Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Max-relative Entropy of Coherence: An Operational Coherence Measure

by

Kaifeng Bu, Uttam Singh, Shao-Ming Fei,
Arun Kumar Pati, and Junde Wu

Preprint no.: 7 2018

Max- relative entropy of coherence : an operational coherence measure

Kaifeng Bu,[1,*] Uttam Singh,[2,3,†] Shao-Ming Fei,[4,5,‡] Arun Kumar Pati,[2,§] and Junde Wu[1,¶]

[1] *School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, PR China*
[2] *Harish-Chandra Research Institute, Allahabad, 211019, India*
[3] *Homi Bhabha National Institute, Training School Complex, Anushakti Nagar, Mumbai 400085, India*
[4] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, PR China*
[5] *Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany*

The operational characterization of quantum coherence is the corner stone in the development of resource theory of coherence. We introduce a new coherence quantifier based on max-relative entropy. We prove that max-relative entropy of coherence is directly related to the maximum overlap with maximally coherent states under a particular class of operations, which provides an operational interpretation of max-relative entropy of coherence. Moreover, we show that, for any coherent state, there are examples of subchannel discrimination problems such that this coherent state allows for a higher probability of successfully discriminating subchannels than that of all incoherent states. This advantage of coherent states in subchannel discrimination can be exactly characterized by the max-relative entropy of coherence. By introducing suitable smooth max-relative entropy of coherence, we prove that the smooth max-relative entropy of coherence provides a lower bound of one-shot coherence cost, and the max-relative entropy of coherence is equivalent to the relative entropy of coherence in asymptotic limit. Similar to max-relative entropy of coherence, min-relative entropy of coherence has also been investigated. We show that the min-relative entropy of coherence provides an upper bound of one-shot coherence distillation, and in asymptotic limit the min-relative entropy of coherence is equivalent to the relative entropy of coherence.

I. INTRODUCTION

Quantumness in a single system is characterized by quantum coherence, namely, the superposition of a state in a given reference basis. The coherence of a state may quantify the capacity of a system in many quantum manipulations, ranging from metrology [1] to thermodynamics [2, 3] . Recently, various efforts have been made to develop a resource theory of coherence [4–10]. One of the earlier resource theories is that of quantum entanglement [11], which is a basic resource for various quantum information processing protocols such as superdense coding [12], remote state preparation [13, 14] and quantum teleportation [15]. Other notable examples include the resource theories of asymmetry [16–22], thermodynamics [23], and steering [24]. One of the main advantages that a resource theory offers is the lucid quantitative and operational description as well as the manipulation of the relevant resources at ones disposal, thus operational characterization of quantum coherence is required in the resource theory of coherence.

A resource theory is usually composed of two basic elements: free states and free operations. The set of allowed states (operations) under the given constraint is what we call the set of free states (operations). Given a fixed basis, say $\{\,|i\rangle\}_{i=0}^{d-1}$ for a d-dimensional system, any quantum state which is diagonal in the reference basis is called an incoherent state and is a free state in the resource theory of coherence. The set of incoherent states is denoted by $\mathscr{I}$. Any quantum state can be mapped into an incoherent state by a full dephasing operation Δ, where $\Delta(\rho) := \sum_{i=0}^{d-1} \langle i|\rho|i\rangle\, |i\rangle\langle i|$. However, there is no general consensus on the set of free operations in the resource theory of coherence. We refer the following types of free operations in this work: maximally incoherent operations (MIO) [25], incoherent operations (IO) [4], dephasing-covariant operations (DIO) [25] and strictly incoherent operations (SIO) [10, 25]. By maximally incoherent operation (MIO), we refer to the maximal set of quantum operations Φ which maps the incoherent states into incoherent states, i.e., $\Phi(\mathscr{I}) \subset \mathscr{I}$ [25]. Incoherent operations (IO) is the set of all quantum operations Φ that admit a set of Kraus operators $\{K_i\}$ such that $\Phi(\cdot) = \sum_i K_i(\cdot)K_i^\dagger$ and $K_i \mathscr{I} K_i^\dagger \subset \mathscr{I}$ for any i [4]. Dephasing-covariant operations (DIO) are the quantum operations Φ with $[\Delta, \Phi] = 0$ [25]. Strictly incoherent operations (SIO) is the set of all quantum operations Φ admitting a set of Kraus operators $\{K_i\}$ such that $\Phi(\cdot) = \sum_i K_i(\cdot)K_i^\dagger$ and $\Delta(K_i\rho K_i^\dagger) = K_i\Delta(\rho)K_i^\dagger$ for any i and any quantum state ρ. Both IO and DIO are subsets of MIO , and SIO is a subset of both IO and DIO [25]. However, IO and DIO are two different types of free operations and there is no inclusion relationship between them (The operational gap between them can be seen in [26]).

Several operational coherence quantifiers have been introduced as candidate coherence measures, subjecting to physical requirements such as monotonicity under certain type of free operations in the resource theory of coherence. One canonical measure to quantify coherence is the relative entropy of coherence, which is defined as $C_r(\rho) = S(\Delta(\rho)) - S(\rho)$, where $S(\rho) = -\mathrm{Tr}\,[\rho \log \rho]$ is the von Neumann entropy [4]. The relative entropy of coherence plays an important role in the process of coherence distillation, in which it can be interpreted as the optimal rate to distill maximally coherent state from a given state ρ by IO in the asymptotic limit [7]. Besides, the l_1 norm of coherence [4], which is defined

* bkf@zju.edn.cn
† uttamsingh@hri.res.in
‡ feishm@cnu.edn.cn
§ akpati@hri.res.in
¶ wjd@zju.edn.cn

as $C_{l_1}(\rho)=\sum_{i\neq j}|\rho_{ij}|$ with $\rho_{ij}=\langle i|\rho|j\rangle$, has also attracted lots of discussions about its operational interpretation [27]. Recently, an operationally motivated coherence measure- robustness of coherence (RoC) - has been introduced, which quantifies the minimal mixing required to erase the coherence in a given quantum state [28, 29]. There is growing concern about the operational characterization of quantum coherence and further investigations are needed to provide an explicit and rigorous operational interpretation of coherence.

In this letter, we introduce a new coherence measure based on max-relative entropy and focus on its operational characterizations. Max- and min- relative entropies have been introduced and investigated in [30–33]. The well-known (conditional and unconditional) max- and min- entropies [34, 35] can be obtained from these two quantities. It has been shown that max- and min-entropies are of operational significance in the applications ranging from data compression [34, 36] to state merging [37] and security of key [38, 39]. Besides, max- and min- relative entropies have been used to define entanglement monotone and their operational significance in the manipulation of entanglement has been provided in [30–33]. Here, we define max-relative of coherence $C_{\max}$ based on max-relative entropy and investigate the properties of $C_{\max}$. We prove that max-relative entropy of coherence for a given state ρ is the maximum achievable overlap with maximally coherent states under DIO, IO and SIO, which gives rise to an operational interpretation of $C_{\max}$ and shows the equivalence among DIO, IO and SIO in an operational task. Besides, we show that max-relative entropy of coherence characterizes the role of quantum states in an operational task: subchannel discrimination. Subchannel discrimination is an important quantum information task which distinguishes the branches of a quantum evolution for a quantum system to undergo [40]. It has been shown that every entangled or steerable state is a resource in some instance of subchannel discrmination problems [40, 41]. Here, we prove that that every coherent state is useful in the subchannel discrimination of certain instruments, where the usefulness can be quantified by the max-relative entropy of coherence of the given quantum state. By smoothing the max-relative entropy of coherence, we introduce $\varepsilon-$smoothed max-relative entropy of coherence $C^{\varepsilon}_{\max}$ for any fixed $\varepsilon>0$ and show that the smooth max-relative entropy gives an lower bound of coherence cost in one-shot version. Moreover, we prove that for any quantum state, max-relative entropy of coherence is equivalent to the relative entropy of coherence in asymptotic limit.

Corresponding to the max-relative entropy of coherence, we also introduce the min-relative entropy of coherence $C_{\min}$ by min-relative entropy, which is not a proper coherence measure as it may increase on average under IO. However, it gives an upper bound for the maximum overlap between the given states and the set of incoherent states. This implies that min-relative entropy of coherence also provides a lower bound of a well-known coherence measure, geometry of coherence [6]. By smoothing the min-relative entropy of coherence, we introduce $\varepsilon-$smoothed min-relative entropy of coherence $C^{\varepsilon}_{\min}$ for any fixed $\varepsilon>0$ and show that the smooth max-relative entropy gives an upper bound of coherence distillation in one-shot version. Furthermore, we show that the min-relative of coherence is also equivalent to distillation of coherence in asymptotic limit. The relationship among $C_{\min}$, $C_{\max}$ and other coherence measures has also been investigated.

II. MAIN RESULTS

Let $\mathscr{H}$ be a d-dimensional Hilbert space and $\mathscr{D}(\mathscr{H})$ be the set of density operators acting on $\mathscr{H}$. Given two operators ρ and σ with $\rho\geq 0$, $\mathrm{Tr}[\rho]\leq 1$ and $\sigma\geq 0$, the max-relative entropy of ρ with respect to σ is defined by [30, 31],

$$D_{\max}(\rho||\sigma):=\min\{\lambda:\rho\leq 2^{\lambda}\sigma\}. \tag{1}$$

We introduce a new coherence quantifier by max-relative entropy: max-relative entropy of coherence $C_{\max}$,

$$C_{\max}(\rho):=\min_{\sigma\in\mathscr{I}} D_{\max}(\rho||\sigma), \tag{2}$$

where $\mathscr{I}$ is the set of incoherent states in $\mathscr{D}(\mathscr{H})$.

We now show that $C_{\max}$ satisfies the conditions a coherence measure needs to fulfil. First, it is obvious that $C_{\max}(\rho)\geq 0$. And since $D_{\max}(\rho||\sigma)=0$ iff $\rho=\sigma$ [30], we have $C_{\max}(\rho)=0$ if and only if $\rho\in\mathscr{I}$. Besides, as $D_{\max}$ is monotone under CPTP maps [30], we have $C_{\max}(\Phi(\rho))\leq C_{\max}(\rho)$ for any incoherent operation Φ. Moreover, $C_{\max}$ is nonincreasing on average under incoherent operations, that is, for any incoherent operation $\Phi(\cdot)=\sum_i K_i(\cdot)K_i^{\dagger}$ with $K_i\mathscr{I}K_i^{\dagger}\subset\mathscr{I}$, $\sum_i p_iC_{\max}(\tilde{\rho}_i)\leq C_{\max}(\rho)$, where $p_i=\mathrm{Tr}\left[K_i\rho K_i^{\dagger}\right]$ and $\tilde{\rho}_i=K_i\rho K_i^{\dagger}/p_i$, see proof in Supplemental Material [42].

Remark We have proven that the max-relative entropy of coherence $C_{\max}$ is a bona fide measure of coherence. Since $D_{\max}$ is not jointly convex, we may not expect that $C_{\max}$ has the convexity, which is a desirable (although not a fundamental) property for a coherence quantifier. However, we can prove that for $\rho=\sum_i^n p_i\rho_i$, $C_{\max}(\rho)\leq\max_i C_{\max}(\rho_i)$. Suppose that $C_{\max}(\rho_i)=D_{\max}(\rho_i||\sigma_i^*)$ for some σ_i^*, then from the fact that $D_{\max}(\sum_i p_i\rho_i||\sum_i p_i\sigma_i)\leq\max_i D_{\max}(\rho_i||\sigma_i)$ [30], we have $C_{\max}(\rho)\leq D_{\max}(\sum_i p_i\rho_i||\sum_i p_i\sigma_i^*)\leq\max_i D_{\max}(\rho_i||\sigma_i^*)=\max_i C_{\max}(\rho_i)$. Besides, although $C_{\max}$ is not convex, we can obtain a proper coherence measure with convexity from $C_{\max}$ by the approach of convex roof extension, see Supplemental Material [42].

In the following, we concentrate on the operational characterization of the max-relative entropy of coherence, and provide operational interpretations of $C_{\max}$.

Maximum overlap with maximally coherent states.—At first we show that $2^{C_{\max}}$ is equal to the maximum overlap with the maximally coherent state that can be achieved by DIO, IO and SIO.

Theorem 1. *Given a quantum state $\rho\in\mathscr{D}(\mathscr{H})$, we have*

$$2^{C_{\max}(\rho)}=d\max_{\mathscr{E},|\Psi\rangle}F(\mathscr{E}(\rho),|\Psi\rangle\langle\Psi|)^2, \tag{3}$$

where $F(\rho,\sigma)=\mathrm{Tr}\left[|\sqrt{\rho}\sqrt{\sigma}|\right]$ is the fidelity between states ρ and σ [43], $|\Psi\rangle\in\mathscr{M}$ and $\mathscr{M}$ is the set of maximally coherent states in $\mathscr{D}(\mathscr{H})$, $\mathscr{E}$ belongs to either DIO or IO or SIO.

(See proof in Supplemental Material [42].)

Here although IO, DIO and SIO are different types of free operations in resource theory of coherence [25, 26], they have the same behavior in the maximum overlap with the maximally coherent states. From the view of coherence distillation [7], the maximum overlap with maximally coherent states can be regarded as the distillation of coherence from given states under IO, DIO and SIO. As fidelity can be used to define certain distance, thus $C_{\max}(\rho)$ can also be viewed as the distance between the set of maximally coherent state and the set of $\{\mathscr{E}(\rho)\}_{\mathscr{E}\in\theta}$, where $\theta = DIO, IO$ or SIO.

Besides distillation of coherence, another kind of coherence manipulation is the coherence cost [7]. Now we study the one-shot version of coherence cost under MIO based on smooth max-relative entropy of coherence. We define the one-shot coherence cost of a quantum state ρ under MIO as

$$C^{(1),\varepsilon}_{C,MIO}(\rho) := \min_{\substack{\mathscr{E}\in MIO\\ M\in\mathbb{Z}}} \{\log M : F(\rho, \mathscr{E}(\,|\Psi^M_+\rangle\langle\Psi^M_+|))^2 \geq 1-\varepsilon\},$$

where $|\Psi^M_+\rangle = \frac{1}{\sqrt{M}}\sum_{i=1}^M |i\rangle$, $\mathbb{Z}$ is the set of integer and $\varepsilon > 0$. The ε-smoothed max-relative entropy of coherence of a quantum state ρ is defined by,

$$C^{\varepsilon}_{\max}(\rho) := \min_{\rho'\in B_\varepsilon(\rho)} C_{\max}(\rho'), \tag{4}$$

where $B_\varepsilon(\rho) := \{\rho' \geq 0 : \|\rho'-\rho\|_1 \leq \varepsilon, \mathrm{Tr}[\rho'] \leq \mathrm{Tr}[\rho]\}$. We find that the smooth max-relative entropy of coherence gives a lower bound of one-shot coherence cost. Given a quantum state $\rho \in \mathscr{D}(\mathscr{H})$, for any $\varepsilon > 0$,

$$C^{\varepsilon'}_{\max}(\rho) \leq C^{(1),\varepsilon}_{C,MIO}(\rho), \tag{5}$$

where $\varepsilon' = 2\sqrt{\varepsilon}$, see proof in Supplemental Material [42].

Besides, in view of smooth max-relative entropy of coherence, we can obtain the equivalence between max-relative entropy of coherence and relative entropy of coherence in the asymptotic limit. Since relative entropy of coherence is the optimal rate to distill maximally coherent state from a given state under certain free operations in the asymptotic limit [7], the smooth max-relative entropy of coherence in asymptotic limit is just the distillation of coherence. That is, given a quantum state $\rho \in \mathscr{D}(\mathscr{H})$, we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} C^{\varepsilon}_{\max}(\rho^{\otimes n}) = C_r(\rho). \tag{6}$$

(The proof is presented in Supplemental Material [42].)

Maximum advantage achievable in subchannel discrimination.– Now, we investigate another quantum information processing task: subchannel discrimination, which can also provide an operational interpretation of $C_{\max}$. Subchannel discrimination is an important quantum information task which is used to identify the branch of a quantum evolution to undergo. We consider some special instance of subchannel discrimination problem to show the advantage of coherent states.

A linear completely positive and trace non-increasing map $\mathscr{E}$ is called a subchannel. If a subchannel $\mathscr{E}$ is trace preserving, then $\mathscr{E}$ is called a channel. An instrument $\mathfrak{I} = \{\mathscr{E}_a\}_a$ for a channel $\mathscr{E}$ is a collection of subchannels $\mathscr{E}_a$ with $\mathscr{E} = \sum_a \mathscr{E}_a$ and every instrument has its physical realization [40]. A dephasing covariant instrument $\mathfrak{I}^D$ for a DIO $\mathscr{E}$ is a collection of subchannels $\{\mathscr{E}_a\}_a$ such that $\mathscr{E} = \sum_a \mathscr{E}_a$. Similarly, we can define incoherent instrument $\mathfrak{I}^I$ and strictly incoherent instrument $\mathfrak{I}^S$ for channel $\mathscr{E} \in IO$ and $\mathscr{E} \in SIO$ respectively.

Given an instrument $\mathfrak{I} = \{\mathscr{E}_a\}_a$ for a quantum channel $\mathscr{E}$, let us consider a Positive Operator Valued Measurement (POVM) $\{M_b\}_b$ with $\sum_b M_b = \mathbb{I}$. The probability of successfully discriminating the subchannels in the instrument $\mathfrak{I}$ by POVM $\{M_b\}_b$ for input state ρ is given by

$$p_{\mathrm{succ}}(\mathfrak{I}, \{M_b\}_b, \rho) = \sum_a \mathrm{Tr}[\mathscr{E}_a(\rho)M_a]. \tag{7}$$

The optimal probability of success in subchannel discrimination of $\mathfrak{I}$ over all POVMs is given by

$$p_{\mathrm{succ}}(\mathfrak{I}, \rho) = \max_{\{M_b\}_b} p_{\mathrm{succ}}(\mathfrak{I}, \{M_b\}_b, \rho). \tag{8}$$

If we restrict the input states to be incoherent ones, then the optimal probability of success among all incoherent states is given by

$$p^{ICO}_{\mathrm{succ}}(\mathfrak{I}) = \max_{\sigma\in\mathscr{I}} p_{\mathrm{succ}}(\mathfrak{I}, \sigma). \tag{9}$$

We have the following theorem.

Theorem 2. *Given a quantum state ρ, $2^{C_{\max}(\rho)}$ is the maximal advantage achievable by ρ compared with incoherent states in all subschannel discrimination problems of dephasing-covariant, incoherent and strictly incoherent instruments,*

$$2^{C_{\max}(\rho)} = \max_{\mathfrak{I}} \frac{p_{succ}(\mathfrak{I}, \rho)}{p^{ICO}_{succ}(\mathfrak{I})}, \tag{10}$$

where $\mathfrak{I}$ is either $\mathfrak{I}^D$ or $\mathfrak{I}^I$ or $\mathfrak{I}^S$, denoting the dephasing-covariant, incoherent and strictly incoherent instrument, respectively.

The proof of Theorem 2 is presented in Supplemental Material [42]. This result shows that the advantage of coherent states in certain instances of subchannel discrimination problems can be exactly captured by $C_{\max}$, which provides another operational interpretation of $C_{\max}$ and also shows the equivalence among DIO, IO and SIO in the information processing task of subchannel discrimination.

Min-relative entropy of coherence $C_{\min}(\rho)$.–Given two operators ρ and σ with $\rho \geq 0, \mathrm{Tr}[\rho] \leq 1$ and $\sigma \geq 0$, max- and min- relative entropy of ρ relative to σ are defined as

$$D_{\min}(\rho||\sigma) := -\log \mathrm{Tr}\left[\Pi_\rho \sigma\right] \tag{11}$$

where Π_ρ denotes the projector onto suppρ, the support of ρ. Corresponding to $C_{\max}(\rho)$ defined in (2), we can similarly introduce a quantity defined by min-relative entropy,

$$C_{\min}(\rho) := \min_{\sigma\in\mathscr{I}} D_{\min}(\rho||\sigma). \tag{12}$$

Since $D_{\min}(\rho||\sigma) = 0$ if $\mathrm{supp}\,\rho = \mathrm{supp}\,\sigma$ [30], we have $\rho \in \mathscr{I} \Rightarrow C_{\min}(\rho) = 0$. However, converse direction may not be true, for example, let $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$ with $|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$, then ρ is coherent but $C_{\min}(\rho) = 0$.

Besides, as $D_{\min}$ is monotone under CPTP maps [30], we have $C_{\min}(\Phi(\rho)) \leq C_{\min}(\rho)$ for any $\Phi \in IO$. However, $C_{\min}$ may increase on average under IO (see Supplemental Material [42]). Thus, $C_{\min}$ is not be a proper coherence measure as $C_{\max}$.

Although $C_{\min}$ is not a good coherence quantifier, it still has some interesting properties in the manipulation of coherence. First, $C_{\min}$ gives upper bound of the maximum overlap with the set of incoherent states for any given quantum state $\rho \in \mathscr{D}(\mathscr{H})$,

$$2^{-C_{\min}(\rho)} \geq \max_{\sigma\in\mathscr{I}} F(\rho,\sigma)^2. \tag{13}$$

Moreover, if ρ is pure state $|\psi\rangle$, then above equality holds, that is,

$$2^{-C_{\min}(\psi)} = \max_{\sigma\in\mathscr{I}} F(\psi,\sigma)^2, \tag{14}$$

see proof in Supplemental Material [42].

Moreover, for geometry of coherence defined by $C_g(\rho) = 1-\max_{\sigma\in\mathscr{I}} F(\rho,\sigma)^2$ [6], $C_{\min}$ also provides a lower bound for C_g as follows

$$C_g(\rho) \geq 1-2^{-C_{\min}(\rho)}. \tag{15}$$

Now let us consider again the one-shot version of distillable coherence under MIO by modifying and smoothing the min-relative entropy of coherence $C_{\min}$. We define the one-shot distillable coherence of a quantum state ρ under MIO as

$$C^{(1),\varepsilon}_{D,MIO}(\rho) := \max_{\substack{\mathscr{E}\in MIO\\ M\in\mathbb{Z}}} \{\log M : F(\mathscr{E}(\rho), |\Psi^M_+\rangle\langle\Psi^M_+|)^2 \geq 1-\varepsilon\},$$

where $|\Psi^M_+\rangle = \frac{1}{\sqrt{M}}\sum_{i=1}^M |i\rangle$ and $\varepsilon > 0$.

For any $\varepsilon > 0$, we define the smooth min-relative entropy of coherence of a quantum state ρ as follows

$$C^{\varepsilon}_{\min}(\rho) := \max_{\substack{0\leq A\leq\mathbb{I}\\ \mathrm{Tr}[A\rho]\geq 1-\varepsilon}} \min_{\sigma\in\mathscr{I}} -\log\mathrm{Tr}[A\sigma], \tag{16}$$

where $\mathbb{I}$ denotes the identity. It can be shown that $C^{\varepsilon}_{\min}$ is a upper bound of one-shot distillable coherence,

$$C^{(1),\varepsilon}_{D,MIO}(\rho) \leq C^{\varepsilon}_{\min}(\rho) \tag{17}$$

for any $\varepsilon > 0$, see proof in Supplemental Material [42].

The distillation of coherence in asymptotic limit can be expressed as

$$C_{D,MIO} = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}C^{(1),\varepsilon}_{D,MIO}(\rho).$$

It has been proven that $C_{D,MIO}(\rho) = C_r(\rho)$ [7]. Here we show that the equality in inequality (17) holds in the asymptotic limit as the $C_{\min}$ is equivalent to C_r in the asymptotic limit. Given a quantum state $\rho \in \mathscr{D}(\mathscr{H})$, then

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}C^{\varepsilon}_{\min}(\rho^{\otimes n}) = C_r(\rho). \tag{18}$$

(The proof is presented in Supplemental Material [42].)

We have shown that $C_{\min}$ gives rise to the bounds for maximum overlap with the incoherent states and for one-shot distillable coherence. Indeed the exact expression of $C_{\min}$ for some special class of quantum states can be calculated. For pure state $|\psi\rangle = \sum_{i=1}^d \psi_i |i\rangle$ with $\sum_{i=1}^d |\psi_i|^2 = 1$, we have $C_{\min}(\psi) = -\log\max_i |\psi_i|^2$. For maximally coherent state $|\Psi\rangle = \frac{1}{\sqrt{d}}\sum_{j=1}^d e^{i\theta_j}|j\rangle$, we have $C_{\min}(\Psi) = \log d$, which is the maximum value for $C_{\min}$ in d-dimensional space.

Relationship between $C_{\max}$ and other coherence measures.– First, we investigate the relationship among $C_{\max}$, $C_{\min}$ and C_r. Since $D_{\min}(\rho||\sigma) \leq S(\rho||\sigma) \leq D_{\max}(\rho||\sigma)$ for any quantum states ρ and σ [30], one has

$$C_{\min}(\rho) \leq C_r(\rho) \leq C_{\max}(\rho). \tag{19}$$

Moreover, as mentioned before, these quantities are all equal in the asymptotic limit.

Above all, $C_{\max}$ is equal to the logarithm of robustness of coherence, as $RoC(\rho) = \min_{\sigma\in\mathscr{I}}\{s\geq 0 \mid \rho \leq (1+s)\sigma\}$ and $C_{\max}(\rho) = \min_{\sigma\in\mathscr{I}}\min\{\lambda : \rho \leq 2^{\lambda}\sigma\}$ [25], that is, $2^{C_{\max}(\rho)} = 1+RoC(\rho)$. Thus, the operational interpretations of $C_{\max}$ in terms of maximum overlap with maximally coherent states and subchannel discrimination, can also be viewed as the operational interpretations of robustness of coherence RoC. It is known that robustness of coherence plays an important role in a phase discrimination task, which provides an operational interpretation for robustness of coherence [28]. This phase discrimination task investigated in [28] is just a special case of the subchannel discrimination in depasing-covariant instruments. Due to the relationship between $C_{\max}$ and RoC, we can obtain the closed form of $C_{\max}$ for some special class of quantum states. As an example, let us consider a pure state $|\psi\rangle = \sum_{i=1}^d \psi_i|i\rangle$. Then $C_{\max}(\psi) = \log((\sum_{i=1}^d |\psi_i|)^2)) = 2\log(\sum_{i=1}^d |\psi_i|)$. Thus, for maximally coherent state $|\Psi\rangle = \frac{1}{\sqrt{d}}\sum_{j=1}^d e^{i\theta_j}|j\rangle$, we have $C_{\max}(\Psi) = \log d$, which is the maximum value for $C_{\max}$ in d-dimensional space.

Since $RoC(\rho) \leq C_{l_1}(\rho)$ [28] and $1+RoC(\rho) = 2^{C_{\max}(\rho)}$, then $C_{\max}(\rho) \leq \log(1+C_{l_1}(\rho))$. We have the relationship among these coherence measures,

$$\begin{aligned} C_{\min}(\rho) \leq C_r(\rho) \leq C_{\max}(\rho) &= \log(1+RoC(\rho)) \\ &\leq \log(1+C_{l_1}(\rho)), \end{aligned}$$

which implies that $2^{C_r(\rho)} \leq 1+C_{l_1}(\rho)$ (See also [27]).

III. CONCLUSION

We have investigated the properties of max- and min-relative entropy of coherence, especially the operational interpretation of the max-relative entropy of coherence. It has been found that the max-relative entropy of coherence characterizes the maximum overlap with the maximally coherent states under DIO, IO and SIO, as well as the maximum advantage achievable by coherent states compared with all incoherent states in subchannel discrimination problems of all dephasing-covariant, incoherent and strictly incoherent instruments, which also provides new operational interpretations of robustness of coherence and illustrates the equivalence of DIO, IO and SIO in these two operational taks. The study of $C_{\max}$ and $C_{\min}$ also makes the relationship between the operational coherence measures (e.g. C_r and C_{l_1}) more clear. These results may highlight the understanding to the operational resource theory of coherence.

Besides, the relationships among smooth max- and min-relative relative entropy of coherence and one-shot coherence cost and distillation have been investigated explicitly. As both smooth max- and min- relative entropy of coherence are equal to relative entropy of coherence in the asymptotic limit and the significance of relative entropy of coherence in the distillation of coherence, further studies are desired on the one-shot coherence cost and distillation.

ACKNOWLEDGMENTS

This work is supported by the Natural Science Foundation of China (Grants No. 11171301, No. 10771191, No. 11571307 and No. 11675113) and the Doctoral Programs Foundation of the Ministry of Education of China (Grant No. J20130061).

[1] V. Giovannetti, S. Lloyd, and L. Maccone, Nat. Photon. **5**, 222 (2011).
[2] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, Phys. Rev. X **5**, 021001 (2015).
[3] M. Lostaglio, D. Jennings, and T. Rudolph, Nat. Commun. **6** (2015).
[4] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014).
[5] D. Girolami, Phys. Rev. Lett. **113**, 170401 (2014).
[6] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Phys. Rev. Lett. **115**, 020403 (2015).
[7] A. Winter and D. Yang, Phys. Rev. Lett. **116**, 120404 (2016).
[8] N. Killoran, F. E. S. Steinhoff, and M. B. Plenio, Phys. Rev. Lett. **116**, 080402 (2016).
[9] E. Chitambar, A. Streltsov, S. Rana, M. N. Bera, G. Adesso, and M. Lewenstein, Phys. Rev. Lett. **116**, 070402 (2016).
[10] E. Chitambar and G. Gour, Phys. Rev. Lett. **117**, 030401 (2016).
[11] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[12] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[13] A. K. Pati, Phys. Rev. A **63**, 014302 (2000).
[14] C. H. Bennett, D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and W. K. Wootters, Phys. Rev. Lett. **87**, 077902 (2001).
[15] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[16] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Rev. Mod. Phys. **79**, 555 (2007).
[17] G. Gour and R. W. Spekkens, New J. Phys. **10**, 033023 (2008).
[18] G. Gour, I. Marvian, and R. W. Spekkens, Phys. Rev. A **80**, 012307 (2009).
[19] I. Marvian, *Symmetry, Asymmetry and Quantum Information* (PhD thesis, University of Waterloo, 2012).
[20] I. Marvian and R. W. Spekkens, New J. Phys. **15**, 033001 (2013).
[21] I. Marvian and R. W. Spekkens, Phys. Rev. A **90**, 062110 (2014).
[22] I. Marvian and R. W. Spekkens, Nat. Commun. **5**, 3821 (2014).
[23] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Phys. Rev. Lett. **111**, 250404 (2013).
[24] R. Gallego and L. Aolita, Phys. Rev. X **5**, 041008 (2015).
[25] E. Chitambar and G. Gour, Phys. Rev. A **94**, 052336 (2016).
[26] K. Bu and C. Xiong, arXiv:1604.06524.
[27] S. Rana, P. Parashar, A. Winter, and M. Lewenstein, arXiv:1612.09234.
[28] C. Napoli, T. R. Bromley, M. Cianciaruso, M. Piani, N. Johnston, and G. Adesso, Phys. Rev. Lett. **116**, 150502 (2016).
[29] M. Piani, M. Cianciaruso, T. R. Bromley, C. Napoli, N. Johnston, and G. Adesso, Phys. Rev. A **93**, 042107 (2016).
[30] N. Datta, IEEE Trans. Inf. Theory **55**, 2816 (2009).
[31] N. Datta, Int. J. Quant. Inf. **07**, 475 (2009).
[32] F. G. S. L. Brandao and N. Datta, IEEE Trans. Inf. Theory **57**, 1754 (2011).
[33] F. Buscemi and N. Datta, IEEE Trans. Inf. Theory **56**, 1447 (2010).
[34] R. Renner and S. Wolf, in *Proc. Int. Symp. Inf. Theory* (2004) p. 233.
[35] R. Renner, *Security of quantum key distribution* (ETH Zurich, 2005).
[36] J. M. Renes and R. Renner, IEEE Trans. Inf. Theory **58**, 1985 (2012).
[37] M. Horodecki, J. Oppenheim, and A. Winter, Commun. Math. Phys. **269**, 107 (2007).
[38] H. Buhrman, M. Christandl, P. Hayden, H.-K. Lo, and S. Wehner, Phys. Rev. Lett. **97**, 250501 (2006).
[39] R. Konig, R. Renner, and C. Schaffner, IEEE Trans. Inf. Theory **55**, 4337 (2009).
[40] M. Piani and J. Watrous, Phys. Rev. Lett. **114**, 060404 (2015).
[41] M. Piani and J. Watrous, Phys. Rev. Lett. **102**, 250501 (2009).
[42] See Supplemental Material [url] for the details of the proof, which includes Refs. [44–54].
[43] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2010).
[44] V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).
[45] L. Vandenberghe and S. Boyd, SIAM review **38**, 49 (1996).
[46] J. Watrous, Theory of Computing **5**, 217 (2009).
[47] F. G. Brandao and K. Svore, arXiv:1609.05537.
[48] Y. Peng, Y. Jiang, and H. Fan, Phys. Rev. A **93**, 032326 (2016).
[49] K. M. R. Audenaert, J. Phys. A Math. Theor. **40**, 8127 (2007).
[50] T. Ogawa and H. Nagaoka, IEEE Trans. Inf. Theory **46**, 2428 (2000).
[51] S. Du, Z. Bai, and X. Qi, arXiv:1504.02862.
[52] M. Hayashi, *Quantum Information:An introduction* (Springer, 2006).
[53] F. G. S. L. Brandão and M. B. Plenio, Commun. Math. Phys. **295**, 791 (2010).
[54] F. G. S. L. Brandão and G. Gour, Phys. Rev. Lett. **115**, 070503 (2015).

Appendix A: strong monotonicity under IO for $C_{\max}$

We prove this property based on the method in [44] and the basic facts of $D_{\max}$ [30]. Due to the definition of $C_{\max}$, there exists an optimal $\sigma_* \in \mathscr{I}$ such that $C_{\max}(\rho) = D_{\max}(\rho||\sigma_*)$. Let $\tilde{\sigma}_i = K_i\sigma_* K_i^\dagger / \mathrm{Tr}\left[K_i\sigma_* K_i^\dagger\right]$, then we have

$$
\begin{aligned}
&\sum_i p_i D_{\max}(\tilde{\rho}_i||\tilde{\sigma}_i)\\
\leq &\sum_i D_{\max}(K_i\rho K_i^\dagger||K_i\sigma_* K_i^\dagger)\\
\leq &\sum_i D_{\max}(\mathrm{Tr}_E\left[\mathbb{I}\otimes|i\rangle\langle i|U\rho\otimes|\alpha\rangle\langle\alpha|U^\dagger\mathbb{I}\otimes|i\rangle\langle i|\right]||\\
&\times\mathrm{Tr}_E\left[\mathbb{I}\otimes|i\rangle\langle i|U\sigma_*\otimes|\alpha\rangle\langle\alpha|U^\dagger\mathbb{I}\otimes|i\rangle\langle i|\right])\\
\leq &\sum_i D_{\max}(\mathbb{I}\otimes|i\rangle\langle i|U\rho\otimes|\alpha\rangle\langle\alpha|U^\dagger\mathbb{I}\otimes|i\rangle\langle i|||\\
&\times\mathbb{I}\otimes|i\rangle\langle i|U\sigma_*\otimes|\alpha\rangle\langle\alpha|U^\dagger\mathbb{I}\otimes|i\rangle\langle i|)\\
= &D_{\max}(U\rho\otimes|\alpha\rangle\langle\alpha|U^\dagger||U\sigma_*\otimes|\alpha\rangle\langle\alpha|U^\dagger)\\
= &D_{\max}(\rho\otimes|\alpha\rangle\langle\alpha|||\sigma_*\otimes|\alpha\rangle\langle\alpha|)\\
= &D_{\max}(\rho||\sigma_*)\\
= &C_{\max}(\rho),
\end{aligned}
$$

where the first inequality comes from the proof of Theorem 1 in [30], the second inequality comes from the fact that there exists an extended Hilbert space $\mathscr{H}_E$, a pure $|\alpha\rangle \in \mathscr{H}_E$ and a global unitary U on $\mathscr{H}\otimes\mathscr{H}_E$ such that $\mathrm{Tr}_E\left[\mathbb{I}\otimes|i\rangle\langle i|U\rho\otimes|\alpha\rangle\langle\alpha|U^\dagger\mathbb{I}\otimes|i\rangle\langle i|\right] = K_i\rho K_i^\dagger$ [44], the third inequality comes from the fact that $D_{\max}$ is monotone under partial trace [30], the last inequality comes from the fact that for any set of mutually orthogonal projectors $\{P_k\}$, $D_{\max}(\sum_k P_k\rho_1 P_k||\sum_k P_k\rho_2 P_k) = \sum_k D_{\max}(P_k\rho_1 P_k||P_k\rho_2 P_k)$ [30] and the first equality comes from the fact that $D_{\max}$ is invariant under unitary operation and $D_{\max}(\rho_1\otimes P||\rho_2\otimes P) = D_{\max}(\rho_1||\rho_2)$ for any projector P [30]. Besides, since $C_{\max}(\tilde{\rho}_i) = \min_{\tau\in\mathscr{I}} D_{\max}(\tilde{\rho}_i||\tau) \leq D_{\max}(\tilde{\rho}_i||\tilde{\sigma}_i)$, we have $\sum_i p_i C_{\max}(\tilde{\rho}_i) \leq C_{\max}(\rho)$.

Appendix B: Coherence measure induced from $C_{\max}$

Here we introduce a proper coherence measure from $C_{\max}$ by the method of convex roof and prove that it satisfies all the conditions (including convexity) a coherence measure need to fulfil. We define the convex roof of $C_{\max}$ as follows

$$\tilde{C}_{\max}(\rho) = \min_{\rho=\sum\lambda_i|\psi_i\rangle\langle\psi_i|}\sum_i \lambda_i C_{\max}(\psi_i), \tag{B1}$$

where the minimum is taken over all the pure state decompositions of state ρ. Due to the definition of $\tilde{C}_{\max}$ and the properties of $C_{\max}$, the positivity and convexity of $\tilde{C}_{\max}$ are obvious. We only need to prove that it is nonincreasing on average under IO.

Proposition 3. *Given a quantum state $\rho \in \mathscr{D}(\mathscr{H})$, for any incoherent operation $\Phi(\cdot) = \sum_\mu K_\mu(\cdot)K_\mu^\dagger$ with $K_\mu\mathscr{I}K_\mu^\dagger \subset \mathscr{I}$,*

$$\sum_\mu p_\mu \tilde{C}_{\max}(\tilde{\rho}_\mu) \leq \tilde{C}_{\max}(\rho), \tag{B2}$$

where $p_\mu = \mathrm{Tr}\left[K_\mu\rho K_\mu^\dagger\right]$ and $\tilde{\rho}_\mu = K_\mu\rho K_\mu^\dagger/p_\mu$.

Proof. Due to the definition of $\tilde{C}_{\max}(\rho)$, there exists a pure state decomposition of state $\rho = \sum_j \lambda_j|\psi_j\rangle\langle\psi_j|$ such that $\tilde{C}_{\max}(\rho) = \sum_j \lambda_j C_{\max}(\psi_j)$. Then

$$
\begin{aligned}
\tilde{\rho}_\mu &= \frac{K_\mu\rho K_\mu^\dagger}{p_\mu}\\
&= \sum_j \frac{\lambda_j}{p_u} K_\mu|\psi_j\rangle\langle\psi_j|K_\mu^\dagger\\
&= \sum_j \frac{\lambda_j q_j^{(\mu)}}{p_\mu}|\phi_j^{(\mu)}\rangle\langle\phi_j^{(\mu)}|,
\end{aligned}
$$

where $|\phi_j^{(\mu)}\rangle = K_\mu|\psi_j\rangle/\sqrt{q_j^{(\mu)}}$ and $q_j^{(\mu)} = \mathrm{Tr}\left[K_\mu|\psi_j\rangle\langle\psi_j|K_\mu^\dagger\right]$. Thus, $\tilde{C}_{\max}(\tilde{\rho}_\mu) \leq \sum_j \frac{\lambda_j q_j^{(\mu)}}{p_\mu} C_{\max}(\phi_j^{(\mu)})$ and

$$
\begin{aligned}
\sum_\mu p_\mu \tilde{C}_{\max}(\tilde{\rho}_\mu) &\leq \sum_{j,\mu} \lambda_j q_j^{(\mu)} C_{\max}(\phi_j^{(\mu)})\\
&= \sum_j \lambda_j \sum_\mu q_j^{(\mu)} C_{\max}(\phi_j^{(\mu)})\\
&= \sum_j \lambda_j \sum_\mu q_j^{(\mu)} \log(1 + C_{l_1}(\phi_j^{(\mu)}))\\
&\leq \sum_j \lambda_j \log(1 + \sum_\mu q_j^{(\mu)} C_{l_1}(\phi_j^{(\mu)}))\\
&\leq \sum_j \lambda_j \log(1 + C_{l_1}(\psi_j))\\
&= \sum_j \lambda_j C_{\max}(\psi_j)\\
&= \tilde{C}_{\max}(\rho),
\end{aligned}
$$

where the third line comes from the fact that for pure state ψ, $C_{\max}(\psi) = \log(1 + C_{l_1}(\psi))$, the forth line comes from the concavity of logarithm and the fifth lines comes from the fact that monotonicity of C_{l_1} under IO as $\Phi(\psi_j) = \sum_\mu K_\mu|\psi_j\rangle\langle\psi_j|K_\mu^\dagger = \sum_\mu q_j^{(\mu)}|\phi_j^{(\mu)}\rangle\langle\phi_j^{(\mu)}|$. $\square$

Appendix C: The operational interpretation of $C_{\max}$

To prove the results, we need some preparation. First of all, Semidefinite programming (SDP) is a powerful tool in this work—which is a generalization of linear programming problems [45]. A SDP over $\mathscr{X} = \mathbb{C}^N$ and $\mathscr{Y} = \mathbb{C}^M$ is a triple (Φ, C, D), where Φ is a Hermiticity-preserving map from $\mathscr{L}(\mathscr{X})$ (linear operators on $\mathscr{X}$) to $\mathscr{L}(\mathscr{Y})$ (linear operators on $\mathscr{Y}$), $C \in \mathrm{Herm}(\mathscr{X})$ (Hermitian operators over $\mathscr{X}$), and $D \in \mathrm{Herm}(\mathscr{Y})$ (Hermitian operators over $\mathscr{Y}$). There is a pair of optimization problems associated with every SDP (Φ, C, D), known as the primal and the dual problems. The standard form of an SDP (that is typically followed for general conic programming) is [46]

Primal problem	Dual problem
minimize: $\langle C, X\rangle$,	maximize: $\langle D, Y\rangle$,
subject to: $\Phi(X) \geq D$,	subject to: $\Phi^*(Y) \geq C$,
$X \in Pos(\mathscr{X})$.	$Y \in Pos(\mathscr{Y})$.

(C1)

SDP forms have interesting and ubiquitous applications in quantum information theory. For example, it was recently shown by Brandao *et. al* [47] that there exists a quantum algorithm for solving SDPs that gives an unconditional square-root speedup over any existing classical method.

Lemma 4. *Given a quantum state $\rho \in \mathscr{D}(\mathscr{H})$,*

$$\min_{\substack{\sigma \geq 0 \\ \Delta(\sigma) \geq \rho}} \operatorname{Tr}[\sigma] = \max_{\substack{\tau \geq 0 \\ \Delta(\tau) = \mathbb{I}}} \operatorname{Tr}[\rho\tau]. \tag{C2}$$

Proof. First, we prove that

$$\max_{\substack{\tau \geq 0 \\ \Delta(\tau) = \mathbb{I}}} \operatorname{Tr}[\rho\tau] = \max_{\substack{\tau \geq 0 \\ \Delta(\tau) \leq \mathbb{I}}} \operatorname{Tr}[\rho\tau]. \tag{C3}$$

For any positive operator $\tau \geq 0$ with $\Delta(\tau) \leq \mathbb{I}$, define $\tau' = \tau + \mathbb{I} - \Delta(\tau) \geq 0$, then $\Delta(\tau) = \mathbb{I}$ and $\operatorname{Tr}[\rho\tau'] \geq \operatorname{Tr}[\rho\tau]$. Thus we obtain the above equation.

Now, we prove that

$$\min_{\substack{\sigma \geq 0 \\ \Delta(\sigma) \geq \rho}} \operatorname{Tr}[\sigma] = \max_{\substack{\tau \geq 0 \\ \Delta(\tau) \leq \mathbb{I}}} \operatorname{Tr}[\rho\tau]. \tag{C4}$$

The left side of equation (C4) can be expressed as the following semidefinite programming (SDP)

$$\begin{aligned} &\min \operatorname{Tr}[B\sigma], \\ \text{s.t. } &\Lambda(\sigma) \geq C, \\ &\sigma \geq 0, \end{aligned}$$

where $B = \mathbb{I}$, $C = \rho$ and $\Lambda = \Delta$. Then the dual SDP is given by

$$\begin{aligned} &\max \operatorname{Tr}[C\tau], \\ \text{s.t. } &\Lambda^\dagger(\tau) \leq B, \\ &\tau \geq 0. \end{aligned}$$

That is,

$$\begin{aligned} &\max \operatorname{Tr}[\rho\tau], \\ \text{s.t. } &\Delta(\tau) \leq \mathbb{I}, \\ &\tau \geq 0. \end{aligned}$$

Note that the dual is strictly feasible as we only need to choose $\sigma = 2\lambda_{\max}(\rho)\mathbb{I}$, where $\lambda_{\max}(\rho)$ is the maximum eigenvalue of ρ. Thus, strong duality holds, and the equation (C4) is proved. □

Lemma 5. *For maximally coherent state $|\Psi_+\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^d |i\rangle$, we have the following facts,*

(i) For any $\mathscr{E} \in DIO$, $\tau = d\mathscr{E}^\dagger(|\Psi_+\rangle\langle\Psi_+|)$ satisfies $\tau \geq 0$ and $\Delta(\tau) = \mathbb{I}$.

(ii) For any operator $\tau \geq 0$ with $\Delta(\tau) = \mathbb{I}$, there exists a quantum operation $\mathscr{E} \in DIO$ such that $\tau = d\mathscr{E}^\dagger(|\Psi_+\rangle\langle\Psi_+|)$.

(iii) For any $\mathscr{E} \in IO$, $\tau = d\mathscr{E}^\dagger(|\Psi_+\rangle\langle\Psi_+|)$ satisfies $\tau \geq 0$ and $\Delta(\tau) = \mathbb{I}$.

(iv) For any operator $\tau \geq 0$ with $\Delta(\tau) = \mathbb{I}$, there exists a quantum operation $\mathscr{E} \in IO$ such that $\tau = d\mathscr{E}^\dagger(|\Psi_+\rangle\langle\Psi_+|)$.

(v) For any $\mathscr{E} \in SIO$, $\tau = d\mathscr{E}^\dagger(|\Psi_+\rangle\langle\Psi_+|)$ satisfies $\tau \geq 0$ and $\Delta(\tau) = \mathbb{I}$.

(vi) For any operator $\tau \geq 0$ with $\Delta(\tau) = \mathbb{I}$, there exists a quantum operation $\mathscr{E} \in SIO$ such that $\tau = d\mathscr{E}^\dagger(|\Psi_+\rangle\langle\Psi_+|)$.

Proof. (i) Since $\mathscr{E}$ is a CPTP map, $\mathscr{E}^\dagger$ is unital. Besides, as $\mathscr{E} \in DIO$, $[\mathscr{E}, \Delta] = 0$ implies that $[\mathscr{E}^\dagger, \Delta] = 0$. Thus $\Delta(\tau) = d\mathscr{E}^\dagger(\Delta(|\Psi_+\rangle\langle\Psi_+|)) = \mathscr{E}^\dagger(\mathbb{I}) = \mathbb{I}$.

(ii) For any positive operator $\tau \geq 0$ with $\Delta(\tau) = \mathbb{I}$, $\operatorname{Tr}[\tau] = d$, thus $\tau = d\hat{\tau}$ with $\hat{\tau} \in \mathscr{D}(\mathscr{H})$ and $\Delta(\hat{\tau}) = \frac{1}{d}\mathbb{I}$. Consider the spectral decomposition of $\hat{\tau} = \sum_{i=1}^d \lambda_i |\psi_i\rangle\langle\psi_i|$ with $\sum_{i=1}^d \lambda_i = 1, \lambda_i \geq 0$ for any $i \in \{1, .., d\}$. Besides, for any $i \in \{1, ..., d\}$, $|\psi_i\rangle$ can be written as $|\psi_i\rangle = \sum_{j=1}^d c_j^{(i)} |j\rangle$ with $\sum_{j=1}^d |c_j^{(i)}|^2 = 1$. Let us define $K_n^{(i)} = \sum_{j=1}^d c_j^{(i)} |j\rangle\langle j|$ for any $n \in \{1, ..., d\}$, then $K_n^{(i)} |\Psi_+\rangle = \frac{1}{\sqrt{d}} |\psi_i\rangle$ and $\sum_{n=1}^d K_n^{(i)} |\Psi_+\rangle\langle\Psi_+| K_n^{(i)\dagger} = |\psi_i\rangle\langle\psi_i|$. Let $M_{i,n} = \sqrt{\lambda_i} K_n^{(i)\dagger}$, then

$$\begin{aligned} \sum_{i,n} M_{i,n}^\dagger M_{i,n} &= \sum_{i,n} \lambda_i K_n^{(i)} K_n^{(i)\dagger} \\ &= d\sum_{i=1}^d \lambda_i K_1^{(i)} K_1^{(i)\dagger} \\ &= d\sum_{i=1}^d \lambda_i \sum_{j=1}^d |c_j^{(i)}|^2 |j\rangle\langle j| \\ &= d\sum_{j=1}^d \sum_{i=1}^d \lambda_i |c_j^{(i)}|^2 |j\rangle\langle j| \\ &= d\sum_{j=1}^d \frac{1}{d} |j\rangle\langle j| = \mathbb{I}, \end{aligned}$$

where $\sum_{i=1}^d \lambda_i |c_j^{(i)}|^2 = \sum_i \lambda_i |\langle\psi_i|j\rangle|^2 = \langle j|\hat{\tau}|j\rangle = \frac{1}{d}$. Then $\mathscr{E}(\cdot) = \sum_{i,n} M_{i,n}(\cdot)M_{i,n}^\dagger$ is a CPTP map. Since $M_{i,n}$ is diagonal, the quantum operation $\mathscr{E}(\cdot) = \sum_{i,n} M_{i,n}(\cdot)M_{i,n}^\dagger$ is a DIO. Moreover, $\mathscr{E}^\dagger(|\Psi_+\rangle\langle\Psi_+|) = \sum_{i,n} M_{i,n}^\dagger |\Psi_+\rangle\langle\Psi_+| M_{i,n} = \sum_{i,n} \lambda_i K_n^{(i)} |\Psi_+\rangle\langle\Psi_+| K_n^{(i)\dagger} = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i| = \hat{\tau}$.

(iii) If $\mathscr{E}$ is an incoherent operation, then there exists a set of Kraus operators $\{K_\mu\}$ such that $\mathscr{E}(\cdot) = \sum_\mu K_\mu(\cdot)K_\mu^\dagger$ and

$K_\mu \mathscr{I} K_\mu^\dagger \in \mathscr{I}$. Thus

$$\begin{aligned} d\left\langle i\right|\mathscr{E}^\dagger(|\Psi_+\rangle\langle\Psi_+|)\left|i\right\rangle &= d\sum_\mu \left\langle i\right|K_\mu^\dagger|\Psi_+\rangle\langle\Psi_+|K_\mu\left|i\right\rangle \\ &= \sum_\mu\sum_{m,n}\left\langle i\right|K_\mu^\dagger\left|m\right\rangle\left\langle n\right|K_\mu\left|i\right\rangle \\ &= \sum_\mu\sum_{m}\left\langle i\right|K_\mu^\dagger\left|m\right\rangle\left\langle m\right|K_\mu\left|i\right\rangle \\ &= \sum_\mu \left\langle i\right|K_\mu^\dagger K_\mu\left|i\right\rangle = 1, \end{aligned}$$

where the third line comes from the fact that for any K_μ, there exists at most one nonzero term in each column which implies that $\langle i|K_\mu^\dagger|m\rangle\langle n|K_\mu|i\rangle \neq 0$ only if $m=n$, and the forth line comes from the fact that $\sum_\mu K_\mu^\dagger K_\mu = \mathbb{I}$. Therefore, $\Delta(d\mathscr{E}^\dagger(|\Psi_+\rangle\langle\Psi_+|)) = \mathbb{I}$.

(iv) This is obvious, as the DIO $\mathscr{E}$ given in (ii) is also an incoherent operation.

(v) This is obvious as $SIO \subset DIO$.

(vi) This is obvious as the DIO $\mathscr{E}$ given in (ii) also belongs to SIO.

□

Lemma 6. *Given a quantum state $\rho \in \mathscr{D}(\mathscr{H})$, one has*

$$\max_{\mathscr{E}\in DIO} F(\mathscr{E}(\rho),|\Psi_+\rangle\langle\Psi_+|)^2 = \max_{\substack{\mathscr{E}\in DIO\\ |\Psi\rangle\in\mathscr{M}}} F(\mathscr{E}(\rho),|\Psi\rangle\langle\Psi|)^2.$$

where $|\Psi_+\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^d |i\rangle$ and $\mathscr{M}$ is the set of maximally coherent states.

Proof. Due to [48], every maximally coherent can be expressed as $|\Psi\rangle = \frac{1}{\sqrt{d}}\sum_{j=1}^d e^{i\theta_j}|j\rangle$, that is, $|\Psi\rangle = U_\Psi|\Psi_+\rangle$ where $U_\Psi = \sum_{j=1}^d e^{i\theta_j}|j\rangle\langle j|$. Obviously, $[U_\Psi,\Delta]=0$, thus $U_\Psi \in DIO$ and

$$\begin{aligned} F(\mathscr{E}(\rho),|\Psi\rangle\langle\Psi|)^2 &= F(\mathscr{E}(\rho),U_\Psi|\Psi_+\rangle\langle\Psi_+|U_\Psi^\dagger)^2 \\ &= F(U_\Psi^\dagger\mathscr{E}(\rho)U_\Psi,|\Psi\rangle\langle\Psi|)^2 \\ &= F(\mathscr{E}'(\rho),|\Psi_+\rangle\langle\Psi_+|)^2, \end{aligned}$$

where $\mathscr{E}'(\cdot) = U_\Psi^\dagger\mathscr{E}(\cdot)U_\Psi \in DIO$ as $\mathscr{E}, U_\Psi \in DIO$. □

After these preparation, we begin to prove Theorem 1.

Proof of Theorem 1. If $\mathscr{E}$ belongs to DIO, that is, we need to prove

$$2^{C_{\max}(\rho)} = d\max_{\substack{\mathscr{E}\in DIO\\ |\Psi\rangle\in\mathscr{M}}} F(\mathscr{E}(\rho),|\Psi\rangle\langle\Psi|)^2, \tag{C5}$$

where $\mathscr{M}$ is the set of maximally coherent states. In view of Lemma 6, we only need to prove

$$2^{C_{\max}(\rho)} = d\max_{\mathscr{E}\in DIO} F(\mathscr{E}(\rho),|\Psi_+\rangle\langle\Psi_+|)^2, \tag{C6}$$

where $|\Psi_+\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^d |i\rangle$.

First of all,

$$\begin{aligned} 2^{C_{\max}(\rho)} &= \min_{\sigma\in\mathscr{I}}\min\{\lambda \mid \rho\leq\lambda\sigma\} \\ &= \min_{\sigma\geq 0}\{\mathrm{Tr}[\sigma] \mid \rho\leq\Delta(\sigma)\} \\ &= \min_{\substack{\sigma\geq 0\\ \Delta(\sigma)\geq\rho}} \mathrm{Tr}[\sigma]. \end{aligned}$$

Second,

$$\begin{aligned} dF(\mathscr{E}(\rho),|\Psi_+\rangle\langle\Psi_+|)^2 &= d\,\mathrm{Tr}[\mathscr{E}(\rho)|\Psi_+\rangle\langle\Psi_+|] \\ &= d\,\mathrm{Tr}\left[\rho\mathscr{E}^\dagger(|\Psi_+\rangle\langle\Psi_+|)\right] \\ &= \mathrm{Tr}[\rho\tau], \end{aligned}$$

where $\tau = d\mathscr{E}^\dagger(|\Psi_+\rangle\langle\Psi_+|$. According to Lemma 5, there is one to one correspondence between *DIO* and the set $\{\tau\geq 0 \mid \Delta(\tau)=\mathbb{I}\}$. Thus we have

$$d\max_{\mathscr{E}\in DIO} F(\mathscr{E}(\rho),|\Psi_+\rangle\langle\Psi_+|)^2 = \max_{\substack{\tau\geq 0\\ \Delta(\tau)=\mathbb{I}}} \mathrm{Tr}[\rho\tau]. \tag{C7}$$

Finally, according to Lemma 4, we get the desired result (C5). Similarly, we can prove the case where $\mathscr{E}$ belongs to either IO or SIO based on Lemma 5. ■

Appendix D: Subchannel discrimination in dephasing covariant instrument

Proof of Theorem 2. First, we consider the case where instrument $\mathfrak{I}$ is dephasing-covariant instrument $\mathfrak{I}^D$. Due to the definition of $C_{\max}(\rho)$, there exists an incoherent state σ such that $\rho \leq 2^{C_{\max}(\rho)}\sigma$. Thus, for any dephasing-covariant instrument $\mathfrak{I}^D$ and POVM $\{M_b\}_b$,

$$p_{\mathrm{succ}}(\mathfrak{I}^D,\{M_b\}_b,\rho) \leq 2^{C_{\max}(\rho)}p_{\mathrm{succ}}(\mathfrak{I}^D,\{M_b\}_b,\sigma),$$

which implies that

$$p_{\mathrm{succ}}(\mathfrak{I}^D,\rho) \leq 2^{C_{\max}(\rho)}p_{\mathrm{succ}}^{ICO}(\mathfrak{I}^D). \tag{D1}$$

Next, we prove that there exists a dephasing-covariant instrument $\mathfrak{I}^D$ such that the equality in (D1) holds. In view of Theorem 1, there exists a DIO $\mathscr{E}$ such that

$$2^{C_{\max}(\rho)} = d\,\mathrm{Tr}[\mathscr{E}(\rho)|\Psi_+\rangle\langle\Psi_+|], \tag{D2}$$

where $|\Psi_+\rangle = \frac{1}{\sqrt{d}}\sum_{j=1}^d |j\rangle$. Let us consider the following diagonal unitaries

$$U_k = \sum_{j=1}^d e^{i\frac{jk}{d}2\pi}|j\rangle\langle j|, k\in\{1,..,d\}. \tag{D3}$$

The set $\{U_k \mid \Psi_+\rangle\}_{k=1}^d$ forms a basis of the Hilbert space and $\sum_{k=1}^d U_k|\Psi_+\rangle\langle\Psi_+|U_k^\dagger = \mathbb{I}$. Let us define subchannels $\{\mathscr{E}_k\}_k$ as $\mathscr{E}_k(\rho) = \frac{1}{d}U_k\mathscr{E}(\rho)U_k^\dagger$. Then the channel $\widetilde{\mathscr{E}} = \sum_{k=1}^d \mathscr{E}_k$ is a DIO. That is, the instrument $\widetilde{\mathfrak{I}}^D = \{\mathscr{E}_k\}_k$ is a dephasing-covariant instrument.

For any POVM $\{M_k\}_k$ and any incoherent state σ, the probability of success is

$$\begin{aligned}p_{\text{succ}}(\widetilde{\mathfrak{I}}^D, \{M_k\}_k, \sigma) &= \sum_k \mathrm{Tr}\,[\mathscr{E}_k(\sigma)M_k] \\ &= \frac{1}{d}\,\mathrm{Tr}\left[\mathscr{E}(\sigma)\sum_k U_k^\dagger M_k U_k\right].\end{aligned}$$

Since $\{M_k\}_k$ is a POVM, then $\sum_k M_k = \mathbb{I}$. As $\{U_k\}_k$ are all diagonal unitaries , we have

$$\begin{aligned}\Delta(\sum_k U_k^\dagger M_k U_k) &= \sum_k U_k^\dagger \Delta(M_k) U_k \\ &= \sum_k \Delta(M_k) \\ &= \Delta(\sum_k M_k) \\ &= \Delta(\mathbb{I}) = \mathbb{I}.\end{aligned}$$

Thus,

$$\begin{aligned}p_{\text{succ}}(\widetilde{\mathfrak{I}}^D, \{M_k\}_k, \sigma) &= \frac{1}{d}\,\mathrm{Tr}\left[\mathscr{E}(\sigma)\sum_k U_k^\dagger M_k U_k\right] \\ &= \frac{1}{d}\,\mathrm{Tr}\left[\Delta(\mathscr{E}(\sigma))\sum_k U_k^\dagger M_k U_k\right] \\ &= \frac{1}{d}\,\mathrm{Tr}\left[\mathscr{E}(\sigma)\Delta(\sum_k U_k^\dagger M_k U_k)\right] \\ &= \frac{1}{d}\,\mathrm{Tr}\,[\mathscr{E}(\sigma)] = \frac{1}{d},\end{aligned}$$

where the second equality comes from the fact that $\mathscr{E}(\sigma) \in \mathscr{I}$ for any incoherent state σ, and the second last equality comes from that fact that $\Delta(\sum_k U_k^\dagger M_k U_k) = \mathbb{I}$. That is,

$$p_{\text{succ}}^{ICO}(\widetilde{\mathfrak{I}}^D) = \frac{1}{d}. \tag{D4}$$

Besides, taking the POVM $\{N_k\}_k$ with $N_k = U_k|\Psi_+\rangle\langle\Psi_+|U_k^\dagger$, one has $\mathrm{Tr}\,[\mathscr{E}_k(\rho)N_k] = \frac{1}{d}\mathrm{Tr}\,[\mathscr{E}(\rho)|\Psi_+\rangle\langle\Psi_+|]$ and

$$\begin{aligned}p_{\text{succ}}(\widetilde{\mathfrak{I}}^D, \{N_k\}_k, \rho) &= \sum_k \mathrm{Tr}\,[\mathscr{E}_k(\rho)N_k] \\ &= \sum_k \frac{1}{d}\,\mathrm{Tr}\,[\mathscr{E}(\rho)|\Psi_+\rangle\langle\Psi_+|] \\ &= \mathrm{Tr}\,[\mathscr{E}(\rho)|\Psi_+\rangle\langle\Psi_+|] \\ &= \frac{2^{C_{\max}(\rho)}}{d} \\ &= 2^{C_{\max}(\rho)} p_{\text{succ}}^{ICO}(\widetilde{\mathfrak{I}}^D).\end{aligned}$$

Thus, for this depasing-covariant instrument $\widetilde{\mathfrak{I}}^D = \{\mathscr{E}_k\}_k$,

$$\frac{p_{\text{succ}}(\widetilde{\mathfrak{I}}^D, \rho)}{p_{\text{succ}}^{ICO}(\widetilde{\mathfrak{I}}^D)} \geq 2^{C_{\max}(\rho)}. \tag{D5}$$

Finally it is easy to see that the above proof is also true for $\mathfrak{I}$ is $\mathfrak{I}^I$ or $\mathfrak{I}^S$.

■

Note that the phasing discrimination game studied in [28] is just a special case of the subchannel discrimination in the dephasing-covariant instruments. In the phasing discrimination game, the phase ϕ_k is encoded into a diagonal unitary $U_{\phi_k} = \sum_j e^{ij\phi_k}|j\rangle\langle j|$. Thus the discrimination of a collection of phase $\{\phi_k\}$ with a prior probability distribution $\{p_k\}$ is equivalent to the discrimination of the set of subchannel $\{\mathscr{E}_k\}_k$, where $\mathscr{E}_k = p_k\mathbf{U}_k$ and $\mathbf{U}_k(\cdot) = U_{\phi_k}(\cdot)U_{\phi_k}^\dagger$.

Appendix E: $C_{\max}^{\varepsilon}$ as a lower bound of one-shot coherence cost

The ε-smoothed max-relative entropy of coherence of a quantum state ρ is defined by,

$$C_{\max}^{\varepsilon}(\rho) := \min_{\rho' \in B_\varepsilon(\rho)} C_{\max}(\rho'), \tag{E1}$$

where $B_\varepsilon(\rho) := \{\rho' \geq 0 : \|\rho' - \rho\|_1 \leq \varepsilon, \mathrm{Tr}\,[\rho'] \leq \mathrm{Tr}\,[\rho]\}$. Then

$$\begin{aligned}C_{\max}^{\varepsilon}(\rho) &= \min_{\rho' \in B_\varepsilon(\rho)} \min_{\sigma \in \mathscr{I}} D_{\max}(\rho'||\sigma) \\ &= \min_{\sigma \in \mathscr{I}} D_{\max}^{\varepsilon}(\rho||\sigma),\end{aligned}$$

where $D_{\max}^{\varepsilon}(\rho||\sigma)$ is the smooth max-relative entropy [30–32] and defined as

$$D_{\max}^{\varepsilon}(\rho||\sigma) = \inf_{\rho' \in B_\varepsilon(\rho)} D_{\min}(\rho'||\sigma).$$

Proof of Equation (5). Suppose $\mathscr{E}$ is MIO such that $F(\mathscr{E}(|\Psi_+^M\rangle\langle\Psi_+^M|), \rho)^2 \geq 1 - \varepsilon$ and $C_{C,MIO}^{(1),\varepsilon}(\rho) = \log M$. Since $F(\rho,\sigma)^2 \leq 1 - \frac{1}{4}\|\rho - \sigma\|_1^2$ [43], then $\left\|\mathscr{E}(|\Psi_+^M\rangle\langle\Psi_+^M|) - \rho\right\|_1 \leq 2\sqrt{\varepsilon}$. Thus $\mathscr{E}(|\Psi_+^M\rangle\langle\Psi_+^M|) \in B_{\varepsilon'}(\rho)$, where $\varepsilon' = 2\sqrt{\varepsilon}$. As $C_{\max}$ is monotone under MIO, we have $C_{\max}^{\varepsilon'}(\rho) \leq C_{\max}(\mathscr{E}(|\Psi_+^M\rangle\langle\Psi_+^M|)) \leq C_{\max}(|\Psi_+^M\rangle\langle\Psi_+^M|) = \log M = C_{C,MIO}^{(1),\varepsilon}(\rho)$. ■

Appendix F: Equivalence between $C_{\max}$ and C_r in asymptotic case

We introduce several lemmas first to prove the result. For any self-adjoint operator Q on a finite-dimensional Hilbert space, Q has the spectral decomposition as $Q = \sum_i \lambda_i P_i$, where P_i is the orthogonal projector onto the eigenspace of Q. Then we define the positive operator $\{Q \geq 0\} = \sum_{\lambda_i \geq 0} P_i$, and $\{Q > 0\}$, $\{Q \leq 0\}$, $\{Q < 0\}$ are defined in a similar way. Moreover, for any two operators Q_1 and Q_2, $\{Q_1 \geq Q_2\}$ is defined as $\{Q_1 - Q_2 \geq 0\}$.

Lemma 7. *[31] Given two quantum states $\rho, \sigma \in \mathscr{D}(\mathscr{H})$, then*

$$D_{\max}^{\varepsilon}(\rho||\sigma) \leq \lambda \tag{F1}$$

for any $\lambda \in \mathbb{R}$ and $\varepsilon = \sqrt{8\,\mathrm{Tr}\left[\{\rho > 2^\lambda\sigma\}\rho\right]}$.

Note that in [31], the above lemma is proved for bipartite states. However, this lemma also holds for any state.

Lemma 8. *(Fannes-Audenaert Inequality [49]) For any two quantum states ρ and σ with $\varepsilon=\frac{1}{2}\|\rho-\sigma\|_1$, the following inequality holds:*

$$|S(\rho)-S(\sigma)|\leq\varepsilon\log(d-1)+H_2(\varepsilon), \tag{F2}$$

where d is the dimension of the system and $H_2(\varepsilon)=-\varepsilon\log\varepsilon-(1-\varepsilon)\log(1-\varepsilon)$ is the binary Shannon entropy.

Based on these lemmas, we can prove the equivalence between $C_{\max}$ and C_r in asymptotic limit.

Proof of Equation (6). First, we prove that

$$C_r(\rho)\leq\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}C^{\varepsilon}_{\max}(\rho^{\otimes n}). \tag{F3}$$

Since

$$\begin{aligned}C^{\varepsilon}_{\max}(\rho^{\otimes n}) &= \min_{\sigma_n\in\mathscr{I}_n} D^{\varepsilon}_{\max}(\rho^{\otimes n}||\sigma_n)\\ &= D_{\max}(\rho_{n,\varepsilon}||\tilde{\sigma}_n),\end{aligned}$$

where $\rho_{n,\varepsilon}\in B_{\varepsilon}(\rho^{\otimes n})$, $\tilde{\sigma}_n\in\mathscr{I}_n$ and $\mathscr{I}_n$ is the set of incoherent states of $\mathscr{H}^{\otimes n}$.

Due to the definition of $\mathscr{D}_{\max}$, we have

$$\rho_{n,\varepsilon}\leq 2^{C^{\varepsilon}_{\max}(\rho^{\otimes n})}\tilde{\sigma}_n. \tag{F4}$$

Then

$$\begin{aligned}C_r(\rho_{n,\varepsilon}) &\leq S(\rho_{n,\varepsilon}||\tilde{\sigma}_n)\\ &= \mathrm{Tr}\left[\rho_{n,\varepsilon}\log\rho_{n,\varepsilon}\right]-\mathrm{Tr}\left[\rho_{n,\varepsilon}\log\tilde{\sigma}_n\right]\\ &\leq \mathrm{Tr}\left[\rho_{n,\varepsilon}(C^{\varepsilon}_{max}(\rho^{\otimes n})+\log\tilde{\sigma}_n)\right]-\mathrm{Tr}\left[\rho_{n,\varepsilon}\log\tilde{\sigma}_n\right]\\ &\leq C^{\varepsilon}_{\max}(\rho^{\otimes n}),\end{aligned}$$

where $\mathrm{Tr}\left[\rho_{n,\varepsilon}\right]\leq\mathrm{Tr}\left[\rho^{\otimes n}\right]=1$. Besides, as $\|\rho_{n,\varepsilon}-\rho^{\otimes n}\|\leq\varepsilon$, due to the Fannes-Audenaert Inequality (F2), we have

$$\begin{aligned}C_r(\rho_{n,\varepsilon}) &\geq C_r(\rho^{\otimes n})-\varepsilon\log(d-1)-H_2(\varepsilon)\\ &= nC_r(\rho)-\varepsilon\log(d-1)-H_2(\varepsilon).\end{aligned}$$

Thus,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}C^{\varepsilon}_{\max}(\rho^{\otimes n})\geq C_r(\rho). \tag{F5}$$

Next, we prove that

$$C_r(\rho)\geq\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}C^{\varepsilon}_{\max}(\rho^{\otimes n}).$$

Consider the sequence $\hat{\rho}=\{\rho^{\otimes n}\}_{n=1}^{\infty}$ and $\hat{\sigma}_I=\{\sigma_I^{\otimes n}\}_{n=1}^{\infty}$, where $\sigma_I\in\mathscr{I}$ such that $C_r(\rho)=S(\rho||\sigma_I)=\min_{\sigma\in\mathscr{I}}S(\rho||\sigma)$. Denote

$$\overline{D}(\hat{\rho}||\hat{\sigma}):=\inf\{\gamma:\limsup_{n\to\infty}\mathrm{Tr}\left[\{\rho^{\otimes n}\geq 2^{n\gamma}\sigma_I^{\otimes n}\}\rho^{\otimes n}\right]=0\}.$$

Due to the Quantum Stein's Lemma [31, 50],

$$\overline{D}(\hat{\rho}||\hat{\sigma})=S(\rho||\sigma_I)=C_r(\rho).$$

For any $\delta>0$, let $\lambda=\overline{D}(\hat{\rho}||\hat{\sigma})+\delta=S(\rho||\sigma_I)=C_r(\rho)+\delta$. Due to the definition of the quantity $\overline{D}(\hat{\rho}||\hat{\sigma})$, we have

$$\limsup_{n\to\infty}\mathrm{Tr}\left[\{\rho^{\otimes n}\geq 2^{n\lambda}\sigma_I^{\otimes n}\}\rho^{\otimes n}\right]=0.$$

Then for any $\varepsilon>0$, there exists an integer N_0 such that for any $n\geq N_0$, $\mathrm{Tr}\left[\{\rho^{\otimes n}\geq 2^{n\lambda}\sigma_I^{\otimes n}\}\rho^{\otimes n}\right]<\frac{\varepsilon^2}{8}$. According to Lemma 7, we have

$$D^{\varepsilon}_{\max}(\rho^{\otimes n}||\sigma_I^{\otimes n})\leq n\lambda=nC_r(\rho)+n\delta. \tag{F6}$$

for $n\geq N_0$. Hence,

$$C^{\varepsilon}_{\max}(\rho^{\otimes n})\leq nC_r(\rho)+n\delta.$$

Therefore

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}C^{\varepsilon}_{\max}(\rho^{\otimes n})\leq C_r(\rho)+\delta.$$

Since δ is arbitrary, $\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}C^{\varepsilon}_{\max}(\rho^{\otimes n})\leq C_r(\rho)$. ■

Appendix G: $C_{\min}$ may increase on average under IO

For pure state $|\psi\rangle=\sum_{i=1}^{d}\psi_i|i\rangle$, we have $C_{\min}(\psi)=-\log\max_i|\psi_i|^2$. According to [51], if $C_{\min}$ is nonincreasing on average under IO, it requires that $C_{\min}$ should be a concave function of its diagonal part for pure state. However, $C_{\min}(\psi)=-\log\max_i|\psi_i|^2$ is convex on the diagonal part of the pure states, hence $C_{\min}$ may increase on average under IO.

Besides, according the definition of $C_{\min}$ for pure state $|\psi\rangle$, one has

$$2^{-C_{\min}(\psi)}=\max_{\sigma\in\mathscr{I}}F(\psi,\sigma)^2. \tag{G1}$$

However, this equality does not hold for any states. For any quantum state, the inequality (13) in the main context holds.

Proof of Equation (13). There exists a $\sigma_*\in\mathscr{I}$ such that $F(\rho,\sigma_*)^2=\max_{\sigma\in\mathscr{I}}F(\rho,\sigma)^2$. Let us consider the spectrum decomposition of the quantum state ρ, $\rho=\sum_i\lambda_i|\psi_i\rangle\langle\psi_i|$ with $\lambda_i>0$ and $\sum_i\lambda_i=1$. Then the projector Π_ρ onto the support of ρ can be written as $\Pi_\rho=\sum_i|\psi_i\rangle\langle\psi_i|$ and $2^{-C_{\min}(\rho)}=\max_{\sigma\in\mathscr{I}}\mathrm{Tr}\left[\Pi_\rho\sigma\right]$.

Besides, there exists pure state decomposition of $\sigma_*=\sum_i\mu_i|\phi_i\rangle\langle\phi_i|$ such that $F(\rho,\sigma_*)=\sum_i\sqrt{\lambda_i\mu_i}\langle\psi_i|\phi_i\rangle$ [52]. Thus

$$\begin{aligned}F(\rho,\sigma_*)^2 &= (\sum_i\sqrt{\lambda_i\mu_i}\langle\psi_i|\phi_i\rangle)^2\\ &\leq (\sum_i\lambda_i)(\sum_i\mu_i\mathrm{Tr}\left[|\psi_i\rangle\langle\psi_i||\phi_i\rangle\langle\phi_i|\right])\\ &\leq \sum_i\mu_i\mathrm{Tr}\left[\Pi_\rho|\phi_i\rangle\langle\phi_i|\right]\\ &= \mathrm{Tr}\left[\Pi_\rho\sigma_*\right]\\ &\leq 2^{-C_{\min}(\rho)},\end{aligned}$$

where the first inequality is due to the Cauchy-Schwarz inequality and the second inequality comes from the fact that $\sum_i\lambda_i=1$ and $|\psi_i\rangle\langle\psi_i|\leq\Pi_\rho$ for any i. ■

Appendix H: Equivalence between $C_{\min}$ and C_r in asymptotic case

For any $\varepsilon > 0$, the smooth min-relative entropy of coherence of a quantum state ρ is defined as follows

$$C^{\varepsilon}_{\min}(\rho) := \max_{\substack{0\leq A\leq \mathbb{I}\\ \mathrm{Tr}[A\rho]\geq 1-\varepsilon}} \min_{\sigma\in\mathscr{I}} -\log \mathrm{Tr}\,[A\sigma], \tag{H1}$$

where $\mathbb{I}$ denotes the identity. Then

$$\begin{aligned} C^{\varepsilon}_{\min}(\rho) &= \max_{\substack{0\leq A\leq \mathbb{I}\\ \mathrm{Tr}[A\rho]\geq 1-\varepsilon}} \min_{\sigma\in\mathscr{I}} -\log \mathrm{Tr}\,[A\sigma] \\ &= \min_{\sigma\in\mathscr{I}} \max_{\substack{0\leq A\leq \mathbb{I}\\ \mathrm{Tr}[A\rho]\geq 1-\varepsilon}} -\log \mathrm{Tr}\,[A\sigma] \\ &= \min_{\sigma\in\mathscr{I}} D^{\varepsilon}_{\min}(\rho||\sigma), \end{aligned}$$

where $D^{\varepsilon}_{\min}(\rho||\sigma)$ is the smooth min-relative entropy [32] and defined as

$$D^{\varepsilon}_{\min}(\rho||\sigma) = \sup_{\substack{0\leq A\leq \mathbb{I}\\ \mathrm{Tr}[A\rho]\geq 1-\varepsilon}} -\log \mathrm{Tr}\,[A\sigma].$$

Lemma 9. *Given a quantum state $\rho \in \mathscr{D}(\mathscr{H})$, for any $\varepsilon > 0$,*

$$C^{\varepsilon}_{\min}(\rho) \leq C^{\varepsilon}_{\max}(\rho) - \log(1-2\varepsilon). \tag{H2}$$

Proof. Since

$$\begin{aligned} C^{\varepsilon}_{\max}(\rho) &= \min_{\sigma\in\mathscr{I}} D^{\varepsilon}_{\max}(\rho||\sigma), \\ C^{\varepsilon}_{\min}(\rho) &= \min_{\sigma\in\mathscr{I}} D^{\varepsilon}_{\min}(\rho||\sigma), \end{aligned}$$

we only need to prove that for any two states ρ and σ,

$$D^{\varepsilon}_{\min}(\rho||\sigma) \leq D^{\varepsilon}_{\max}(\rho||\sigma) - \log(1-2\varepsilon). \tag{H3}$$

First, there exists a $\rho_\varepsilon \in B_\varepsilon(\rho)$ such that $D^{\varepsilon}_{\max}(\rho||\sigma) = D_{\max}(\rho_\varepsilon||\sigma) = \log\lambda$. Hence $\lambda\sigma - \rho_\varepsilon \geq 0$.

Second, let $0 \leq A \leq \mathbb{I}$, $\mathrm{Tr}\,[A\rho] \geq 1-\varepsilon$ such that $D^{\varepsilon}_{\min}(\rho||\sigma) = -\log \mathrm{Tr}\,[A\sigma]$. Since for any two positive operators A and B, $\mathrm{Tr}\,[AB] \geq 0$. Therefore $\mathrm{Tr}\,[(\lambda\sigma-\rho_\varepsilon)A] \geq 0$, that is,

$$\mathrm{Tr}\,[A\rho_\varepsilon] \leq \lambda\,\mathrm{Tr}\,[A\sigma].$$

Since $\|\rho-\rho_\varepsilon\|_1 = \mathrm{Tr}\,[|\rho-\rho_\varepsilon|] < \varepsilon$ and $\mathrm{Tr}\,[A\rho] \geq 1-\varepsilon$, one gets

$$\begin{aligned} |\,\mathrm{Tr}\,[A\rho_\varepsilon] - \mathrm{Tr}\,[A\rho]\,| &\leq \mathrm{Tr}\,[A|\rho-\rho_\varepsilon|] \\ &\leq \mathrm{Tr}\,[|\rho-\rho_\varepsilon|] < \varepsilon. \end{aligned}$$

Thus,

$$\mathrm{Tr}\,[A\rho_\varepsilon] \geq \mathrm{Tr}\,[A\rho] - \varepsilon \geq 1-2\varepsilon,$$

which implies that

$$1-2\varepsilon \leq \lambda\,\mathrm{Tr}\,[A\sigma].$$

Take logarithm on both sides of the above inequality, we have

$$-\log \mathrm{Tr}\,[A\sigma] \leq \log\lambda - \log(1-2\varepsilon).$$

That is,

$$D^{\varepsilon}_{\min}(\rho||\sigma) \leq D^{\varepsilon}_{\max}(\rho||\sigma) - \log(1-2\varepsilon). \tag{H4}$$

□

The following lemma is a kind of generalization of the Quantum Stein' Lemma [53] for the special case of the incoherent state set $\mathscr{I}$, as the the set of incoherent states satisfies the requirement in [53]. Note that this lemma can be generalized to any quantum resource theory which satisfies some postulates [54] and it is called the exponential distinguishability property (EDP) (see [54]).

Lemma 10. *Given a quantum state $\rho \in \mathscr{D}(\mathscr{H})$,*

(Direct part) For any $\varepsilon > 0$, there exists a sequence of POVMs $\{A_n, \mathbb{I}-A_n\}_n$ such that

$$\lim_{n\to\infty} \mathrm{Tr}\left[(\mathbb{I}-A_n)\rho^{\otimes n}\right] = 0, \tag{H5}$$

and for every integer n and incoherent state $w_n \in \mathscr{I}_n$ with $\mathscr{I}_n$ is the set of incoherent states on $\mathscr{H}^{\otimes n}$,

$$-\frac{\log \mathrm{Tr}\,[A_n w_n]}{n} + \varepsilon \geq C_r(\rho). \tag{H6}$$

(Strong converse) If there exists $\varepsilon > 0$ and a sequence of POVMs $\{A_n, \mathbb{I}-A_n\}_n$ such that for every integer $n > 0$ and $w_n \in \mathscr{I}_n$,

$$-\frac{\log \mathrm{Tr}\,[A_n w_n]}{n} - \varepsilon \geq C_r(\rho), \tag{H7}$$

then

$$\lim_{n\to\infty} \mathrm{Tr}\left[(\mathbb{I}-A_n)\rho^{\otimes n}\right] = 1. \tag{H8}$$

Now, we are ready to prove the equivalence between $C_{\min}$ and C_r in asymptotic limit.

Proof of Equation (18). First we prove that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} C^{\varepsilon}_{\min}(\rho^{\otimes n}) \leq C_r(\rho).$$

Since $C_r(\rho) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} C^{\varepsilon}_{\max}(\rho^{\otimes n})$ and $C^{\varepsilon}_{\min}(\rho) \leq C^{\varepsilon}_{\max}(\rho) - \log(1-2\varepsilon)$, then we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} C^{\varepsilon}_{\min}(\rho^{\otimes n}) \leq C_r(\rho).$$

Now we prove that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} C^{\varepsilon}_{\min}(\rho^{\otimes n}) \geq C_r(\rho)$$

According to Lemma 10, for any $\varepsilon > 0$, there exists a sequence of POVMs $\{A_n\}$ such that for sufficient large integer n, $\mathrm{Tr}\,[\rho^{\otimes n} A_n] \geq 1-\varepsilon$, and thus

$$C^{\varepsilon}_{\min}(\rho^{\otimes n}) \geq \min_{\sigma\in\mathscr{I}} -\log \mathrm{Tr}\,[A_n\sigma] \geq n(C_r(\rho)-\varepsilon),$$

where the last inequality comes from the direct part of Lemma 10. Therefore,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} C^{\varepsilon}_{\min}(\rho^{\otimes n}) \geq C_r(\rho).$$

■

Appendix I: $C_{\min}^{\varepsilon}$ as an upper bound of one-shot distillable coherence

Lemma 11. *Given a quantum state $\rho \in \mathscr{D}(\mathscr{H})$, then for any $\mathscr{E} \in MIO$,*

$$C_{\min}^{\varepsilon}(\mathscr{E}(\rho)) \leq C_{\min}^{\varepsilon}(\rho). \tag{I1}$$

Proof. Let $0 \leq A \leq \mathbb{I}$ and $\mathrm{Tr}\,[A\mathscr{E}(\rho)] \geq 1-\varepsilon$ such that

$$C_{\min}^{\varepsilon}(\mathscr{E}(\rho)) = \min_{\sigma\in\mathscr{I}} -\log \mathrm{Tr}\,[A\sigma] = -\log \mathrm{Tr}\,[A\sigma_*]\,.$$

Then

$$\begin{aligned} C_{\min}^{\varepsilon}(\rho) &\geq -\log \mathrm{Tr}\left[\mathscr{E}^{\dagger}(A)\sigma_*\right] \\ &= -\log \mathrm{Tr}\,[A\mathscr{E}(\sigma_*)] \\ &\geq \min_{\sigma\in\mathscr{I}} -\log \mathrm{Tr}\,[A\sigma] \\ &= C_{\min}^{\varepsilon}(\mathscr{E}(\rho)), \end{aligned}$$

where the first inequality comes from the fact that $\mathrm{Tr}\left[\mathscr{E}^{\dagger}(A)\rho\right] = \mathrm{Tr}\,[A\mathscr{E}(\rho)] \geq 1-\varepsilon$ and $0 \leq \mathscr{E}^{\dagger}(A) \leq \mathbb{I}$ as $0 \leq A \leq \mathbb{I}$ and $\mathscr{E}^{\dagger}$ is unital. □

Proof of Equation (17). Suppose that $\mathscr{E}$ is the optimal MIO such that $F(\mathscr{E}(\rho), |\Psi_+^M\rangle\langle\Psi_+^M|)^2 \geq 1-\varepsilon$ with $\log M = C_{D,MIO}^{(1),\varepsilon}(\rho)$. By Lemma 11, we have

$$\begin{aligned} C_{\min}^{\varepsilon}(\rho) &\geq C_{\min}^{\varepsilon}(\mathscr{E}(\rho)) \\ &= \max_{\substack{0\leq A\leq\mathbb{I} \\ \mathrm{Tr}[A\mathscr{E}(\rho)]\geq 1-\varepsilon}} \min_{\sigma\in\mathscr{I}} -\log \mathrm{Tr}\,[A\sigma] \\ &\geq \min_{\sigma\in\mathscr{I}} -\log \mathrm{Tr}\left[|\Psi_+^M\rangle\langle\Psi_+^M|\sigma\right] \\ &= \log M = C_{D,MIO}^{(1),\varepsilon}(\rho), \end{aligned}$$

where the second inequality comes from the fact that $0 \leq |\Psi_+^M\rangle\langle\Psi_+^M| \leq \mathbb{I}$ and $\mathrm{Tr}\left[|\Psi_+^M\rangle\langle\Psi_+^M|\mathscr{E}(\rho)\right] = F(\mathscr{E}(\rho), |\Psi_+^M\rangle\langle\Psi_+^M|)^2 \geq 1-\varepsilon$.

■