Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Quantum steerability based on jointly measurability

by

Zhi-Hua Chen, Xiangjun Ye, and Shao-Ming Fei

Preprint no.: 9 2018

Quantum steerability based on jointly measurability

Zhihua Chen[1], **Xiangjun Ye**[2,3], **and Shao-Ming Fei**[4,5,*]

[1]Department of Mathematics, College of Science, Zhejiang University of Technology, Hangzhou 310023, China
[2]Key Laboratory of Quantum Information, University of Science and Technology of China, CAS, Hefei 230026, China
[3]Synergetic Innovation Center of Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei, 230026, China
[4]School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
[5]Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany
[*]corresponding.author@feishm@cnu.edu.cn

ABSTRACT

Lying in between entanglement and Bell nonlocality, the Einstein-Podolsky-Rosen (EPR) steering has received increasing attention in recent years. To characterize the EPR-steering, many criteria have been proposed and experimentally implemented. Nevertheless, only a few results are given to quantify the steerability with analytical results. In this work, we propose a method to quantify the steerability for two-qubit quantum states in the two-setting EPR-steering senario, by using the connection between joint measurability and the steerability. We derive the analytical formula of the steerability for a class of X-states. The sufficient and necessary conditions for two-setting EPR-steering are presented. Based on these results, a class of asymmetric states: one-way steerable states are obtained.

Introduction

Quantum nonlocality, EPR-steering and quantum entanglement are important quantum correlations. EPR-steering, originally given by Schrodinger in the context of famous Einstein-Podolsky-Rosen (EPR) paradox [1], lies in between quantum nonlocality and quantum entanglement, which means that one observer, by performing a local measurement on one's subsystem, can nonlocally steer the state of the other subsystem. Recently EPR-steering was reformulated by Wiseman et al who showed the hierarchy among Bell nonlocality, EPR-steering and quantum entanglement [2]. EPR-steering has shown to be of advantages for the quantum tasks such as randomness generation, subchannel discrimination, quantum information processing and one-sided device-independent processing in quantum key distribution [3, 4, 5, 6, 7] etc..

Many efforts have been made to detect and measure EPR-steering. Some steering inequalities based on uncertainty relations [8, 9, 10, 11, 12, 13], inequalities based on steering witnesses and Clauser-Horne-Shimony-Holt (CHSH)-like inequality, and geometric Bell-like inequalities et al [16, 18, 19, 20, 14, 15, 17] are constructed to diagnose the steerability, which usually are necessary conditions. Besides inequalities, all-versus-nothing proof without inequalities, were also presented to detect the steerability [21]. But only a few methods are given to quantify EPR-steering based on maximal violation of steering inequalities [22], steering weight [23] and steering robustness. In these cases semi-definite programming are needed to calculate the measures. Recently, the radius of super quantum hidden state model was proposed to evaluate the steerability [25] by finding the optimal super local hidden states. Nevertheless, it is formidably difficult to find the optimal super quantum hidden states. A critical radius was proposed through the geometrical method and the critical radius of T-states was calculated explicitly [24]. The closed formulas for steering were derived in the two and three measurement scenarios [26], however, which is the case when Alice and Bob are both allowed to measure the observables in their sites. It has been proven that there is a one to one mapping between joint measurability and the steerability of any assemblage [27, 28, 29, 30]. By using the connection between steering and joint measurability, the closed formula of the measure for two setting EPR-steering of Bell-diagonal states was given [31]. However, for any two-qubit quantum states, one still lacks the closed formula for the steerability problem even for 2-setting scenario.

Different from the Bell nonlocality and quantum entanglement, steering exhibits asymmetric features, proposed by Wiseman et al [2]. There exist quantum states ρ_{AB}, for which Alice can steer Bob's state but Bob can not steer Alice's state, or vice versus. This distinguished feature would be useful for some one-way quantum information tasks such as quantum cryptography. But until recently only a few asymmetric states are proposed and experimentally demonstrated [33, 34, 25, 32].

In this work we aim to investigate the analytical formula for the quantification of EPR-steering and get the necessary and sufficient condition of steerability for a class of quantum states. Then the asymmetric feature of EPR-steering will be also investigated.

Setting up the stage

Consider a bipartite qubit system ρ_{AB} shared by Alice and Bob, with reduced density states ρ_A and ρ_B. Alice performs positive-operator-valued measures (POVMs) $\Pi_{\kappa|\vec{n}}$ on subsystem A, where $\Pi_{\kappa|\vec{n}} = \frac{1}{2}(I_2 + (-1)^\kappa \vec{n}\cdot\vec{\sigma})$, I_2 is the identity matrix and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. Alice gets the result κ ($\kappa = 0,1$) when measuring along the direction $\vec{n}$. Bob's unnormalized conditional state is $\tilde{\rho}_{\kappa|\vec{n}} = \mathrm{Tr}_A[\rho_{AB}(\Pi_{\kappa|\vec{n}} \otimes I)]$. Bob's unconditional state $\rho_B = \mathrm{Tr}_A \rho_{AB} = \sum_\kappa \tilde{\rho}_{\kappa|\vec{n}}$ remains unchanged under any measurement direction. A state assemblage $\tilde{\rho}_{\kappa|\vec{n}}$ is unsteerable if there exists a local hidden state model (LHSM) with the state ensemble of $p_i\rho_i$ satisfying $\tilde{\rho}_{\kappa|\vec{n}} = \sum_i P(\kappa|\vec{n},i)p_i\rho_i$, where $\rho_B = \sum_i p_i\rho_i$ and $\sum_\kappa P(\kappa|\vec{n},i) = 1$. The quantum state ρ_{AB} is unsteerable from A to B if for all the local POVMs, the state assemblages are all unsteerable. The quantum state ρ_{AB} is steerable from A to B if there exist measurements in Alice's part that produce an assemblage that demonstrates the steerability.

The corresponding local hidden state model and the joint measurement observables are connected through $O_{\kappa|\vec{n}} = \frac{1}{\sqrt{\rho_B}}\tilde{\rho}_{\kappa,\vec{n}}\frac{1}{\sqrt{\rho_B}}$ and $G_i = \frac{1}{\sqrt{\rho_B}} p_i\rho_i \frac{1}{\sqrt{\rho_B}}$ by the one to one mapping between the joint measurement problem and the steerability problem, whenever ρ_B is invertible [27]. The steerability can be detected through the joint measurability of the observables.

Two setting steering scenario: Any two-qubit quantum state can be expressed by $\rho_{AB} = (I_4 + \vec{a}\cdot\vec{\sigma}\otimes I_2 + I_2\otimes\vec{b}\cdot\vec{\sigma} + \sum_i^3 c_i\sigma_i\otimes\sigma_i)/4$ under local unitary equivalence, where $\vec{a},\vec{b},\vec{c} \in R^3$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$, $\vec{\sigma} = \{\sigma_1,\sigma_2,\sigma_3\}$, $C = \mathrm{Diag}\{\mathrm{c}_1,\mathrm{c}_2,\mathrm{c}_3\}$ is the correlation matrix.

When Alice performs two sets of POVMs $\Pi_{\kappa|\vec{n}_i} = (I_2 + (-1)^\kappa\vec{n}_i\cdot\vec{\sigma})/2$ ($i = 0,1$, $\kappa = 0,1$) on A with $\vec{n}_i = (\sin\alpha_i\cos\beta_i, \sin\alpha_i\sin\beta_i, \cos\alpha_i)$, Bob's unnormalized conditional states are $\tilde{\rho}_{\kappa|\vec{n}_i} = \mathrm{Tr}[\tilde{\rho}_{\kappa|\vec{n}_i}](I_2 + (-1)^\kappa\vec{s}_{\kappa,i}\cdot\vec{\sigma})/2$, where $\mathrm{Tr}[\tilde{\rho}_{\kappa|\vec{n}_i}] = (1+(-1)^\kappa\vec{a}\cdot\vec{n}_i)/2$ and $\vec{s}_{\kappa,i} = (\vec{b} + (-1)^\kappa C\cdot\vec{n}_i)/(2\mathrm{Tr}[\tilde{\rho}_{\kappa|\vec{n}_i}])$. Then when $|b| \neq 1$, the measurement assemblages

$$O_\kappa(x_i,\vec{g}_i) \quad = \quad \frac{1}{\sqrt{\rho_B}}\tilde{\rho}_{\kappa|\vec{n}_i}\frac{1}{\sqrt{\rho_B}} = \frac{1}{2}((1+(-1)^\kappa x_i)I_2 + (-1)^\kappa\vec{g}_i\cdot\vec{\sigma}),$$

where $\vec{g}_i = U\vec{n}_i$, $x_i = V\vec{n}_i$ with

$$U = \frac{\vec{b}\vec{a}^T}{|b|^2-1} + \frac{(-1+\sqrt{1-|b|^2})\vec{b}\vec{b}^TC}{|b|^2(|b|^2-1)} + \frac{C}{\sqrt{1-|b|^2}},$$

and $V = \frac{\vec{a}^T - \vec{b}^TC}{1-|b|^2}$. Then $\{\tilde{\rho}_{\kappa|\vec{n}_i}\}_{\kappa,i}$ are unsteerable assemblages if and only if $\{O_\kappa(x_i,\vec{g}_i)\}_{\kappa,i}$ are jointly measurable [37, 38, 39], namely,

$$(1 - F_{x_0}^2 - F_{x_1}^2)(1 - \frac{x_0^2}{F_{x_0}^2} - \frac{x_1^2}{F_{x_1}^2}) - (\vec{g}_0\cdot\vec{g}_1 - x_0x_1)^2 \leqslant 0, \tag{1}$$

where $F_{x_i} = \frac{1}{2}(\sqrt{(1+x_i)^2 - g_i^2} + \sqrt{(1-x_i)^2 - g_i^2})$, $g_i = |\vec{g}_i|$.

(1) gives rise to the condition for Alice to steer Bob's state. If Bob performs two sets of POVMs $\Pi_{\kappa|\vec{n}_i}$ on his system to steer Alice's state, the corresponding condition can be similarly written by changing $\vec{a} \to \vec{b}$, $\vec{b} \to \vec{a}$ and $C \to C^T$ in (1).

However, generally it is quite difficult to deal with the condition (1) and get explicit conditions to judge the steerability for an arbitrary given two-qubit state. For Bell-diagonal states, a necessary and sufficient condition of steerability has been derived from the relations between steerability and joint measurable problem [31]. In the following we study the steerability of any arbitrary given two-qubit states. We present analytical steerability conditions for classes of two-qubit X-state.

Results

Steerability of two-qubit states

First, based on the jointly measurability condition (1) of $\{O_\kappa(x_i,\vec{g}_i)\}_{\kappa,i}$ for two-setting steering scenario we define the steerability of two-qubit states ρ_{AB} by

$$S = \max\{\max_{\alpha_i,\beta_i}(S_1 - S_2), 0\}, \tag{2}$$

where $S_1 = (1 - F_{x_0}^2 - F_{x_1}^2)(1 - \frac{x_0^2}{F_{x_0}^2} - \frac{x_1^2}{F_{x_1}^2})$, $S_2 = (\vec{g}_0\cdot\vec{g}_1 - x_0x_1)^2$, and the maximization runs over all the measurements $\Pi_{\kappa|\vec{n}_i}$, namely, over the parameters α_i and β_i, $i = 0,1$. It is obvious that S lies between 0 and 1. ρ_{AB} is steerable if and only if $S > 0$.

For general two-qubit states, global search can be used to get the global minimum values of S. We give matlab code in the supplementary material.

Due to the relationship between joint measurements and steerability, local hidden states $\tilde{\rho}_{\kappa|\vec{n}_i}$ are represented as $\sqrt{\rho_B}G_{\mu\nu}\sqrt{\rho_B}$ $(\mu=\pm1,\nu=\pm1)$, where $G_{\mu\nu}=\frac{1}{4}(1+\mu x_0+\nu x_1+\mu\nu Z+(\mu\nu\vec{z}+\mu\vec{g}_0+\nu\vec{g}_1)\vec{\sigma})$ which are all possible sets of four measurements satisfying the marginal constraints for any two jointly measurable observables $\{O_\kappa(x_i,\vec{g}_i)\}_{\kappa,i}$ [37, 38, 39]. The steering radius $R(\rho_{AB})$ [25] can be calculated by optimizing $\vec{z}$ and Z.

In the following we calculate analytically the steerability S for some X-states ρ_X. We define a class of two-qubit X-states to be zero-states ρ_{zero} if the X-states ρ_X satisfy the condition that the maximum points (stationary points) of S_1 belong to the zero points of S_2 with respect to the measurement parameters α_i and $\beta_i,(i=1,2)$.

For any two-qubit X-state, $\rho_X=\frac{1}{4}(I_4+a_3\sigma_3\otimes I_2+b_3I_2\otimes\sigma_3+\sum_i^3 c_i\sigma_i\otimes\sigma_i)$, we have $U=\mathrm{Diag}\{\mathrm{u}_1,\mathrm{u}_2,\mathrm{u}_3\}$, $V=[0,0,t_3]$, where $u_1=c_1/\sqrt{1-b_3^2}$, $u_2=c_2/\sqrt{1-b_3^2}$, $u_3=(a_3b_3-c_3)/(-1+b_3^2)$ and $t_3=(a_3-b_3c_3)/(1-b_3^2)$. We have the following results:

Theorem. For the zero-states ρ_{zero}, the analytical formula of the steerability is given by

$$S=\max\{\Delta_1,\Delta_2,\Delta_3,0\}, \tag{3}$$

where $\Delta_1=u_1^2+u_2^2-1$, $\Delta_2=\frac{1}{2}[u_1^2(u_3^2-t_3^2)+u_1^2+u_3^2+t_3^2-1-(1-u_1^2)\sqrt{((1-t_3)^2-u_3^2)((1+t_3)^2-u_3^2)}]$, $\Delta_3=\frac{1}{2}[u_2^2(u_3^2-t_3^2)+u_2^2+u_3^2+t_3^2-1-(1-u_2^2)\times\sqrt{((1-t_3)^2-u_3^2)((1+t_3)^2-u_3^2)}]$. When $S>0$, the optimal measurements which give rise to maximal S are σ_x and σ_y if $\Delta_1>\max\{\Delta_2,\Delta_3,0\}$, σ_x and σ_z if $\Delta_2>\max\{\Delta_1,\Delta_3,0\}$, σ_y and σ_z if $\Delta_3>\max\{\Delta_1,\Delta_2,0\}$, respectively.

See proof in supplementary material.

It is obvious that any X-state with $t_3=0$ belongs to ρ_{zero}, e.g. $|\phi\rangle=a|00\rangle+\sqrt{1-a^2}|11\rangle$ $(0<|a|<1)$ and the Bell-diagonal state $\rho=\frac{1}{4}(I+c_1\sigma_1\otimes\sigma_1+c_2\sigma_2\otimes\sigma_2+c_3\sigma_3\otimes\sigma_3)$ are all the zero states. For $|\phi\rangle$, we have $S=1$.

For the Bell-diagonal state, interestingly the steerability S is given by the non-locality characterized by the maximal violation of the CHSH inequality. Let $\mathscr{B}_{CHSH}$ denote the Bell operator for the CHSH inequality [35], $\mathscr{B}_{CHSH}=A_1\otimes B_1+A_1\otimes B_2+A_2\otimes B_1-A_2\otimes B_2$, where $A_i=\vec{a}_i\cdot\vec{\sigma}$, $B_i=\vec{b}_i\cdot\vec{\sigma}$, $\vec{a}_i$ and $\vec{b}_i$, $i=1,2$, are unit vectors. Then the maximal violation of the CHSH inequality is given by [36]

$$N=\max_{\mathscr{B}_{\mathscr{CHSH}}}|\langle\mathscr{B}_{\mathscr{CHSH}}\rangle_\rho|=2\sqrt{\tau_1+\tau_2}, \tag{4}$$

where τ_1 and τ_2 are the two largest eigenvalues of the matrix $T^\dagger T$, T is the matrix with entries $T_{\alpha\beta}=tr[\rho\,\sigma_\alpha\otimes\sigma_\beta]$, $\alpha,\beta=1,2,3$, $\dagger$ stands for transpose and conjugation. For the Bell-diagonal state, we have $N=2\sqrt{c_1^2+c_2^2+c_3^2-\min\{c_1^2,c_2^2,c_3^2\}}$. From (3) we get that the steerability of Bell-diagonal state is given by $S=\frac{N^2}{4}-1$.

For $t_3\neq0$, we give the explicit conditions of zero states in supplementary material.

In the following we present the maximum value of the steerability S for given N of ρ_{zero}.

Corollary 1: For zero-states ρ_{zero} with given N, $0\leqslant N\leqslant2$, we have $S\leqslant\frac{N}{2}$. Moreover, $S=N/2$ is attained when $a_3=1-c_3+b_3$, $b_3\to-1$, $c_1=\sqrt{(1+b_3)(c_3-b_3)}$, $c_2=-c_1$, i.e. ρ_{zero} has the following form,

$$\rho_{X_0}=\begin{pmatrix}\frac{1+b_3}{2} & 0 & 0 & \pm\frac{\sqrt{(1+b_3)(c_3-b_3)}}{2}\\ 0 & \frac{1-c_3}{2} & 0 & 0\\ 0 & 0 & 0 & 0\\ \pm\frac{\sqrt{(1+b_3)(c_3-b_3)}}{2} & 0 & 0 & \frac{c_3-b_3}{2}\end{pmatrix}. \tag{5}$$

The following corollary gives the conditions at which one gets the minimal value of S for given N.

Corollary 2: For zero-states ρ_{zero} with given CHSH value N, S gets the minimal value when $a_3=0$ and $b_3=0$ or $|a_3+b_3|=\sqrt{(1+c_3)^2-(c_1-c_2)^2}$ or $|a_3-b_3|=\sqrt{(1-c_3)^2-(c_1+c_2)^2}$.

The proofs of Corollary 1 and Corollary 2 are given in supplementary material. In Fig. 1, we give a description for the boundaries of the steerability S for given value of N. From Fig. 1 we see that for any given N with $0\leqslant N\leqslant2$, the lower bound of S is always 0 and the upper bound of S is always less than 2 (light blue), and for $N>2$ the lower bound of S is always greater than 0 and the upper bound of S is always 2 (dark blue).

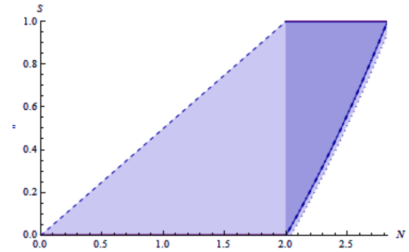

Figure 1. The regions of the values taking by the steerability S for given N.

For zero-states ρ_{zero} the steering radius $R(\rho_{zero})$ can be obtained when Alice measures her qubit along the directions σ_x and σ_y, or σ_x and σ_z or σ_y and σ_z. Actually, from the construction of joint measurements [37], when Alice measures her qubit along the directions of σ_x and σ_z, the local hidden states can be expressed as

$$\frac{1}{2}(I_2+\frac{m_x\sigma_x+m_z\sigma_z}{1+\mu a_3+\nu(b_3z_3+Z)}),$$

where $m_x=\mu\nu(c_1+\mu\sqrt{1-b_3^2}z_1)$, $m_z=b_3+\mu c_3+\nu(z_3+b_3Z)$, $\mu=\pm1,\nu=\pm1$. Therefore

$$R(\rho_{zero})=\max\{r(\rho_x)_{xy},r(\rho_x)_{xz},r(\rho_x)_{yz}\}, \tag{6}$$

where

$$r(\rho_{zero})_{xy}=\sqrt{c_1^2+c_2^2+b_3^2}; \qquad r(\rho_{zero})_{xz}=\min_{z_1,z_3,Z}\max_{\mu,\nu}\sqrt{r_{\mu,\nu}^{xz}}; \qquad r(\rho_{zero})_{yz}=\min_{z_1,z_3,Z}\max_{\mu,\nu}\sqrt{r_{\mu,\nu}^{yz}};$$

$$r_{\mu,\nu}^{xz}=\frac{(c_1+\mu\sqrt{1-b_3^2}z_1)^2+(b_3+\mu c_3+\nu(z_3+b_3Z))^2}{(1+\mu a_3+\nu(b_3z_3+Z))^2}; \qquad r_{\mu,\nu}^{yz}=\frac{(c_2+\mu\sqrt{1-b_3^2}z_1)^2+(b_3+\mu c_3+\nu(z_3+b_3Z))^2}{(1+\mu a_3+\nu(b_3z_3+Z))^2}.$$

It is not easy to calculate $r(\rho_{zero})_{xz}$ and $r(\rho_{zero})_{yz}$ analytically. We give the analytical results for $R(\rho_{zero})$ for some special states in the following.

Asymmetric two-setting EPR-steering

Different from Bell-nonlocality and quantum entanglement, EPR-steering has the asymmetric property - one-way EPR steering: Alice may steer Bob's state but not the vice versa. The demonstration of asymmetric steerabiliy has practical implications in quantum communication networks [40]. Till now only a few asymmetric steering states are found [33, 34, 25, 32]. Here we present a class of asymmetric steering states of the form ρ_{X_0} in (5).

If Alice performs measurements on her qubit, the steerability is given by $S(\rho_{X_0})=\max\{\frac{2c_3-1-b_3}{1-b_3},0\}$ which approaches c_3 when b_3 approaches to -1 and $c_3>0$. If Bob performs measurements on his qubit, the related steerability is given by

$$S(\rho_{X_0})=\max\{\frac{(1+b_3)(b_3+c_3)}{(2+b_3-c_3)^2},0\}$$

which equals to zero as long as $(1+b_3)(b_3+c_3)\leqslant 0$. Therefore, when $0<c_3<-b_3$ and $b_3\to-1$, Alice can always steer Bob's state, but Bob can never steer Alice's state, see Fig. 2 for the asymmetric EPR-steering for $b_3=-0.999$. We see that Alice can always steer Bob's state, while Bob can not steer Alice's state.

In the following part, we investigate the geometric features of the asymmetric steering state-ρ_{x_0} in terms of the steering ellipsoid [41]. The steering ellipsoid of ρ_{X_0} when Alice performs POVMs is quite different from that of when Bob performs POVMs. The center of the steering ellipsoid ε_B for Alice performing POVMs on her qubit is $(0,0,(b_3-a_3c_3)/(1-a_3^2))$, which goes to $(0,0,-1)$ when $b\to-1$. And the volume of the steering ellipsoid ε_B is

$$\frac{4\pi}{3}\frac{|c_1c_2(c_3-a_3b_3)|}{(1-a_3^2)^2}=\frac{4\pi}{3}\frac{(1+b_3)^2}{(2-c_3+b_3)^2},$$

Here the steering ellipsoid is tangent to the Bolch sphere. The center of the steering ellipsoid ε_A for Bob performing POVMs on his qubit is

$$(0,0,\frac{a_3-b_3c_3}{1-b_3^2}=(0,0,\frac{1-c_3}{1-b_3}),$$

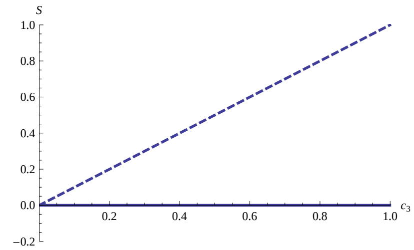

Figure 2. The steerability S versus c_3 for $b_3 = -0.999$: dashed line for Alice steering Bob's state, solid line (horizontal coordinate) for Bob steering Alice's state.

which goes to $(1-c_3)/2$ when $b_3 \to -1$. The volume of the steering ellipsoid ε_A is given by

$$\frac{4\pi}{3}\frac{|c_1c_2(c_3-a_3b_3)|}{(1-b_3^2)^2} = \frac{4\pi(c_3-b_3)^2}{3(1-b_3)^2},$$

which goes to $\frac{\pi(1+c_3)^2}{3}$ when $b_3 \to -1$. The steering ellipsoid is also tangent to the Bolch sphere. Here the ellipsoid represents some peculiar feature, when $b_3 \to -1$ and $c_3 \to 0$, , the ellipsoid ε_B are almost 0, but Alice can still steer Bob; When $b_3 \to -1$ and $c_3 \to -b_3$, , the ellipsoid ε_A are almost the whole Bloch sphere, but Bob can not steer Alice.

As a special case of ρ_{X_0}, we take $a_3 = 1-2\eta(1-\chi)$, $b_3 = 2\eta\chi - 1$, $c_3 = 2\eta - 1$, $c_1 = -c_2 = -2\eta\sqrt{\chi(1-\chi)}$. The state has the following form,

$$W_\eta^\chi = \begin{pmatrix} \eta\chi & 0 & 0 & -\eta\sqrt{\chi(1-\chi)} \\ 0 & 1-\eta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\eta\sqrt{\chi(1-\chi)} & 0 & 0 & \eta(1-\chi) \end{pmatrix}. \tag{7}$$

From the Theorem, we get that when Alice measures her qubit,

$$S(W_\eta^\chi) = \max\{\frac{1+\eta(-2+\chi)}{-1+\eta\chi}, \frac{\eta(1+\eta(-2+\chi))(-1+\chi)}{(1-\eta\chi)^2}, 0\}.$$

The sufficient and necessary condition in the two-setting steering scenario is $\eta > 1/(2-\chi)$ for Alice to steer Bob's state. The corresponding optimal measurements are σ_x and σ_y.

If Bob measures his qubit, the steerability is given by

$$S(W_\eta^\chi) = \max\{\frac{\eta\chi(-1+\eta+\eta\chi)}{(1+\eta(-1+\chi))^2}, \frac{-1+\eta+\eta\chi}{1+\eta(-1+\chi)}, 0\}.$$

The sufficient and necessary condition for Bob to steer Alice's state is $\eta > 1/(1+\chi)$. The related optimal measurements are σ_x and σ_y. The asymmetric property in quantum steering given by this example is shown in Fig. 3 and Fig. 4. The steering radius is $\sqrt{1-4\eta\chi(1-\eta(2-\chi))}$ when Alice measures her qubit, and $\sqrt{1-4\eta(1-\chi)(1-\eta-\eta\chi)}$ when Bob measures his qubit.

As another example showing the asymmetry of quantum steering, we consider the state W_V^θ [25],

$$W_V^\theta = V|\psi_1\rangle\langle\psi_1| + (1-V)|\psi_2\rangle\langle\psi_2|, \tag{8}$$

where $|\psi_1\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$, $|\psi_2\rangle = \cos\theta|10\rangle + \sin\theta|01\rangle$, $\theta \in (0,\pi/2)$, $V \in [0,1/2)\cup(1/2,1]$. W_V^θ is a zero state. From our Theorem, we have that when Alice performs the measurements on her qubit, $S(W_V^\theta) = (1-2V)^2$. The optimal measurements are σ_x, σ_y or σ_x, σ_z. This state is always steerable for Alice except for $V = 1/2$.

When Bob performs two projective measurements on his qubit, we have

$$S(W_V^\theta) = \max\{\frac{(1-2V)^2-\cos^2 2\theta}{1-(1-2V)^2\cos^2 2\theta}, \frac{\sin 2\theta^2((1-2V)^2-\cos^2 2\theta)}{(1-(1-2V)^2\cos^2 2\theta)^2}, 0\}. \tag{9}$$

The sufficient and necessary condition in the two-setting steering scenario for Bob to steer Alice's state is $|\cos 2\theta| < |2V-1|$, with the optimal measurements σ_x and σ_y. For W_V^θ the corresponding steering radius is $\sqrt{1+(1-2V)^2\sin^2 2\theta}$ when Alice

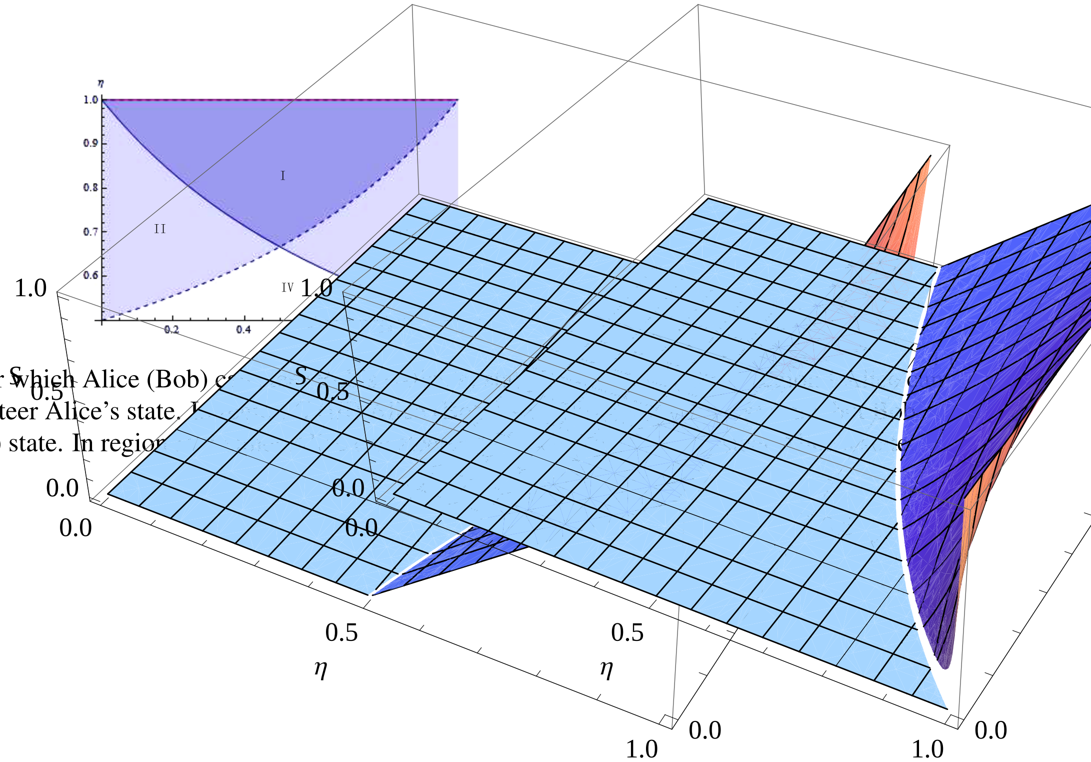

Figure 3. The parameter region for which Alice (Bob) c… steer Bob's state and Bob can also steer Alice's state. … (Alice) can not steer Alice's (Bob's) state. In regio…

Figure 4. Fig. (a) (Fig. (b)): $S(W_\eta^\chi)$ when Alice (Bob) measures her (his) qubit.

measures her qubit, and $\sqrt{(1-2V)^2+\sin^2 2\theta}$ when Bob measuring his qubit. From Fig. 5 we see that Alice can always steer Bob's state except for $V = 1/2$. While Bob can only steer Alice's state for some V depending on θ.

From our Theorem, analytical results of steerability can be obtained for more detailed zero states. And the asymmetric property of steering can be readily studied. In the following we give two examples of symmetric two-setting EPR-steering.

Example 1. The two-qubit nonmaximally entangled state mixed with color noise,

$$\rho_{\rm cn} = V|\psi(\theta)\rangle\langle\psi(\theta)| + \frac{1-V}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|),$$

where $|\psi(\theta)\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$, $\theta \in (0, \pi/2)$, $V \in (0,1]$. The steerability is given by $S(\rho_{\rm cn}) = V^2\sin^2 2\theta/(1-V^2\cos 2\theta^2)$. Therefore $\rho_{\rm cn}$ is steerable if and only if $V\sin 2\theta \neq 0$.

Example 2. The generalized isotropic state, $\rho_{gi} = V|\psi(\theta)\rangle\langle\psi(\theta)| + (1-V)I/4$, where $|\psi(\theta)\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$, $\theta \in (0, \pi/2)$, $V \in (0,1]$. The state reduces to the usual isotropic state when $\theta = \pi/4$. By our theorem, we get the analytical steerability of ρ_{gi},

$$S(\rho_{gi}) = \frac{1-V^2\cos^2 4\theta + (1-V)\sqrt{(1+V)^2-4V^2\cos^2 2\theta}}{4(1-V^2\cos^2 2\theta)} \times \frac{V^2(1+2\sin^2 2\theta) - 1 - (1-V)\sqrt{(1+V)^2-4V^2\cos^2 2\theta}}{1-V^2\cos^2 2\theta}.$$

Hence, the sufficient and necessary condition of steerability is $1 + (1-V)\sqrt{(1+V)^2 - 4V^2\cos^2 2\theta} < V^2(1+2\sin^2 2\theta)$.

Discussions

Based on the one-to-one correspondence between EPR-steering and the joint measurability, we have investigated the steerability for any two-qubit systems in the two-setting measurement scenario. The steerability we introduced is invariant under local unitary operations. Analytical formula of the steerability for a class of X-states has been derived, and the sufficient and necessary conditions for two-setting EPR-steering has been presented. For general two-qubit states, it has been shown that the lower and upper bounds of the steerability are explicitly connected to the non-locality of the states given by the CHSH values of maximal violation. Moreover, we have also presented a class of asymmetric steering states by investigating the steerability with respect to the measurements from Alice's and Bob's sides. Our strategy may be also used to study the quantification of steerability for multi-setting scenarios, especially for three-setting scenarios since the joint measurability problem of three qubit observables has already been investigated [42, 43]. Our method may also be used to the continuous variable steering,

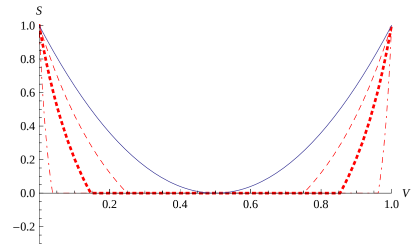

Figure 5. $S(W_V^\theta)$ versus θ: blue solid line when Alice measures her qubit; red dashed line ($\theta = \frac{\pi}{6}$), red dotted line ($\theta = \frac{\pi}{8}$) and red dot-dashed line ($\theta = \frac{\pi}{16}$) when Bob measures his qubit.

temporal and channel steering. The steerability of the state assemblages or the instruments assemblages can be connected to the incompatibility problems of the quantum measurement assemblages [44, 45], so the steerability of the quantum states or the quantum channel may also be studied by investigating all their corresponding incompatibility problems through over all the measurement parameters.

References

1. Einstein, A., Podolsky, B. and Rosen, N. Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? *Phys. Rev.* **47**, 777-780(1935).
2. Wiseman, H. M., Jones, S. J. and Doherty, A. C. Steering, Entanglement, Nonlocality, and the Einstein-Podolsky-Rosen Paradox. *Phys. Rev. Lett.* **98**, 140402(2007).
3. Law, Y. Z., Thinh, L. P., Bancal, J. D. and Scarani V. Quantum randomness extraction for various levels of characterization of the devices. *J. Phys. A* **47**, 42(2014).
4. Piani, M. and Watrous J. Necessary and Sufficient Quantum Information Characterization of Einstein-Podolsky-Rosen Steering, *Phys. Rev. Lett.* **114**, 060404(2015).
5. Branciard, C. and Gisin, N. Quantifying the Nonlocality of Greenberger-Horne-Zeilinger Quantum Correlations by a Bounded Communication Simulation Protocol, *Phys. Rev. Lett.* **107**, 020401(2011).
6. Branciard, C., Cavalcanti, E. G., Walborn, S. P., Scarani, V. and Wiseman, H. M. One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering, *Phys. Rev. A* **85**, 010301(R)(2012).
7. Chen, S. L., Lambert, N., Li, C .M., Miranowicz, A., Chen, Y. N. and Nori, F. Quantifying Non-Markovianity with Temporal Steering, *Phys. Rev. Lett.* **116**, 020503(2016).
8. Reid, M. D. Demonstration of the Einstein-Podolsky- Rosen paradox using nondegenerate parametric amplification, *Phys. Rev. A* **40**, 913(1989).
9. Cavalcanti, E. G., Jones, S. J., Wiseman, H. M. and Reid, M. D. Experimental criteria for steering and the Einstein-Podolsky- Rosen paradox, *Phys. Rev. A* **80**, 032112(2009).
10. Schneeloch, J., Broadbent, C. J., Walborn, S. P., Cavalcanti,E. G. and Howell, J. C. Einstein-Podolsky-Rosen steering inequalities from entropic uncertainty relations, *Phys. Rev. A* **87**, 062103(2013).
11. Pramanik, T., Kaplan, M. and Majumdar, A. S. Fine-grained Einstein-Podolsky-Rosen-steering inequalitie, *Phys. Rev. A* **90**, 050305(2014).
12. Schneeloch, J., Dixon, P. B., Howland, G. A., Broadbent, C. J. and Howell, J. C. Violation of continuous-variable Einstein-Podolsky-Rosen steering with discrete measurements, *Phys. Rev. Lett.* **110**, 130407(2013).
13. Kogias, I., Skrzypczyk, P., Cavalcanti, D., Acín, A. and Adesso, G. Hierarchy of Steering Criteria Based on Moments for All Bipartite Quantum Systems, *Phys. Rev. Lett.* **115**, 210401(2015).
14. Saunders, D. J., Jones, S. J., Wiseman, H. M. and Pryde, G. J. Experimental EPR-steering using Bell-local states, *Nat. Phys.* **6**, 845-849(2010).
15. Walborn, S. P., Salles, A., Gomes, R. M., Toscano, F. and SoutoRibeiro, P. H. Revealing Hidden Einstein-Podolsky-Rosen Nonlocality, *Phys. Rev. Lett.* **106**, 130402(2011).
16. Cavalcanti, E. G., Foster, C. J., Fuwa, M. and Wiseman, H. M. Analog of the Clauser-Horne-Shimony-Holt inequality for steering, *J. Opt. Soc. Am. B* **32**, A74(2015).

17. Ji, S. W., Lee, J., Park, J. and Nha, H. Steeing criteria via covariance matrices of local observables in arvitary-dimensional quantum systems, *Phys. Rev. A.* **92**, 062130(2015).

18. Roy, A., Bhattacharya, S. S., Mukherjee, A. and Banik, M. Optimal quantum violation of Clauser-Horne-Shimony-Holt like steering inequality, *J. Phys. A* **48**, 415302(2015).

19. Żukowski, M., Dutta, A. and Yin, Z. Geometric Bell-like inequalities for steering, *Phys. Rev. A* **91**, 032107(2015).

20. Cavalcanti, D. and Skrzypczyk, P. Quantum steering: a review with focus on semidefinite programming, *Prog. Phys.* **80**, 024001(2017).

21. Sun, K., Xu, J. S., Ye, X. J., Wu, Y. C., Chen, J. L., Li, C. F. and Guo, G. C. Experimental demonstration of Einstein-Podolsky-Rosen steering game based on the all-versus-nothing proof, *Phys. Rev. Lett.* **113**, 140402(2014).

22. Hsieh, C. Y., Liang, Y. C. and Lee, R. K. Quantum steerability: characterization, quantification, superactivation, and unbounded amplification, *Phys. Rev. A* **94**, 062120(2016).

23. Skrzypczyk, P., Navascués, M. and Cavalcanti, D. Quantifying Einstein-Podolsky-Rosen steering, *Phys. Rev. Lett.* **112**, 180404(2014).

24. Nguyen, C., Vu, T., Necessary and sufficient condition for steerability of two-qubit states by the geometry of steering outcomes, Europhys. Lett. 115,10003(2016).

25. Sun, K., Ye, X. J., Xu, J. S., Xu, X. Y., Tang, J. S., Wu, Y. C., Chen, J. L., Li, C. F. and Guo, G. C. Experimental quantification of asymmetric Einstein-Podolsky-Rosen steering, *Phys. Rev. Lett.* **116**, 160404(2016).

26. Costa, A. C. S. and Angelo, R. M. Quantification of Einstein-Podolski-Rosen steering for two-qubit state, *Phys. Rev. A* **93**, 020103(R)(2016).

27. Uola, R., Budroni, C., G*ü*hne, O. and Pellonp*ää* J. P. One-to-One Mapping between Steering and Joint Measurability Problems, *Phys. Rev. Lett.* **115**, 230402(2015).

28. Quintino, M. T., V*é*rtesi, T. and Brunner, N. Joint measurability, Einstein-Podolsky-Rosen steering and Bell nonlocality, *Phys. Rev. Lett.* **113**, 160402(2014).

29. Uola, R., Moroder, T. and G*ü*hne, O. Joint Measurability of Generalized Measurements Implies Classicality, *Phys. Rev. Lett.* **113**, 160403(2014).

30. Cavalcanti, D. and Skrzypczyk, P. Quantitative relations between measurement incompatibility, quantum steering, and nonlocality, *Phys. Rev. A* **93**, 052112(2016).

31. Quan, Q., Zhu, H. J., Liu, S. Y., Fei, S. M., Fan, H. and Yang, W. L. Steering Bell-diagonal states, *Sci. Rep.* **6**, 22025(2016).

32. Xiao, Y., Ye, X. J., Sun, K., Xu, J. S., Li, C. F. and Guo, G. C. Demonstration of multiSetting one-way Einstein-Podolsky-Rosen steering in two-qubit systems, *Phys. Rev. Lett.* **118**, 140404(2017).

33. H*ä*ndchen, V., Eberle, T., Steinlechner, S., Samblowski, A., Franz, T., Werner, R. F. and Schnabel, R. Observation of one-way Einstein-Podolsky-Rosen Steering, *Nat. Photon* **6**, 596-599(2012).

34. Bowles, J., V*é*rtesi T., Quintino, M. T. and Brunner, N. One-way Einstein-Podolsky-Rosen Steering, *Phys. Rev. Lett.* **112**, 200402(2014).

35. Clauser, J. F., Horne, M. A., Shimony, A. and Holt, R. A. Proposed Experiment to Test Local Hidden-Variable Theories, *Phys. Rev. Lett.* **23**, 880(1969).

36. Horodecki, R.,Horodecki, P. and Horodecki, M. Violating Bell inequality by mixed spin-12 states: necessary and sufficient condition, *Phys. Lett. A* **200**, 340(1995).

37. Yu, S. X., Liu, N. L., Li, L. and Oh, C. H. Joint measurement of two unsharp observables of a qubit, *Phys. Rev. A* **81**, 062116, (2010).

38. Stano, P., Peitzner, D. and Heinosaari, T., Coexistence of qubit effects, *Quan. Inf. Proc.* **9**, 143-169(2010).

39. Stano, P., Peitzner, D. and Heinosaari, T., Coexistence of qubit effects, *Phys. Rev. A* **78**, 012315, (2008).

40. Wollmann, S., Walk, N., Bennet, A. J., Wiseman, H. M. and Pryde, G. J. Observation of Genuine One-Way Einstein-Podolsky-Rosen Steering, *Phys. Rev. Lett.* **116**, 160403(2016).

41. Jevtic, S., Pusey, M., Jennings, D. and Rudolph, T. Quantum Steering Ellipsoids, *Phys. Rev. Lett.* **113**, 020402(2014).

42. Yu, S. X., Oh, C. H., Quantum contextuality and joint measurement of three observables of a qubit, *arXiv:* 1312.6470.

43. Pau, R., Ghosh, S., Approximate joint measurement of qubit observables through an Arthur-Kelly model, *J. Phys. A: Math. Theor.* **44**, 485303(2011).

44. Kiukas, J., Budroni, C., Uola, R., and Pellonp*ää*, J. P., Uola, R., Lever, F., G*ü*hne, O, and Pellonp*ää*, J. P. Continuous variable steering and incompatibility via state-channel duality, *arXiv:* 1704.05734.

45. Uola, R., Lever, F., G*ü*hne, O, and Pellonp*ää*, J. P. Unified picture for spatial, temporal and channel steering, *arXiv:* 1707.09237.

1 Acknowledgements

This work is supported by the NSFC under no. 11571313,11475089, 11675113.

2 Author contributions

Z.C. and X.Y initiated the research, Z.C. proved the main theorems and developed the numerical codes, Z.C. X.Y. and S.F. wrote the manuscript.

3 Additional information

Competing financial interests: The authors declare no competing financial interests.

4 Supplementary Information

4.1 Proof of the Theorem

Denote $\delta=\sqrt{\frac{((1+x_1)^2-g_1^2)((1-x_1)^2-g_1^2)}{((1+x_0)^2-g_0^2)((1-x_0)^2-g_0^2)}}$, $\delta_1=1-x_1^2+g_1^2+\delta(1+x_0^2-g_0^2)$, $\delta_2=1+x_1^2-g_1^2+\delta(1-x_0^2+g_0^2)$, $\delta_3=\frac{\delta_2}{\delta}$ and $\delta_4=\frac{\delta_1}{\delta}$. To calculate the term $\max\limits_{\alpha_i,\beta_i} S_1$ of the steerability, we compute the derivations of S_1 with respect to the variables α_i and β_i, $i=1,2$,

$$\begin{cases}\dfrac{\partial S_1}{\partial\alpha_1}= & \sin\alpha_1\cos\alpha_1[\delta_1u_1^2\cos^2\beta_1+\delta_1u_2^2\sin^2\beta_1-\delta_1u_3^2-\delta_2t_3^2], \qquad \dfrac{\partial S_1}{\partial\beta_1}=\delta_1\sin^2\alpha_1\sin\beta_1\cos\beta_1(u_2^2-u_1^2),\\ \dfrac{\partial S_1}{\partial\alpha_2}= & \sin\alpha_2\cos\alpha_2[\delta_3u_1^2\cos^2\beta_2+\delta_3u_2^2\sin^2\beta_2-\delta_3u_3^2-\delta_4t_3^2], \qquad \dfrac{\partial S_1}{\partial\beta_2}=\delta_3\sin^2\alpha_2\sin\beta_2\cos\beta_2(u_2^2-u_1^2).\end{cases}$$

From $\frac{\partial S_1}{\partial\alpha_1}=\frac{\partial S_1}{\partial\beta_1}=\frac{\partial S_1}{\partial\alpha_2}=\frac{\partial S_1}{\partial\beta_2}=0$, we have the following solutions,

$$\begin{cases}\sin\alpha_1\cos\alpha_1=0 \quad \text{or} \quad \Delta=0,\\ \sin^2\alpha_1\sin\beta_1\cos\beta_1=0,\\ \sin\alpha_2\cos\alpha_2=0 \quad \text{or} \quad \Omega=0,\\ \sin^2\alpha_2\sin\beta_2\cos\beta_2=0,\end{cases}$$

where $\Delta=\delta_1(u_1^2\cos^2\beta_1+u_2^2\sin^2\beta_1-u_3^2)-\delta_2t_3^2$ and $\Omega=\delta_3(u_1^2\cos^2\beta_2+u_2^2\sin^2\beta_2-u_3^2)-\delta_4t_3^2$. Therefore we have either

$$\begin{cases}\sin\alpha_1\cos\alpha_1=0,\\ \sin^2\alpha_1\sin\beta_1\cos\beta_1=0,\\ \sin\alpha_2\cos\alpha_2=0,\\ \sin^2\alpha_2\sin\beta_2\cos\beta_2=0,\end{cases} \tag{10}$$

or

$$\begin{cases}\sin\alpha_1\cos\alpha_1=0,\\ \sin^2\alpha_1\sin\beta_1\cos\beta_1=0,\\ \sin\alpha_2\cos\alpha_2\neq0 \quad \text{but} \quad \Omega=0,\\ \sin^2\alpha_2\sin\beta_2\cos\beta_2=0,\end{cases} \tag{11}$$

or

$$\begin{cases} \sin\alpha_1 \cos\alpha_1 \neq 0 \quad \text{but} \quad \Delta = 0, \\ \sin^2\alpha_1 \sin\beta_1 \cos\beta_1 = 0, \\ \sin\alpha_2 \cos\alpha_2 = 0, \\ \sin^2\alpha_2 \sin\beta_2 \cos\beta_2 = 0, \end{cases} \tag{12}$$

or

$$\begin{cases} \sin\alpha_1 \cos\alpha_1 \neq 0 \quad \text{but} \quad \Delta = 0, \\ \sin^2\alpha_1 \sin\beta_1 \cos\beta_1 = 0, \\ \sin\alpha_2 \cos\alpha_2 \neq 0 \quad \text{but} \quad \Omega = 0, \\ \sin^2\alpha_2 \sin\beta_2 \cos\beta_2 = 0. \end{cases} \tag{13}$$

Actually, (11) is equivalent to (12). Hence we only need to consider (10), (11) and (13). From (11), we have

$$\begin{cases} \cos\alpha_1 = 0, \\ \sin\beta_1 \cos\beta_1 = 0, \\ \Omega = 0, \\ \sin\beta_2 \cos\beta_2 = 0, \end{cases} \quad \text{or} \quad \begin{cases} \sin\alpha_1 = 0, \\ \Omega = 0, \\ \sin\beta_2 \cos\beta_2 = 0, \end{cases} \tag{14}$$

which gives rise to

$$\begin{cases} \alpha_1 = \dfrac{\pi}{2}, \\ \beta_1 = \dfrac{(i-1)\pi}{2}, \\ \Omega = 0, \\ \beta_2 = \dfrac{(j-1)\pi}{2}, \end{cases} \quad \text{or} \quad \begin{cases} \alpha_1 = 0, \\ \Omega = 0, \\ \beta_2 = \dfrac{(j-1)\pi}{2}. \end{cases} \tag{15}$$

(13) is equivalent to

$$\begin{cases} \Delta = 0, \\ \sin\beta_1 \cos\beta_1 = 0, \\ \Omega = 0, \\ \sin\beta_2 \cos\beta_2 = 0, \end{cases} \Longrightarrow \begin{cases} \Delta = 0, \\ \beta_1 = \dfrac{(i-1)\pi}{2}, \\ \Omega = 0, \\ \beta_2 = \dfrac{(j-1)\pi}{2}. \end{cases} \tag{16}$$

Here $i = 1,2$ and $j = 1,2$. Form (15), given $\alpha_1 = 0$, $\beta_2 = \frac{(j-1)\pi}{2}$ or $\alpha_1 = \frac{\pi}{2}$, $\beta_1 = \frac{(i-1)\pi}{2}$, $\beta_2 = \frac{(j-1)\pi}{2}$, $\Omega = 0$ is an equation satisfied by α_2. From (16), given $\beta_1 = \frac{(i-1)\pi}{2}, \beta_2 = \frac{(j-1)\pi}{2}$, then $\Delta = 0$ and $\Omega = 0$ are equations satisfied by the variables α_1 and α_2. Hence we have the following conditions:

(I) For $\alpha_1 = \frac{\pi}{2}$, $\beta_1 = \frac{(i-1)\pi}{2}$ and $\beta_2 = \frac{(j-1)\pi}{2}$, if the equation $\Omega = 0$

(a) does not have a solution, or

(b) only has the solution $\alpha_2 = \frac{m\pi}{2}$ $(m = 0,1)$, or

(c) has the solutions $\alpha_2 = \alpha_2^0 \neq \frac{m\pi}{2}$, but this solution $\alpha_2 = \alpha_2^0$, together with $\alpha_1 = \frac{\pi}{2}$, $\beta_1 = \frac{(i-1)\pi}{2}$ and $\beta_2 = \frac{(j-1)\pi}{2}$, are not the maximum points of S_1.

(II) For $\alpha_1 = 0$, $\beta_2 = \frac{(j-1)\pi}{2}$, if the equation $\Omega = 0$

(a) does not have a solution, or

(b) only has the solutions $\alpha_2 = \frac{m\pi}{2}$, $m = 0,1$, or

(c) has the solutions $\alpha_2 = \alpha_2^1 \neq \frac{m\pi}{2}$, but $\alpha_2 = \alpha_2^1$, together with $\alpha_1 = 0$, $\beta_2 = \frac{(j-1)\pi}{2}$, are not the maximum points of S_1.

(III) For $\beta_1 = \frac{(i-1)\pi}{2}$ and $\beta_2 = \frac{(j-1)\pi}{2}$, the equations $\Delta = 0$ and $\Omega = 0$ are satisfied simultaneously if and only if $\alpha_1 = \frac{m\pi}{2}$, $\alpha_2 = \frac{n\pi}{2}$, $m = 0,1, n = 0,1$.

It is obvious that if ρ_X satisfies all the conditions (I) to (III), the candidates of the maximal points of S_1 are $\alpha_1=\frac{\pi}{2}$, $\alpha_2=0$, $\beta_1=0$ or $\alpha_1=\frac{\pi}{2}$, $\alpha_2=0$, $\beta_1=\frac{\pi}{2}$ or $\alpha_1=0$, $\alpha_2=\frac{\pi}{2}$, $\beta_2=0$ or $\alpha_1=0$, $\alpha_2=\frac{\pi}{2}$, $\beta_2=\frac{\pi}{2}$ or $\alpha_1=0$, $\alpha_2=0$ or $\alpha_1=\frac{\pi}{2}$, $\alpha_2=\frac{\pi}{2}$, $\beta_1=0$, $\beta_2=\frac{\pi}{2}$ or $\alpha_1=\frac{\pi}{2}$, $\alpha_2=\frac{\pi}{2}$, $\beta_1=\frac{\pi}{2}$, $\beta_2=0$, therefore the maximum points of S_1 are all the zero points of S_2, i.e. the states satisfying (I)-(III) are zero-states ρ_{zero}. We do not need to consider the case $\alpha_1=\alpha_2=0$, since when $\alpha_1=\alpha_2=0$, $S_1-S_2\leqslant 0$. Therefore, $S=\max\{\Delta_1,\Delta_2,\Delta_3,0\}$. □

4.2 Conditions of ρ_{zero} for X-state

For any given two-qubit X-state, it is difficult to check if the state belongs to zero-state or not. Here we study further the conditions that a X-state needs to satisfy to be a zero-state ρ_{zero}. In the following we denote $cond_{zero}$ the conditions such that ρ_X satisfying $cond_{zero}$ is a zero state.

We have already classified the problem by conditions (I)-(III). For conditions (I): $\alpha_1=\frac{\pi}{2}$, $\beta_1=\frac{(i-1)\pi}{2}$ and $\beta_2=\frac{(j-1)\pi}{2}$, $(i,j=1,2)$, $\Omega=0$ is actually an equation satisfied by $\cos\alpha_2$. We can prove that the following conditions are equivalent to (I),

1a1).

$$\begin{cases} u_j^2<u_3^2 \quad \text{or} \quad [(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)(u_j^2-u_3^2)][u_j^2(u_j^2-u_3^2+t_3^2)-u_j^2+u_3^2]<0, i=1,j=1\\ u_j^2<u_3^2 \quad \text{or} \quad [(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)(u_j^2-u_3^2)][u_j^2(u_j^2-u_3^2+t_3^2)-u_j^2+u_3^2]<0, i=1,j=2\\ u_j^2<u_3^2 \quad \text{or} \quad [(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)(u_j^2-u_3^2)][u_j^2(u_j^2-u_3^2+t_3^2)-u_j^2+u_3^2]<0, i=2,j=1\\ u_j^2<u_3^2 \quad \text{or} \quad [(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)(u_j^2-u_3^2)][u_j^2(u_j^2-u_3^2+t_3^2)-u_j^2+u_3^2]<0, i=2,j=2 \end{cases}$$

1a2). if the conditions in 1a1) are not satisfied, that is, at least one of the four inequalities in 1a1) is not satisfied, i.e. for the i and j which satisfy $u_j^2\geqslant u_3^2$ and $[(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)(u_j^2-u_3^2)][u_j^2(u_j^2-u_3^2+t_3^2)-u_j^2+u_3^2]\geqslant 0$, we have

$$\begin{cases} \dfrac{(u_3^2-u_j^2+t_3^2+u_j^2(u_j^2-u_3^2+t_3^2))}{(u_j^2-u_3^2+t_3^2)^2}+\dfrac{|t_3||u_i^2(u_j^2-u_3^2+t_3^2)+u_j^2-u_3^2-t_3^2|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2-u_j^2+u_j^2(u_j^2-u_3^2+t_3^2)}{(u_j^2-u_3^2)(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)}}>1\\ \dfrac{(u_3^2-u_j^2+t_3^2+u_j^2(u_j^2-u_3^2+t_3^2))}{(u_j^2-u_3^2+t_3^2)^2}-\dfrac{|t_3||u_i^2(u_j^2-u_3^2+t_3^2)+u_j^2-u_3^2-t_3^2|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2-u_j^2+u_j^2(u_j^2-u_3^2+t_3^2)}{(u_j^2-u_3^2)(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)}}<0. \end{cases}$$

1a3). if the conditions in 1a1) and 1a2) are not satisfied, i.e. for the i and j which satisfy $u_j^2\geqslant u_3^2$ and $[(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)(u_j^2-u_3^2)][u_j^2(u_j^2-u_3^2+t_3^2)-u_j^2+u_3^2]\geqslant 0$, we have

$$\begin{cases} \cos^2\alpha_2=\dfrac{(u_3^2-u_j^2+t_3^2+u_j^2(u_j^2-u_3^2+t_3^2))}{(u_j^2-u_3^2+t_3^2)^2}+\dfrac{|t_3||u_i^2(u_j^2-u_3^2+t_3^2)+u_j^2-u_3^2-t_3^2|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2-u_j^2+u_j^2(u_j^2-u_3^2+t_3^2)}{(u_j^2-u_3^2)(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)}}\leqslant 1\\ \dfrac{(-1+u_i^2)(2u_j^2-u_j^4-2u_3^2+u_3^4-2(1+u_3^2)t_3^2+t_3^4+(u_j^2-u_3^2+t_3^2)^2)(2\cos\alpha_2^2-1)}{2(1+u_i^2)(u_j^2-u_3^2)-2(1-u_i^2)t_3^2}<0 \end{cases}$$

or

$$\begin{cases} \cos^2\alpha_2=\dfrac{(u_3^2-u_j^2+t_3^2+u_j^2(u_j^2-u_3^2+t_3^2))}{(u_j^2-u_3^2+t_3^2)^2}-\dfrac{|t_3||u_i^2(u_j^2-u_3^2+t_3^2)+u_j^2-u_3^2-t_3^2|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2-u_j^2+u_j^2(u_j^2-u_3^2+t_3^2)}{(u_j^2-u_3^2)(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)}}\geqslant 0\\ \dfrac{(-1+u_i^2)(2u_j^2-u_j^4-2u_3^2+u_3^4-2(1+u_3^2)t_3^2+t_3^4+(u_j^2-u_3^2+t_3^2)^2)(2\cos\alpha_2^2-1)}{2(1+u_i^2)(u_j^2-u_3^2)-2(1-u_i^2)t_3^2}<0 \end{cases}$$

If ρ_X satisfies conditions 1a1) or 1a2) or 1a3), we have for $\alpha_1=\frac{\pi}{2}$, $\beta_1=\frac{(i-1)\pi}{2}$ and $\beta_2=\frac{(j-1)\pi}{2}$, $\Omega=0$ does not have solutions.

If both 1a1),1a2) and 1a3) are not satisfied, then

1b) for the i and j which satisfy $u_j^2 \geqslant u_3^2$ and $[(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)(u_j^2-u_3^2)][u_j^2(u_j^2-u_3^2+t_3^2)-u_j^2+u_3^2] \geqslant 0$ we have

$$\begin{cases} \cos^2\alpha_2 = \dfrac{(u_3^2-u_j^2+t_3^2+u_j^2(u_j^2-u_3^2+t_3^2))}{(u_j^2-u_3^2+t_3^2)^2} + \dfrac{|t_3||u_i^2(u_j^2-u_3^2+t_3^2)+u_j^2-u_3^2-t_3^2|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2-u_j^2+u_j^2(u_j^2-u_3^2+t_3^2)}{(u_j^2-u_3^2)(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)}} = 1 \\ \dfrac{(-1+u_i^2)(2u_j^2-u_j^4-2u_3^2+u_3^4-2(1+u_3^2)t_3^2+t_3^4+(u_j^2-u_3^2+t_3^2)^2)}{2(1+u_i^2)(u_j^2-u_3^2)-2(1-u_i^2)t_3^2} \geqslant 0. \end{cases}$$

or

$$\begin{cases} \cos^2\alpha_2 = \dfrac{(u_3^2-u_j^2+t_3^2+u_j^2(u_j^2-u_3^2+t_3^2))}{(u_j^2-u_3^2+t_3^2)^2} - \dfrac{|t_3||u_i^2(u_j^2-u_3^2+t_3^2)+u_j^2-u_3^2-t_3^2|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2-u_j^2+u_j^2(u_j^2-u_3^2+t_3^2)}{(u_j^2-u_3^2)(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)}} = 0 \\ \dfrac{(-1+u_i^2)(2u_j^2-u_j^4-2u_3^2+u_3^4-2(1+u_3^2)t_3^2+t_3^4-(u_j^2-u_3^2+t_3^2)^2)}{2(1+u_i^2)(u_j^2-u_3^2)-2(1-u_i^2)t_3^2} \geqslant 0. \end{cases}$$

i.e. for $\alpha_1 = \frac{\pi}{2}$, $\beta_1 = \frac{(i-1)\pi}{2}$ and $\beta_2 = \frac{(j-1)\pi}{2}$, $\Omega = 0$ has the solution $\alpha_2 = \frac{m\pi}{2}$ ($m = 0$ or 1).

1c) for the i and j which satisfy $u_j^2 \geqslant u_3^2$ and $[(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)(u_j^2-u_3^2)][u_j^2(u_j^2-u_3^2+t_3^2)-u_j^2+u_3^2] \geqslant 0$, we have

$$\begin{cases} \cos^2\alpha_2 = \dfrac{(u_3^2-u_j^2+t_3^2+u_j^2(u_j^2-u_3^2+t_3^2))}{(u_j^2-u_3^2+t_3^2)^2} + \dfrac{|t_3||u_i^2(u_j^2-u_3^2+t_3^2)+u_j^2-u_3^2-t_3^2|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2-u_j^2+u_j^2(u_j^2-u_3^2+t_3^2)}{(u_j^2-u_3^2)(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)}} < 1 \\ \dfrac{(-1+u_i^2)(2u_j^2-u_j^4-2u_3^2+u_3^4-2(1+u_3^2)t_3^2+t_3^4+(u_j^2-u_3^2+t_3^2)^2(2\cos^2\alpha_2-1))}{2(1+u_i^2)(u_j^2-u_3^2)-2(1-u_i^2)t_3^2} \geqslant 0 \end{cases}$$

or

$$\begin{cases} \cos^2\alpha_2 = \dfrac{(u_3^2-u_j^2+t_3^2+u_j^2(u_j^2-u_3^2+t_3^2))}{(u_j^2-u_3^2+t_3^2)^2} - \dfrac{|t_3||u_i^2(u_j^2-u_3^2+t_3^2)+u_j^2-u_3^2-t_3^2|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2-u_j^2+u_j^2(u_j^2-u_3^2+t_3^2)}{(u_j^2-u_3^2)(u_i^2(u_j^2-u_3^2+t_3^2)-t_3^2)}} > 0 \\ \dfrac{(-1+u_i^2)(2u_j^2-u_j^4-2u_3^2+u_3^4-2(1+u_3^2)t_3^2+t_3^4+(u_j^2-u_3^2+t_3^2)^2(2\cos^2\alpha_2-1))}{2(1+u_i^2)(u_j^2-u_3^2)-2(1-u_i^2)t_3^2} \geqslant 0. \end{cases}$$

i.e. for some i and j, $\Omega = 0$ has the solutions $\alpha_2 = \alpha_2^0 \neq \frac{m\pi}{2}$ ($m = 1, 2$), but we require that $\alpha_1 = \frac{\pi}{2}$, $\beta_1 = \frac{(i-1)\pi}{2}$, $\beta_2 = \frac{(j-1)\pi}{2}$, $\alpha_2 = \alpha_2^0$ are not the maximum points of S_1.

For condition (II): when $\alpha_1 = 0$, $\beta_2 = \frac{(j-1)\pi}{2}$ ($j = 1, 2$), $\Omega = 0$ is actually the equation of $\cos\alpha_2$.

Let $r_1 = \sqrt{((1+t_3)^2-u_3^2)((1-t_3)^2-u_3^2)}$, $r_2 = u_3^2+t_3^2+(u_3^2-t_3^2)^2+u_j^2(-1-u_3^2+t_3^2)$, we can prove that the following conditions are equivalent to (II).

2a1)

$$\begin{cases} u_j^2 < u_3^2 \quad \text{or} \quad (u_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2))(r_2^2-r_1^2(u_j^2-u_3^2+t_3^2)) < 0,\ j=1 \\ u_j^2 < u_3^2 \quad \text{or} \quad (u_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2))(r_2^2-r_1^2(u_j^2-u_3^2+t_3^2)) < 0,\ j=2 \end{cases}$$

If the conditions in 2a1) are not satisfied, i.e.

2a2) $u_j^2 \geqslant u_3^2$ and $(u_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2))(r_2^2-r_1^2(u_j^2-u_3^2+t_3^2)) \geqslant 0$, $j=1$ or $j=2$ or $j=1,2$, but

$$\begin{cases} \dfrac{u_3^2+t_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{(u_j^2-u_3^2+t_3^2)^2} + \dfrac{2|r_2t_3|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{r_2^2-r_1^2(u_j^2-u_3^2+t_3^2)}} > 1 \\ \dfrac{u_3^2+t_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{(u_j^2-u_3^2+t_3^2)^2} + \dfrac{2|r_2t_3|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{r_2^2-r_1^2(u_j^2-u_3^2+t_3^2)}} < 0 \end{cases}$$

If the conditions in 2a1) and 2a2) are not satisfied, i.e.
2a3) $u_j^2 \geqslant u_3^2$ and $(u_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2))(r_2^2-r_1^2(u_j^2-u_3^2+t_3^2)) \geqslant 0$, $j=1$ or $j=2$ or $j=1,2$, but

$$\begin{cases} \cos^2\alpha_2 = \dfrac{u_3^2+t_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{(u_j^2-u_3^2+t_3^2)^2} + \dfrac{2|r_2t_3|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{r_2^2-r_1^2(u_j^2-u_3^2+t_3^2)}} \leqslant 1 \\ \dfrac{(1-u_j^2)(u_j^2-u_3^2)-(1+u_j^2)t_3^2+(u_j^2-u_3^2+t_3^2)^2\cos\alpha_2^2}{u_3^2+t_3^2+(u_3^2-t_3^2)^2+u_j^2(-1-u_3^2+t_3^2)} < 0 \end{cases}$$

or

$$\begin{cases} \cos^2\alpha_2 = \dfrac{u_3^2+t_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{(u_j^2-u_3^2+t_3^2)^2} - \dfrac{2|r_2t_3|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{r_2^2-r_1^2(u_j^2-u_3^2+t_3^2)}} \geqslant 0 \\ \dfrac{(1-u_j^2)(u_j^2-u_3^2)-(1+u_j^2)t_3^2+(u_j^2-u_3^2+t_3^2)^2\cos\alpha_2^2}{u_3^2+t_3^2+(u_3^2-t_3^2)^2+u_j^2(-1-u_3^2+t_3^2)} < 0 \end{cases}$$

If ρ_X satisfies conditions in 2a1) or 2a2) or 2a3), we have for $\alpha_1=0$, $\beta_2=\frac{(j-1)\pi}{2}$, $\Omega=0$ does not have solutions.
If both 2a1), 2a2) and 2a3) are not satisfied, i.e.
2b) $u_j^2 \geqslant u_3^2$ and $(u_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2))(r_2^2-r_1^2(u_j^2-u_3^2+t_3^2)) \geqslant 0$, $j=1$ or $j=2$ or $j=1,2$, but

$$\begin{cases} \cos^2\alpha_2 = \dfrac{u_3^2+t_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{(u_j^2-u_3^2+t_3^2)^2} + \dfrac{2|r_2t_3|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{r_2^2-r_1^2(u_j^2-u_3^2+t_3^2)}} = 1 \\ \dfrac{(1-u_j^2)(u_j^2-u_3^2)-(1+u_j^2)t_3^2+(u_j^2-u_3^2+t_3^2)^2}{u_3^2+t_3^2+(u_3^2-t_3^2)^2+u_j^2(-1-u_3^2+t_3^2)} \geqslant 0 \end{cases}$$

or

$$\begin{cases} \cos^2\alpha_2 = \dfrac{u_3^2+t_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{(u_j^2-u_3^2+t_3^2)^2} - \dfrac{2|r_2t_3|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{r_2^2-r_1^2(u_j^2-u_3^2+t_3^2)}} = 0 \\ \dfrac{(1-u_j^2)(u_j^2-u_3^2)-(1+u_j^2)t_3^2}{u_3^2+t_3^2+(u_3^2-t_3^2)^2+u_j^2(-1-u_3^2+t_3^2)} \geqslant 0 \end{cases}$$

i.e. for $\alpha_1=0$, $\beta_2=\frac{(j-1)\pi}{2}$, $\Omega=0$ only has the solution $\alpha_2=\frac{m\pi}{2}$ $(m=0,1)$.
2c) $u_j^2 \geqslant u_3^2$ and $(u_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2))(r_2^2-r_1^2(u_j^2-u_3^2+t_3^2)) \geqslant 0$, $j=1$ or $j=2$ or $j=1,2$, but

$$\begin{cases} \cos^2\alpha_2 = \dfrac{u_3^2+t_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{(u_j^2-u_3^2+t_3^2)^2} + \dfrac{2|r_2t_3|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{r_2^2-r_1^2(u_j^2-u_3^2+t_3^2)}} < 1 \\ \dfrac{(1-u_j^2)(u_j^2-u_3^2)-(1+u_j^2)t_3^2+(u_j^2-u_3^2+t_3^2)^2\cos\alpha_2^2}{u_3^2+t_3^2+(u_3^2-t_3^2)^2+u_j^2(-1-u_3^2+t_3^2)} \geqslant 0 \end{cases}$$

or

$$\begin{cases} \cos^2\alpha_2 = \dfrac{u_3^2+t_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{(u_j^2-u_3^2+t_3^2)^2} - \dfrac{2|r_2t_3|}{(u_j^2-u_3^2+t_3^2)^2}\sqrt{\dfrac{u_3^2+u_j^2(u_j^2-1-u_3^2+t_3^2)}{r_2^2-r_1^2(u_j^2-u_3^2+t_3^2)}} > 0 \\ \dfrac{(1-u_j^2)(u_j^2-u_3^2)-(1+u_j^2)t_3^2+(u_j^2-u_3^2+t_3^2)^2\cos\alpha_2^2}{u_3^2+t_3^2+(u_3^2-t_3^2)^2+u_j^2(-1-u_3^2+t_3^2)} \geqslant 0 \end{cases}$$

i.e. for $\alpha_1=0$, $\beta_2=\frac{(j-1)\pi}{2}$, $\Omega=0$ has the solution $\alpha_2=\alpha_2^1\neq\frac{m\pi}{2}$, but we require that $\alpha_1=0$, $\beta_2=\frac{(j-1)\pi}{2}$ and $\alpha_2=\alpha_2^1$ are not the maximum of S_1.

For condition (III): If $t_3^4\neq(u_1^2-u_3^2)(u_2^2-u_3^2)$, when $\beta_1=\frac{k\pi}{2}$, $\beta_2=\frac{k\pi}{2}$, Δ and Ω can not be 0 simultaneously, then (13) does not have solutions. □

4.3 Proof of Corollaries

Proof of Corollary 1: For the states ρ_{zero}, the positivity of density matrix gives the conditions $(a_3-b_3)^2+(c_1+c_2)^2 \leqslant (1-c_3)^2$ and $(a_3+b_3)^2+(c_1-c_2)^2 \leqslant (1+c_3)^2$.

Case I: The maximal value of $\Delta_1 = u_1^2+u_2^2-1$

From the condition $(a_3-b_3)^2+(c_1+c_2)^2 \leqslant (1-c_3)^2$ and $(a_3+b_3)^2+(c_1-c_2)^2 \leqslant (1+c_3)^2$, we have that $c_1^2+c_2^2 \leqslant 1+c_3^2-a_3^2-b_3^2$. Hence, $u_1^2+u_2^2-1 \leqslant \frac{c_3^2-a_3^2}{1-b_3^2}$. $\frac{c_3^2-a_3^2}{1-b_3^2}$ gets the maximum value for small a_3^2 and large b_3^2. Due to $(a_3-b_3)^2+(c_1+c_2)^2 \leqslant (1-c_3)^2$ and $(a_3+b_3)^2+(c_1-c_2)^2 \leqslant (1+c_3)^2$, we get $|a_3-b_3| \leqslant 1-c_3$ and $|a_3+b_3| \leqslant 1+c_3$. When $a_3=0$, b_3 attains its maximum value, $\min\{1-c_3, 1+c_3\}$. Therefore,

$$\frac{c_3^2-a_3^2}{1-b_3^2} \leqslant \frac{c_3^2}{1-\min\{(1-c_3)^2,(1+c_3)^2\}} \leqslant |c_3|.$$

Actually when $c_3 \geqslant 0$, if $b_3 \to -1$ and $a_3-b_3 = 1-c_3$, we have $u_1^2+u_2^2-1 \to c_3$. When $c_3<0$, if $b_3 \to 1$ and $a_3+b_3 = 1+c_3$, we have $u_1^2+u_2^2-1 \to |c_3|$.

Case II: The maximal value of $\Delta_2 = \frac{1}{2}[u_1^2(u_3^2-t_3^2)+u_1^2+u_3^2+t_3^2-1-(1-u_1^2)\sqrt{((1-t_3)^2-u_3^2)((1+t_3)^2-u_3^2)}]$

For any given a_3, b_3 and c_3, Δ_2 increases with $|c_1|$. The maximum value of c_1 is attained when $(a_3-b_3)^2+(c_1+c_2)^2 = (1-c_3)^2$ and $(a_3+b_3)^2+(c_1-c_2)^2 = (1+c_3)^2$. We only need to consider the parameters a_3, b_3, c_1, c_2 and c_3 which satisfy $(a_3-b_3)^2+(c_1+c_2)^2 = (1-c_3)^2$ and $(a_3+b_3)^2+(c_1-c_2)^2 = (1+c_3)^2$. Let $\Gamma_1 = \sqrt{(1-c_3)^2-(a_3-b_3)^2}$, $\Gamma_2 = \sqrt{(1+c_3)^2-(a_3+b_3)^2}$. We assume $c_1 \geqslant c_2$, and $c_1 \geqslant 0$, then $c_2 \leqslant \frac{\Gamma_1}{2}$. Set $c_2 = \frac{\Gamma_1}{2}-x$, $c_1 = \frac{\Gamma_1}{2}+x$, for $0 \leqslant x \leqslant \frac{\Gamma_2}{2}$. We have that Δ_2 is an increasing function of x. Hence

$$\Delta_2 \leqslant \frac{c_3^2-a_3^2}{2(1-b_3^2)} \frac{1-b_3^2+c_3^2-a_3^2+\Gamma_1\Gamma_2}{1-b_3^2}.$$

$\Gamma_1\Gamma_2/(1-b_3^2)$ attain the maximum value when $c_3 \geqslant 0$ ($\leqslant 0$) and $a_3b_3 \geqslant 0$ ($\leqslant 0$). By the optimization method, one can find $\frac{c_3^2-a_3^2}{2(1-b_3^2)} \frac{1-b_3^2+c_3^2-a_3^2+\Gamma_1\Gamma_2}{1-b_3^2}$ attain the maximum value when $1+c_3 = |a_3+b_3|$ or $1-c_3 = |a_3-b_3|$. So we have when b_3 approaches to -1 and $a_3-b_3 = 1-c_3$, or $b_3 \to 1$ and $a_3+b_3 = 1+c_3$, we have the maxima of $\Delta_2 = |c_3|$.

We have the steerability $S \leqslant |c_3|$ when $b_3 \to -1$ and $a_3-b_3 = 1-c_3$, or $b_3 \to 1$ and $a_3+b_3 = 1+c_3$. Then either $c_1 = -c_2 = \pm\sqrt{(1+b_3)(c_3-b_3)}$ or $c_1 = c_2 = \pm\sqrt{(1-b_3)(b_3-c_3)}$, and $S \leqslant \frac{N}{2}$. This completes the proof. □

Proof of Corollary 2: Due to the positivity of density matrix ρ_X, a_3, b_3 and c_i $(i=1,2,3)$ satisfy the conditions $|a_3+b_3| \leqslant \sqrt{(1+c_3)^2-(c_1-c_2)^2}$ and $|a_3-b_3| \leqslant \sqrt{(1-c_3)^2-(c_1+c_2)^2}$. Let

$$\begin{aligned}\Omega = \{&|a_3+b_3| \leqslant \sqrt{(1+c_3)^2-(c_1-c_2)^2},\\ &|a_3-b_3| \leqslant \sqrt{(1-c_3)^2-(c_1+c_2)^2}\}\end{aligned}$$

The minimum of $\max\limits_{\alpha_i,\beta_i} S_1$ is attained at the interior points or the boundary of Ω for given CHSH value N:

(1) For the interior points of Ω, when $t_3=0$, $u_3=c_3$, and $b_3=0$, we have the minimal steerability $S = \frac{N^2}{4}-1$.

(2) For the boundary of Ω, the minimal value of S is attained at the extreme points of Δ_2 or Δ_3, or the points solving the equation $\Delta_2 = \Delta_1$, $\Delta_3 = \Delta_1$, or $\Delta_2 = \Delta_3$. By numerical simulations, we find that when $N \geqslant 2$, the lower bound is very close to $\frac{N^2}{4}-1$ but smaller than $\frac{N^2}{4}-1$. □

4.4 Matlab program for computing steerability of general two-qubit states

opts=optimoptions(@fmincon,'Algorithm','interior
−point');
problem=createOptimProblem(opt);
gs=GlobalSearch;
[x,f]=run(gs,problem).
Here
opt='fmincon','objective',... @$(x)(-S(x_1,x_2,x_3,x_4))$, 'x_0', x^*, 'lb', lb^*, 'ub', un^*, 'options',opts
with $x_1=\alpha_1, x_2=\beta_1, x_3=\alpha_2, x_4=\beta_2$, $x^*=[0,0,0,0]$, $lb^*=[0,0,0,0]$, and $ub^*=[\pi,2\pi,\pi,2\pi]$.