# Max-Planck-Institut <br> für Mathematik <br> in den Naturwissenschaften Leipzig 

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# Improved monogamy relations with concurrence of assistance and negativity of assistance for multiqubit W-class states 

Zhi-Xiang Jin ${ }^{1}$, Shao-Ming Fei ${ }^{1,2}$, and Xianqing Li-Jost ${ }^{2,3}$<br>${ }^{1}$ School of Mathematical Sciences, Capital Normal University, Beijing 100048, China<br>${ }^{2}$ Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany<br>${ }^{3}$ School of Mathematics and Statistics, Hainan Normal University, Haikou 571158, China


#### Abstract

Monogamy relations characterize the distributions of entanglement in multipartite systems. We investigate the monogamy relations satisfied by the concurrence of assistance and the negativity of assistance for multiqubit generalized $W$-class states. Analytical monogamy inequalities are presented for both concurrence of assistance and negativity of assistance, which are shown to be tighter than the existing ones. Detailed examples have been presented.


## INTRODUCTION

Quantum entanglement [1-8] is an essential feature of quantum mechanics. As one of the fundamental differences between quantum entanglement and classical correlations, a key property of entanglement is that a quantum system entangled with one of the subsystems limits its entanglement with the remaining subsystems. The monogamy relations characterize the distribution of quantum entanglement in multipartite systems. Monogamy property is also an essential feature allowing for security in quantum key distribution [9].

For a tripartite system $A, B$ and $C$, the authors [10] show that there is a trade-off between $A^{\prime} s$ entanglement with $B$ and its entanglement with $C$. In [11], the authors present a simple identity which captures the trade-off between entanglement and classical correlation, which can be used to derive rigorous monogamy relations. They also proved various monogamous trade-off relations for other entanglement measures and correlation measures. In [12], the author proved the longstanding conjecture of Coffman, Kundu, and Wootters [10] that the distribution of bipartite quantum entanglement, measured by the tangle $\tau$, amongst $n$ qubits satisfies a tight inequality: $\tau \rho_{A_{1} A_{2}}+\tau \rho_{A_{1} A_{3}}+\cdots+\tau \rho_{A_{1} A_{n}} \leq \tau \rho_{A_{1} \mid A_{2} \cdots A_{n}}$, where $\tau \rho_{A_{1} \mid A_{2} \cdots A_{n}}$ denotes the bipartite quantum entanglement measured by the tangle under the bipartition $A_{1}$ and $A_{2} A_{3} \cdots A_{n}$.

Recently, the monogamy of entanglement for multiqubit $W$-class states has been investigated, and the monogamy relations for tangle and the squared concurrence have been presented in Ref. [13, 14]. In Ref. [15], the general monogamy inequalities of the $\alpha$-th power of concurrence and entanglement of formation are presented for N -qubit states. However, the concurrence of assistance does not satisfy monogamy relations for general quantum states. Therefore, special classes of quantum states have been taken into account for monogamy relations satisfied by the concurrence of assistance. The monogamy relations for the $x$-power of concurrence of assistance for the generalized multiqubit $W$-class states have been derived in [16]. In [17], a tighter monogamy relation of quantum entnglement for multiqubit $W$-class states has been presented.

In this paper, we show that the monogamy inequalities for concurrence of assistance obtained so far can be further improved. We present entanglement monogamy relations for the $x$-th $(x \geq 2)$ power of the concurrence of assistance, which are tighter than those in $[16,17]$ and give rise to finer characterizations of the entanglement distributions among the
multipartite $W$-class states. Moreover, we present the general monogamy relations for the $x$-power of negativity of assistance for the generalized multiqubit $W$-class states, which are also better than that in $[16,17]$.

## IMPROVED MONOGAMY RELATIONS FOR CONCURRENCE OF ASSISTANCE

Let $H_{X}$ denote a discrete finite dimensional complex vector space associated with a quantum subsystem $X$. For a bipartite pure state $|\psi\rangle_{A B}$ in vector space $H_{A} \otimes H_{B}$, the concurrence is given by $C\left(|\psi\rangle_{A B}\right)=\sqrt{2\left[1-\operatorname{Tr}\left(\rho_{A}^{2}\right)\right]}[18-20]$, where $\rho_{A}$ is the reduced density matrix, $\rho_{A}=\operatorname{Tr}_{B}\left(|\psi\rangle_{A B}\langle\psi|\right)$. The concurrence for a bipartite mixed state $\rho_{A B}$ is defined by the convex roof extension $C\left(\rho_{A B}\right)=\min _{\left\{p_{i}\left|\psi_{i}\right\rangle\right\}} \sum_{i} p_{i} C\left(\left|\psi_{i}\right\rangle\right)$, where the minimum is taken over all possible pure state decompositions of $\rho_{A B}=\sum_{i} p_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|$, with $p_{i} \geq 0$ and $\sum_{i} p_{i}=1$ and $\left|\psi_{i}\right\rangle \in H_{A} \otimes H_{B}$.

For an $N$-qubit state $\rho_{A B_{1} \cdots B_{N-1}} \in H_{A} \otimes H_{B_{1}} \otimes \cdots \otimes H_{B_{N-1}}$, the concurrence $C\left(\rho_{A \mid B_{1} \cdots B_{N-1}}\right)$ of the state $\rho_{A \mid B_{1} \cdots B_{N-1}}$, viewed as a bipartite state under the bipartition $A$ and $B_{1}, B_{2}, \cdots, B_{N-1}$, satisfies [15]

$$
C^{x}\left(\rho_{A \mid B_{1}, B_{2} \cdots, B_{N-1}}\right) \geq C^{x}\left(\rho_{A B_{1}}\right)+C^{x}\left(\rho_{A B_{2}}\right)+\cdots+C^{x}\left(\rho_{A B_{N-1}}\right),
$$

for $\alpha \geq 2$, where $\rho_{A B_{i}}=\operatorname{Tr}_{B_{1} \cdots B_{i-1} B_{i+1} \cdots B_{N-1}}\left(\rho_{A B_{1} \cdots B_{N-1}}\right)$. The above monogamy relation is improved such that for $\alpha \geq 2$, if $C\left(\rho_{A B_{i}}\right) \geq C\left(\rho_{A \mid B_{i+1} \cdots B_{N-1}}\right)$ for $i=1,2, \cdots, m$, and $C\left(\rho_{A B_{j}}\right) \leq C\left(\rho_{A \mid B_{j+1} \cdots B_{N-1}}\right)$ for $j=m+1, \cdots, N-2, \forall 1 \leq m \leq N-3, N \geq 4$, then [21],

$$
\begin{align*}
& C^{x}\left(\rho_{A \mid B_{1} B_{2} \cdots B_{N-1}}\right) \geq \\
& C^{x}\left(\rho_{A B_{1}}\right)+\frac{x}{2} C^{x}\left(\rho_{A B_{2}}\right)+\cdots+\left(\frac{x}{2}\right)^{m-1} C^{x}\left(\rho_{A B_{m}}\right) \\
& +\left(\frac{x}{2}\right)^{m+1}\left(C^{x}\left(\rho_{A B_{m+1}}\right)+\cdots+C^{x}\left(\rho_{A B_{N-2}}\right)\right) \\
& +\left(\frac{x}{2}\right)^{m} C^{x}\left(\rho_{A B_{N-1}}\right) . \tag{1}
\end{align*}
$$

In [22], (1) is further improved such that for $x \geq 2$, one has

$$
\begin{align*}
& C^{x}\left(\rho_{A \mid B_{1} B_{2} \cdots B_{N-1}}\right) \geq \\
& C^{x}\left(\rho_{A B_{1}}\right)+h C^{x}\left(\rho_{A B_{2}}\right)+\cdots+h^{m-1} C^{x}\left(\rho_{A B_{m}}\right) \\
& +h^{m+1}\left(C^{x}\left(\rho_{A B_{m+1}}\right)+\cdots+C^{x}\left(\rho_{A B_{N-2}}\right)\right) \\
& +h^{m} C^{x}\left(\rho_{A B_{N-1}}\right) \tag{2}
\end{align*}
$$

for all $x \geq 2$, where $h=2^{\frac{x}{2}}-1$.
For a tripartite pure state $|\psi\rangle_{A B C}$, the concurrence of assistance is defined by [23, 24]

$$
C_{a}\left(|\psi\rangle_{A B C}\right) \equiv C_{a}\left(\rho_{A B}\right)=\max _{\left\{p_{i},\left|\psi_{i}\right\rangle\right\}} \sum_{i} p_{i} C\left(\left|\psi_{i}\right\rangle\right),
$$

where the maximum is taken over all possible decompositions of $\rho_{A B}=\operatorname{Tr}_{C}\left(|\psi\rangle_{A B C}\langle\psi|\right)=$ $\sum_{i} p_{i}\left|\psi_{i}\right\rangle_{A B}\left\langle\psi_{i}\right|$. When $\rho_{A B}$ is a pure state, then one has $C\left(|\psi\rangle_{A B}\right)=C_{a}\left(\rho_{A B}\right)$.

Different from the Coffman-Kundu-Wootters inequality satisfied by the concurrence, the concurrence of assistance does not satisfy the monogamy relations in general. However, for $N$-qubit generalized $W$-class state, $|\psi\rangle_{A B_{1} \cdots B_{N-1}} \in H_{A} \otimes H_{B_{1}} \otimes \cdots \otimes H_{B_{N-1}}$ defined by

$$
\begin{equation*}
|\psi\rangle_{A B_{1} \cdots B_{N-1}}=a|10 \cdots 0\rangle+b_{1}|01 \cdots 0\rangle+\cdots+b_{N-1}|00 \cdots 1\rangle \tag{3}
\end{equation*}
$$

with $|a|^{2}+\sum_{i=1}^{N-1}\left|b_{i}\right|^{2}=1$, one has [16],

$$
\begin{equation*}
C\left(\rho_{A B_{i}}\right)=C_{a}\left(\rho_{A B_{i}}\right), \quad i=1,2, \ldots, N-1, \tag{4}
\end{equation*}
$$

where $\rho_{A B_{i}}=\operatorname{Tr}_{B_{1} \cdots B_{i-1} B_{i+1} \cdots B_{N-1}}\left(|\psi\rangle_{A B_{1} \cdots B_{N-1}}\langle\psi|\right)$, and the concurrence of assistance $C_{a}\left(|\psi\rangle_{A \mid B_{1} \cdots B_{N-1}}\right)$ satisfies the monogamy inequality [16],

$$
\begin{equation*}
C_{a}^{x}\left(|\psi\rangle_{A \mid B_{1}, B_{2} \cdots, B_{N-1}}\right) \geq C_{a}^{x}\left(\rho_{A B_{1}}\right)+C_{a}^{x}\left(\rho_{A B_{2}}\right)+\cdots+C_{a}^{x}\left(\rho_{A B_{N-1}}\right), \tag{5}
\end{equation*}
$$

for $x \geq 2$. (5) has been further improved such that for $x \geq 2$, if $C\left(\rho_{A B_{i}}\right) \geq C\left(\rho_{A \mid B_{i+1} \cdots B_{N-1}}\right)$ for $i=1,2, \cdots, m$, and $C\left(\rho_{A B_{j}}\right) \leq C\left(\rho_{A \mid B_{j+1} \cdots B_{N-1}}\right)$ for $j=m+1, \cdots, N-2, \forall 1 \leq m \leq$ $N-3, N \geq 4$, then [17],

$$
\begin{align*}
& C_{a}^{x}\left(|\psi\rangle_{A \mid B_{1} B_{2} \cdots B_{N-1}}\right) \geq \\
& C_{a}^{x}\left(\rho_{A B_{1}}\right)+\frac{x}{2} C_{a}^{x}\left(\rho_{A B_{2}}\right)+\cdots+\left(\frac{x}{2}\right)^{m-1} C_{a}^{x}\left(\rho_{A B_{m}}\right) \\
& +\left(\frac{x}{2}\right)^{m+1}\left(C_{a}^{x}\left(\rho_{A B_{m+1}}\right)+\cdots+C_{a}^{x}\left(\rho_{A B_{N-2}}\right)\right) \\
& +\left(\frac{x}{2}\right)^{m} C_{a}^{x}\left(\rho_{A B_{N-1}}\right) . \tag{6}
\end{align*}
$$

In fact, as a kind of characterization of the entanglement distribution among the subsystems, the monogamy inequalities satisfied by the concurrence of assistance can be further refined and become tighter.
[Theorem 1]. Let $\rho_{A B_{j_{1}} \cdots B_{j_{m-1}}}$ denote the $m$-qubit reduced density matrix of the $N$ qubit generalized $W$-class state $|\psi\rangle_{A B_{1} \cdots B_{N-1}} \in H_{A} \otimes H_{B_{1}} \otimes \cdots \otimes H_{B_{N-1}}$. If $C\left(\rho_{A B_{j_{i}}}\right) \geq$
$C\left(\rho_{A B_{j_{i+1}} \cdots B_{j_{m-1}}}\right)$ for $i=1,2, \cdots t$, and $C\left(\rho_{A B_{j_{k}}}\right) \leq C\left(\rho_{A B_{j_{k+1}} \cdots B_{j_{m-1}}}\right)$ for $k=t+$ $1, \cdots, m-2, \forall 1 \leq t \leq m-3, m \geq 4$, the concurrence of assistance satisfies

$$
\begin{align*}
& C_{a}^{x}\left(\rho_{A \mid B_{j_{1}} \cdots B_{j_{m-1}}}\right) \geq C_{a}^{x}\left(\rho_{A B_{j_{1}}}\right) \\
& +h C_{a}^{x}\left(\rho_{A B_{j_{2}}}\right)+\cdots+h^{t-1} C_{a}^{x}\left(\rho_{A B_{j_{t}}}\right) \\
& +h^{t+1}\left(C_{a}^{x}\left(\rho_{A B_{j_{t+1}}}\right)+\cdots+C_{a}^{x}\left(\rho_{A B_{j_{m-2}}}\right)\right) \\
& +h^{t} C_{a}^{x}\left(\rho_{A B_{j_{m-1}}}\right) \tag{7}
\end{align*}
$$

for all $x \geq 2$, where $h=2^{\frac{x}{2}}-1$.
[Proof]. For the $N$-qubit generalized $W$-class states $|\psi\rangle_{A B_{1} \cdots B_{N-1}}$, according to the definitions of $C(\rho)$ and $C_{a}(\rho)$, one has $C_{a}\left(\rho_{A \mid B_{j_{1}} \cdots B_{j_{m-1}}}\right) \geq C\left(\rho_{A \mid B_{j_{1}} \cdots B_{j_{m-1}}}\right)$. When $x \geq 2$, we have

$$
\begin{align*}
C_{a}^{x}\left(\rho_{A \mid B_{j_{1}} \cdots B_{j_{m-1}}}\right) & \geq C^{x}\left(\rho_{A \mid B_{j_{1}} \cdots B_{j_{m-1}}}\right) \geq C^{x}\left(\rho_{A B_{j_{1}}}\right) \\
& +h C^{x}\left(\rho_{A B_{j_{2}}}\right)+\cdots+h^{t-1} C^{x}\left(\rho_{A B_{j_{t}}}\right) \\
& +h^{t+1}\left(C^{x}\left(\rho_{A B_{j_{t+1}}}\right)+\cdots+C^{x}\left(\rho_{A B_{j_{m-2}}}\right)\right) \\
& +h^{t} C^{x}\left(\rho_{A B_{j_{m-1}}}\right) \\
& =C_{a}^{x}\left(\rho_{A B_{j_{1}}}\right)+h C_{a}^{x}\left(\rho_{A B_{j_{2}}}\right)+\cdots+h^{t-1} C_{a}^{x}\left(\rho_{A B_{j_{t}}}\right) \\
& +h^{t+1}\left(C_{a}^{x}\left(\rho_{A B_{j_{t+1}}}\right)+\cdots+C_{a}^{x}\left(\rho_{A B_{j_{m-2}}}\right)\right) \\
& +h^{t} C_{a}^{x}\left(\rho_{A B_{j_{m-1}}}\right), \tag{8}
\end{align*}
$$

where we have used in the first inequality the relation $a^{x} \geq b^{x}$ for $a \geq b \geq 0, x \geq 2$. The second inequality is due to (2). The equality is due to (4).

As for $x \geq 2, h^{t} \geq(x / 2)^{t}$ for all $1 \leq t \leq m-3$, comparing with the monogamy relations for concurrence of assistance (5) and (6), our formula (7) in Theorem 1 gives a tighter monogamy relation with larger lower bound. In Theorem 1 we have assumed that some $C\left(\rho_{A B_{j_{i}}}\right) \geq C\left(\rho_{A j_{j_{i+1}} \cdots B_{j_{m-1}}}\right)$ and some $C\left(\rho_{A B_{j_{k}}}\right) \leq C\left(\rho_{A B_{j_{k+1}} \cdots B_{j_{m-1}}}\right)$ for the $N$-qubit generalized $W$-class states. If all $C\left(\rho_{A B_{j_{i}}}\right) \geq C\left(\rho_{A B_{j_{i+1}} \cdots B_{j_{m-1}}}\right)$ for $i=1,2, \cdots, m-2$, then we have the following conclusion:
[Theorem 2]. If $C\left(\rho_{A B_{j_{i}}}\right) \geq C\left(\rho_{A B_{j_{i+1}} \cdots B_{j_{m-1}}}\right)$ for $i=1,2, \cdots, m-2$, we have

$$
\begin{equation*}
C_{a}^{x}\left(\rho_{A \mid B_{j_{1}} \cdots B_{j_{m-1}}}\right) \geq C_{a}^{x}\left(\rho_{A B_{j_{1}}}\right)+h C_{a}^{x}\left(\rho_{A B_{j_{2}}}\right)+\cdots+h^{m-2} C_{a}^{x}\left(\rho_{A B_{j_{m-1}}}\right) \tag{9}
\end{equation*}
$$

for all $x \geq 2$, where $h=2^{\frac{x}{2}}-1$.


FIG. 1: $y$ is the value of $C_{a}\left(|W\rangle_{A \mid B_{1} B_{2} B_{3}}\right)$. Solid (red) line is the exact value of $C_{a}\left(|W\rangle_{A \mid B_{1} B_{2} B_{3}}\right)$, dashed (blue) line is the lower bound of $C_{a}\left(|W\rangle_{A \mid B_{1} B_{2} B_{3}}\right)$ in (7), dot dashed (green) line is the lower bound in [17], and dotted (black) line is the lower bound in [16] for $x \geq 2$.

Example 1. Let us consider the 4-qubit generlized $W$-class state,

$$
\begin{equation*}
|W\rangle_{A B_{1} B_{2} B_{3}}=\frac{1}{2}(|1000\rangle+|0100\rangle+|0010\rangle+|0001\rangle) . \tag{10}
\end{equation*}
$$

We have $C_{a}^{x}\left(|W\rangle_{A \mid B_{1} B_{2} B_{3}}\right)=\left(\frac{\sqrt{3}}{2}\right)^{x}$. From our result (7) we have $C_{a}^{x}\left(|W\rangle_{A \mid B_{1} B_{2} B_{3}}\right) \geq$ $\left[2 \cdot 2^{\frac{x}{2}}-1\right]\left(\frac{1}{2}\right)^{x}$, from (6) one has $C_{a}^{x}\left(|W\rangle_{A \mid B_{1} B_{2} B_{3}}\right) \geq(x+1)\left(\frac{1}{2}\right)^{x}$, and from (5) one has $C_{a}^{x}\left(|W\rangle_{A \mid B_{1} B_{2} B_{3}}\right) \geq 3\left(\frac{1}{2}\right)^{x}, x \geq 2$. One can see that our result is better than that in [16] and [17] for $x \geq 2$, see Fig. 1 .

## IMPROVED MONOGAMY RELATIONS FOR NEGATIVITY OF ASSISTANCE

Another well-known quantifier of bipartite entanglement is the negativity. Given a bipartite state $\rho_{A B}$ in $H_{A} \otimes H_{B}$, the negativity is defined by [25], $N\left(\rho_{A B}\right)=\left(\left\|\rho_{A B}^{T_{A}}\right\|-1\right) / 2$, where $\rho_{A B}^{T_{A}}$ is the partially transposed $\rho_{A B}$ with respect to the subsystem $A,\|X\|$ denotes the trace norm of $X$, i.e $\|X\|=\operatorname{Tr} \sqrt{X X^{\dagger}}$. Negativity is a computable measure of entanglement, and is a convex function of $\rho_{A B}$. It vanishes if and only if $\rho_{A B}$ is separable for the $2 \otimes 2$ and $2 \otimes 3$
systems [26]. For the purpose of discussion, we use the following definition of negativity, $N\left(\rho_{A B}\right)=\left\|\rho_{A B}^{T_{A}}\right\|-1$. For any bipartite pure state $|\psi\rangle_{A B}$, the negativity $N\left(\rho_{A B}\right)$ is given by $N\left(|\psi\rangle_{A B}\right)=2 \sum_{i<j} \sqrt{\lambda_{i} \lambda_{j}}=\left(\operatorname{Tr} \sqrt{\rho_{A}}\right)^{2}-1$, where $\lambda_{i}$ are the eigenvalues for the reduced density matrix $\rho_{A}$ of $|\psi\rangle_{A B}$. For a mixed state $\rho_{A B}$, the convex-roof extended negativity (CREN) is defined by

$$
\begin{equation*}
N_{c}\left(\rho_{A B}\right)=\min \sum_{i} p_{i} N\left(\left|\psi_{i}\right\rangle_{A B}\right), \tag{11}
\end{equation*}
$$

where the minimum is taken over all possible pure state decompositions $\left\{p_{i},\left|\psi_{i}\right\rangle_{A B}\right\}$ of $\rho_{A B}$. CREN gives a perfect discrimination of positive partial transposed bound entangled states and separable states in any bipartite quantum systems [27, 28]. For a mixed state $\rho_{A B}$, the convex-roof extended negativity of assistance (CRENOA) is defined as [29]

$$
\begin{equation*}
N_{a}\left(\rho_{A B}\right)=\max \sum_{i} p_{i} N\left(\left|\psi_{i}\right\rangle_{A B}\right) \tag{12}
\end{equation*}
$$

where the maximum is taken over all possible pure state decompositions $\left\{p_{i},\left|\psi_{i}\right\rangle_{A B}\right\}$ of $\rho_{A B}$.

For an $N$-qubit state $\rho_{A B_{1} \cdots B_{N-1}} \in H_{A} \otimes H_{B_{1}} \otimes \cdots \otimes H_{B_{N-1}}$, we denote $N_{c}\left(\rho_{A \mid B_{1} \cdots B_{N-1}}\right)$ the negativity of the state $\rho_{A \mid B_{1} \cdots B_{N-1}}$, viewed as a bipartite state under the partition $A$ and $B_{1}, B_{2}, \cdots, B_{N-1}$. If $N_{c}\left(\rho_{A B_{i}}\right) \geq N_{c}\left(\rho_{A \mid B_{i+1} \cdots B_{N-1}}\right)$ for $i=1,2, \cdots, m$, and $N_{c}\left(\rho_{A B_{j}}\right) \leq$ $N_{c}\left(\rho_{A \mid B_{j+1} \cdots B_{N-1}}\right)$ for $j=m+1, \cdots, N-2, \forall 1 \leq m \leq N-3, N \geq 4$, then [17]

$$
\begin{align*}
& N_{c}^{x}\left(\rho_{A \mid B_{1} B_{2} \cdots B_{N-1}}\right) \\
& \geq N_{c}^{x}\left(\rho_{A B_{1}}\right)+\frac{x}{2} N_{c}^{x}\left(\rho_{A B_{2}}\right)+\cdots+\left(\frac{x}{2}\right)^{m-1} N_{c}^{x}\left(\rho_{A B_{m}}\right) \\
& +\left(\frac{x}{2}\right)^{m+1}\left(N_{c}^{x}\left(\rho_{A B_{m+1}}\right)+\cdots+N_{c}^{x}\left(\rho_{A B_{N-2}}\right)\right) \\
& +\left(\frac{x}{2}\right)^{m} N_{c}^{x}\left(\rho_{A B_{N-1}}\right), \tag{13}
\end{align*}
$$

for all $x \geq 2$. The inequality (13) is further improved that for $x \geq 2$ [22]

$$
\begin{align*}
& N_{c}^{x}\left(\rho_{A \mid B_{1} B_{2} \cdots B_{N-1}}\right) \\
& \geq N_{c}^{x}\left(\rho_{A B_{1}}\right)+h N_{c}^{x}\left(\rho_{A B_{2}}\right)+\cdots+h^{m-1} N_{c}^{x}\left(\rho_{A B_{m}}\right) \\
& +h^{m+1}\left(N_{c}^{x}\left(\rho_{A B_{m+1}}\right)+\cdots+N_{c}^{x}\left(\rho_{A B_{N-2}}\right)\right) \\
& +h^{m} N_{c}^{x}\left(\rho_{A B_{N-1}}\right) \tag{14}
\end{align*}
$$

where $h=2^{\frac{x}{2}}-1$.

The negativity of assistance does not satisfy a monogamy relation in general. However, for an $N$-qubit generlized $W$-class state $|\psi\rangle_{A B_{1} \cdots B_{N-1}} \in H_{A} \otimes H_{B_{1}} \otimes \cdots \otimes H_{B_{N-1}}$, if $N_{c}\left(\rho_{A B_{i}}\right) \geq N_{c}\left(\rho_{A \mid B_{i+1} \cdots B_{N-1}}\right)$ for $i=1,2, \cdots, m$, and $N_{c}\left(\rho_{A B_{j}}\right) \leq N_{c}\left(\rho_{A \mid B_{j+1} \cdots B_{N-1}}\right)$ for $j=m+1, \cdots, N-2, \forall 1 \leq m \leq N-3, N \geq 4$, then the negativity of assistance $N_{a}\left(|\psi\rangle_{A \mid B_{1} \cdots B_{N-1}}\right)$ of the state $|\psi\rangle_{A B_{1} \cdots B_{N-1}}$ satisfies the inequality [17],

$$
\begin{align*}
& N_{a}^{x}\left(|\psi\rangle_{A \mid B_{1} B_{2} \cdots B_{N-1}}\right) \geq \\
& N_{a}^{x}\left(\rho_{A B_{1}}\right)+\frac{x}{2} N_{a}^{x}\left(\rho_{A B_{2}}\right)+\cdots+\left(\frac{x}{2}\right)^{m-1} N_{a}^{x}\left(\rho_{A B_{m}}\right) \\
& +\left(\frac{x}{2}\right)^{m+1}\left(N_{a}^{x}\left(\rho_{A B_{m+1}}\right)+\cdots+N_{a}^{x}\left(\rho_{A B_{N-2}}\right)\right) \\
& +\left(\frac{x}{2}\right)^{m} N_{a}^{x}\left(\rho_{A B_{N-1}}\right) \tag{15}
\end{align*}
$$

for all $x \geq 2$.
In fact, to have a better characterization of the entanglement distribution among the subsystems, the monogamy inequalities satisfied by the negativity of assistance can be further refined and become tighter. Taking into account the fact that for $N$-qubit generlized $W$-class states (3) [17],

$$
\begin{equation*}
N_{c}\left(\rho_{A B_{i}}\right)=N_{a}\left(\rho_{A B_{i}}\right), \quad i=1,2, \cdots, N-1 \tag{16}
\end{equation*}
$$

where $\rho_{A B_{i}}=\operatorname{Tr}_{B_{1} \cdots B_{i-1} B_{i+1} \cdots B_{N-1}}\left(|\psi\rangle_{A B_{1} \cdots B_{N-1}}\langle\psi|\right)$, we have the following relations:
[Theorem 3]. For the $N$-qubit generalized $W$-class states $|\psi\rangle_{A B_{1} \cdots B_{N-1}} \in H_{A} \otimes H_{B_{1}} \otimes$ $\cdots \otimes H_{B_{N-1}}$, if $N_{c}\left(\rho_{A B_{j_{i}}}\right) \geq N_{c}\left(\rho_{A B_{j_{i+1}} \cdots B_{j_{m-1}}}\right)$ for $i=1,2, \cdots, t$, and $N_{c}\left(\rho_{A B_{j_{k}}}\right) \leq$ $N_{c}\left(\rho_{A B_{j_{k+1}} \cdots B_{j_{m-1}}}\right)$ for $k=t+1, \cdots, m-2, \forall 1 \leq t \leq m-3, m \geq 4$, then the CRENOA satisfies

$$
\begin{align*}
& N_{a}^{x}\left(\rho_{A \mid B_{j_{1}} \cdots B_{j_{m-1}}}\right) \geq N_{a}^{x}\left(\rho_{A B_{j_{1}}}\right) \\
& +h N_{a}^{x}\left(\rho_{A B_{j_{2}}}\right)+\cdots+h^{t-1} N_{a}^{x}\left(\rho_{A B_{j_{t}}}\right) \\
& +h^{t+1}\left(N_{a}^{x}\left(\rho_{A B_{j_{t+1}}}\right)+\cdots+N_{a}^{x}\left(\rho_{A B_{j_{m-2}}}\right)\right) \\
& +h^{t} N_{a}^{x}\left(\rho_{A B_{j_{m-1}}}\right) \tag{17}
\end{align*}
$$

for all $x \geq 2$, where $h=2^{\frac{x}{2}}-1$.
[Proof]. For the $N$-qubit generalized $W$-class states $|\psi\rangle_{A B_{1} \cdots B_{N-1}}$, according to the definitions of $N_{c}(\rho)$ and $N_{a}(\rho)$, one has $N_{a}\left(\rho_{A \mid B_{j_{1}} \cdots B_{j_{m-1}}}\right) \geq N_{c}\left(\rho_{A \mid B_{j_{1}} \cdots B_{j_{m-1}}}\right)$. When $x \geq 2$, we
have

$$
\begin{align*}
N_{a}^{x}\left(\rho_{A \mid B_{j_{1}} \cdots B_{j_{m-1}}}\right) & \geq N_{c}^{x}\left(\rho_{A \mid B_{j_{1}} \cdots B_{j_{m-1}}}\right) \geq N_{c}^{x}\left(\rho_{A B_{j_{1}}}\right) \\
& +h N_{c}^{x}\left(\rho_{A B_{j_{2}}}\right)+\cdots+h^{t-1} N_{c}^{x}\left(\rho_{A B_{j_{t}}}\right) \\
& +h^{t+1}\left(N_{c}^{x}\left(\rho_{A B_{j_{t+1}}}\right)+\cdots+N_{c}^{x}\left(\rho_{A B_{j_{m-2}}}\right)\right) \\
& +h^{t} N_{c}^{x}\left(\rho_{A B_{j_{m-1}}}\right) \\
& =N_{a}^{x}\left(\rho_{A B_{j_{1}}}\right)+h N_{a}^{x}\left(\rho_{A B_{j_{2}}}\right)+\cdots+h^{t-1} N_{a}^{x}\left(\rho_{A B_{j_{t}}}\right) \\
& +h^{t+1}\left(N_{a}^{x}\left(\rho_{A B_{j_{t+1}}}\right)+\cdots+N_{a}^{x}\left(\rho_{A B_{j_{m-2}}}\right)\right) \\
& +h^{t} N_{a}^{x}\left(\rho_{A B_{j_{m-1}}}\right), \tag{18}
\end{align*}
$$

where we have used in the first inequality the relation $a^{x} \geq b^{x}$ for $a \geq b \geq 0, x \geq 2$. Due to the (14), one gets the second inequality. The equality is due to relation (16).

As for $x \geq 2, h^{t} \geq(x / 2)^{t}$ for all $1 \leq t \leq m-3$, comparing with the monogamy relations for CRENOA in (15), our formula (17) in Theorem 3 gives a tighter monogamy relation with larger lower bounds. In Theorem 3 we have assumed that some $N_{c}\left(\rho_{A B_{j_{i}}}\right) \geq$ $N_{c}\left(\rho_{A B_{j_{i+1}} \cdots B_{j_{m-1}}}\right)$ and some $N_{c}\left(\rho_{A B_{j_{k}}}\right) \leq N_{c}\left(\rho_{A B_{j_{k+1}} \cdots B_{j_{m-1}}}\right)$ for the $N$-qubit generalized $W$ class states. If all $N_{c}\left(\rho_{A B_{j_{i}}}\right) \geq N_{c}\left(\rho_{A B_{j_{i+1}} \cdots B_{j_{m-1}}}\right)$ for $i=1,2, \cdots, m-2$, then we have the following conclusion:
[Theorem 4]. If $N_{c}\left(\rho_{A J_{j_{i}}}\right) \geq N_{c}\left(\rho_{A B_{j_{i+1}} \cdots B_{j_{m-1}}}\right)$ for $i=1,2, \cdots, m-2$, we have

$$
\begin{equation*}
N_{a}^{x}\left(\rho_{A \mid B_{j_{1}} \cdots B_{j_{m-1}}}\right) \geq N_{a}^{x}\left(\rho_{A B_{j_{1}}}\right)+h N_{a}^{x}\left(\rho_{A B_{j_{2}}}\right)+\cdots+h^{m-2} N_{a}^{x}\left(\rho_{A B_{j_{m-1}}}\right) \tag{19}
\end{equation*}
$$

for all $x \geq 2$, where $h=2^{\frac{x}{2}}-1$.
Example 2. Let us consider the $N$-qubit pure $W$-class state,

$$
\begin{equation*}
|W\rangle_{A B_{1} \cdots B_{N-1}}=\frac{1}{\sqrt{N}}(|10 \cdots 0\rangle+|01 \cdots 0\rangle+\cdots+|00 \cdots 1\rangle) . \tag{20}
\end{equation*}
$$

It is straightforword to check: $N_{a}^{x}\left(|W\rangle_{A \mid B_{1} \cdots B_{N-1}}\right)=\left(\frac{2 \sqrt{N-1}}{N}\right)^{x}, N_{a}^{x}\left(\rho_{A B_{1}}\right)=N_{a}^{x}\left(\rho_{A B_{2}}\right)=$ $\cdots=N_{a}^{x}\left(\rho_{A B_{N-1}}\right)=\left(\frac{2}{N}\right)^{x}$. Let us choose $N=5$. Then $N_{a}^{x}\left(|W\rangle_{A \mid B_{1} \cdots B_{4}}\right)=\left(\frac{4}{5}\right)^{x}$. From our result (17) we have $N_{a}^{x}\left(|W\rangle_{A \mid B_{1} \cdots B_{4}}\right) \geq\left[3 \cdot 2^{\frac{x}{2}}-2\right]\left(\frac{1}{2}\right)^{x}$, while from (15) one has $N_{a}^{x}\left(|W\rangle_{A \mid B_{1} \cdots B_{4}}\right) \geq\left(\frac{3 x}{2}+1\right)\left(\frac{1}{2}\right)^{x}, x \geq 2$. Obviously, our result is better than that in [17] with $x \geq 2$, see Fig. 2 .


FIG. 2: $y$ is the value of $N_{a}^{x}\left(|W\rangle_{A \mid B_{1} \cdots B_{4}}\right)$. Solid (red) line is the exact value of $N_{a}^{x}\left(|W\rangle_{A \mid B_{1} \cdots B_{4}}\right)$, dashed (blue) line is the lower bound of $N_{a}^{x}\left(|W\rangle_{A \mid B_{1} \cdots B_{4}}\right)$ in (17), dot dashed (green) line is the lower bound in [17] for $x \geq 2$.

## CONCLUSION

Entanglement monogamy is a fundamental property of multipartite entangled states. We have presented tighter monogamy inequalities for the $x$-power of concurrence of assistance $C_{a}^{x}\left(\rho_{A \mid B_{j_{1}} \cdots B_{j_{m-1}}}\right)$ of the $m$-qubit reduced density matrices, $4 \leq m \leq N$, for the $N$-qubit generalized $W$-class states, when $x \geq 2$. The monogamy relations for the $x$-power of negativity of assistance for the $N$-qubit generalized $W$-class states have been also investigated for $x \geq 2$. These relations give rise to the restrictions of entanglement distribution among the qubits in generalized $W$-class states. It should be noted that entanglement of assistances like concurrence of assistance and negativity of assistance are not genuine measures of quantum entanglement. They quantify the maximum average amount of entanglement between two parties, Alice and Bob, which can be extracted given assistance from a third party, Charlie, by performing a measurement on his system and reporting the measurement outcomes to Alice and Bob. Nevertheless, similar to quantum entanglement, we see that the entanglement of assistances also satisfy certain monogamy relations.

Acknowledgments This work is supported by the NSF of China under Grant No. 11675113.
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    Zhi-Xiang Jin, Shao-Ming Fei, and Xianqing Li-Jost

