Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Quantifying quantum coherence based on the generalized α-z-relative Rényi entropy

by

Xue-Na Zhu, Zhi-Xiang Jin, and Shao-Ming Fei

Preprint no.: 59 2019

Quantifying quantum coherence based on the generalized $\alpha-z-$relative Rényi entropy

Xue-Na Zhu[1], Zhi-Xiang Jin[2], and Shao-Ming Fei[3,4]
[1]*School of Mathematics and Statistics Science, Ludong University, Yantai 264025, China*
[2]*School of Physics, University of Chinese Academy of Sciences, Yuquan Road 19A, Beijing 100049, China*
[3]*School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*
[4]*Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany*

We present a family of coherence quantifiers based on the generalized $\alpha-z-$relative Rényi entropy. These quantifiers satisfy all the standard criteria for well-defined measures of coherence, and include some existing coherence measures as special cases.

PACS numbers: 03.67.Mn, 03.65.Ud

I. INTRODUCTION

Coherence, being at the heart of interference phenomena, plays a central role in quantum physics as it enables applications that are impossible within classical mechanics or ray optics. Coherence is also a vital physical resource with various applications in biology [1–3], thermodynamical systems [4, 5], transport theory [6, 7] and nanoscale physics [8]. Recent developments in our understanding of quantum coherence [9–14] and nonclassical correlation have come from the burgeoning field of quantum information science. One important pillar of the field is the study on quantification of coherence.

In Ref. [15] the authors established a rigorous framework (BCP framework) for quantifying coherence. The BCP framework consists of the following postulates that any quantifier of coherence C should fulfill:

(C_1) Faithfulness: $C(\rho)\geq 0$, with equality if and only if ρ is incoherent.

(C_2) Monotonicity: C does not increase under the action of an incoherent operation, i.e.,

$$C[\Phi_{\mathcal{I}}(\rho)]\leq C(\rho),$$

for any incoherent operation $\Phi_{\mathcal{I}}$.

(C_3) Convexity: C is a convex function of the state, i.e.,

$$\sum_n p_n C(\rho_n)\geq C(\sum_n p_n\rho_n),$$

where $p_n\geq 0,\sum_n p_n=1$.

(C_4) Strong monotonicity: C does not increase on average under selective incoherent operations, i.e,

$$C(\rho)\geq\sum_n p_n C(\varrho_n),$$

with probabilities $p_n=tr(\mathcal{K}_n\rho\mathcal{K}_n^\dagger)$, post measurement states $\rho_n=\frac{\mathcal{K}_n\rho\mathcal{K}_n^\dagger}{p_n}$, and incoherent operators $\mathcal{K}_n$.

The authors of Ref. [16] provided a simple and interesting condition to replace (C3) and (C4) with the additivity of coherence for block-diagonal states,

$$C(p\rho_1\oplus(1-p)\rho_2)=pC(\rho_1)+(1-p)C(\rho_2), \tag{1}$$

for any $p\in[0,1]$, $\rho_i\in\varepsilon(\mathcal{H}_i)$, $i=1,2$, and $p\rho_1\oplus(1-p)\rho_2\in\varepsilon(\mathcal{H}_1\oplus\mathcal{H}_2)$, where $\varepsilon(\mathcal{H})$ denotes the set of density matrices on the Hilbert space $\mathcal{H}$.

For a given d-dimensional Hilbert space $\mathcal{H}$, let us fix an orthonormal basis $\{|i\rangle\}_{i=1}^d$. We call all density matrices that are diagonal in this basis incoherent and label this set of quantum states by $\mathcal{I}\in\mathcal{H}$. All density operators $\delta\in\mathcal{I}$ are of the form:

$$\delta=\sum_i p_i|i\rangle\langle i|,$$

where $p_i\geq 0$ and $\sum_i p_i=1$. Otherwise the states are coherent. Let Λ be a completely positive trace preserving (CPTP) map:

$$\Lambda(\rho)=\sum_i\mathcal{K}_n\rho\mathcal{K}_n^\dagger,$$

where $\{\mathcal{K}_n\}$ is a set of Kraus operators satisfying $\sum_n \mathcal{K}_n^\dagger \mathcal{K}_n = I_d$, with I_d the identity operator. If $\mathcal{K}_n^\dagger \mathcal{I} K_n \in \mathcal{I}$ for all n, we call $\{\mathcal{K}_n\}$ a set of incoherent Kraus operators, and the corresponding operation Λ an incoherent operational one.

II. THE FUNCTION $f_{\alpha,z}(\rho,\sigma)$

Quantifying coherence is a key task in both quantum mechanical theory and practical applications. In Ref. [17, 18] the following function has been presented,

$$f_{\alpha,z}(\rho,\sigma) = Tr(\sigma^{\frac{1-\alpha}{2z}}\rho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{2z}})^z, \tag{2}$$

for arbitrary two density matrices ρ and σ. Here, α, $z \in R$. To study the limit when $\alpha \to 1$ and $z \to 0$, the authors in Ref. [18] parameterized z in terms of α as $z = r(\alpha - 1)$, where r is a non-zero finite real number, and considered the limit when $\alpha \to 1$: $\lim_{\alpha\to 1} f_{\alpha,r(\alpha-1)}(\rho,\sigma) = \rho$. For fixed $\alpha \neq 1$, $z \to 0$ is exactly related to the anti Lie-Trotter problem [19].

For a finite dimensional Hilbert space $\mathcal{H}$, the set of linear operators is denoted by $\mathcal{L}(\mathcal{H})$. The adjoint of $X \in \mathcal{L}(\mathcal{H})$ is denoted by $X^\dagger$. For $X \in \mathcal{L}(\mathcal{H})$ and real $p \neq 0$, the norm $||X||_p$ is defined by [20],

$$||X||_p = (tr|X|^p)^{\frac{1}{p}},$$

where $|X| = \sqrt{X^\dagger X}$. Here, for a self-adjoint operator X, X^{-1} means the inverse restricted to $supp(X)$, so $X^{-1}X = XX^{-1}$ equals to the orthogonal projection on $supp(X)$.

The $H\ddot{o}lder's$ inequality belongs to a richer family of inequalities. For every $p_1, ..., p_k, r > 0$ with $\frac{1}{r} = \frac{1}{p_1} + ... + \frac{1}{p_k}$ one has [20]:

$$||X_1...X_k||_r \leq ||X_1||_{p_1}...||X_k||_{p_k}. \tag{3}$$

From this inequality and the fact that $||X^{-1}||_{-p} = ||X||_p^{-1}$, the following reverse $H\ddot{o}lder's$ inequality is derived. Let $r > 0$ and $p_1, ..., p_k$ be such that $\frac{1}{r} = \frac{1}{p_1} + ... + \frac{1}{p_k}$ and that exactly one of $p_i's$ is positive and the rests are negative [20]:

$$||X_1...X_k||_r \geq ||X_1||_{p_1}...||X_k||_{p_k}. \tag{4}$$

Moreover, equalities holds in (3) and (4) if and only if $|X_i|^{p_i}, i = 1, 2, ..., k$, are proportional.

Lemma 1 *For states ρ and σ,*

(1) If $0 < \alpha < 1$ and $z > 0$, we have

$$f_{\alpha,z}(\rho,\sigma) \leq 1;$$

(2) If $\alpha > 1$ and $z > 0$, we have

$$f_{\alpha,z}(\rho,\sigma) \geq 1.$$

(3) $f_{\alpha,z}(\rho,\sigma) = 1$ if and only if $\rho = \sigma$, for $\alpha \in (0,1) \cup (1,+\infty)$ and $z > 0$.

[Proof] Let $r = z$, $p_1 = \frac{2z}{1-\alpha}$, $p_2 = \frac{z}{\alpha}$, $X_1 = \sigma^{\frac{1}{p_1}}$, $X_2 = \rho^{\frac{1}{p_2}}$. When $\alpha \in (0,1)$ and $z > 0$, we have

$$\begin{aligned} f_{\alpha,z}(\rho,\sigma) &= tr(X_1X_2X_1)^z \\ &= tr(|X_1X_2X_1|)^r \\ &= (||X_1X_2X_1||_r)^r \\ &\leq (||X_1||_{p_1}||X_2||_{p_2}||X_1||_{p_1})^r \\ &= 1, \end{aligned} \tag{5}$$

where the second equality is due to $X_i^\dagger = X_i$ for $i = 1, 2$. From (3), we obtain the first inequality.

When $\alpha > 1$ and $z > 0$, we have

$$\begin{aligned} f_{\alpha,z}(\rho,\sigma) &= (||X_1X_2X_1||_r)^r \\ &\geq (||X_1||_{p_1}||X_2||_{p_2}||X_1||_{p_1})^r \\ &= 1, \end{aligned} \tag{6}$$

where the first inequality is due to (4).

In the above proof of inequalities (5) and (6), $||X_1X_2X_1||_r = ||X_1||_{p_1}||X_2||_{p_2}||X_1||_{p_1}$ if and only if $|X_1|^{p_1}$ and $|X_2|^{p_2}$ are proportional, i.e, there is a number k which satisfies $\sigma = k\rho$. Since $tr(\rho) = tr(\sigma) = 1$, then we obtain $k = 1$. □

Let $\mathcal{P}(\mathcal{H})$ be the set of positive semidefinite operators on $\mathcal{H}$. For non-normalized states ρ: $\forall\rho, \sigma \in \mathcal{P}(\mathcal{H})$ with $supp\,\rho \subseteq supp\,\sigma$, it has been defined in Ref. [18],

$$D_{\alpha,z}(\rho||\sigma) := \frac{1}{\alpha-1}\log\frac{f_{\alpha,z}(\rho,\sigma)}{tr\rho}. \tag{7}$$

For any states ρ, σ such that $supp\,\rho \subseteq supp\,\sigma$, and for any CPTP map Λ: $D_{\alpha,z}(\Lambda(\rho)||\Lambda(\sigma)) \leq D_{\alpha,z}(\rho||\sigma)$ holds in each of the following cases [18]:

- $\alpha \in (0,1]$ and $z \geq \max\{\alpha, 1-\alpha\}$;
- $\alpha \in [1,2]$ and $z = 1$;
- $\alpha \in [1,2]$ and $z = \frac{\alpha}{2}$.
- $\alpha \geq 1$ and $z = \alpha$.

For two states ρ and σ, one has $f_{\alpha,z}(\rho,\sigma) = e^{(\alpha-1)D_{\alpha,z}(\rho||\sigma)}$. Hence $f_{\alpha,z}(\rho,\sigma)$ has the following properties:

Lemma 2 *For any quantum states ρ and σ, such that $supp\,\rho \subseteq supp\,\sigma$, and for any CPTP map Λ, we have*

- *If $\alpha \in (0,1]$ and $z \geq \max\{\alpha, 1-\alpha\}$, then*

$$f_{\alpha,z}(\Lambda(\rho),\Lambda(\sigma)) \geq f_{\alpha,z}(\rho,\sigma);$$

- *If $\alpha \in [1,2]$ and $z \in \{1, \frac{\alpha}{2}\}$; or $\alpha \geq 1$ and $z = \alpha$, then*

$$f_{\alpha,z}(\Lambda(\rho),\Lambda(\sigma)) \leq f_{\alpha,z}(\rho,\sigma).$$

III. COHERENCE QUANTIFICATION

The coherence $C(\rho)$ in Ref. [21] can be expressed as

$$C(\rho) = 1 - \left[\max_{\sigma\in\mathcal{I}} f_{\frac{1}{2},1}(\rho,\sigma)\right]^2. \tag{8}$$

In Ref. [22] a bona fide measure of quantum coherence $C(\rho)$ has been presented by utilizing the Hellinger distance: $D_H(\rho,\sigma) = Tr(\sqrt{\rho}-\sqrt{\sigma})^2$,

$$\begin{aligned} C(\rho) &= \min_{\sigma\in\mathcal{I}} D_H(\rho,\sigma) \\ &= 2\left[1 - \max_{\sigma\in\mathcal{I}} f_{\frac{1}{2},1}(\rho,\sigma)\right], \end{aligned} \tag{9}$$

which is the coherence $C_{\frac{1}{2}}(\varepsilon|\rho)$ of Theorem 3 in Ref. [23].

In Ref. [23] the coherence has been quantified based on the Tsallis relative α entropy,

$$D'_\alpha(\rho||\sigma) = \frac{1}{\alpha-1}(f_{\alpha,1}(\rho,\sigma)-1). \tag{10}$$

But it was shown that it to violates the strong monotonicity, even though it can unambiguously distinguish the coherent state from the incoherent ones with the monotonicity. In Ref. [24] a family of coherence quantifiers has been presented, which are closely related to the Tsallis relative α entropy:

$$C'_\alpha(\rho) = \min_{\sigma\in\mathcal{I}} \frac{1}{\alpha-1}\left(f^{\frac{1}{\alpha}}_{\alpha,1}(\rho,\sigma)-1\right), \tag{11}$$

where $\alpha \in (0,2]$.

In the following we define a generalized $\alpha-z-$relative *Rényi* entropy:

$$D_{\alpha,z}(\rho,\sigma) = \frac{f^{\frac{1}{\alpha}}_{\alpha,z}(\rho,\sigma)-1}{\alpha-1}. \tag{12}$$

It is worthwhile noting that several coherence measures like relative entropy [15], geometric coherence [25], the sandwiched *Rényi* relative entropy [26] and max-relative entropy [9] are related to the generalized $\alpha-z-$relative *Rényi* entropy.

Based on the relation $f_{\alpha,z}(\rho,\sigma)$ and $D_{\alpha,z}(\rho,\sigma)$, and Lemma 2, we have

Corollary 1 *For any quantum states ρ and σ for which $supp\,\rho \subseteq supp\,\sigma$, and for any CPTP map Λ: $D_{\alpha,z}(\Lambda(\rho),\Lambda(\sigma)) \leq D_{\alpha,z}(\rho,\sigma)$ holds in each of the following case:*
- $\alpha \in (0,1]$ *and* $z \geq \max\{\alpha, 1-\alpha\}$;
- $\alpha \in [1,2]$ *and* $z = 1$;
- $\alpha \in [1,2]$ *and* $z = \frac{\alpha}{2}$;
- $\alpha \geq 1$ *and* $z = \alpha$.

With the above properties, based on the generalized $\alpha-z-$relative *Rényi* entropy we define the quantity: $C_{\alpha,z}(\rho) = \min_{\sigma\in\mathcal{I}} D_{\alpha,z}(\rho,\sigma)$. The following statement takes place.

Theorem 1 *The quantum coherence $C_{\alpha,z}(\rho)$ of a state ρ given by*

$$C_{\alpha,z}(\rho) = \min_{\sigma\in\mathcal{I}} D_{\alpha,z}(\rho,\sigma) \tag{13}$$

is a well-defined measure of coherence for the following case:
- $\alpha \in (0,1)$ *and* $z \geq \max\{\alpha, 1-\alpha\}$;
- $\alpha \in (1,2]$ *and* $z = 1$;
- $\alpha \in (1,2]$ *and* $z = \frac{\alpha}{2}$;
- $\alpha > 1$ *and* $z = \alpha$.

[Proof] Because of (2), (12) and (13), we have

$$C_{\alpha,z}(\rho) = \begin{cases} \frac{1-\max_{\sigma\in\mathcal{I}} f_{\alpha,z}^{\frac{1}{\alpha}}(\rho,\sigma)}{1-\alpha}, & 0<\alpha<1, \\ \frac{\min_{\sigma\in\mathcal{I}} f_{\alpha,z}^{\frac{1}{\alpha}}(\rho,\sigma)-1}{\alpha-1}, & \alpha>1. \end{cases}$$

From Lemma 1, we have $C_{\alpha,z}(\rho) \geq 0$, and $C_{\alpha,z}(\rho) = 0$ if and only if $\rho = \sigma$. Let σ be the optimal incoherent state such that $C_{\alpha,z}(\rho) = D_{\alpha,z}(\rho,\sigma)$. Taking into account Corollary 1, we have that $C_{\alpha,z}(\rho)$ does not increase under any incoherent operations.

Next we prove that $C_{\alpha,z}(\rho)$ satisfies Eq. (1). Suppose ρ is block-diagonal in the reference basis $\{|j\rangle\}_{j=1}^{d}$, $\rho = p_1\rho_1 \oplus p_2\rho_2$ with $p_1 \geq 0, p_2 \geq 0$, $p_1+p_2 = 1$, ρ_1 and ρ_2 are density operators. Let $\sigma = q_1\sigma_1 \oplus q_2\sigma_2$ with $q_1 \geq 0, q_2 \geq 0$, $q_1+q_2 = 1$, and σ_1,σ_2 are diagonal states similar to ρ_1,ρ_2, respectively.

Denote Δ either max or min. Set $t_i = \Delta_{\sigma_i} tr(\sigma_i^{\frac{1-\alpha}{2z}}\rho_i^{\frac{\alpha}{z}}\sigma_i^{\frac{1-\alpha}{2z}})^z$, $i = 1,2$. We have

$$\begin{aligned} &\Delta_{\sigma\in\mathcal{I}} tr(\sigma^{\frac{1-\alpha}{2z}}\rho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{2z}})^z \\ =\ &\Delta_{q_1,q_2}(q_1^{1-\alpha}p_1^{\alpha}t_1 + q_2^{1-\alpha}p_2^{\alpha}t_2). \end{aligned} \tag{14}$$

Due to the Hölder inequality with $0<\alpha<1$, we have

$$q_1^{1-\alpha}p_1^{\alpha}t_1 + q_2^{1-\alpha}p_2^{\alpha}t_2 \leq (\sum_{i=1,2} p_i t_i^{\frac{1}{\alpha}})^{\alpha},$$

where the equality holds if and only $q_1 = lp_1t_1^{\frac{1}{\alpha}}$ and $q_2 = lp_2t_2^{\frac{1}{\alpha}}$ with $l = \left[p_1t_1^{\frac{1}{\alpha}} + p_2t_2^{\frac{1}{\alpha}}\right]^{-1}$, i.e,

$$\max_{q_1,q_2}(q_1^{1-\alpha}p_1^{\alpha}t_1 + q_2^{1-\alpha}p_2^{\alpha}t_2) = (\sum_{i=1,2} p_i t_i^{\frac{1}{\alpha}})^{\alpha}. \tag{15}$$

Similarly, for the inequality with $\alpha>1$, we have

$$q_1^{1-\alpha}p_1^{\alpha}t_1 + q_2^{1-\alpha}p_2^{\alpha}t_2 \geq (\sum_{i=1,2} p_i t_i^{\frac{1}{\alpha}})^{\alpha}.$$

When $q_1 = lp_1t_1^{\frac{1}{\alpha}}$ and $q_2 = lp_2t_2^{\frac{1}{\alpha}}$, we obtain

$$\min_{q_1,q_2}(q_1^{1-\alpha}p_1^{\alpha}t_1 + q_2^{1-\alpha}p_2^{\alpha}t_2) = (\sum_{i=1,2} p_i t_i^{\frac{1}{\alpha}})^{\alpha}. \tag{16}$$

Combining (14), (15) and (16), we have

$$\begin{aligned}\Delta_{\sigma\in\mathcal{I}} f^{\frac{1}{\alpha}}_{\alpha,z}(\rho,\sigma) = \quad & p_1\Delta_{\sigma_1\in\mathcal{I}} f^{\frac{1}{\alpha}}_{\alpha,z}(\rho_1,\sigma_1)\\ & +p_2\Delta_{\sigma_2\in\mathcal{I}} f^{\frac{1}{\alpha}}_{\alpha,z}(\rho_2,\sigma_2).\end{aligned}$$

Thus, $C_{\alpha,z}$ satisfies additivity of coherence for block-diagonal states: $C_{\alpha,z}(p_1\rho_1\oplus p_1\rho_1) = p_1C_{\alpha,z}(\rho_1)+p_2C_{\alpha,z}(\rho_2)$. □

$C_{\alpha,z}(\rho)$ actually defines a family of coherence measures which includes several typical coherence measures.

• The coherence $C_{\alpha,z}(\rho)$ with $\alpha=\frac{1}{2}, z=1$, i.e, $C_{\frac{1}{2},1}(\rho)$ is the coherence $C(\rho)$ of (8) in Ref. [21].

• $\alpha\in(0,1)$ and $z=1$ the coherence $C_{\alpha,1}(\rho)$ is the coherence $C^{\alpha}_{a}(\rho)$ in Ref. [17], where the difference of a constant factor $\frac{1}{1-\alpha}$ in defining the coherence has already been taken into account.

• $\alpha\in(0,1)\cup(1,2]$ and $z=1$, the coherence $C_{\alpha,1}(\rho)$ is the coherence $C(\rho)$ in Ref. [24].

• $\alpha\in[\frac{1}{2},1)$ and $z=\alpha$; $\alpha>1$ and $z=\alpha$, the coherence $C_{\alpha,z}(\rho)$ is the coherence $C_{s,\alpha}(\rho)$ in Ref. [26].

In particular, from the relation between the α affinity of coherence [17] and $C_{\alpha,z}$, we have that $\frac{1}{2}C_{\frac{1}{2},1}(\rho)$ is just the error probability to discriminate $\{|\varphi\rangle_i,\eta_i\}_{i=1}^d$ with von Neumann measurement, where $|\varphi\rangle_i=\eta_i^{-\frac{1}{2}}\sqrt{\rho}|i\rangle$, $\eta_i=\rho_{ii}$ and $d=\sqrt{\rho}$. Furthermore, if ρ is an incoherent state, the coherence $C_{\frac{1}{2},1}(\rho)=0$, which means that a set of linearly independent pure states can be perfectly discriminated by the least square measurement.

IV. THE PROPERTIES OF $C_{\alpha,z}(\rho)$

From Theorem 1, $C_{a,1}(\rho)$ is a well-defined measure of coherence for $\alpha\in(0,1)\cup(1,2]$,

$$C_{\alpha,1} = \min_{\sigma\in\mathcal{I}}\left[\frac{f^{\frac{1}{\alpha}}(\rho,\sigma)-1}{\alpha-1}\right],$$

where $f(\rho,\sigma)=tr(\rho^{\alpha}\sigma^{1-\alpha})$, since for any pair of square matrices A and B, the eigenvalues of AB and BA are the same. For any incoherent state $\sigma=\sum_{k=1}^d\delta_{kk}|k\rangle\langle k|$, we have

$$\begin{aligned}tr(\sigma^{1-\alpha}\rho^{\alpha}) &= \sum_{k=1}^{d}\delta_{kk}^{1-\alpha}\langle k|\rho^{\alpha}|k\rangle\\ &= Q\sum_{k=1}^{d}\frac{\langle k|\rho^{\alpha}|k\rangle}{Q}\delta_{kk}^{1-\alpha},\end{aligned}$$

where $Q=\left(\sum_{k=1}^d\langle k|\rho^{\alpha}|k\rangle^{\frac{1}{\alpha}}\right)^{\alpha}$. Denote

$$\varepsilon(\alpha)=\begin{cases}-1, & 0<\alpha<1,\\ 1, & 1<\alpha.\end{cases}$$

According to the Hölder inequality and the converse Hölder inequality, we have

$$\begin{aligned}&\varepsilon(\alpha)\sum_{k=1}^{d}\frac{\langle k|\rho^{\alpha}|k\rangle}{Q}\delta_{kk}^{1-\alpha} \qquad (17)\\ \geq\ &\varepsilon(\alpha)\left(\sum_{k=1}^{d}\delta_{kk}\right)\left[\sum_{k=1}^{d}\left(\frac{\langle k|\rho^{\alpha}|k\rangle}{Q}\right)^{\frac{1}{\alpha}}\right]^{\alpha}\\ =\ &\varepsilon(\alpha),\end{aligned}$$

where the equality is attained when $\delta_{kk}^{1-\alpha}=\frac{\langle k|\rho^{\alpha}|k\rangle}{Q}$. Then one finds the following conclusion.

Corollary 2 *For $\alpha\in(0,1)\cup(1,2]$,*

$$C_{\alpha,1}(\rho)=\frac{\sum_{k=1}^d\langle k|\rho^{\alpha}|k\rangle^{\frac{1}{\alpha}}-1}{\alpha-1}.$$

And the maximal coherence can be achieved by the maximally coherent states.

That the maximal coherence can be achieved by the maximally coherent states for $C_{\alpha,1}(\rho)$, with $\alpha\in(0,1)\cup(1,2]$, can been seen in the following. Based on the eigen-decomposition of a d-dimensional state $\rho=\sum_{j=1}^{d}\lambda_j|\varphi\rangle_j\langle\varphi|$, with λ_j and $|\varphi\rangle_j$ representing the eigenvalue and eigenvectors, we have:

$$\begin{aligned}\varepsilon(\alpha)\sum_{k=1}^{d}\langle k|\rho^\alpha|k\rangle^{\frac{1}{\alpha}} &= \varepsilon(\alpha)\sum_{k=1}^{d}\left(\sum_{j=1}^{d}\lambda_j^\alpha|\langle\varphi_j|k\rangle|^2\right)^{\frac{1}{\alpha}}\\ &\le \varepsilon(\alpha)d^{\frac{\alpha-1}{\alpha}}\left[\sum_{k,j=1}^{d}\lambda_j^\alpha|\langle\varphi_j|k\rangle|^2\right]^{\frac{1}{\alpha}}\\ &= \varepsilon(\alpha)d^{\frac{\alpha-1}{\alpha}}\left[\sum_{j=1}^{d}\lambda_j^\alpha\right]^{\frac{1}{\alpha}},\end{aligned}$$

where the first inequality is due to

$$\sum_{k=1}^{n}\lambda_k x_k^p\begin{cases}\le\left(\sum_{k=1}^{n}\lambda_k\right)^{1-p}\left(\sum_{k=1}^{n}\lambda_k x_k\right)^p, & 0<p\le 1,\\ \ge\left(\sum_{k=1}^{n}\lambda_k\right)^{1-p}\left(\sum_{k=1}^{n}\lambda_k x_k\right)^p, & p>1,\end{cases}$$

with $x_k=\sum_{j=1}^{d}\lambda_j^\alpha|\langle\varphi_j|k\rangle|^2\ge 0, \lambda_k=1$ $(k=1,2,...,n)$ and $p=\frac{1}{\alpha}$. Then one can easily find that the upper bound of the coherence can be attained by the maximally coherent states $\rho_d=|\varphi\rangle\langle\varphi|$ with $|\varphi\rangle=\frac{1}{\sqrt{d}}\sum_j e^{i\phi_j}|j\rangle$, $C_{\alpha,1}(\rho_d)=\frac{d^{\frac{\alpha-1}{\alpha}}-1}{\alpha-1}$. □

Theorem 2 *For $\alpha\in(0,1)$, $\beta\in(1,2]$, $\gamma>1$, $\max\{\alpha,1-\alpha\}\le z_1\le 1$, $z_2\ge 1$, we have*

$$C_{\alpha,z_1}(\rho)\le C_{\alpha,1}(\rho)\le C_{\alpha,z_2}(\rho);\tag{18}$$

$$C_{\beta,\beta}(\rho)\le C_{\beta,1}(\rho)\le C_{\beta,\frac{\beta}{2}}(\rho);\tag{19}$$

And

$$C_{\gamma,\gamma}(\rho)\le\frac{\sum_{k=1}^{d}\langle k|\rho^\gamma|k\rangle^{\frac{1}{\gamma}}-1}{\gamma-1}.\tag{20}$$

[Proof] Set

$$\varepsilon(z_i)=\begin{cases}-1, & 0\le z_i\le 1,\\ 1, & z_i>1,\end{cases}$$

where $i=1,2$. According to the Araki-Lieb-Thirring inequality, for matrixes $A,B\ge 0$, $q\ge 0$ and for $0\le r\le 1$, the following inequality holds [28],

$$tr(A^rB^rA^r)^q\le tr(ABA)^{rq}.\tag{21}$$

While for $r\ge 1$, the inequality is reversed [28],

$$tr(A^rB^rA^r)^q\ge tr(ABA)^{rq}.\tag{22}$$

From (21) and (22), we have

$$\begin{aligned}\varepsilon(z_i)f_{\alpha,z_i}(\rho,\sigma) &= \varepsilon(z_i)tr(\sigma^{\frac{1-\alpha}{2z_i}}\rho^{\frac{\alpha}{z_i}}\sigma^{\frac{1-\alpha}{2z_i}})^{z_i}\\ &\le \varepsilon(z_i)tr(\sigma^{\frac{1-\alpha}{2}}\rho^\alpha\sigma^{\frac{1-\alpha}{2}})\\ &= \varepsilon(z_i)tr(\rho^\alpha\sigma^{1-\alpha})\\ &= \varepsilon(z_i)f_{\alpha,1}(\rho,\sigma).\end{aligned}$$

Combining (13) and $\alpha \in (0,1)$, we have $C_{\alpha,z_1}(\rho) \le C_{\alpha,1}(\rho) \le C_{\alpha,z_2}(\rho)$. (19) can be obtained in a similar way.

Since $\gamma > 1$, we have $f_{\gamma,\gamma}(\rho,\sigma) \le tr(\rho^\gamma \sigma^{1-\gamma})$. Similar to the proof of (17), $\min_{\sigma\in I} tr^{\frac{1}{r}}(\rho^\gamma\sigma^{1-\gamma}) = \sum_{k=1}^d \langle k|\rho^\gamma|k\rangle^{\frac{1}{\gamma}}$, we obtain (20). □

Example 1: Let us consider a single-qubit pure state,

$$\rho = \frac{1}{2}(I_2 + \sum_i c_i\sigma_i),$$

where $\sum_i c_i^2 = 1$, I_2 is the 2×2 identity matrix and σ_i $(i = 1,2,3)$ are Pauli matrices. By Ref. [17], one has

$$\max_{\sigma\in\mathcal{I}} tr^2(\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}) = \frac{1}{2}(1+|c_3|),$$

and

$$\max_{\sigma\in\mathcal{I}} tr^2(\sqrt{\rho}\sqrt{\sigma}) = \frac{1}{2}(1+c_3^2).$$

For the single-qubit pure state ρ, one has

$$\rho^{\frac{1}{4}} = \rho = \begin{pmatrix} \frac{1+c_3}{2} & \frac{c_1-ic_2}{2} \\ \frac{c_1+ic_2}{2} & \frac{1-c_3}{2} \end{pmatrix}. \tag{23}$$

Since $tr(\sigma^{\frac{1}{8}}\rho^{\frac{1}{4}}\sigma^{\frac{1}{8}})^2 = tr(\sigma^{\frac{1}{4}}\rho^{\frac{1}{4}})^2$, we now compute $\max_{\sigma\in\mathcal{I}}\left[tr(\sigma^{\frac{1}{4}}\rho^{\frac{1}{4}})^2\right]^2$. Suppose that $\sigma = \sum_i p_i|i\rangle\langle i|$ with $p_1+p_2 = 1$ and $0 \le p_1, p_2 \le 1$. We have

$$\begin{aligned}\sqrt{tr(\sigma^{\frac{1}{4}}\rho^{\frac{1}{4}})^2} &= \frac{1+c_3}{2}p_1^{\frac{1}{4}} + \frac{1-c_3}{2}p_2^{\frac{1}{4}} \\ &\le \left[\left(\frac{1+c_3}{2}\right)^{\frac{4}{3}} + \left(\frac{1-c_3}{2}\right)^{\frac{4}{3}}\right]^{\frac{3}{4}},\end{aligned}$$

by using the Hölder inequality and that the equality holds if and only $p_1 = c(\frac{1+c_3}{2})^{\frac{4}{3}}$ and $p_2 = c(\frac{1-c_3}{2})^{\frac{4}{3}}$ with $c = \left[(\frac{1-c_3}{2})^{\frac{4}{3}} + (\frac{1+c_3}{2})^{\frac{4}{3}}\right]^{-1}$. Therefore we have

$$\max_{\sigma\in\mathcal{I}}\left[tr(\sigma^{\frac{1}{4}}\rho^{\frac{1}{4}})^2\right]^2 = \left[\left(\frac{1+c_3}{2}\right)^{\frac{4}{3}} + \left(\frac{1-c_3}{2}\right)^{\frac{4}{3}}\right]^3.$$

Due to (13), we obtain

$$C_{\frac{1}{2},z}(\rho) = 2\left[1 - \max_{\sigma\in\mathcal{I}} tr^2(\sigma^{\frac{1}{4z}}\rho^{\frac{1}{2z}}\sigma^{\frac{1}{4z}})^z\right],$$

then we have

$$C_{\frac{1}{2},\frac{1}{2}}(\rho) = 1 - |c_3|,$$

$$C_{\frac{1}{2},1}(\rho) = 1 - c_3^2$$

and

$$C_{\frac{1}{2},2}(\rho) = 2 - 2\left[\left(\frac{1+c_3}{2}\right)^{\frac{4}{3}} + \left(\frac{1-c_3}{2}\right)^{\frac{4}{3}}\right]^3.$$

It is obvious that $C_{\frac{1}{2},\frac{1}{2}}(\rho) \le C_{\frac{1}{2},1}(\rho) \le C_{\frac{1}{2},2}(\rho)$, see Fig. 1.

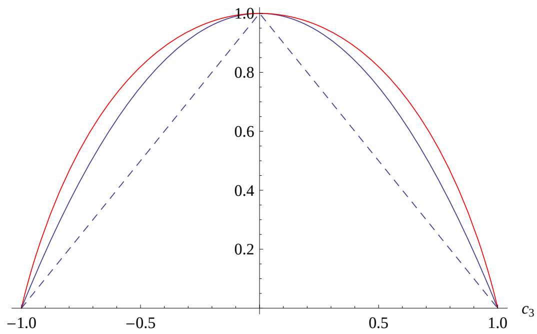

FIG. 1: The red dotted line is the vale of $C_{\frac{1}{2},2}(\rho)$; The blue solid line is the vale of $C_{\frac{1}{2},1}(\rho)$; The dashed line is the vale of $C_{\frac{1}{2},\frac{1}{2}}(\rho)$.

V. CONCLUSION

In summary, we have proposed four classes of coherence $C_{\alpha,z}(\rho)$ measures based on the generalized $\alpha - z-$relative R*é*nyi entropy. It has been proven that these coherence measures satisfy all the required criteria for a satisfactory coherence measure. Moreover, we have obtained the analytical formulas for special quantifiers with $z = 1$ and also studied relations among the four classes of coherence $C_{\alpha,z}(\rho)$.

Acknowledgments This work is supported by NSFC under numbers 11675113, 11605083, and Beijing Municipal Commission of Education (KZ201810028042).

---

[1] Lloyd, S.: Quantum coherence in biological systems. J. Phys. Conf. Ser. **302**, 012037 (2011)
[2] Li, C.M., Lambert, N., Chen, Y.N., Chen, G.Y., Nori, F.: Witnessing Quantum Coherence: from solid-state to biological systems. Sci. Rep. **2**, 885 (2012)
[3] Huelga, S.F., Plenio, M.B.: Vibrations, quanta and biology. Contemp. Phys. **54**, 181 (2013)
[4] Rybak, L., Amaran, S., Levin, L., Tomza, M., Moszynski, R., Kosloff, R., Koch, C.P., Amitay, Z.: Generating Molecular Rovibrational Coherence by Two-Photon Femtosecond Photoassociation of Thermally Hot Atoms. Phys. Rev. Lett. **107**, 273001 (2011)
[5] Lostaglio, M., Jennings, D., Rudolph, T.: Description of quantum coherence in thermodynamic processes requires constraints beyond free energy. Nat. Commun. **6**, 6383 (2015)
[6] Rebentrost, P., Mohseni, M., Aspuru-Guzik, A.: Role of Quantum Coherence and Environmental Fluctuations in Chromophoric Energy Transport. J. Phys. Chem. B **113**, 9942 (2009)
[7] Witt, B., Mintert, F.: Stationary quantum coherence and transport in disordered networks. New J. Phys. **15**, 093020 (2013)
[8] Karlströ*m*, O., Linke, H., Karlströ*m*, G., Wacker, A.: Increasing thermoelectric performance using coherent transport. Phys. Rev. B **84**, 113415 (2011)
[9] Bu, K.F., Singh, U., Fei, S.M., Pati, A.K., Wu, J.D.: Maximum Relative Entropy of Coherence: An Operational Coherence Measure. Phys. Rev. Lett. **119**, 150405 (2017)
[10] Winter, A., Yang, D.: Operational Resource Theory of Coherence. Phys. Rev. Lett. **116**, 120404 (2016)
[11] Yuan, X., Zhou, H., Cao, Z., Ma, X.: Intrinsic randomness as a measure of quantum coherence. Phys. Rev. A **92**, 022124 (2015)
[12] Napoli, C., Bromley, T.R., Cianciaruso, M., Piani, M., Johnston, N., Adesso, G.: Robustness of Coherence: An Operational and Observable Measure of Quantum Coherence. Phys. Rev. Lett. **116**, 150502 (2016)
[13] Qi, X., Gao, T., Yan, F.: Measuring coherence with entanglement concurrence. J. Phys. A: Math. Theor. **50**, 285301 (2017)
[14] Du, S., Bai, S., Qi, X.: Coherence measures and optimal conversion for coherent states. Quantum Inf. Comput. **15**, 1307 (2015)
[15] Baumgratz, T., Cramer, M., Plenio, M.B.: Quantifying Coherence. Phys. Rev. Lett. **113**, 140401 (2014)
[16] Yu, X.D., Zhang, D.J., Xu, G.F., Tong, D. M.: Alternative framework for quantifying coherence. Phys.Rev. A **94**, 060302 (2016)
[17] Xiong, C.H., Kumar, A., Wu, J.D.: Family of coherence measures and duality between quantum coherence and path distinguishability. Phys. Rev. A **98**, 032324 (2018)
[18] Audenaert, K.M.R., Datta, N.: $\alpha - z-$relative *Ŕ*enyi entropies. J. Math. Phys. **56**, 022202 (2015)
[19] Audenaert, K.M.R., Hiai, F.: Anti-Lie-Trotter formula. e-print arXiv:1412.7905 (2014)

[20] Salman, B.G.: Sandwiched Rényi Divergence Satisfies Data Processing Inequality. J. Math. Phys. **54**, 122202(2013)
[21] Yu, C.S.: Quantum Coherence via Skew Information and Its Polygamy. Phys. Rev. A **95**, 042337 (2017)
[22] Jin, Z.X., Fei, S. M.: Quantifying quantum coherence and non-classical correlation based on Hellinger distance. Phys. Rev. A **97**,062342 (2018)
[23] Rastegin, A. E.: Quantum-coherence quantifiers based on the Tsallis relative α entropies. Phys. Rev. A **93**, 032136 (2016)
[24] Zhao, H.Q., Yu, C.S.: Remedying the strong monotonicity of the coherence measure in terms of the Tsallis relative α entropy. Sci. Rep.**8**, b299(2018)
[25] Streltsov, A., Singh, U., Dhar, H.S., Bera, M.N., Adesso, G.: Measuring Quantum Coherence with Entanglement. Phys. Rev. Lett. **115**, 020403 (2015)
[26] Xu, J.W.: Cohence measures based on sandwiched Rényi relative entropy.(arXiv:1808.04662v2)
[27] Luo, S.L., Zhang, Q.: Informational distance on quantum-state space. Phys. Rev. A **69**, 032106 (2004)
[28] Audenaert, K.M.R.: On the Araki-Lieb-Thirring inequality. Int. J. Inf. Syst. Sci. 4, 78 (2008)