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On Sundaram's Full Bijection

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ON SUNDARAM'S FULL BIJECTION

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Abstract. We highlight refinement of a bijection given by Sheila Sundaram in her PhD thesis. The framework allows for comparison to a late conjecture of Naito-Sagaki. We give an injection of the set of dominant paths for this model into the corresponding set of Littlewood-Richardson Sundaram tableaux, which is not surjective. We relate this to the refinement of the aforementioned bijection.

1. Introduction

In her PhD thesis [9], Sundaram provided an extension of the Littlewood branching rule for the restriction of representations from the special linear group to the symplectic group. Sundaram's result dates back to 1986, while the Littlewood branching rule was first stated in 1953. This extension is in terms of certain generalised Littlewood-Richardson tableaux, which are fillings of skew tableaux with entries constituting a lattice path of even content. While the Littlewood branching rule is only true for representations with a stable highest weight, Sundaram's result holds in full generality.

In [7] in 2005 Naito and Sagaki conjectured a branching rule for the same decomposition using certain Littlemann paths. They conjectured that, when considering the embedding of the symplectic Lie algebra in the special linear Lie algebra as the set of fixed points of the endomorphism induced by the folding of the Dynkin diagram of type A_{2n-1}, the irreducible summands of the restriction of a simple module are indexed by the set of Littelmann paths (in a special choice of model) which, when restricted, become dominant.

In recent work with Schumann [8], this conjecture was proven by showing its connection to Sundaram's rule. The goal of this paper is two-fold: on the one hand, we observe that, the context in which Sundaram's rule comes about allows for an equivalent formulation of the conjecture of Naito-Sagaki. On the other hand, we analyse a bijection by Sundaram, whose existence implies the rule itself, highlighting a refinement thereof. This refinement allows one to realise that the equivalent conjecture does not hold in this case.

It is worth mentioning that Sundaram's rule has also appeared in the recent work of Lecouvey and Lenart [5], where they formulate their results in its terms, and relate it to the rule by Kwon [4].

1.1. **Formulation of results.** The bijection by Sundaram relates on the one hand semi standard Young tableaux T of shape λ in the alphabet

$$\mathcal{K} = 1 < \bar{1} < \cdots < n < \bar{n}$$

and on the other pairs (P, L), where P is a symplectic King tableau of shape μ, and L is a generalised Littlewood-Richardson tableau of skew shape λ/μ and content an even partition, also referred to as a Littlewood-Richardson Sundaram tableau. Now, the mentioned set of semi standard Young tableaux, which we denote by $\mathrm{SSYT}_{\mathcal{K}}(\lambda)$ can be considered as the set of Littelmann paths for the simple module of $\mathrm{SL}(2n, \mathbb{C})$ when restricted to $\mathrm{Sp}(2n, \mathbb{C})$ with respect to the natural embedding and under an appropriate choice of basis. Let us denote by $\mathrm{domres}_{\mathcal{K}}(\lambda, \mu)$ the set of those tableaux with content μ and which, when considered as such paths, are dominant. Let us denote the relevant set of semi standard Young tableaux by $\mathrm{SSYT}_{\mathcal{K}}(\lambda)$, the set of King tableaux of shape μ by $\mathrm{King}_n(\mu)$, and the set of Littlewood-Richardson Sundaram tableaux of skew shape λ/μ and even weight by $\mathrm{LRS}(\lambda/\mu)$. Sundaram's bijection will be then denoted by

$$\mathrm{SSYT}_{\mathcal{K}}(\lambda) \xrightarrow{\Phi} \bigcup_{\mu \subset \lambda} \mathrm{King}_n(\mu) \times \mathrm{LRS}(\lambda/\mu)$$
$$\mathrm{T} \mapsto (\Phi_1(\mathrm{T}), \Phi_2(\mathrm{T})).$$

Before we continue with this narrative, let us already point out the analogy to the case of the Naito-Sagaki conjecture. In Theorem 17 we show that Φ restricts to a well-defined injective map

$$\mathrm{domres}_{\mathcal{K}}(\lambda, \mu) \longrightarrow \mathrm{King}_n(\mu, \mu) \times \bigcup_{\nu \mathrm{even}} \mathrm{LRS}(\lambda/\mu, \nu)$$

which is however not surjective. The proof of the Naito-Sagaki conjecture consists precisely in finding such a bijection, substituting the set $\mathrm{domres}_{\mathcal{K}}(\lambda, \mu)$ by the analogous set in that context.

We now highlight a refinement of the bijection Φ. Given $\mathrm{L} \in \mathrm{LRS}(\lambda/\mu)$, consider the set

$$\Phi^{-1}(\mathrm{King}_n \times \{\mathrm{L}\}).$$

Since the map Φ is a bijection, these sets partition the set $\mathrm{SSYT}_{\mathcal{K}}(\lambda)$. From a representation theoretical point of view, one may ask: Given $\mathrm{T} \in \mathrm{SSYT}(\lambda)$, how to find $\mathrm{dom}(\mathrm{T}) \in \mathrm{SSYT}_{\mathcal{K}}(\lambda)$ such that $\Phi_1(\mathrm{T})$ is the unique element in $\mathrm{King}_n(\mu, \mu)$? The answer to this question is simply to trace back the definition of Φ. The resulting element of $\mathrm{SSYT}_{\mathcal{K}}(\lambda)$ does not always belong to the set $\mathrm{domres}_{\mathcal{K}}(\lambda)$, as we show in several examples. We conclude this introduction by noting that an analogue of the map Φ in the context of the Naito-Sagaki conjecture is unknown.

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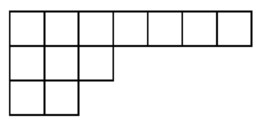
2. King Tableaux and the Berele Algorithm

We will make no distinction between the following words: partition, shape, weight, Young diagram. They will all mean a non-increasing sequence of positive integers, and will be thought of as the highest weight of a representation, or as an arrangement of top-left aligned empty boxes, with as many rows as entries in the sequence; the number of boxes of each is determined by the corresponding number of the sequence.



$(7, 3, 2)$.

2.1. **King tableaux.** Let n be a positive integer and λ a Young diagram of at most $2n$ rows. We will use the same alphabet as King:

$$\mathcal{K}_n = \{1 < \bar{1} < 2 < \bar{2} < 3 < \bar{3} < \cdots < \cdots < n < \bar{n}\} \tag{1}$$

and denote the set of all semi-standard Young tableaux in this alphabet by $\mathrm{SSYT}_{\mathcal{K}_n}(\lambda)$. Recall that this means that the entries are weakly increasing along the rows and weakly increasing along the columns. A word $w = w_1 \cdots w_r$ in this alphabet is a **lattice path** if, for each $1 \leq j \leq r$, we have

$$\#i \text{ in } w_1 \cdots w_j - \#\bar{i} \text{ in } w_1 \cdots w_j \geq \#i+1 \text{ in } w_1 \cdots w_j - \#\overline{i+1} \text{ in } w_1 \cdots w_j.$$

Example 1. The word

$$1\bar{1}111222\bar{2}33$$

is a lattice path, while

$$1\bar{1}\bar{1}\bar{1}1222\bar{2}33$$

is not.

Definition 2. A semistandard Young tableau in the alphabet $\mathcal{K}_n$ of shape a Young diagram of at most n rows, is a **King tableau**, if, in row i, all appearing entries are greater than or equal to i. We will denote the set of all King tableaux of shape ν by $\mathrm{King}_n(\nu)$. For $\mathrm{T} \in \mathrm{SSYT}_{\mathcal{K}_n}(\lambda)$, we say that all entries k in row j smaller than j are **symplectic violations**. For a semi-standard tableau $\mathrm{T} \in \mathrm{SSYT}_{\mathcal{K}_n}(\lambda)$ of shape of at most $2n$ rows, we will use the same terminology, and by convention will call symplectic violations any entries in a row $j > n$ larger than n.

Remark 3. Let V be a complex vector space of even dimension $2n$ with a skew-symmetric bilinear form $\langle -,-\rangle$, and let

$$\mathrm{Sp}(2n,\mathbb{C}) = \{\mathrm{A} \in \mathrm{GL}(\mathrm{V}) : \langle \mathrm{A}v, \mathrm{A}v\rangle = \langle v, v\rangle\}$$

be the symplectic group of transformations invariant under the form $\langle -,-\rangle$. Under an appropriate choice of basis, the set

$$\mathrm{H} = \mathrm{diag}(x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}) \in \mathrm{Sp}(2n,\mathbb{C}), x_i \in \mathbb{C}^{\times}$$

of diagonal matrices is a maximal torus in $\mathrm{Sp}(2n,\mathbb{C})$. Finite dimensional irreducible representations of $\mathrm{Sp}(2n,\mathbb{C})$ are parametrised by shapes of at most n parts/rows. King tableaux are used to express the characters of these representations: if $\rho : \mathrm{Sp}(2n,\mathbb{C}) \to \mathrm{L}(\lambda)$ is the irreducible representation of highest weight the shape λ, then is character, the trace $\mathrm{sp}_\lambda = \mathrm{tr}(\rho)$ can be expressed as a polynomial function on H as

$$\mathrm{sp}_\lambda = \sum_{\mathrm{T} \in \mathrm{King}_n(\lambda)} \mathrm{wt}(\mathrm{T}) \tag{2}$$

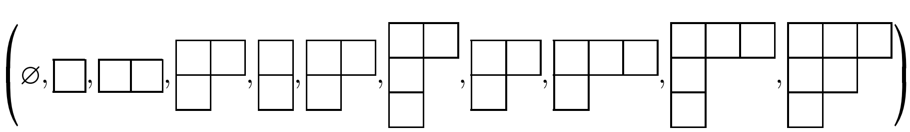
where, if $w(\mathrm{T}) = w_1 \cdots w_r$ denotes the word of T, read from right to left and top to bottom,

$$\mathrm{wt}(\mathrm{T}) = \mathrm{sgn}(w_1) \cdots \mathrm{sgn}(w_r)$$

with $\mathrm{sgn}(i) = x_i$ and $\mathrm{sgn}(\bar{i}) = x_i^{-1}$ for $i = 1, \cdots, n$. It is in this sense that King tableaux are analogues of usual semi-standard Young tableaux, which are used in the representation theory of $\mathrm{SL}(n,\mathbb{C})$. The formula (2) is the analogue of the original formula for the usual Schur functions/characters of representations of $\mathrm{SL}(n,\mathbb{C})$, where $\mathrm{SSYT}_{\mathcal{K}_\mathrm{n}}(\lambda)$ is replaced by the usual set of semi-standard Young tableaux in the alphabet $1 < \cdots < n$.

An **up-down sequence of shapes** is a sequence $(\varnothing = \mu_0, \mu_1, \cdots, \mu_k)$ where each μ_i is a Young diagram and μ_i/μ_{i-1} differ by exactly one box for every $1 \leq i \leq n$, and such that each μ_i has, at most, n rows.

Example 4. An up-down sequence of shapes of length 10:



2.2. **The Berele algorithm.** Berele's theorem reads as follows.

Theorem 5 (Berele)**.** *There is a bijection between the set of words of length k in the alphabet $\mathcal{K}_n$ and pairs $(\mathrm{P}_\mu, \mathrm{S})$, where P_μ is a King tableau of shape μ and S is an up-down sequence of length k and final shape μ.*

Proof. The Berele algorithm asigns, to each word in the alphabet $\mathcal{K}_n$, a King tableau and an up-down sequence of shapes in the following way. The idea is to follow the usual Robinson-Schensted-Knuth algorithm, as long as there are no symplectic violations. Set $i = 0$, $\mu^0 = \varnothing$ and $\mathrm{P}^0 = \varnothing$. Repreat the following process until $i = r$.

(1) Set $i \to i+1$ and $\mathrm{P} = w_i \to \mathrm{P}^{i-1}$.
(2) If P is a King tableau, then set $\mathrm{P}^i = \mathrm{P}$, $\mu^i = \mathrm{sh}(\mathrm{P})$ and go to step (1). Otherwise, there must be a symplectic violation. It is not complicated to realise that the first time such a violation is introduced in the tableau will happen precisely when a letter i bumps an $\bar{i}$ out of row i and into row $i+1$. If this is the case, bump the entries that you can without producing a symplectic violation, and then, when you reach the first point at which an i is about to bump a $\bar{i}$ out of row i and into row $i+1$, stop the bumping, and, instead, do the following:
   (a) Replace the $\bar{i}$ by an i.
   (b) Replace the first i in that column by a black dot.
   (c) Apply jeu-de taquin to the dot, sliding it east and south, out of the tableau.

   Set $\mathrm{P}^i = \mathrm{P}$, $\mu^i = \mathrm{sh}(\mathrm{P})$. If $i < k$, go back to Step (1). Else stop the procedure.

For a full proof we refer the reader to Sundaram's thesis [9] or to the original paper by Berele [2]. □

Example 6. Let $w = \bar{1}321\bar{2}21211$. If we apply the Berele algorithm to w, the resulting sequence of King tableaux/ up down sequence of shapes are as follows.

$$\begin{ytableau} \bar{1} \end{ytableau}, \begin{ytableau} \bar{1} & 3 \end{ytableau}, \begin{ytableau} \bar{1} & 2 \\ 3 \end{ytableau}, \begin{ytableau} 2 \\ 3 \end{ytableau}, \begin{ytableau} 2 & \bar{2} \\ 3 \end{ytableau}, \begin{ytableau} 2 & 2 \\ \bar{2} \\ 3 \end{ytableau}, \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}, \begin{ytableau} 1 & 2 & 2 \\ 3 \end{ytableau}, \begin{ytableau} 1 & 1 & 2 \\ 2 \\ 3 \end{ytableau}, \begin{ytableau} 1 & 1 & 1 \\ 2 & 2 \\ 3 \end{ytableau}$$

$$\ydiagram{1}, \ydiagram{2}, \ydiagram{2,1}, \ydiagram{1,1}, \ydiagram{2,1}, \ydiagram{2,1,1}, \ydiagram{2,1}, \ydiagram{3,1}, \ydiagram{3,1,1}, \ydiagram{3,2,1}$$

3. Sundaram's refinement

In her thesis, Sundaram gave a very useful refinement of Berele's bijection. The idea is as follows. To each up-down tableau, one may associate two things: a standard Young tableau and a Littlewood-Richardson Sundaram tableau of the same shape. The contents of this section are in their majority already exposed in [8] but we provide a quick description of the comfort of the reader.

3.1. **The Burge correspondence.** We recall some facts about the combinatorics of two-line arrangements which we need in this next section.

Definition 7. A *special two-line array* is a two-line array of distinct positive integers

$$\begin{bmatrix} j_1 & \cdots & j_r \\ i_1 & \cdots & i_r \end{bmatrix}$$

such that

1. $j_1 < \cdots < j_r$
2. $j_s > i_s, s \in \{1, \cdots, r\}$

Consider a special two-line array as above and let $s_1, \cdots, s_r$ be the reordering of the index set $\{1, \cdots, r\}$ such that $i_{s_1} < \cdots < i_{s_r}$. We can obtain a standard Young tableau from our array by column bumping its entries:

$$j_{s_1} \to \cdots \to j_{s_r} \to i_1 \to \cdots \to i_r \to \varnothing.$$

Example 8. Consider the two-line arrangement

$$\begin{bmatrix} 6 & 10 \\ 3 & 1 \end{bmatrix}$$

We have

$$1 \to \varnothing = \begin{array}{|c|}\hline 1 \\ \hline\end{array},\ 3 \to \begin{array}{|c|}\hline 1 \\ \hline\end{array} = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array},\ 6 \to \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array} = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 6 \\ \hline\end{array},\ 10 \to \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 6 \\ \hline\end{array} = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 6 \\ \hline 10 \\ \hline\end{array}$$

which is the same as

$$10 \to 6 \to 3 \to 1 \to \varnothing = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 6 \\ \hline 10 \\ \hline\end{array}$$

Definition 9. A partition is *even* if every column in its corresponding Young diagram/shape has an even number of boxes. If the partition shape of a (semi) standard Young tableau is even, we will say that the tableau has *even shape*.

Example 10. The standard Young tableau in Example 8 has even shape.

The following theorem follows is known as the Burge correspondence [1] (see also Theorem 3.31 in [9]) and Lemma 10.7 in Sundaram's thesis [9].

Theorem 11. The assignment above defines a bijection between special two-line arrangements and standard Young tableaux of even shape.

3.2. **Sundaram's refinement.** We may associate a standard Young tableaux to every shape in a given up-down sequence, ending up with a standard Young tableau of shape μ and a standard Young tableau of even shape ν (this means every column has an even number of boxes). We do this in the spirit of the Robinson-Schensted-Knuth correspondence: At each step j of the sequence of shapes, write a "j" inside the new box if a new box was added at this step, and, if a box was removed, column-bump the entry corresponding to that box out of the standard Young tableau that you had at step $j-1$. This procedure defines a standard Young tableau of shape μ which will be obtained at the end of the sequence. We will denote it by $\mathrm{Q}^p_{\mathrm{T}}$ and call it the partial Q-symbol. The tableau at step j in the sequence will be referred to as *the partial* Q*-symbol at step j.* To obtain the even permutation we do the following. Save the step j together with the bumped-out entry r_j in a special two-line array as $\begin{bmatrix} j \\ r_j \end{bmatrix}$ and concatenate the two-line arrays obtained every time there is a box-removal, with the first one always left-most. In the end we get a special two-line array which we will denote by $\mathrm{Ł}_{\mathrm{T}}$. Now use the method described in Section 3.1 to produce a standard Young tableau of even shape; denote it by E_{T}. With notation as in Section 3.1 for $\mathrm{Ł}_{\mathrm{T}}$, we have:

$$\mathrm{E}_{\mathrm{T}} = j_{s_1} \to \cdots \to j_{s_r} \to i_1 \to \cdots \to i_r \to \varnothing.$$

Now we produce one last standard Young tableau, the "final" Q-symbol:

$$\mathrm{Q}_{\mathrm{T}} = j_{s_1} \to \cdots \to j_{s_r} \to i_1 \to \cdots \to i_r \to \mathrm{Q}^p_{\mathrm{T}}.$$

Proposition 12 (Sundaram)**.** Let $\mathrm{T} \in \mathrm{domres}(\lambda, \mu)$. Then Q_{T} has shape λ. Moreover, any two elements in $\mathrm{domres}(\lambda, \mu)$ have the same final Q-symbol.

Recall that Littlewood-Richardson Sundaram tableaux are certain skew tableaux, filled with numbers from 1 to n, and such that its word is a dominant path, with weight, a shape consisting only of even columns. Such a shape is said to be **even**. We will denote the set of Littlewood-Richardson Sundaram tableaux of skew shape λ/μ and weight ν by $\mathrm{LRS}(\lambda/\mu, \nu)$. When ν need not need be specified we denote the larger set by $\mathrm{LRS}(\lambda/\mu)$. We also say that elements in this set have shape λ, when μ is not immediatly relevant, and denote the corresponding set by $\mathrm{LRS}(\lambda)$. The first step towards the refinement is the following theorem.

Theorem 13. There is a one-to-one correspondence between the set of up-down sequences of shapes $\underline{\mu}$ with final shape μ, and the union $\bigcup_{\lambda \supset \mu} \mathrm{LRS}(\lambda/\mu)$.

To assign a Littlewood-Richardson tableau to a final Q-symbol, we "fill up" the skew shape λ/μ as follows: for each entry j in E_{T}, let r_j be the row to which it belongs (in E_T). Write this number in the skew shape λ/μ in the row of Q_T where j lies. In the proof of Theorem 8.11 in [9] (and Theorem 9.4) it is shown that this process yields a Littlewood Richardson Sundaram tableau $\phi(T) \in \mathrm{LRS}(\lambda/\mu,\eta)$. In fact, the following Theorem holds (it is stated and proven in Theorems 8.14 and 9.4) of [9].

Theorem 14 (Sundaram)**.** Berele's algorithm gives a bijection between the set $\mathrm{SSYT}_{\mathcal{K}}(\lambda)$ of semistandard Young tableaux of shape λ in the alphabet $1 < \bar{1} \cdots < n < \bar{n}$ and the set $\bigcup_{\substack{\mu\subset\lambda \\ \nu \text{ even}}} \mathrm{King}_n(\mu) \times \mathrm{LRS}(\lambda/\mu,\nu)$.

The details of the proof can be found in [9]. We recall from there the inverse map (see [9]); we will need it in the next section.

Start off with $L \in \mathrm{LRS}(\lambda/\mu)$ and $P \in \mathrm{King}_n(\mu)$. The deleted boxes are added to P in pairs, in the inverse order in which they were deleted. First, identify all pairs in L of the form $(2i+1, 2i+2)$, starting with $i = 0$. Identify the upper-most and left-most such pair. Order all such pairs in this fashion, and when all the pairs run out, continue to $i = 1$ and so on. Now, if the first such marked pair occurred in rows $(i, i+1)$ add a pair of blank boxes at the same location as the pair, and perform jeu-de taquin until the first column; add the pair $(i, \bar{i})$ to P where the boxes are. It is possible to do this such that the boxes are in two consecutive rows. Call this new tableau P_1 continue in this fashion, but now, if a pair $(k, \bar{k})$ has already been added in rows $(j-2, j-1)$, add a $(k+1, \overline{k+1})$. Otherwise, add a $(j, \bar{j})$.

Example 15. Let $L = \begin{array}{|c|c|}\hline & 1 \\ \hline & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$ and $P = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$. We perform the procedure described above:

$$P \to \begin{array}{|c|c|}\hline 1 & \\ \hline 2 & \\ \hline\end{array} \to \begin{array}{|c|c|}\hline & 1 \\ \hline & 2 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \bar{1} & 2 \\ \hline\end{array}$$

$$\to \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \bar{1} & 2 \\ \hline \\ \cline{1-1} \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \bar{1} & 2 \\ \hline 2 \\ \cline{1-1} \bar{2} \\ \cline{1-1}\end{array}$$

4. Dominant classes

A tableau $T \in \mathrm{SSYT}_{\mathcal{K}}(\lambda)$ is **dominant** if its word, read from right to left and top to bottom, is a lattice path.

Lemma 16. Let $T \in \mathrm{SSYT}_{\mathcal{K}}(\lambda)$ be dominant. Then if an unbarred entry i appears in T in row $j \leq n$, then $i \leq j$. Moreover all barred letters are cancellations, and all cancellations, symplectic violations.

Proof. If there is a 2 in row 1, the tableau cannot possibly be dominant. If there is a $j > 2$ in row $k < j$, then, by dominance, there must exist a $j-1$

in row $l < k < j$; in particular, $l < j - 1$, which contradicts the induction hypothesis. The first part of the Lemma then follows by induction. Now, that all barred letters are cancellations follows directly from the definition of dominance. Let $\bar{i}$ be a barred letter appearing in row $j \leq n$. We know that $i < \bar{i}$ and that $\bar{i}$ is a cancellation, that is, there exists a row $r < j$ containing an i in the north-eastern quadrant determined by $\bar{i}$. By the first part of the lemma, $i \leq r < j$. Now, if $\bar{i} > j$, we would have $i \leq r < j < \bar{i}$, which is impossible. We conclude that $\bar{i} < j$ since $\bar{i} = j$ is impossible (i and j are both unbarred), i.e. that $\bar{i}$ is a symplectic violation. $\square$

We denote the set of all dominant tableaux in $\mathrm{SSYT}(\lambda)$ by $\mathrm{domres}_{\mathcal{K}}(\lambda)$, and by $\mathrm{domres}_{\mathcal{K}}(\lambda, \mu)$ those therein of weight μ. Recall that the Berele algorithm gives a bijection

$$\mathrm{SSYT}_{\mathcal{K}}(\lambda) \xrightarrow{\Phi} \bigcup_{\substack{\mu \subset \lambda \\ \nu \text{ even}}} \mathrm{King}_n(\mu) \times \mathrm{LRS}(\lambda/\mu, \nu)$$
$$\mathrm{T} \mapsto (\Phi_1(\mathrm{T}), \Phi_2(\mathrm{T})).$$

Theorem 17. The Berele algoritm defines an injective map

$$\mathrm{domres}_{\mathcal{K}}(\lambda, \mu) \xrightarrow{\tilde{\Phi}} \mathrm{King}_n(\mu, \mu) \times \bigcup_{\nu \text{even}} \mathrm{LRS}(\lambda/\mu, \nu)$$

where $\mathrm{King}_n(\mu, \mu)$ is the set containing the unique King tableau of shape μ with only i's in row i.

Proof. First we show that the Berele algorithm indeed defines such a map. Let $\mathrm{T} \in \mathrm{domres}_{\mathcal{K}}(\lambda, \mu)$. We need to show that $\Phi_1(\mathrm{T})$ is the tableau in $\mathrm{King}_n(\mu, \mu)$. We call a **deletion** a pair $(i, \bar{i})$ which is eliminated from the (inverse) word of T during the process of performing the Berele algorithm. Therefore, Lemma 16 guarantees that every barred letter in T will contribute to a deletion. Moreover, since it is impossible to have a deletion without a barred letter, it follows that deletions can only arise in this way. Thus $\Phi_1(\mathrm{T})$ is a symplectic tableau with no barred letters and weight μ. Now we observe that whenever a deletion occurs, we inevitably delete an i from row i. Moreover, all the remaining letters r in the tableau will, at this step, be moved to the left or up. By Lemma 16, these letters satisfy $r \leq j$, where j is their row. But since the procedure will move them up a row, at most, and since the final result $\Phi_1(\mathrm{T})$ contains no symplectic violations, then $\Phi_1(\mathrm{T})$ has only i's in row i. And since there is only one tableau with these properties, we conclude the desired result. Now, Theorem 14 affirms that any $\mathrm{T} \in \mathrm{SSYT}_{\mathcal{K}}(\lambda)$ is mapped by Φ_2 to a Littlewood-Richardson Sundaram tableau of shape λ, so our map is well defined. The injectivity of the map also follows from Sundaram's results (Proposition 12). $\square$

The map is, however, not surjective.

Example 18. Let $n > 2, \lambda = (1,1,1,1)$ and $\mu = \varnothing$. Then also $\operatorname{domres}_{\mathcal{K}}(\lambda,\mu) = \varnothing$, however, the tableau $\begin{array}{|c|}\hline 1\\\hline \bar{1}\\\hline 2\\\hline \bar{2}\\\hline\end{array}$, which is the result of performing the procedure described above to obtain $\Phi^{-1}(\varnothing, \begin{array}{|c|}\hline \ \\\hline \ \\\hline \ \\\hline \ \\\hline\end{array})$ is not dominant. In contrast, in the set-up ([8]) of the Naito-Sagaki conjecture, where the ordered alphabet considered is

$$\mathcal{C}_n = \{1 < \cdots < n < \bar{n} < \cdots \bar{1}\},$$

the column $\begin{array}{|c|}\hline 1\\\hline 2\\\hline \bar{2}\\\hline \bar{1}\\\hline\end{array}$ is dominant and is in this case the missing element needed for surjectivity. Note that in Example 15, the tableau constructed is also not dominant.

4.1. **A comparison.** In the following example we illustrate an analogy to the set-up of the Naito-Sagaki conjecture. Compare to Example 19. in [8].

Example 19. Let $n = 3$, $\lambda = (4,3,2,1)$ and $\mu = (3,2,1)$. We have

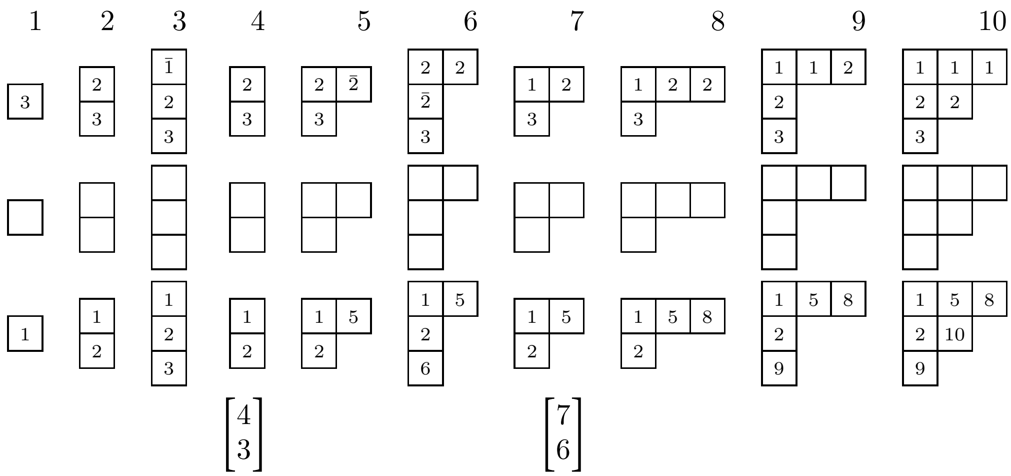
$$\mathcal{K} = \{1 < \bar{1} < 2 < \bar{2} < 3 < \bar{3}\}$$

and

$$\operatorname{domres}_{\mathcal{K}}(\lambda,\mu) = \left\{ \mathrm{T}_1 := \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1\\\hline \bar{1} & 2 & 2 \\\cline{1-3} 3 & 3 \\\cline{1-2} \bar{3} \\\cline{1-1}\end{array}, \mathrm{T}_2 := \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1\\\hline \bar{1} & 2 & 2 \\\cline{1-3} 2 & 3 \\\cline{1-2} \bar{2} \\\cline{1-1}\end{array}, \mathrm{T}_3 := \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1\\\hline \bar{1} & 2 & 2 \\\cline{1-3} 2 & \bar{2} \\\cline{1-2} 3 \\\cline{1-1}\end{array} \right\}.$$

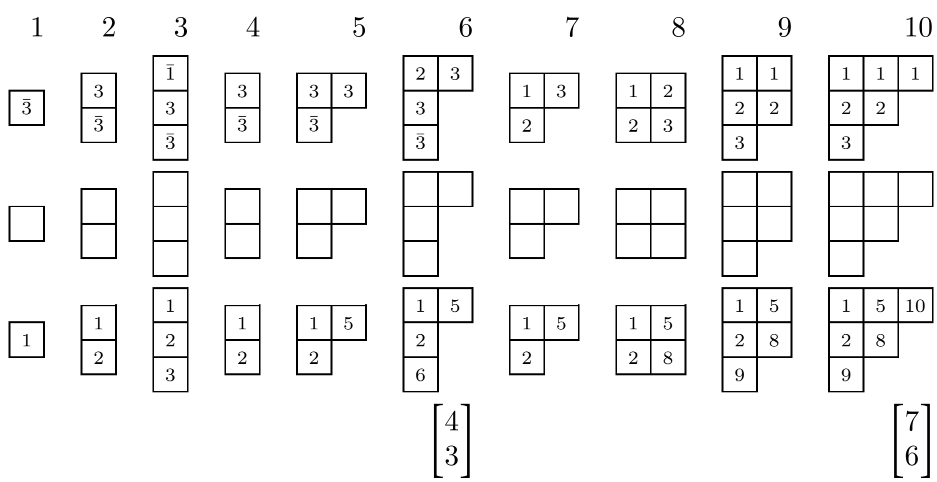
Now, applying Sundaram's refinement of the Berele algorithm using the Burge correspondence, we get the following data:

$$\mathrm{T}_1 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1\\\hline \bar{1} & 2 & 2 \\\cline{1-3} 2 & \bar{2} \\\cline{1-2} 3 \\\cline{1-1}\end{array}$$



$$\mathrm{E}_{\mathrm{T}_1} = \begin{array}{|c|c|}\hline 3 & 6 \\ \hline 4 & 7 \\ \hline \end{array}, \qquad \mathrm{Q}_{\mathrm{T}_1} = \begin{array}{|c|c|c|c|}\hline 1 & 5 & 8 & 10 \\ \hline 2 & 6 & 9 \\ \cline{1-3} 3 & 7 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}, \qquad \phi(\mathrm{T}_1) = \begin{array}{|c|c|c|c|}\hline \ & \ & \ & 1 \\ \hline \ & \ & 1 \\ \cline{1-3} \ & 2 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array}, \qquad Ł_{\mathrm{T}_1} = \begin{bmatrix} 4 & 7 \\ 3 & 6 \end{bmatrix}.$$

$$\mathrm{T}_2 := \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline \bar{1} & 2 & 2 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} \bar{3} \\ \cline{1-1} \end{array}$$



$$\mathrm{E}_{\mathrm{T}_2} = \begin{array}{|c|c|}\hline 3 & 6 \\ \hline 4 & 7 \\ \hline \end{array}, \qquad \mathrm{Q}_{\mathrm{T}_2} = \begin{array}{|c|c|c|c|}\hline 1 & 5 & 8 & 10 \\ \hline 2 & 6 & 9 \\ \cline{1-3} 3 & 7 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}, \qquad \phi(\mathrm{T}_2) = \begin{array}{|c|c|c|c|}\hline \ & \ & \ & 1 \\ \hline \ & \ & 2 \\ \cline{1-3} \ & 1 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array}, \qquad Ł_{\mathrm{T}_2} = \begin{bmatrix} 4 & 7 \\ 3 & 6 \end{bmatrix}.$$

$$\mathrm{T}_3 = \begin{ytableau} 1 & 1 & 1 & 1 \\ \bar{1} & 2 & 2 \\ 2 & 3 \\ \bar{2} \end{ytableau}$$

1	2	3	4	5	6	7	8	9	10
$\begin{ytableau} \bar{2} \end{ytableau}$	$\begin{ytableau} 2 \\ \bar{2} \end{ytableau}$	$\begin{ytableau} \bar{1} \end{ytableau}$	$\varnothing$	$\begin{ytableau} 3 \end{ytableau}$	$\begin{ytableau} 2 \\ 3 \end{ytableau}$	$\begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau}$	$\begin{ytableau} 1 & 2 \\ 2 \\ 3 \end{ytableau}$	$\begin{ytableau} 1 & 1 \\ 2 & 2 \\ 3 \end{ytableau}$	$\begin{ytableau} 1 & 1 & 1 \\ 2 & 2 \\ 3 \end{ytableau}$
$\ydiagram{1}$	$\ydiagram{1,1}$	$\ydiagram{1}$	$\varnothing$	$\ydiagram{1}$	$\ydiagram{1,1}$	$\ydiagram{1,1,1}$	$\ydiagram{2,1,1}$	$\ydiagram{2,2,1}$	$\ydiagram{3,2,1}$
$\begin{ytableau} 1 \end{ytableau}$	$\begin{ytableau} 1 \\ 2 \end{ytableau}$	$\begin{ytableau} 1 \end{ytableau}$	$\varnothing$	$\begin{ytableau} 5 \end{ytableau}$	$\begin{ytableau} 5 \\ 6 \end{ytableau}$	$\begin{ytableau} 5 \\ 6 \\ 7 \end{ytableau}$	$\begin{ytableau} 5 & 8 \\ 6 \\ 7 \end{ytableau}$	$\begin{ytableau} 5 & 8 \\ 6 & 9 \\ 7 \end{ytableau}$	$\begin{ytableau} 5 & 8 & 10 \\ 6 & 9 \\ 7 \end{ytableau}$
		$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$						

$$\mathrm{E}_{\mathrm{T}_3} = \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau}, \qquad \mathrm{Q}_{\mathrm{T}_3} = \begin{ytableau} 1 & 5 & 8 & 10 \\ 2 & 6 & 9 \\ 3 & 7 \\ 4 \end{ytableau}, \qquad \phi(\mathrm{T}_3) = \begin{ytableau} \ & \ & \ & 1 \\ \ & \ & 2 \\ \ & 3 \\ 4 \end{ytableau}, \qquad \text{Ł}_{\mathrm{T}_3} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}.$$

References

[1] W. H. Burge. Four correspondences between graphs and generalized young tableaux. *J. Combinatorial Theory(A)*, 17:12–30, 1974.

[2] A. Berele A Schensted-type correspondence for the Symplectic Group. *J. Combinatorial Theory(A)*, 43(2):320-328, 1986.

[3] R. Howe and S. T. Lee. Why should the Littlewood-Richardson rule be true? *Bull. Amer. Math. Soc. (N.S.)*, 49(2):187–236, 2012.

[4] J-H. Kwon Combinatorial extension of stable branching rules for classical groups To appear in Transactions of the American Mathematical Society arXiv:1512.01877, 2015.

[5] C. Lecouvey and C. Lenart Combinatorics of generalized exponents arXiv:1707.03314 2017.

[6] D. E. Littlewood. *The Theory of Group Characters*. Oxford University Press, 1950. 2nd. ed.

[7] S. Naito and D. Sagaki. An approach to the branching rule from $\mathfrak{sl}_{2n}(\mathbb{C})$ to $\mathfrak{sp}_{2n}(\mathbb{C})$ via Littelmann's path model. *J. Algebra*, 286(1):187–212, 2005.

[8] B. Schumann and J. Torres. A non-Levi branching rule in terms of Littelmann paths, arxiv:arXiv:1607.08225 2016.

[9] S. Sundaram. *ON THE COMBINATORICS OF REPRESENTATIONS OF THE SYMPLECTIC GROUP*. ProQuest LLC, Ann Arbor, MI, 1986. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[10] S. Sundaram. Tableaux in the representation theory of the classical Lie groups. In *Invariant theory and tableaux (Minneapolis, MN, 1988)*, volume 19 of *IMA Vol. Math. Appl.*, pages 191–225. Springer, New York, 1990.

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