects a genus 3 curve at one point;
δ_2: two genus 2 curves intersect at one point.

Our discriminants Δ_i are the inverse images of known irreducible divisors in the moduli space under the rational map $\mathbb{P}^9 \times \mathbb{P}^{19} \dashrightarrow \overline{\mathcal{M}}_4$.

First, Δ_2 is the pull-back of the divisor $D_4 \subset \overline{\mathcal{M}}_4$ of curves with degenerate odd spin structures. It follows from [7, Theorem 0.5] that

$$[D_4] \;=\; 1440\lambda - 152\delta_0 - \alpha\delta_1 - \beta\delta_2 \quad \text{for some } \alpha, \beta \in \mathbb{N}. \qquad (8)$$

For any curve $\gamma \subset \overline{\mathcal{M}}_4$, the sum $\sum_{i=0}^{2} \gamma \cdot \delta_i$ counts points on γ whose associated curve is singular. Write h resp. v for the curve γ that represents *line* $\times$ *point* resp. *point* $\times$ *line* in $\mathbb{P}^9 \times \mathbb{P}^{19}$. We saw

$$(h \cdot \delta_0, v \cdot \delta_0) \;=\; \operatorname{bidegree}(\Delta_1) \;=\; (33, 34).$$

Moreover, it can be shown that

$$h \cdot \lambda = v \cdot \lambda = 4 \qquad \text{and} \qquad h \cdot \delta_i = v \cdot \delta_i = 0 \quad \text{for } i = 1, 2.$$

This implies the assertion about the bidegree of our discriminant:

$$\operatorname{bidegree}(\Delta_2) = (h{\cdot}[D_4], v{\cdot}[D_4]) = (1440{\cdot}4{-}152{\cdot}33, 1440{\cdot}4{-}152{\cdot}34).$$

Similarly, Δ_3 is the pull-back of the *common contact divisor* $Q_4 \subset \overline{\mathcal{M}}_4$ studied by Sertöz. It follows from [16, Theorem II.2.43] that

$$[Q_4] \;=\; 32130\lambda - 3626\delta_0 - \alpha\delta_1 - \beta\delta_2 \quad \text{for some } \alpha, \beta \in \mathbb{N}. \quad (9)$$

Replacing (8) with (9) in our argument, we find that bidegree(Δ_3) is

$$(h \cdot [Q_4], v \cdot [Q_4]) \;=\; (32130 \cdot 4 - 3626 \cdot 33,\; 32130 \cdot 4 - 3626 \cdot 34).$$

This completes our derivation of the bidegrees in Theorem 5.4.

The irreducibility of the loci Δ_i is shown by a standard double-projection argument. One marks the relevant special point(s) on C. Then Δ_i becomes a family of linear spaces of fixed dimension. □

6 WHAT NEXT?

In this paper, we initiated the computational study of totally real tritangents of space sextics in $\mathbb{P}^3$. These objects are important in algebraic geometry because they represent odd theta characteristics of canonical curves of genus 4. We developed systematic tools for constructing curves all of whose tritangents are defined over algebraic extensions of $\mathbb{Q}$, and we used this to answer the longstanding question whether the upper bound of 120 totally real tritangent planes can be attained. We argued that computing the tritangents directly from the representation $C = Q \cap K$ is hard, and we characterized the discriminants for these polynomial systems.

This article leads to many natural directions to be explored next. We propose the following eleven specific problems for further study.

(1) Decide whether every integer between 0 and 120 is realizable.

(2) Determine the correct upper and lower bounds in Table 1. In particular, is 84 the lower bound for curves with five ovals?

(3) A smooth quadric Q is either an ellipsoid or a hyperboloid. Degtyarev and Zvonilov [4] characterized the topological types of real space sextics on these surfaces. What are the possible numbers of totally real tritangents for their types?

(4) What does [4] tell us about space sextics on a singular quadric Q? Which types arise on Q, how do they deform to those on a hyperboloid, and what does this imply for tritangents?

(5) Given a space sextic C whose quadric Q is singular, how to best compute a configuration $\mathcal{P} \in (\mathbb{P}^2)^8$ such that $C = C_{\mathcal{P}}$? Our idea is to design an algorithm based on the constructions described in [11, Proposition 4.8 and Remark 4.12].

(6) Lehavi [14] shows that a general space sextic C can be reconstructed from its 120 tritangents. How to do this in practice?

(7) Let $C_{\mathcal{P}}$ be the space sextic of a configuration $\mathcal{P} \in (\mathbb{P}^2)^8$. How to see the ovals of $C_{\mathcal{P}}$ in $\mathbb{P}^2$? For each tritangent as in Figure 3, 4 or 5, how to see the number of ovals it touches?

(8) Design a custom-tailored *homotopy algorithm* for numerically computing the 120 tritangents from the pair (Q, K).

(9) The *tropical limit* of a space sextic has 15 classes of tritangents, each of size eight [9, Theorem 5.2]. This is realized classically by a $K_{3,3}$*-curve*, obtained by taking K as three planes tangent to a smooth quadric Q. How many totally real tritangents are possible in the vicinity of (Q, K) in $\mathbb{P}^9_{\mathbb{R}} \times \mathbb{P}^{19}_{\mathbb{R}}$?

(10) The 28 bitangents of a plane quartic are the off-diagonal entries of a symmetric 8×8-matrix, known as the *bitangent matrix* [3]. How to generalize this to genus 4? Is there such a canonical matrix (or tensor) for the 120 tritangents?

(11) What is maximal number of 2-dimensional faces in the convex hull of a space sextic in $\mathbb{R}^3$? There are at most 120 such facets. In addition, there are infinitely many edges. These form a ruled surface of degree 54, by [15, Theorem 2.1].

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