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by

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THE h^*-POLYNOMIAL OF THE ORDER POLYTOPE OF THE ZIG-ZAG POSET

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ABSTRACT. We describe a family of shellings for the canonical triangulation of the order polytope of the zig-zag poset. This gives a combinatorial interpretation for the coefficients in the numerator of the generating functions for OEIS A050446 in terms of the swap statistic on alternating permutations.

1. Introduction and Preliminaries

The *zig-zag poset* $\mathcal{Z}_n$ on ground set $\{z_1, \ldots, z_n\}$ is the poset with exactly the cover relations $z_1 < z_2 > z_3 < z_4 > \ldots$. That is, this partial order satisfies $z_{2i-1} < z_{2i}$ and $z_{2i} > z_{2i+1}$ for all i between 1 and $\lfloor \frac{n-1}{2} \rfloor$. The order polytope of $\mathcal{Z}_n$, denoted $\mathcal{O}(\mathcal{Z}_n)$ is the set of all n-tuples $(x_1, \ldots, x_n) \in \mathbb{R}^n$ that satisfy $0 \leq x_i \leq 1$ for all i and $x_i \leq x_j$ whenever $z_i < z_j$ in $\mathcal{Z}_n$. We aim to understand the Ehrhart series of $\mathcal{O}(\mathcal{Z}_n)$.

The *Ehrhart function* of a polytope P, denoted $i_P(m)$ is the function that counts the number of lattice points in the m-th dilate of P for any positive integer m. That is,

$$i_P(m) = \#(\mathbb{Z}^n \cap mP)$$

where $mP = \{m\mathbf{v} \mid \mathbf{v} \in P\}$. The *Ehrhart series* of P is the formal power series

$$\mathrm{Ehr}_P(t) = \sum_{m=0}^{\infty} i_P(m) t^m.$$

The Ehrhart series of a polytope with integer vertices is a rational function of the form

$$\mathrm{Ehr}_P(t) = \frac{h_P^*(t)}{(1-t)^{d+1}}$$

where d is the dimension of P. The numerator polynomial in this rational expression is called the h^*-*polynomial* of P.

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Our goal in this paper is to understand the h^*-polynomials of the $\mathcal{O}(\mathcal{Z}_n)$. For $n = 1, 2, 3, 4, 5, 6$, these have the following form

$$1,\ 1,\ 1+t,\ 1+3t+t^2,\ 1+7t+7t^2+t^3,\ 1+14t+31t^2+14t^3+t^4$$

and appear in the Online Encyclopedia of Integer Sequences with reference number A205497 [4]. We began studying this problem because the Ehrhart polynomial of $\mathcal{O}(\mathcal{Z}_n)$ is equal to that of the CFN-MC polytope of any rooted binary tree on $n+1$ leaves [3]. Therefore, it is also equal to the Hilbert series of the toric ideal of phylogenetic invariants of the CFN-MC model on such a tree.

To understand the h^* polynomial of $\mathcal{O}(\mathcal{Z}_n)$, we will interpret its coefficients in terms of a permutation statistic on alternating permutations. An *alternating permutation* on n letters is a permutation $a_1 a_2 \dots a_n$ such that $a_1 < a_2 > a_3 < a_4 > \dots$. Notice that alternating permutations coincide with bijective labelings of $\mathcal{Z}_n$ with the numbers $1, \dots, n$ that agree with the partial order on $\mathcal{Z}_n$. We define the *swap* permutation statistic on an alternating permutation σ to be the number of integers i such that $\sigma^{-1}(i) < \sigma^{-1}(i+1)$ and swapping i and $i+1$ in σ yields another alternating permutation. In other words, i to the left of $i+1$ and there is at least one other character between them. The goal of this paper is to prove the following theorem relating the h^*-polynomial of $\mathcal{O}(\mathcal{Z}_n)$ and the swap statistic.

Theorem 1.1. *The numerator of the Ehrhart series of $\mathcal{O}(\mathcal{Z}_n)$ is*

$$h^*_{\mathcal{O}(\mathcal{Z}_n)}(t) = \sum_{\sigma} t^{swap(\sigma)}$$

where this sum ranges over all alternating permutations of length n.

In Section 2, we provide further background information on zig-zag posets and their order polytopes. We relate these to the theory of alternating permutations. Then we discuss the necessary background on Ehrhart theory. In Section 3, we prove our main result, Theorem 1.1, by giving a shelling of the canonical triangulation of the order polytope of the zig-zag poset. In Section 4, we give an alternate proof of Theorem 1.1 by counting chains in the lattice of order ideals of the zig-zag poset. This proof makes use of the theory of Jordan-Hölder sets of general posets developed in Chapter 3 of [7].

2. Preliminaries

The *zig-zag poset* $\mathcal{Z}_n$ on ground set $\{z_1, \dots, z_n\}$ is the poset with the cover relations $z_{2i-1} < z_{2i}$ and $z_{2i} > z_{2i+1}$ for $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$. Linear extensions of the zig-zag poset are in bijection with *alternating* *permutations* of length n; that is, permutations $a_1 \dots a_n$ for which the a_i's satisfy $a_{2i-1} < a_{2i}$ and $a_{2i} > a_{2i+1}$ for $1 \le i \le \lfloor \frac{n-1}{2} \rfloor$.

The number of alternating permutations of length n is the nth *Euler zig-zag number* E_n. The sequence of Euler zig-zag numbers starting with E_0 begins $1, 1, 1, 2, 5, 16, 61, 272, \dots$. This sequence can be found in the Online Encyclopedia of Integer Sequences with identification number A000111 [4]. The exponential generating function for the Euler zig-zag numbers satisfies

$$\sum_{n \ge 0} E_n \frac{x^n}{n!} = \tan x + \sec x.$$

Furthermore, the Euler zig-zag numbers satisfy the recurrence

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}$$

for $n \ge 1$ with initial values $E_0 = E_1 = 1$. A thorough background on the combinatorics of alternating permutations can be found in [6]. The following new permutation statistic on alternating permutations is central to our results.

Definition 2.1. Let σ be an alternating permutation. The permutation statistic $\mathrm{swap}(\sigma)$ is the number of $i < n$ such that $\sigma^{-1}(i) < \sigma^{-1}(i+1) - 1$. Equivalently, this is the number of $i < n$ such that i is to the left of $i+1$ and swapping i and $i+1$ in σ yields another alternating permutation. The swap-set $\mathrm{Swap}(\sigma)$ is the set of all $i < n$ for which we can perform this operation. We say that σ *swaps to* τ if τ can be obtained from σ by performing this operation a single time.

For example, the alternating permutation 15342 has $\mathrm{swap}(15342) = 1$ and $\mathrm{Swap}(15342) = \{1\}$. Hence, 15342 swaps to 25341 and to no other alternating permutation.

To every finite poset on n elements one can associate a polytope in $\mathbb{R}^n$ by viewing the cover relations on the poset as inequalities on Euclidean space.

Definition 2.2. The *order polytope* $\mathcal{O}(P)$ of any poset P on ground set $p_1, \dots, p_n$ is the set of all $\mathbf{v} \in \mathbb{R}^n$ that satisfy $0 \le v_i \le 1$ for all i and $v_i \le v_j$ if $p_i < p_j$ is a cover relation in P.

Order polytopes for arbitrary posets have been the object of considerable study, and are discussed in detail in [5]. In the case of $\mathcal{O}(\mathcal{Z}_n)$, the facet defining inequalities are those of the form

$$\begin{aligned} -v_i &\leq 0 \text{ for } i \leq n \text{ odd} \\ v_i &\leq 1 \text{ for } i \leq n \text{ even} \\ v_i - v_{i+1} &\leq 0 \text{ for } i \leq n-1 \text{ odd, and} \\ -v_i + v_{i+1} &\leq 0 \text{ for } i \leq n-1 \text{ even.} \end{aligned}$$

Note that the inequalities of the form $-v_i \leq 0$ for i even and $v_i \leq 1$ for i odd are redundant. The order polytope of $\mathcal{Z}_n$ is also the convex hull of all $(v_1, \ldots, v_n) \in \{0,1\}^n$ that correspond to labelings of $\mathcal{Z}_n$ that are weakly consistent with the partial order on $\{p_1, \ldots, p_n\}$.

In [5], Stanley gives the following canonical unimodular triangulation of the order polytope of any poset P on ground set $\{p_1, \ldots, p_n\}$. Let $\sigma : P \to [n]$ be a linear extension of P. Denote by $\mathbf{e}_i$ the ith standard basis vector in $\mathbb{R}^n$. The simplex Δ^σ is the convex hull of $\mathbf{v}_0^\sigma, \ldots, \mathbf{v}_n^\sigma$ where $\mathbf{v}_0^\sigma$ is the all 1's vector and $\mathbf{v}_i^\sigma = \mathbf{v}_{i-1}^\sigma - \mathbf{e}_{\sigma^{-1}(i)}$. Letting σ range over all linear extensions of P yields a unimodular triangulation of $\mathcal{O}(P)$. Hence, the normalized volume of $\mathcal{O}(P)$ is the number of linear extensions of P. In particular, this means that the volume of $\mathcal{O}(\mathcal{Z}_n)$ is the Euler zig-zag number, E_n.

Example 2.3. Consider the case when $n = 4$. The zig-zag poset $\mathcal{Z}_4$ is pictured in Figure 2.1. The order polytope $\mathcal{O}(\mathcal{Z}_4)$ has facet defining inequalities

$$\begin{aligned} -v_1 &\leq 0 & v_2 &\leq 1 \\ -v_3 &\leq 0 & v_4 &\leq 1 \\ v_1 - v_2 &\leq 0 & -v_2 + v_3 &\leq 0 \\ v_3 - v_4 &\leq 0. \end{aligned}$$

The vertices of $\mathcal{O}(\mathcal{Z}_4)$ are the columns of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The alternating permutations on 4 elements, which correspond to linear extensions of $\mathcal{Z}_4$ are 1324, 1423, 2314, 2413, and 3412. Note that there are $E_4 = 5$ such alternating permutations, so the normalized volume of $\mathcal{O}(\mathcal{Z}_4)$ is 5. The simplex in the canonical triangulation of $\mathcal{O}(\mathcal{Z}_n)$

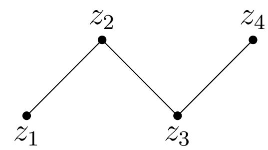

FIGURE 2.1. The zig-zag poset $\mathcal{Z}_4$

corresponding to 1324 is

$$\Delta^{1324} = \operatorname{conv} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

We turn our attention to the study of Ehrhart functions and series of lattice polytopes. Let $P \subset \mathbb{R}^n$ be any polytope with integer vertices. Recall that the *Ehrhart function*, $i_P(m)$, counts the integer points in dilates of P; that is,

$$i_P(m) = \#(\mathbb{Z}^n \cap mP),$$

where $mP = \{m\mathbf{v} \mid \mathbf{v} \in P\}$ denotes the mth dilate of P. The Ehrhart function is, in fact, a polynomial in m [1, Chapter 3]. We further define the *Ehrhart series* of P to be the generating function

$$\operatorname{Ehr}_P(t) = \sum_{m \geq 0} i_P(m) t^m.$$

The Ehrhart series is of the form

$$\operatorname{Ehr}_P(t) = \frac{h_P^*(t)}{(1-t)^{d+1}},$$

where d is the dimension of P and $h_P^*(t)$ is a polynomial in t of degree at most n. Often we just write $h^*(t)$ when the particular polytope is clear. The coefficients of $h^*(t)$ have an interpretation in terms of a *shelling* of a unimodular triangulation of P, if such a triangulation exists.

Definition 2.4. Let $\mathcal{T}$ be the collection of maximal dimensional simplices in a pure simplicial complex of dimension d with $\#\mathcal{T} = s$. An ordering $\Delta_1, \Delta_2, \ldots, \Delta_s$ on the simplices in $\mathcal{T}$ is a *shelling order* if for all $1 < r \leq s$,

$$\bigcup_{i=1}^{r-1} \left(\Delta_i \cap \Delta_r \right)$$

is a union of facets of Δ_r.

Equivalently, the order $\Delta_1, \Delta_2, \ldots, \Delta_s$ is a shelling order if and only if for all $r \leq s$ and $k < r$, there exists an $i < r$ such that $\Delta_k \cap \Delta_r \subset \Delta_i \cap \Delta_r$ and $\Delta_i \cap \Delta_r$ is a facet of Δ_r. This means that when we build our simplicial complex by adding facets in the order prescribed by the shelling order, we add each simplex along its highest dimensional facets. Keeping track of the number of facets that each simplex is added along gives the following relationship between shellings of a triangulation of an integer polytope and the Ehrhart series of the polytope, which is proved in [1, Chapter 3].

Theorem 2.5. *Let P be an integer polytope. Let $\Delta_1, \ldots, \Delta_s$ be a unimodular triangulation of P using no new vertices. Denote by h_j^* the coefficient of t^j in the h^* polynomial of P. If $\Delta_1, \ldots, \Delta_s$ is a shelling order, then h_j^* is the number of Δ_i that are added along j of their facets in this shelling. Equivalently,*

$$h^*(t) = \sum_{i=1}^{s} t^{a_i},$$

where $a_i = \#\{k < i \mid \Delta_k \cap \Delta_i$ is a facet of $\Delta_i\}$.

Example 2.6. Consider the order polytope $\mathcal{O}(\mathcal{Z}_4)$ with its canonical triangulation by alternating permutations

$$\Delta^{3412}, \Delta^{2413}, \Delta^{2314}, \Delta^{1423}, \Delta^{1324}.$$

This particular ordering of the facets in the canonical triangulation is a special case of the shelling order that will be established and proved in the next section. The fact that this is a shelling order can be checked directly in this example, for instance:

$$\Delta^{2314} \cap (\Delta^{3412} \cup \Delta^{2413}) = \mathbf{conv} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

which is a facet of Δ^{2314}. Since the intersection consists of a single facet, it will contribute a 1 to the coefficient of t in $h^*_{\mathcal{O}(\mathcal{Z}_4))}(t) = 1 + 3t + t^2$.

3. Shelling the Canonical Triangulation of the Order Polytope

In this section we describe a family of shelling orders on the simplices of the canonical triangulation of $\mathcal{O}(\mathcal{Z}_n)$. Let σ be an alternating permutation. We will denote by $\mathrm{vert}(\sigma)$ the set of all vertices of the simplex Δ^σ. Note that this is the set of all 0/1 vectors $\mathbf{v}$ of length n that have $v_i \leq v_j$ whenever $\sigma(i) < \sigma(j)$.

Proposition 3.1. *The simplices Δ^σ and Δ^τ are joined along a facet if and only if σ swaps to τ or τ swaps to σ.*

Proof. Simplices Δ^σ and Δ^τ are joined along a facet if and only if $\mathrm{vert}(\sigma)$ and $\mathrm{vert}(\tau)$ differ by a single element. Since every simplex in the canonical triangulation of $\mathcal{O}(\mathcal{Z}_n)$ has exactly one vertex with the sum of its components equal to i for $0 \leq i \leq n$ and the all 0's and all 1's vector are in every simplex in this triangulation, this occurs if and only if there exists an i with $1 \leq i \leq n-1$ such that $\mathrm{vert}(\sigma) - \{\mathbf{v}_i^\sigma\} = \mathrm{vert}(\tau) - \{\mathbf{v}_i^\tau\}$. By definition of each $\mathbf{v}_j^\sigma$ and $\mathbf{v}_j^\tau$, this occurs if and only if $\sigma^{-1}(j) = \tau^{-1}(j)$ for all $j \neq i, i+1$ and $e_{\sigma^{-1}(i)} + e_{\sigma^{-1}(i+1)} = e_{\tau^{-1}(i)} + e_{\tau^{-1}(i+1)}$. This is true if and only if swapping the positions of i and $i+1$ in σ yields τ, as needed. ☐

Denote by $\mathrm{inv}(\sigma)$ the number of inversions of a permutation σ; that is, $\mathrm{inv}(\sigma)$ is the number of pairs $i < j$ such that $\sigma(i) > \sigma(j)$. We similarly define a *non-inversion* to be a pair $i < j$ with $\sigma(i) < \sigma(j)$. We call an inversion or non-inversion (i, j) *relevant* if $i < j - 1$; that is, if it is not required by the structure of an alternating permutation. Note that performing a swap on an alternating permutation always decreases its inversion number by exactly one. Theorem 1.1 follows as a corollary of Proposition 3.1 and the following theorem.

Theorem 3.2. *Let $\sigma_1, \ldots, \sigma_{E_n}$ be an order on the alternating permutations such that*

- *if $i < j$ then $\mathrm{inv}(\sigma_i) \geq \mathrm{inv}(\sigma_j)$ and*
- *if σ_j swaps to σ_i then $i < j$.*

Then the order $\Delta^{\sigma_1}, \ldots, \Delta^{\sigma_{E_n}}$ on the simplices of the canonical triangulation of $\mathcal{O}(\mathcal{Z}_n)$ is a shelling order.

For any alternating permutation σ, define the *exclusion set* of σ, $\mathrm{excl}(\sigma)$ to be the set of all $\mathbf{v}_k^\sigma \in \mathrm{vert}(\sigma)$ such that k is a swap in σ. In other words,

$$\mathrm{excl}(\sigma) = \{\mathbf{v} \mid \mathbf{v} \in \Delta^\sigma - \Delta^\tau \text{ for some } \tau \text{ such that } \sigma \text{ swaps to } \tau\}.$$

Proposition 3.1 implies that in order to prove Theorem 3.2, it suffices to check that if $\mathrm{inv}(\sigma) \leq \mathrm{inv}(\tau)$, then $\mathrm{excl}(\sigma) \not\subset \mathrm{vert}(\tau)$. This fact will follow from the next two propositions.

Proposition 3.3. *An alternating permutation σ maximizes inversion number over all alternating permutations τ with $\mathrm{excl}(\sigma) \subset \mathrm{excl}(\tau)$.*

Proof. Consider a vertex $\mathbf{v}_k^\sigma \in \mathrm{vert}(\sigma)$. Note that we may read all of the non-inversions (i, j) with $\sigma(i) \leq k < \sigma(j)$ from $\mathbf{v}_k^\sigma$ since these correspond to pairs of positions in $\mathbf{v}_k^\sigma$ with a 0 in the first position and a 1 in the second. That is to say, we have $\mathbf{v}_k^\sigma(i) = 0$, $\mathbf{v}_k^\sigma(j) = 1$, and $i < j$.

We claim that every relevant non-inversion of σ can be read from an element of $\mathrm{excl}(\sigma)$ in this way. To prove this, it suffices to show that if (i, j) is a relevant non-inversion of σ, then there exists a k with $\sigma(i) \leq k < \sigma(j)$ such that k is a swap in σ. We will prove this by induction on $\sigma(j) - \sigma(i)$.

If $\sigma(j) - \sigma(i) = 1$, then since (i, j) is a relevant non-inversion, $\sigma(i)$ is a swap in σ.

Let $\sigma(j) - \sigma(i) > 1$. Consider the position of $\sigma(i) + 1$ in σ. If $\sigma^{-1}(\sigma(i)+1) < j-1$, then $(\sigma^{-1}(\sigma(i)+1), j)$ is a relevant non-inversion, and we are done by induction. If $\sigma^{-1}(\sigma(i)+1) > j$, then $\sigma(i)$ is a swap in σ. If $\sigma^{-1}(\sigma(i)+1) = j-1$, then note that $i < \sigma^{-1}(\sigma(i)+1)-1$ since otherwise, $\sigma(i), \sigma(i)+1, \sigma(j)$ would be an adjacent increasing sequence in σ, which would contradict that σ is alternating. So $\sigma(i)$ is a swap in σ.

Therefore, there exists a swap k in σ with $\sigma(i) \leq k < \sigma(j)$, and the relevant non-inversion (i, j) can be read from $\mathbf{v}_k^\sigma$ in the manner described above. Therefore, all relevant non-inversions in σ can be found as a non-adjacent $0 - 1$ pair in a vertex in $\mathrm{excl}(\sigma)$. In particular, we can count the number of relevant non-inversions in σ from the vertices in $\mathrm{excl}(\sigma)$. Furthermore, if $\mathrm{excl}(\sigma) \subset \mathrm{vert}(\tau)$, then all non-inversions in σ must also be non-inversions in τ, though τ can contain more non-inversions as well. So σ minimizes the number of non-inversions, and therefore maximizes the number of inversions, over all τ with $\mathrm{excl}(\sigma) \subset \mathrm{vert}(\tau)$. $\square$

Proposition 3.4. *Let $S \subset \mathit{vert}(\mathcal{O}(\mathcal{Z}_n))$ be contained in $\mathit{vert}(\sigma)$ for some alternating σ. Then there exists a unique alternating $\hat{\sigma}$ that maximizes inversion number over all alternating permutations whose vertex set contains S.*

Proof. Let $S = \{\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_r\}$ ordered by decreasing coordinate sum. We can assume that S contains both the all zeroes and all ones vectors since those vectors belong to the simplex Δ^σ for any alternating permutation σ. Since $S \subset \mathrm{vert}(\sigma)$ for some alternating σ, if $s_i(j) = 0$, then $s_k(j) = 0$ for all $k > i$. For $i = 1, \ldots, r$, let m_i be the number of positions in $\mathbf{s}_i$ that are equal to zero, and let $n_i = m_i - m_{i-1}$ (with $n_1 = m_1$).

Let τ be any alternating permutation such that $S \subseteq \mathrm{vert}(T)$. The 0-pattern of each $\mathbf{s}_i$ partitions the entries of all τ with $S \subset \mathrm{vert}(\tau)$ as follows: For $1 \leq k \leq r$, the n_k positions j such that $\mathbf{s}_k(j) = 0$ and $\mathbf{s}_{k-1}(j) = 1$ are the positions of τ such that $\tau(j) \in \{m_{k-1}+1, \ldots, m_k\}$.

The positions of inversions and non-inversions across these groups are fixed for all τ with $S \subset \mathrm{vert}(\tau)$. We can build an alternating permutation $\hat{\sigma}$ that maximizes the inversions within each group as follows. For $1 \leq k \leq r$, let $j_1^k, \ldots, j_{n_k}^k$ be the positions of $\hat{\sigma}$ that must take values in $\{m_{k-1}+1, \ldots, m_k\}$, as described above. We place these values in reverse; i.e. map j_l^k to $m_k - l + 1$. The permutation obtained in this way need not be alternating, so we switch adjacent positions that need to contain non-descents in order to make the permutation alternating. Note that we never need to make such a switch between groups, since the partition given by S respects the structure of an alternating permutation.

This permutation is unique because within the kth group, arranging the values in this way is equivalent to finding the permutation on n_k elements with some fixed non-descent positions that maximizes inversion number. To obtain this permutation, we begin with the permutation $(m_k, m_k - 1 \ldots m_{k-1}+1)$ and switch all the positions that must be non-descents. The alternating structure of the original permutation implies that none of these non-descent positions can be adjacent, so these transpositions commute and give a unique permutation. $\square$

Example 3.5. Let $n = 7$ and let

$$S = \left\{ \begin{bmatrix}1\\1\\1\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix} \right\}$$

We will construct $\hat{\sigma}$, the alternating permutation that maximizes inversion number overall alternating permuations whose vertex set contains S. The second and third vertices in S are the only one that gives information about the position of each character; we will denote them $\mathbf{w}_1$ and $\mathbf{w}_2$, respectively. Since $\mathbf{w}_1$ has 0's in exactly the first, third and seventh positions, we know that $1, 2$ and 3 are in these positions. We insert them into these positions in decreasing order, so that $\hat{\sigma}$ has the form

$$3 \_ 2 \_ \_ \_ 1.$$

The zeros added in $\mathbf{w}_2$ are in the fourth, fifth and sixth positions. Placing them in decreasing order yields the permutation

$$3 \_ 2\ 6\ 5\ 4\ 1.$$

However, this permutation cannot be alternating, since there must be an ascent from position 5 to position 6. To create this ascent, we switch the entries in these positions, yielding a permutation of the form

$$3\ \_\ 2\ 6\ 4\ 5\ 1.$$

Finally, the only character missing is 7, which must go in the remaining space. This gives the permutation

$$\hat{\sigma} = 3\ 7\ 2\ 6\ 4\ 5\ 1.$$

Proof of Theorem 3.2. It suffices to show that for any alternating permutations σ and τ, if $\mathrm{inv}(\tau) \geq \mathrm{inv}(\sigma)$ then $\mathrm{excl}(\sigma) \not\subset \mathrm{vert}(\tau)$. If $\mathrm{inv}(\tau) > \mathrm{inv}(\sigma)$, then since σ maximizes inversion number over all alternating permutations that contain the exclusion set of σ by Proposition 3.3, $\mathrm{excl}(\sigma) \not\subset \mathrm{vert}(\tau)$. Furthermore, Proposition 3.4 implies that if $\mathrm{inv}(\tau) = \mathrm{inv}(\sigma)$, then $\mathrm{excl}(\sigma) \not\subset \mathrm{vert}(\tau)$ because σ is the *unique* permutation that maximizes inversion number of all alternating permutation that contain its exclusion set. ☐

Proof of Theorem 1.1. Let $\Delta^{\sigma_1}, \ldots, \Delta^{\sigma_{E_n}}$ be a shelling order as described in Theorem 3.2. Then by Proposition 3.1, each Δ^{σ_i} is added in the shelling along exactly $\mathrm{swap}(\sigma_i)$ facets. Therefore, by Theorem 2.5,

$$h^*_{\mathcal{O}(Z_N)}(t) = \sum_{\sigma} t^{\mathrm{swap}(\sigma)},$$

where σ ranges overall alternating permutations of length n. ☐

4. The Swap Statistic Via Rank Selection

An alternate proof of Theorem 1.1 relies heavily on the concepts of rank selection and flag f-vectors developed for general posets in Sections 3.13 and 3.15 of [7]. We will focus our attention to the zig-zag poset, $\mathcal{Z}_n$. Denote by $J(\mathcal{Z}_n)$ the distributive lattice of order ideals in Z_n ordered by inclusion. Let $S = \{s_1, \ldots, s_k\} \subset [0, n]$, where $[0, n] = \{0, \ldots, n\}$. We always assume that $s_1 < s_2 < \ldots < s_k$. Denote by $\alpha_n(S)$ the number of chains of order ideals $I_1 \subsetneq \cdots \subsetneq I_k$ in $J(\mathcal{Z}_n)$ such that $\#I_j = s_j$ for all j. Define

$$\beta_n(S) = \sum_{T \subset S} (-1)^{\#(S-T)} \alpha_n(T).$$

By the Principle of Inclusion-Exclusion, or equivalently, via Möbius inversion on the Boolean lattice,

$$\alpha_n(S) = \sum_{T \subset S} \beta_n(S).$$

In Section 3.13 of [7], the function $\alpha_n : 2^{[0,n]} \to \mathbb{Z}$ is called the *flag f-vector* of $\mathcal{Z}_n$ and $\beta_n : 2^{[0,n]} \to \mathbb{Z}$ is called the *flag h-vector* of $\mathcal{Z}_n$. For any poset P of size n, let $\omega : P \to [n]$ be an order-preserving bijection that assigns a label to each element of P. Then for any linear extension $\sigma : P \to [n]$, we may define a permutation of the labels by $\omega(\sigma^{-1}(1)), \dots, \omega(\sigma^{-1}(n))$. The *Jordan-Hölder set* $\mathcal{L}(P, \omega)$ is the set of all permutations obtained in this way. The following result for arbitrary finite posets can be found in chapter 3.13 of [7].

Theorem 4.1 ([7], Theorem 3.13.1)**.** *Let $S \subset [n-1]$. Then $\beta_n(S)$ is equal to the number of permutations $\tau \in \mathcal{L}(P, \omega)$ with descent set S.*

Recall that the Ehrhart polynomial of $\mathcal{O}(\mathcal{Z}_n)$ evaluated at m is equal to the order polynomial of $\mathcal{Z}_n$ evaluated at $m+1$ [5]. It follows from this fact and from Theorem 3.15.8 in [7] that the h^*-polynomial of $\mathcal{O}(\mathcal{Z}_n)$ is

$$h^*_{\mathcal{O}(\mathcal{Z}_n)}(t) = \sum_{S \subset [n-1]} \beta_n(S) t^{\# S}. \tag{1}$$

So, Theorem 1.1 will follow from Equation 1 and the following theorem, which is analogous to Theorem 3.13.1 in [7].

Theorem 4.2. *Let $S \subset [n-1]$. Then $\beta_n(S)$ is the number of alternating permutations ω with Swap$(\omega) = S$.*

To prove this theorem, for every $S = \{s_1, \dots, s_n\} \subset [n-1]$, we will find define a function ϕ_S that maps chains of order ideals of sizes $s_1, \dots, s_k$ to alternating permutations whose swap set is contained in S. Let $I_1, \dots, I_k$ be a chain of order ideals in $J(\mathcal{Z}_n)$ with sizes $\# I_j = s_j$. Let $\mathbf{w}_i$ be the vertex of $\mathcal{O}(\mathcal{Z}_n)$ that satisfies

$$\mathbf{w}_i(j) = \begin{cases} 0 & \text{if } j \in I_i \\ 1 & \text{if } j \notin I_i. \end{cases}$$

Define $\phi_S(I_1, \dots, I_k)$ to be the unique alternating permutation that maximizes inversion number over all alternating permutations whose vertex set contains $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$. This map is well-defined by Proposition 3.4.

Let ψ_S be the map that sends an alternating permutation ω with Swap$(\omega) \subset S$ to the chain of order ideals $(I_1, \dots, I_k)$ where each $I_j = \{\omega^{-1}(1), \dots, \omega^{-1}(s_j)\}$. Since every alternating permutation ω is a linear extension of $\mathcal{Z}_n$, each I_j obtained in this way is an order ideal. They form a chain by construction, so the map ψ_S is well-defined. We will show that ψ_S is the inverse of ϕ_S in the proof of Theorem 4.2.

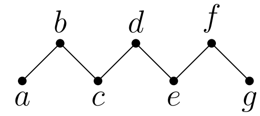

FIGURE 4.1. The zig-zag poset $\mathcal{Z}_7$

Example 4.3. Consider the zig-zag poset on seven elements $\mathcal{Z}_7$ pictured in Figure 4.1. Let $S = \{3, 6\}$, and let $I_1 = \{a, c, g\}$ and $I_2 = \{a, c, d, e, f, g\}$ be the given order ideals of sizes 3 and 6 respectively. Then the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are

$$\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Notice that these are the same vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ as in Example 3.5. So the unique alternating permutation $\phi_S(I_1, I_2)$ that maximizes inversion number over all alternating permutations whose vertex set contains $\{\mathbf{w}_1, \mathbf{w}_2\}$ is the same permutation as in Example 3.5,

$$\phi_S(I_1, I_2) = 3\ 7\ 2\ 6\ 4\ 5\ 1.$$

Note that $\mathrm{Swap}(3726451) = \{3\} \subset \{3, 6\} = S$.

We will recover our original order ideals I_1 and I_2 by finding $\psi_S(\omega)$. For clarity, we will treat ω as a map from $\{a, \ldots, g\}$ to $\{1, \ldots, 7\}$. The first order ideal of $\psi_S(\omega)$ consists of the inverse images of $1, 2$, and 3 in ω. That is,

$$I_1 = \{\omega^{-1}(1), \omega^{-1}(2), \omega^{-1}(3)\} = \{a, c, g\}.$$

The second order ideal of $\psi_S(\omega)$ consists of the inverse images of 1 through 6 in ω. So we obtain

$$I_2 = \{\omega^{-1}(1), \ldots, \omega^{-1}(6)\} = \{a, c, d, e, f, g\}.$$

Note that this is, in fact, the chain of order ideals with which we began.

Proof of Theorem 4.2. Let $S = \{s_1, \ldots, s_k\} \subset [n-1]$. We will show that $\alpha_n(S)$ is the number of alternating permutations whose swap set is contained in S by showing that the map ϕ_S described above is a bijection.

Let $I_1, \ldots, I_k$ be a chain of order ideals in $J(\mathcal{Z}_n)$ with sizes $\#I_j = s_j$. It is clear from the definitions of ϕ_S and ψ_S that

$$\psi_S(\phi_S(I_1, \ldots, I_k)) = (I_1, \ldots, I_k).$$

Since ϕ_S is injective, it suffices to show that ψ_S is also injective. We will show that $\phi_S(I_1, \ldots, I_k)$ is the only alternating permutation that maps to $(I_1, \ldots, I_k)$ under ψ_S. Since $\omega = \phi_S(I_1, \ldots, I_k)$ is the unique alternating permuation that maximizes inversion number over all alternating permutations with $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ in their vertex sets, any other alternating permutation σ that maps to $(I_1, \ldots, I_k)$ under ψ_S must have fewer inversions than ω.

Let σ be such a permutation. Since each inversion between the sets I_1, $\mathcal{Z}_n - I_k$ and $I_j - I_{j-1}$ for all $1 < j \leq k$ are fixed, the additional non-inversion must be contained in one of these sets. Without loss of generality, let this be $R = I_j - I_{j-1}$. Denote by $\sigma|_R$ the restriction of σ to the domain R. Let $(\sigma^{-1}(a), \sigma^{-1}(b))$ be the non-inversion of $\sigma|_R$ that is *not* required by the alternating structure. Then note that $\sigma^{-1}(a) + 1 < \sigma^{-1}(b)$. We claim that $\sigma|_R$ must have at least one swap position. To prove this, we will induct on $b - a$ in a similar manner as in the proof of Proposition 3.3. If $b - a = 1$, then a is a swap in $\sigma|_R$ since $\sigma^{-1}(a) + 1 < \sigma^{-1}(b)$.

Suppose $b - a > 1$. Consider the position of $a + 1$. If $\sigma^{-1}(a + 1) < \sigma^{-1}(a)$, then $(\sigma^{-1}(a + 1), \sigma^{-1}(b))$ is a non-inversion of $\sigma|_R$ that is not required by the alternating structure. Since $b - (a + 1) < b - a$, we are done by induction. If $\sigma^{-1}(a + 1) > \sigma^{-1}(b)$, then a can be swapped in $\sigma|_R$.

Consider the case where $\sigma^{-1}(a) < \sigma^{-1}(a + 1) < \sigma^{-1}(b)$. If $\sigma^{-1}(a) + 1 \neq \sigma^{-1}(a+1)$, then a can be swapped in $\sigma|_R$. Otherwise, if $\sigma^{-1}(a)+1 = \sigma^{-1}(a+1)$, it cannot be the case that $\sigma^{-1}(a+1)+1 = \sigma^{-1}(b)$ due to the alternating structure of σ. So $(\sigma^{-1}(a + 1), \sigma^{-1}(b))$ is a non-inversion of $\sigma|_R$ that is not required by the alternating structure and we are done by induction. So σ must have a swap position that is not equal to $s_1, \ldots, s_k$.

Therefore, ω is the only alternating permutation that can map to $(I_1, \ldots, I_k)$ under ψ_S, and ψ_S is the inverse map of ϕ_S. So $\alpha_n(S)$ is equal to the number of alternating permutations whose swap set is contained in S. By the Principle of Inclusion-Exclusion, $\beta_n(S)$ is the number of alternating permutations whose swap set is equal to S. $\square$

Theorem 1.1 follows as a corollary of Theorem 4.2.

Proof of Theorem 1.1. Equation 1 states that

$$h^*_{\mathcal{O}(\mathcal{Z}_n)}(t) = \sum_{S \subset [n-1]} \beta_n(S) t^{\#S}.$$

Theorem 4.2 tells us that $\beta_n(S)$ is the number of alternating permutations with swap set S. So the sum $\sum_{\#S=k} \beta_n(S)$ is the number of alternating permutations σ with $\mathrm{swap}(\sigma) = k$. So

$$h^*_{\mathcal{O}(\mathcal{Z}_n)}(t) = \sum_{\sigma} t^{\mathrm{swap}(\sigma)},$$

as needed. □

5. Combinatorial Properties of Swap Numbers

Let $s_n(k)$ denote the number of alternating permutations on n letters such that have exactly k swaps. We call these numbers the *swap numbers*. Theorem 1.1 shows that the h^*-polynomial of $\mathcal{O}(\mathcal{Z}_n)$ is

$$\sum_{k=0}^{n-1} s_n(k) t^k.$$

We are interested in interrogating these numbers. For example, it would be interesting to find an explicit formula for $s_n(k)$, though we have not been able to do this yet.

One straightforward property that becomes apparent looking at examples is that $s_n(n-1) = 0$. This is clear because it is not possible that every $k \in [n-1]$ is a swap. Indeed, otherwise k is to the left of $k+1$ for all $k \in [n-1]$ which implies that σ is the identity permutation, which is not alternating. Furthermore, $s_n(n-2) = 1$, since the unique alternating permutation with this many swaps is the one with $1, 2, \ldots, \lceil \frac{n}{2} \rceil$ in order in the odd numbered positions and $\lceil \frac{n}{2} \rceil + 1, \ldots, n$ in order in the even numbered positions. Similarly, $s_n(0) = 1$, because there is a unique alternating permutation with no swaps. It is the permutation $(n-1, n, n-3, n-2, n-5, n-4, \ldots)$.

Another property that is apparent from examples is summarized in the following:

Theorem 5.1. *The sequence $s_n(0), s_n(1), \ldots, s_n(n-2)$ is symmetric and unimodal.*

In fact, Theorem 5.1 and all the preceding properties will follow from the fact that $\mathcal{O}(\mathcal{Z}_n)$ is a Gorenstein polytope of index 3.

Definition 5.2. An integral polytope is *Gorenstein* if there is a positive integer m such that mP contains exactly one lattice point v in its relative interior, and for each facet-defining inequality $a^T x \leq b$, we have that $b - a^T v = 1$. The integer m is called the *index* of P.

The following relevant theorem concerning the h^* polynomials of Gorenstein polytopes with unimodular triangulations appears in [2].

Theorem 5.3. *Suppose that P is a Gorenstein polytope of dimension d and index m. Then $h^*_P(t)$ is a polynomial of degree $d-m+1$, whose coefficients form a symmetric sequence. Furthermore, the constant term of $h^*_P(t)$ is 1. If, in addition, P has a regular unimodular triangulation, then the coefficient sequence is unimodal.*

Proof of Theorem 5.1. It suffices to show that $\mathcal{O}(\mathcal{Z}_n)$ is a Gorenstein polytope of index three with a regular unimodular triangulation. The canonical triangulation of $\mathcal{O}(\mathcal{Z}_n)$ is a regular unimodular triangulation. To see that it satisfies the Gorenstein property with respect to $m = 3$, note that the defining inequalities for $3\mathcal{O}(\mathcal{Z}_n)$ are that $v_i \geq 0$ for i odd, $v_i \leq 3$ for i even, $v_{2i-1} \leq v_{2i}$ and $v_{2i+1} \leq v_{2i}$. The unique interior lattice point of $3\mathcal{O}(\mathcal{Z}_n)$ is the point v where $v_i = 1$ for i odd, and $v_i = 2$ for i even. Finally, this point has lattice distance 1 from each of the facet-defining inequalities. Hence $\mathcal{O}(\mathcal{Z}_n)$ is a Gorenstein polytope of index three with a regular unimodular triangulation and Theorem 5.3 can be applied to deduce that the coefficient sequence is symmetric and unimodal. □

While general principles provide a proof of the symmetry and unimodality of the sequence $s_n(0), s_n(1), \ldots, s_n(n-2)$, it would be interesting to find explicit combinatorial arguments that would produce these results. In particular, we let $A_{n,k}$ denote the set of alternating permutations on n letters with exactly k swaps, then it would be interesting to solve the following problems.

Problem 5.4.
(1) Find a bijection between $A_{n,k}$ and $A_{n,n-k-2}$.
(2) For each $0 \leq k \leq \lfloor (n-4)/2 \rfloor$ find an injective map from $A_{n,k}$ to $A_{n,k+1}$.

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