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by

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Abstract

The coherence-breaking channels play a significant role in quantum information theory. We study the coherence-breaking channels for qubit systems and give a method to amend the coherence-breaking channels by applying unitary operations. For given incoherent channel Φ, we give a necessary and sufficient condition for the channel to be a coherence-breaking channel and amend it via unitary operations. For incoherent channels Φ that are not coherence-breaking ones, we consider $\Phi \circ \Phi$ and give the conditions for coherence-breaking and channel amendment as well.

Key words: coherence-breaking channel, incoherent channel, coherence-breaking index.

1 Introduction

Originating from quantum superposition, quantum coherence has been a cornerstone of quantum information theory. It is of fundamental importance in quantum information processing such as quantum reference frames [1, 2, 3], transport in biological systems [4, 5] and quantum thermodynamics [6, 7]. In recent years, the resource theories of quantum coherence have been rapidly developed [8, 9, 10]. The free operations, the free states and the resource states are three basic ingredients in a quantum resource theory. In the resource theory of quantum coherence, the free states are the incoherent states whose density matrices are diagonal under the reference basis. The free operations are the incoherent operations Φ.

An important aspect in the study of resource theories is related to the evolution under the action of a channel. For example, in the entanglement resource theory, entanglement-breaking channels (EBCs) have been characterized completely [11, 12, 13]. In Ref. [14], Cuevas *et al.* amended EBCs by using unitary operations. Of special interest to us is the coherence-breaking channel and its amendment. Recently, in Ref. [15], Bu *et al.* introduced two kinds of coherence-breaking channels (CBCs). In addition, they devoted to the coherence-breaking indices of incoherent quantum channels

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and presented various examples to elucidate this concept. Based on these works, we give a specific expression of a qubit coherence-breaking channel (CBC) and try to amend it via unitary operations. In many practical situations, a quantum channel can be represented as the consecutive application of a given elementary map Φ repeated n times with n a positive integer. In this case, the full channel is given by $\Phi^n = \underbrace{\Phi \circ \cdots \circ \Phi}_{n\ times}$.

In this paper, we first give explicit expressions for incoherent channels Φ and Φ^2 being CBCs, respectively. Then we amend Φ and Φ^2 via unitary operations.

2 Preliminaries

In this section, we first introduce some relevant basic concepts that are required in presenting our main results.

For a d-dimensional quantum system and a fixed reference basis $\{|i\rangle\}$, the l_1 norm of coherence C_{l_1} of a state ρ is given as $C_{l_1}(\rho) = \sum\limits_{i\neq j} |\langle i|\rho|j\rangle|$. Any qubit state ρ can be written as $\rho = \frac{1}{2}(\mathbb{I} + \overrightarrow{\mathbf{r}} \cdot \overrightarrow{\sigma}) = \frac{1}{2}(\mathbb{I} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)$, where σ_x, σ_y and σ_z are Pauli matrices, $\vec{\mathbf{r}}$ is a 3-dimensional Bloch vector with $|\overrightarrow{\mathbf{r}}| \leq 1$.

A quantum channel Φ is a linear completely positive and trace preserving (CPTP) map [16]. The action of a qubit quantum channel Φ on a state ρ can be expressed by a real 4×4 matrix $\begin{pmatrix} 1 & \mathbf{0}_{1\times 3} \\ \overrightarrow{\mathbf{n}} & M \end{pmatrix}$ which transforms the vector $(1, r_x, r_y, r_z)$ to the one of $\Phi(\rho)$. Obviously, the action of a qubit quantum channel Φ is completely characterized by $(M, \overrightarrow{\mathbf{n}})$ and the iterated channel Φ^n is characterized by $(M^n, (\sum\limits_{k=0}^{n-1} M^k)\overrightarrow{\mathbf{n}})$.

A non-coherence-generating channel (NC) $\tilde{\Phi}$ is a CPTP map from an incoherent state to an incoherent state: $\tilde{\Phi}(\mathcal{I}) \subset \mathcal{I}$, where $\mathcal{I}$ denotes the set of incoherent states [17].

Any quantum channel Φ is called an incoherent channel if there exists a Kraus decomposition $\Phi(\cdot) = \Sigma_n K_n(\cdot)K_n^\dagger$ such that $\rho_n = \frac{K_n(\rho)K_n^\dagger}{\mathrm{Tr}(K_n(\rho)K_n^\dagger)}$ is incoherent for any incoherent state ρ. We call incoherent channel Φ a coherence-breaking channel (CBC) if $\Phi(\rho)$ is an incoherent state for any state ρ [15].

Let Φ be an incoherent channel, the coherence-breaking index $n(\Phi)$ of Φ is defined as [15]

$$n(\Phi) = \min\{n \geq 1 : \Phi^n \text{ is a coherence-breaking channel}\}. \tag{1}$$

A rank-2 qubit channel is an NC if and only if it has the Kraus decomposition either as [17]

$$\Phi^{(1)}(\cdot) = E_1^{(1)}(\cdot)E_1^{(1)\dagger} + E_2^{(1)}(\cdot)E_2^{(1)\dagger} \tag{2}$$

with

$$E_1^{(1)} = \begin{pmatrix} e^{i\eta}\cos\theta\cos\phi & 0 \\ -\sin\theta\sin\phi & e^{i\xi}\cos\phi \end{pmatrix}, \quad E_2^{(1)} = \begin{pmatrix} \sin\theta\cos\phi & e^{i\xi}\sin\phi \\ e^{-i\eta}\cos\theta\sin\phi & 0 \end{pmatrix} \tag{3}$$

or as

$$\Phi^{(2)}(\cdot) = E_1^{(2)}(\cdot)E_1^{(2)\dagger} + E_2^{(2)}(\cdot)E_2^{(2)\dagger} \tag{4}$$

with

$$E_1^{(2)} = \begin{pmatrix} \cos\theta & 0 \\ 0 & e^{i\xi}\cos\phi \end{pmatrix}, \quad E_2^{(2)} = \begin{pmatrix} 0 & \sin\phi \\ e^{i\xi}\sin\theta & 0 \end{pmatrix}, \tag{5}$$

where θ, ϕ, ξ and η are all real numbers. Here we note that $\Phi^{(1)}$ is not an incoherent channel unless $\sin\theta\cos\theta\sin\phi\cos\phi = 0$ and $\Phi^{(2)}$ is an incoherent channel.

3 Coherence-breaking channels

First we give a necessary and sufficient condition for an incoherent channel being a CBC. If an incoherent channel Φ is not a coherence-breaking channel, we can give a consecutive application of Φ repeated 2 times ($n(\Phi) = 2$), i.e., to find a necessary and sufficient condition for Φ^2 being a CBC. For the case of $n(\Phi) \geq 3$, we can get the analogous results similarly.

Lemma 3.1. *Let Φ be an incoherent channel defined by* (2)*, then Φ is a CBC iff* $\cos\theta = 0$.

Proof. Any density operator acting on a two-dimensional quantum system can be generally written as

$$\rho = \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix}, \tag{6}$$

where $|a|^2 + |b|^2 \leq 1$. Substitute (6) into (2), we have

$$\Phi(\rho) = \begin{pmatrix} A & B \\ B^* & 1-A \end{pmatrix}, \tag{7}$$

where $A = a\cos^2\phi + (b^* e^{i\xi} + be^{-i\xi})\sin\theta\sin\phi\cos\phi + (1-a)\sin^2\phi$ and $B = be^{i\eta - i\xi}\cos\theta\cos^2\phi + b^* e^{i\xi + i\eta}\cos\theta\sin^2\phi$. Then we find $\Phi(\rho) \in \mathcal{I}$ iff $B = be^{i\eta - i\xi}\cos\theta\cos^2\phi + b^* e^{i\xi + i\eta}\cos\theta\sin^2\phi = 0$ for arbitrary b. Let $b = |b|e^{\beta}$, where $\beta \in [0, 2\pi)$. Thus, $B = 0$ iff $e^{-i\eta}|b|\cos\theta(e^{i\xi - i\beta}\cos^2\phi + e^{i\beta - i\xi}\sin^2\phi) = 0$ for arbitrary β and $|b|$. Particularly, set $\beta = \xi$, we find $\cos\theta = 0$. It is easy to see that Φ is an incoherent channel when $\cos\theta = 0$. □

Lemma 3.2. *Let Φ be an incoherent channel defined by* (4)*, then Φ is a CBC iff* $\sin\theta = 0,\ \cos\phi = 0$ *or* $\cos\theta = 0,\ \sin\phi = 0$.

Proof. Substitute (6) into (4), we have

$$\Phi(\rho) = \begin{pmatrix} C & D \\ D^* & 1-C \end{pmatrix}, \tag{8}$$

where $C = a\cos^2\theta + (1-a)\sin^2\phi$ and $D = e^{i\xi}(b\cos\theta\cos\phi + b^*\sin\theta\sin\phi)$. Then we find Φ is a CBC iff $D = 0$ for arbitrary b. Let $b = |b|e^{\beta}$. Then Φ is a CBC iff $|b|\sqrt{\cos^2\beta\cos^2(\theta-\phi) + \sin^2\beta\cos^2(\theta+\phi)} = 0$ for arbitrary $|b|$ and β iff $\sin\theta = 0,\ \cos\phi = 0$ or $\cos\theta = 0,\ \sin\phi = 0$. □

Assuming that a given incoherent channel Φ is not a CBC, we give a necessary and sufficient conditions for Φ^2 to be a CBC.

As any qubit channel Φ that maps $\rho = \frac{1}{2}(\mathbb{I}+r_x\sigma_x+r_y\sigma_y+r_z\sigma_z)$ to $\Phi(\rho) = \frac{1}{2}(\mathbb{I}+r'_x\sigma_x+r'_y\sigma_y+r'_z\sigma_z)$ can be give by

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ n_x & M_{11} & M_{12} & M_{13} \\ n_y & M_{21} & M_{22} & M_{23} \\ n_z & M_{31} & M_{32} & M_{33} \end{pmatrix}, \tag{9}$$

such that $\Phi(1, r_x, r_y, r_z)^t = (1, r'_x, r'_y, r'_z)^t$. Thus, $\Phi(\rho) \in \mathcal{I}$ for arbitrary single-qubit state ρ iff $M_{ij} = 0$ for $i = 1, 2$ and $j = 1, 2, 3$.

Lemma 3.3. *Let Φ be an incoherent channel defined by* (2)*, then $n(\Phi) = 2$ iff $\cos 2\phi = 0,\ \sin\theta = 0$ and $\sin\xi \sin\eta + \cos\xi \cos\eta = 0$.*

Proof. It is easy to see $\sin\theta\cos\theta\sin\phi\cos\phi = 0$. Comparing (2),(3) with (9), we find $n_x = n_y = n_z = M_{13} = M_{23} = 0$, $M_{11} = \cos\theta(\cos\eta\cos\xi + \sin\eta\sin\xi\cos 2\phi)$, $M_{12} = \cos\theta(\sin\eta\cos\xi\cos 2\phi - \cos\eta\sin\xi)$, $M_{21} = \cos\theta(\cos\eta\sin\xi\cos 2\phi - \sin\eta\cos\xi)$, $M_{22} = \cos\theta(\cos\eta\cos\xi\cos 2\phi + \sin\eta\sin\xi)$, $M_{31} = 2\sin\theta\sin\phi\cos\phi\cos\xi$, $M_{32} = -2\sin\theta\sin\phi\cos\phi\sin\xi$ and $M_{33} = \cos 2\phi$. Thus, Φ^2 is a CBC iff $M_{11}^2 + M_{12}M_{21} = 0$, $M_{12}(M_{11} + M_{22}) = 0$, $M_{21}(M_{11} + M_{22}) = 0$, $M_{12}M_{21} + M_{22}^2 = 0$ and $\cos\theta \neq 0$ iff $\cos 2\phi = 0,\ \cos\theta \neq 0$ and $\sin\xi\sin\eta + \cos\xi\cos\eta = 0$. Then we have $n(\Phi) = 2$ iff $\cos 2\phi = 0,\ \sin\theta = 0$ and $\sin\xi\sin\eta + \cos\xi\cos\eta = 0$. □

Lemma 3.4. *Let Φ be an incoherent channel defined by* (4)*, then $n(\Phi) = 2$ iff one of the following three conditions holds.*

(i) $\cos 2\xi = 0$ *and* $\cos(\theta + \phi) = \cos(\theta - \phi) \neq 0$*;*

(ii) $\cos(\theta - \phi) = 0,\ \cos\xi = 0$ *and* $\cos(\theta + \phi) \neq 0$*;*

(iii) $\cos(\theta + \phi) = 0,\ \cos\xi = 0$ *and* $\cos(\theta - \phi) \neq 0$.

Proof. Comparing (4),(5) with (9), we find $n_x = n_y = M_{13} = M_{23} = M_{31} = M_{32} = 0$, $n_z = \sin^2\phi - \sin^2\theta$, $M_{11} = \cos\xi\cos(\theta - \phi)$, $M_{12} = \sin\xi\cos(\theta + \phi)$, $M_{21} = -\sin\xi\cos(\theta - \phi)$, $M_{22} = -\cos\xi\cos(\theta+\phi)$ and $M_{33} = \cos^2\theta - \sin^2\phi$. Thus, $n(\Phi) = 2$ iff $M_{11}^2 + M_{12}M_{21} = 0$, $M_{12}(M_{11} + M_{22}) = 0$, $M_{21}(M_{11} + M_{22}) = 0$, $M_{12}M_{21} + M_{22}^2 = 0$, $\sin\theta\cos\phi \neq 0$ and $\cos\theta\sin\phi \neq 0$, which implying one of the three conditions above holds. □

4 Amending coherence-breaking channels

In this section, we discuss the amendment of CBCs. We show that a CBC Φ can be amended via unitary operations Λ_α through $\Lambda_\alpha \circ \Phi$. For the case $n(\Phi) = 2$, the channel can be amended by unitary operations through $\Phi \circ \Lambda_\alpha \circ \Phi$.

Lemma 4.1. *Let Φ be a CBC defined by* (2)*, there always exists a unitary operation Λ_α that can amend Φ by $\Lambda_\alpha \circ \Phi$.*

Proof. Any general unitary operation can be written as $\Lambda_\alpha(\cdot) = \begin{pmatrix} \cos\alpha & -e^{i\alpha_1}\sin\alpha \\ e^{i\alpha_2}\sin\alpha & e^{i\alpha_3}\cos\alpha \end{pmatrix}(\cdot)$ $\begin{pmatrix} \cos\alpha & e^{-i\alpha_2}\sin\alpha \\ -e^{-i\alpha_1}\sin\alpha & e^{-i\alpha_3}\cos\alpha \end{pmatrix}$. Comparing it with (9), we get

$$\Lambda_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & N_{11} & N_{12} & N_{13} \\ 0 & N_{21} & N_{22} & N_{23} \\ 0 & N_{31} & N_{32} & N_{33} \end{pmatrix}, \tag{10}$$

where $N_{11} = \cos^2\alpha\cos\alpha_3 - \sin^2\alpha\cos(\alpha_1-\alpha_2)$, $N_{12} = \sin^2\alpha\sin(\alpha_1-\alpha_2) - \cos^2\alpha\sin\alpha_3$, $N_{13} = \sin 2\alpha\cos\alpha_2$, $N_{21} = \cos^2\alpha\sin\alpha_3 + \sin^2\alpha\sin(\alpha_1-\alpha_2)$, $N_{22} = \sin^2\alpha\cos(\alpha_1-\alpha_2) + \cos^2\alpha\cos\alpha_3$, $N_{23} = \sin 2\alpha\sin\alpha_2$, $N_{31} = -\sin 2\alpha\cos\alpha_1$, $N_{32} = \sin 2\alpha\sin\alpha_1$ and $N_{33} = \cos 2\alpha$. Then we get

$$\Lambda_\alpha \circ \Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & N_{13}M_{31} & N_{13}M_{32} & N_{13}M_{33} \\ 0 & N_{23}M_{31} & N_{23}M_{32} & N_{23}M_{33} \\ 0 & N_{33}M_{31} & N_{33}M_{32} & N_{33}M_{33} \end{pmatrix}, \tag{11}$$

where M_{ij} are defined in Lemma 3.1 with $i, j = 1, 2, 3$. Thus, $\Lambda \circ \Phi$ is not a CBC iff $N_{13} \neq 0$ or $N_{23} \neq 0$ iff $\sin 2\alpha \neq 0$. In other words, Λ can amend Φ iff $\sin 2\alpha \neq 0$. □

Lemma 4.2. *Let Φ be a CBC defined by* (4)*, there always exists a unitary operation Λ_α that can amend Φ^2 by $\Lambda_\alpha \circ \Phi$.*

Proof. Similar to the proof of Lemma 4.1, we find Λ_α can amend Φ iff $\sin 2\alpha \neq 0$. □

Lemma 4.3. *Let Φ be a CBC defined by* (2) *and $n(\Phi) = 2$, there always exists a unitary operation Λ_α that can amend Φ^2 by $\Phi \circ \Lambda_\alpha \circ \Phi$.*

Proof. It is easy to see

$$\Phi \circ \Lambda_\alpha \circ \Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \tilde{M}_{11} & \tilde{M}_{12} & 0 \\ 0 & \tilde{M}_{21} & \tilde{M}_{22} & 0 \\ 0 & \tilde{M}_{31} & \tilde{M}_{32} & 0 \end{pmatrix}, \tag{12}$$

where $\tilde{M}_{11} = M_{11}(M_{11}N_{11} + M_{12}N_{21}) + M_{21}(M_{11}N_{12} + M_{12}N_{22}) + M_{31}(M_{11}N_{13} + M_{12}N_{23})$, $\tilde{M}_{12} = M_{12}(M_{11}N_{11} + M_{12}N_{21}) + M_{22}(M_{11}N_{12} + M_{12}N_{22}) + M_{32}(M_{11}N_{13} + M_{12}N_{23})$, $\tilde{M}_{21} = M_{11}(M_{21}N_{11} + M_{22}N_{21}) + M_{21}(M_{21}N_{12} + M_{22}N_{22}) + M_{31}(M_{21}N_{13} + M_{22}N_{23})$, $\tilde{M}_{22} = M_{12}(M_{21}N_{11} + M_{22}N_{21}) + M_{22}(M_{21}N_{12} + M_{22}N_{22}) + M_{32}(M_{21}N_{13} + M_{22}N_{23})$, $\tilde{M}_{31} = M_{11}(M_{31}N_{11} + M_{32}N_{21}) + M_{21}(M_{31}N_{12} + M_{32}N_{22}) + M_{31}(M_{31}N_{13} + M_{32}N_{23})$, $\tilde{M}_{32} = M_{12}(M_{31}N_{11} + M_{32}N_{21}) + M_{22}(M_{31}N_{12} + M_{32}N_{22}) + M_{32}(M_{31}N_{13} + M_{32}N_{23})$ and M_{ij} and N_{ij} are defined in Lemmas 3.3 and 4.1, respectively. □

Assume $\sin\alpha = 0$, then we find $\tilde{M}_{ij} = 0$ iff $\sin\alpha_3 = 0$, where $i = 1, 2$ and $j = 1, 2, 3$. In other words, if $\sin\alpha = 0$ and $\sin\alpha_3 \neq 0$, Φ^2 is amended.

Lemma 4.4. *Let Φ be a CBC defined by* (4) *and* $n(\Phi) = 2$, *there always exists a unitary operation* Λ_α *that can amend* Φ^2 *by* $\Phi \circ \Lambda_\alpha \circ \Phi$.

Proof. Similar to Lemma 4.3, we obtain

$$\Phi \circ \Lambda_\alpha \circ \Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tilde{n}_x & \tilde{M}_{11} & \tilde{M}_{12} & 0 \\ \tilde{n}_y & \tilde{M}_{21} & \tilde{M}_{22} & 0 \\ \tilde{n}_z & \tilde{M}_{31} & \tilde{M}_{32} & 0 \end{pmatrix}, \tag{13}$$

where $\tilde{n}_x = n_z(M_{11}N_{13} + M_{12}N_{23})$, $\tilde{n}_y = n_z(M_{21}N_{13} + M_{22}N_{23})$, $\tilde{n}_z = n_z M_{33}N_{33}$, $\tilde{M}_{11} = M_{11}(M_{11}N_{11} + M_{12}N_{21}) + M_{21}(M_{11}N_{12} + M_{12}N_{22})$, $\tilde{M}_{12} = M_{12}(M_{11}N_{11} + M_{12}N_{21}) + M_{22}(M_{11}N_{12} + M_{12}N_{22})$, $\tilde{M}_{13} = M_{33}(M_{11}N_{13} + M_{12}N_{23})$, $\tilde{M}_{21} = M_{11}(M_{21}N_{11} + M_{22}N_{21}) + M_{21}(M_{21}N_{12} + M_{22}N_{22})$, $\tilde{M}_{22} = M_{12}(M_{21}N_{11}+M_{22}N_{21})+M_{22}(M_{21}N_{12}+M_{22}N_{22})$, $\tilde{M}_{23} = M_{33}(M_{21}N_{13}+M_{22}N_{23})$, $\tilde{M}_{31} = M_{11}M_{33}N_{31}+M_{21}M_{33}N_{32}$, $\tilde{M}_{32} = M_{12}M_{33}N_{31} + M_{22}M_{33}N_{32}$, $\tilde{M}_{33} = M_{33}^2 N_{33}$ and $M_{ij},\ N_{ij}$ are from Lemmas 3.4 and 4.1, respectively.

(i) $\cos 2\xi = 0$ and $\cos(\theta + \phi) = \cos(\theta - \phi) \neq 0$

Assume $\sin\alpha = 0$, then we find $\tilde{M}_{ij} = 0$ iff $\sin\alpha_3 = 0$, where $i = 1, 2$ and $j = 1, 2, 3$. In other words, if $\sin\alpha = 0$ and $\sin\alpha_3 \neq 0$, Φ^2 is amended.

(ii) $\cos(\theta - \phi) = 0,\ \cos\xi = 0$ and $\cos(\theta + \phi) \neq 0$

In this case,

$$\Phi \circ \Lambda \circ \Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tilde{n}_x & 0 & M_{12}^2 N_{21} & M_{12}M_{33}N_{23} \\ \tilde{n}_y & 0 & 0 & 0 \\ \tilde{n}_z & 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

Thus, $\Phi \circ \Lambda \circ \Phi$ is not a CBC iff $N_{21} \neq 0$ or $N_{23} \neq 0$.

(iii) $\cos(\theta + \phi) = 0,\ \cos\xi = 0$ and $\cos(\theta - \phi) \neq 0$

Similar to (ii), we have for $N_{12} \neq 0$ or $N_{13} \neq 0$, the channel Φ can be amended. □

Now we give examples to illustrate our results about the coherence-breaking channel's amendment.

Example 4.1 Consider an incoherent qubit quantum channel Φ characterized by $(M, \vec{\mathbf{n}})$ with $M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu \end{pmatrix}$, where μ is a real number and $\vec{\mathbf{n}} = (0, 0, 0)^\top$. For $|\mu| \leq 1$, Φ is an incoherent channel [18, 19]. It is easy to see that Φ is a CBC. Set a unitary operation as $\Lambda(\cdot) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}(\cdot)\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ with $\sin 2\alpha \neq 0$. We obtain that $\Lambda \circ \Phi$ is not a CBC, i.e., the channel Φ is amended.

Example 4.2 Consider an incoherent qubit quantum channel Φ characterized by $(M, \vec{\mathbf{n}})$ with

$M = \begin{pmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where γ is a real number and $\vec{\mathbf{n}} = (0,0,0)^\top$. For $|\gamma| \leq 1$, Φ is an incoherent channel [18, 19]. It is easy to see that $n(\Phi) = 2$. Set a unitary operation as $\Lambda(\cdot) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha_3} \end{pmatrix}(\cdot)\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\alpha_3} \end{pmatrix}$ with $\sin\alpha_3 \neq 0$. Then we find $\Phi \circ \Lambda \circ \Phi$ is not a CBC, i.e., the channel is amended.

5 Conclusions

We have discussed the qubit CBCs and give the expressions for the case $n(\Phi) = 1, 2$. Furthermore, for the cases $n(\Phi) = 1$and $n(\Phi) = 2$, we can always find unitary operations Λ_α to amend the channel by $\Lambda_\alpha \circ \Phi$ and $\Phi \circ \Lambda_\alpha \circ \Phi$, respectively. For $n(\Phi) \geq 3$, following similar discussion, there also exist unitary operations to amend the channel Φ by $\underbrace{\Phi \circ \Lambda_\alpha \circ \cdots \circ \Lambda_\alpha \circ \Phi}_{n\ \Phi}$. In addition, for a qudit quantum channel Φ, the channel Φ can be amended by $\underbrace{\Phi \circ \Lambda_\alpha \circ \cdots \circ \Lambda_\alpha \circ \Phi}_{n\ \Phi}$ with proper unitary operations Λ_α when $n(\Phi) = n$ with n a positive integer.

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