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Abstract Uncertainty relation is a core issue in quantum mechanics and quantum information theory. We introduce modified generalized Wigner-Yanase-Dyson (MGWYD) skew information and modified weighted generalized Wigner-Yanase-Dyson (MWGWYD) skew information, and establish new uncertainty relations in terms of the MGWYD skew information and MWGWYD skew information.

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1. Introduction

Let H be a separable complex Hilbert space and $B(H)$, $S(H)$ and $D(H)$ the set of all bounded linear operators, Hermitian operators and density operators on H, respectively. An operator $A \in B(H)$ is called a *trace-class* operator if

$$\|A\|_1 := \sum_{n\in I} \langle e_n||A||e_n\rangle < \infty$$

for some orthonormal basis $\{e_n\}_{n\in I}$ of H, where $|A| = (A^\dagger A)^{\frac{1}{2}}$. In this case the *trace* of A is defined as $\mathrm{Tr}(A) = \sum_{n\in I}\langle e_n|A|e_n\rangle$. We denote the set of all trace-class operators on H by $L^1(H)$. An operator $A \in B(H)$ is called a *Hilbert-Schmidt* operator if

$$\|A\|_2 := (\sum_{n\in I} \langle e_n|A^\dagger A|e_n\rangle)^{\frac{1}{2}} < \infty$$

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for some orthonormal basis $\{e_n\}_{n\in I}$ of H. We denote the set of all Hilbert-Schmidt operators on H by $L^2(H)$.

For a density operator $\rho \in D(H)$ and an observable $A \in S(H)$, the *Wigner-Yanase* (WY) skew information is defined by [1]

$$\mathrm{I}_\rho(A) := -\frac{1}{2}\mathrm{Tr}([\rho^{\frac{1}{2}}, A]^2), \tag{1}$$

where $[X, Y] := XY - YX$ is the commutator of X and Y. A more general quantity was suggested by Dyson,

$$\mathrm{I}_\rho^\alpha(A) := -\frac{1}{2}\mathrm{Tr}([\rho^\alpha, A][\rho^{1-\alpha}, A]), \quad 0 \leq \alpha \leq 1, \tag{2}$$

which is now called the *Wigner-Yanase-Dyson* (WYD) skew information. (2) was further generalized to [2]

$$\mathrm{I}_\rho^{\alpha,\beta}(A) = -\frac{1}{2}\mathrm{Tr}([\rho^\alpha, A][\rho^\beta, A]\rho^{1-\alpha-\beta}), \quad \alpha, \beta \geq 0,\ \alpha + \beta \leq 1, \tag{3}$$

which is termed the *generalized Wigner-Yanase-Dyson* (GWYD) skew information. It is easy to see that when $\alpha + \beta = 1$, Eq. (3) reduces to Eq. (2), and Eq. (2) reduces to Eq. (1) when $\alpha = \frac{1}{2}$.

Another generalization of WYD skew information is given as follows [3]:

$$\mathrm{K}_\rho^\alpha(A) = -\frac{1}{2}\mathrm{Tr}\left(\left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, A_0\right]^2\right), \ 0 \leq \alpha \leq 1, \tag{4}$$

where $A_0 = A - \mathrm{Tr}(\rho A)I$. We call $\mathrm{K}_\rho^\alpha(A)$ the *weighted Wigner-Yanase-Dyson skew information* in the following. Noting that $\mathrm{I}_\rho(A) = \mathrm{I}_\rho(A_0)$, when $\alpha = \frac{1}{2}$, Eq. (4) also reduces to Eq. (1) in this case.

Remarkable properties of WYD skew information and GWYD skew information are revealed, and various types of uncertainty relations based on WY skew information, WYD skew information and GWYD skew information are studied during the past few years [4]-[19]. Particularly, uncertainty relations based on WY skew information and WYD skew information with quantum memory are investigated recently [20]-[21]. Besides skew information, uncertainty relations based on other quantities such as entropy, variance, statistical distance, quantum coherence have been extensively studied with experimental demonstrations. It is well known that the observables and Hamiltonians in quantum mechanics are assumed to be Hermitian operators mathematically. However, it is argued that non-Hermitian quantum mechanics may also be an interesting framework [30]. Moreover, other important operators such as quantum gates [31], generalized quantum gates [38] and the Kraus operators of a quantum channel [31] are not necessarily Hermitian. Therefore, it is natural to consider the corresponding definitions of the different types of the skew information mentioned above for pseudo-Hermitian and/or PT-symmetric quantum mechanics [32, 33, 34, 35, 36, 37].

For a density operator $\rho \in D(H)$ and an operator $A \in L^2(H)$ (not necessarily Hermitian), a generalization of the quantity in Eq. (1) is defined by [39],

$$|\mathrm{I}_\rho|(A) := -\frac{1}{2}\mathrm{Tr}([\rho^{\frac{1}{2}}, A^\dagger][\rho^{\frac{1}{2}}, A]), \tag{5}$$

which we refer to *modified Wigner-Yanase* (MWY) skew information.

Similarly, a generalization of the quantity in Eq. (2) is defined by [40]

$$|\mathrm{I}_\rho^\alpha|(A) := -\frac{1}{2}\mathrm{Tr}([\rho^\alpha, A^\dagger][\rho^{1-\alpha}, A]), \quad 0 \leq \alpha \leq 1, \tag{6}$$

for any $A \in L^2(H)$ and $\rho \in D(H)$, which we call *modified Wigner-Yanase-Dyson* (MWYD) skew information.

And a generalization of the quantity in Eq. (4) is given by [41]

$$|\mathrm{K}_\rho^\alpha|(A) = -\frac{1}{2}\mathrm{Tr}\left(\left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, A_0^\dagger\right]\left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, A_0\right]\right), \quad 0 \leq \alpha \leq 1, \tag{7}$$

for any $A \in L^2(H)$ and $\rho \in D(H)$, which we call *modified weighted Wigner-Yanase-Dyson* (MWWYD) skew information.

In [39], the authors established Heisenberg type uncertainty relation and a Schrödinger-type uncertainty relation based on MWY skew information. The definitions and properties of MWYD skew information were discussed in [40], and the uncertainty relations for MWY skew information and MWYD skew information were extensively studied in [42]-[43]. Moreover, the uncertainty relations for MWWYD skew information was given in [41]. Recently, the authors in [44] introduced some related quantities, and derived some generalizations of Schrödinger's uncertainty and Heisenberg uncertainty relation described by MWYD skew information.

In this paper, we first introduce the concepts of modified generalized Wigner-Yanase-Dyson (MGWYD) skew information and modified weighted generalized Wigner-Yanase-Dyson (MWGWYD) skew information as the generalizations of the quantities defined in Eq. (3) and (7), and discuss their properties in detail in Section 2. Furthermore, we provide new uncertainty relations based on these two new quantities in Section 3. Some concluding remarks are given in Section 4.

2. MGWYD and MWGWYD skew information

We first define the MGWYD skew information for an operator $A \in L^2(H)$ (not necessarily Hermitian) and $\rho \in D(H)$ as follows:

$$|\mathrm{I}_\rho^{\alpha,\beta}|(A) = -\frac{1}{2}\mathrm{Tr}([\rho^\alpha, A^\dagger][\rho^\beta, A]\rho^{1-\alpha-\beta}), \quad \alpha, \beta \geq 0, \ \alpha + \beta \leq 1. \tag{8}$$

Correspondingly we define

$$|\mathrm{J}_\rho^{\alpha,\beta}|(A) = \frac{1}{2}\mathrm{Tr}(\{\rho^\alpha, A_0^\dagger]\}\{\rho^\beta, A_0\}\rho^{1-\alpha-\beta}), \quad \alpha, \beta \geq 0, \ \alpha + \beta \leq 1. \tag{9}$$

where $\{X, Y\} := XY + YX$ is the anti-commutator of X and Y. It follows from the definitions that

$$|\mathrm{I}_\rho^{\alpha,\beta}|(A) = \frac{1}{2}[\mathrm{Tr}(\rho A^\dagger A) + \mathrm{Tr}(\rho^{\alpha+\beta} A \rho^{1-\alpha-\beta} A^\dagger) - \mathrm{Tr}(\rho^{1-\beta} A^\dagger \rho^\beta A) - \mathrm{Tr}(\rho^\alpha A \rho^{1-\alpha} A^\dagger)]$$

and

$$|\mathrm{J}_\rho^{\alpha,\beta}|(A) = \frac{1}{2}[\mathrm{Tr}(\rho A^\dagger A) + \mathrm{Tr}(\rho^{\alpha+\beta} A \rho^{1-\alpha-\beta} A^\dagger) + \mathrm{Tr}(\rho^{1-\beta} A^\dagger \rho^\beta A) + \mathrm{Tr}(\rho^\alpha A \rho^{1-\alpha} A^\dagger)].$$

We also need the following definitions of related quantities.

Definition 1 For $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, $A, B \in L^2(H)$ and $\rho \in D(H)$, we define the following quantities:

(i) $|\mathrm{Cov}_\rho^{\alpha,\beta}|(A, B) = \frac{1}{2}[\mathrm{Tr}(\rho A^\dagger B) + \mathrm{Tr}(\rho^{\alpha+\beta} A \rho^{1-\alpha-\beta} B^\dagger) - \mathrm{Tr}(\rho B) Tr(\rho A^\dagger) - \mathrm{Tr}(\rho A) Tr(\rho B^\dagger)]$;

(ii) $|\mathrm{Var}_\rho^{\alpha,\beta}|(A) = \frac{1}{2}[\mathrm{Tr}(\rho A^\dagger A) + \mathrm{Tr}(\rho^{\alpha+\beta} A \rho^{1-\alpha-\beta} A^\dagger) - \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho A^\dagger) - \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho A^\dagger)]$;

(iii) $|\mathrm{Corr}_\rho^{\alpha,\beta}|(A, B) = \frac{1}{2}[\mathrm{Tr}(\rho A^\dagger B) + \mathrm{Tr}(\rho^{\alpha+\beta} A \rho^{1-\alpha-\beta} B^\dagger) - \mathrm{Tr}(\rho^{1-\beta} A^\dagger \rho^\beta B) - \mathrm{Tr}(\rho^\alpha A \rho^{1-\alpha} B^\dagger)]$;

(iv) $|\mathrm{C}_\rho^{\alpha,\beta}|(A, B) = \frac{1}{2}[\mathrm{Tr}(\rho^{1-\beta} A^\dagger \rho^\beta B) + \mathrm{Tr}(\rho^\alpha A \rho^{1-\alpha} B^\dagger)]$;

(v) $|\mathrm{C}_\rho^{\alpha,\beta}|(A) = |\mathrm{C}_\rho^{\alpha,\beta}|(A, A) = \frac{1}{2}[\mathrm{Tr}(\rho^{1-\beta} A^\dagger \rho^\beta A) + \mathrm{Tr}(\rho^\alpha A \rho^{1-\alpha} A^\dagger)]$;

(vi) $|\mathrm{U}_\rho^{\alpha,\beta}|(A) = \sqrt{|\mathrm{Var}_\rho^{\alpha,\beta}|(A)^2 - [|\mathrm{Var}_\rho^{\alpha,\beta}|(A) - |\mathrm{I}_\rho^{\alpha,\beta}|(A)]^2}$.

The following proposition follows immediately from the above definitions.

Proposition 1 For $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, $A \in L^2(H)$ and $\rho \in D(H)$, it holds that

(i) $|\mathrm{I}_\rho^{\alpha,\beta}|(A) = |\mathrm{I}_\rho^{\beta,\alpha}|(A)$, $|\mathrm{J}_\rho^{\alpha,\beta}|(A) = |\mathrm{J}_\rho^{\beta,\alpha}|(A)$;

(ii) $|\mathrm{I}_\rho^{\alpha,\beta}|(A^\dagger) = |\mathrm{I}_\rho^{\alpha,\beta}|(A)$, $|\mathrm{J}_\rho^{\alpha,\beta}|(A^\dagger) = |\mathrm{J}_\rho^{\alpha,\beta}|(A)$;

(iii) $|\mathrm{I}_\rho^{\alpha,\beta}|(A) = |\mathrm{Corr}_\rho^{\alpha,\beta}|(A, A)$.

In order to obtain the main results in the next section, we first study the properties of the MGWYD skew information and the related quantities defined above.

Proposition 2 Let $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, $A, B \in L^2(H)$, $\rho \in D(H)$, $A_0 = A - \mathrm{Tr}(\rho A)I$, $B_0 = B - \mathrm{Tr}(\rho B)I$. We have

(i) $|\mathrm{Corr}_\rho^{\alpha,\beta}|(A, B) = |\mathrm{Corr}_\rho^{\alpha,\beta}|(A_0, B_0) = |\mathrm{Cov}_\rho^{\alpha,\beta}|(A, B) - |\mathrm{C}_\rho^{\alpha,\beta}|(A_0, B_0)$;

(ii) $|\mathrm{I}_\rho^{\alpha,\beta}|(A) = |\mathrm{I}_\rho^{\alpha,\beta}|(A_0) = |\mathrm{Var}_\rho^{\alpha,\beta}|(A) - |\mathrm{C}_\rho^{\alpha,\beta}|(A_0) = \frac{1}{2}[\mathrm{Tr}(\rho A^\dagger A) + \mathrm{Tr}(\rho^{\alpha+\beta} A \rho^{1-\alpha-\beta} A^\dagger) - \mathrm{Tr}(\rho^{1-\beta} A^\dagger \rho^\beta A) - \mathrm{Tr}(\rho^\alpha A \rho^{1-\alpha} A^\dagger)]$;

(iii) $|\mathrm{J}_\rho^{\alpha,\beta}|(A) = |\mathrm{Var}_\rho^{\alpha,\beta}|(A) + |\mathrm{C}_\rho^{\alpha,\beta}|(A_0) = \frac{1}{2}[\mathrm{Tr}(\rho A^\dagger A) + \mathrm{Tr}(\rho^{\alpha+\beta} A \rho^{1-\alpha-\beta} A^\dagger) + \mathrm{Tr}(\rho^{1-\beta} A^\dagger \rho^\beta A) + \mathrm{Tr}(\rho^\alpha A \rho^{1-\alpha} A^\dagger)]$;

(iv) $|\mathrm{U}_\rho^{\alpha,\beta}|(A) = \sqrt{|\mathrm{I}_\rho^{\alpha,\beta}|(A)|\mathrm{J}_\rho^{\alpha,\beta}|(A)}$;

(v) $0 \leq |\mathrm{I}_\rho^{\alpha,\beta}|(A) \leq |\mathrm{U}_\rho^{\alpha,\beta}|(A) \leq |\mathrm{Var}_\rho^{\alpha,\beta}|(A)$.

Proof. We first prove (i). It is direct to check that

$$
\begin{aligned}
&|\mathrm{Corr}_\rho^{\alpha,\beta}|(A_0,B_0)\\
&=\frac{1}{2}[\mathrm{Tr}(\rho(A-\mathrm{Tr}(\rho A)I)^\dagger B)+\mathrm{Tr}(\rho^{\alpha+\beta}(A-\mathrm{Tr}(\rho A)I)\rho^{1-\alpha-\beta}B^\dagger)\\
&-\mathrm{Tr}(\rho^{1-\beta}(A-\mathrm{Tr}(\rho A)I)^\dagger\rho^\beta B)-\mathrm{Tr}(\rho^\alpha(A-\mathrm{Tr}(\rho A)I)\rho^{1-\alpha}B^\dagger)]\\
&=\frac{1}{2}[\mathrm{Tr}(\rho(A-Tr(\rho A)I)^\dagger B)+\mathrm{Tr}(\rho^{\alpha+\beta}(A-\mathrm{Tr}(\rho A)I)\rho^{1-\alpha-\beta}B^\dagger)\\
&-\mathrm{Tr}(\rho^{1-\beta}(A-\mathrm{Tr}(\rho A)I)^\dagger\rho^\beta B)-\mathrm{Tr}(\rho^\alpha(A-\mathrm{Tr}(\rho A)I)\rho^{1-\alpha}B^\dagger)]\\
&=\frac{1}{2}[(\mathrm{Tr}(\rho A^\dagger B)-\mathrm{Tr}(\rho B)\mathrm{Tr}(\rho A^\dagger)-\overline{\mathrm{Tr}(\rho A)}\mathrm{Tr}(\rho B)+\overline{\mathrm{Tr}(\rho A)}\mathrm{Tr}(\rho B))\\
&+(\mathrm{Tr}(\rho^{\alpha+\beta}A\rho^{1-\alpha-\beta}B^\dagger)-\overline{\mathrm{Tr}(\rho B)}\mathrm{Tr}(\rho A)-\mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B^\dagger)+\mathrm{Tr}(\rho A)\overline{\mathrm{Tr}(\rho B)})\\
&-(\mathrm{Tr}(\rho^{1-\beta}A^\dagger\rho^\beta B)-\mathrm{Tr}(\rho B)\mathrm{Tr}(\rho A^\dagger)-\overline{\mathrm{Tr}(\rho A)}\mathrm{Tr}(\rho B)+\overline{\mathrm{Tr}(\rho A)}\mathrm{Tr}(\rho B))\\
&-(\mathrm{Tr}(\rho^\alpha A\rho^{1-\alpha}B^\dagger)-\overline{\mathrm{Tr}(\rho B)}\mathrm{Tr}(\rho A)-\mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B^\dagger)+\mathrm{Tr}(\rho A)\overline{\mathrm{Tr}(\rho B)})]\\
&=\frac{1}{2}[\mathrm{Tr}(\rho A^\dagger B)+\mathrm{Tr}(\rho^{\alpha+\beta}A\rho^{1-\alpha-\beta}B^\dagger)-\mathrm{Tr}(\rho^{1-\beta}A^\dagger\rho^\beta B)-\mathrm{Tr}(\rho^\alpha A\rho^{1-\alpha}B^\dagger)]\\
&=|\mathrm{Corr}_\rho^{\alpha,\beta}|(A,B).
\end{aligned}
$$

Meanwhile we have that

$$
\begin{aligned}
|\mathrm{Corr}_\rho^{\alpha,\beta}|(A_0,B_0) &= \frac{1}{2}[\mathrm{Tr}(\rho A_0^\dagger B_0)+\mathrm{Tr}(\rho^{\alpha+\beta}A_0\rho^{1-\alpha-\beta}B_0^\dagger)]-|\mathrm{C}_\rho^{\alpha,\beta}|(A_0,B_0)\\
&= |\mathrm{Cov}_\rho^{\alpha,\beta}|(A,B)-|\mathrm{C}_\rho^{\alpha,\beta}|(A_0,B_0).
\end{aligned}
$$

Hence, (i) holds. Then (ii) can be easily obtained. (iii) can be proved analogously. Similar to the proof of Theorem 2 in [44], (iv) and (v) can be deduced similarly. □

Proposition 3 Let $\alpha,\beta\ge 0$, $\alpha+\beta\le 1$, $A,B\in L^2(H)$, $\rho\in D(H)$, $A_0=A-\mathrm{Tr}(\rho A)I$ and $B_0=B-\mathrm{Tr}(\rho B)I$. For a spectral decomposition of $\rho=\sum_m\lambda_m|\psi_m\rangle\langle\psi_m|$, denote $a_{mn}=\langle\psi_m|A_0|\psi_n\rangle$. We have

(i) $|\mathrm{Cov}_\rho^{\alpha,\beta}|(A,B)=\frac{1}{2}\sum_{mn}\lambda_m^{\alpha+\beta}(\lambda_m^{1-\alpha-\beta}\overline{a_{nm}}b_{nm}+\lambda_n^{1-\alpha-\beta}a_{mn}\overline{b_{mn}})$. In particular, $|\mathrm{Var}_\rho^{\alpha,\beta}|(A)=\frac{1}{2}\sum_{mn}\lambda_m^{\alpha+\beta}(\lambda_m^{1-\alpha-\beta}|a_{nm}|^2+\lambda_n^{1-\alpha-\beta}|a_{mn}|^2)$;

(ii) $|\mathrm{Corr}_\rho^{\alpha,\beta}|(A,B)=\frac{1}{2}\sum_{mn}(\lambda_m-\lambda_m^{1-\beta}\lambda_n^\beta)\overline{a_{nm}}b_{nm}+(\lambda_m^{\alpha+\beta}\lambda_n^{1-\alpha-\beta}-\lambda_m^\alpha\lambda_n^{1-\alpha})a_{mn}\overline{b_{mn}}=\frac{1}{2}\sum_{mn}\lambda_m^\alpha(\lambda_m^\beta-\lambda_n^\beta)[\lambda_m^{1-\alpha-\beta}\overline{a_{nm}}b_{nm}+\lambda_n^{1-\alpha-\beta}a_{mn}\overline{b_{mn}}]$;

(iii) $|\mathrm{I}_\rho^{\alpha,\beta}|(A)=\frac{1}{2}\sum_{mn}\lambda_m^\alpha(\lambda_m^\beta-\lambda_n^\beta)[\lambda_m^{1-\alpha-\beta}|a_{nm}|^2+\lambda_n^{1-\alpha-\beta}|a_{mn}|^2]=\frac{1}{2}\sum_{mn}(\lambda_m^\alpha-\lambda_n^\alpha)(\lambda_m^\beta-\lambda_n^\beta)\lambda_n^{1-\alpha-\beta}|a_{mn}|^2=\frac{1}{2}\sum_{m<n}(\lambda_m^\alpha-\lambda_n^\alpha)(\lambda_m^\beta-\lambda_n^\beta)[\lambda_m^{1-\alpha-\beta}|a_{nm}|^2+\lambda_n^{1-\alpha-\beta}|a_{mn}|^2]$;

(iv) $|\mathrm{J}_\rho^{\alpha,\beta}|(A)=\frac{1}{2}\sum_{mn}\lambda_m^\alpha(\lambda_m^\beta+\lambda_n^\beta)[\lambda_m^{1-\alpha-\beta}|a_{nm}|^2+\lambda_n^{1-\alpha-\beta}|a_{mn}|^2]=\frac{1}{2}\sum_{mn}(\lambda_m^\alpha+\lambda_n^\alpha)(\lambda_m^\beta+\lambda_n^\beta)\lambda_n^{1-\alpha-\beta}|a_{mn}|^2\ge\frac{1}{2}\sum_{m<n}(\lambda_m^\alpha+\lambda_n^\alpha)(\lambda_m^\beta+\lambda_n^\beta)[\lambda_m^{1-\alpha-\beta}|a_{nm}|^2+\lambda_n^{1-\alpha-\beta}|a_{mn}|^2]$.

Proof. Direct calculation shows that

$$\mathrm{Tr}(\rho A_0^\dagger B_0)=\sum_{mn}\lambda_m\overline{a_{nm}}b_{nm},$$

$$\mathrm{Tr}(\rho^{\alpha+\beta}A_0^\dagger\rho^{1-\alpha-\beta}B_0)=\sum_{mn}\lambda_m^{\alpha+\beta}\lambda_n^{1-\alpha-\beta}a_{mn}\overline{b_{mn}},$$

$$\mathrm{Tr}(\rho^{1-\beta}A_0^\dagger\rho^\beta B_0) = \sum_{mn}\lambda_m^{1-\beta}\lambda_n^\beta\overline{a_{nm}}b_{nm},$$
$$\mathrm{Tr}(\rho^{\alpha}A_0^\dagger\rho^{1-\alpha} B_0) = \sum_{mn}\lambda_m^{\alpha}\lambda_n^{1-\alpha}a_{mn}\overline{b_{mn}},$$

and we can thus obtain (i) and (ii) immediately. Consequently for (iii), we have

$$|\mathrm{I}_\rho^{\alpha,\beta}|(A) = |\mathrm{Corr}_\rho^{\alpha,\beta}|(A,A) = \frac{1}{2}\sum_{mn}\lambda_m^\alpha(\lambda_m^\beta-\lambda_n^\beta)[\lambda_m^{1-\alpha-\beta}|a_{nm}|^2+\lambda_n^{1-\alpha-\beta}|a_{mn}|^2].$$

Moreover, we can rewrite $|\mathrm{I}_\rho^{\alpha,\beta}|(A)$ as

$$|\mathrm{I}_\rho^{\alpha,\beta}|(A) = \frac{1}{2}\sum_{mn}(\lambda_m^\alpha-\lambda_n^\alpha)(\lambda_m^\beta-\lambda_n^\beta)\lambda_n^{1-\alpha-\beta}|a_{mn}|^2$$

or

$$|\mathrm{I}_\rho^{\alpha,\beta}|(A) = \frac{1}{2}\sum_{m<n}(\lambda_m^\alpha-\lambda_n^\alpha)(\lambda_m^\beta-\lambda_n^\beta)[\lambda_m^{1-\alpha-\beta}|a_{nm}|^2+\lambda_n^{1-\alpha-\beta}|a_{mn}|^2].$$

(iv) can be proved in a similar way. This completes the proof. □

Now, we define the *modified weighted generalized Wigner-Yanase-Dyson* (MWG-WYD) skew information for an operator $A\in L^2(H)$ (not necessarily Hermitian) and $\rho\in D(H)$ as follows:

$$|\mathrm{K}_\rho^{\alpha,\beta}|(A) = -\frac{1}{2}\mathrm{Tr}\left(\left[\frac{\rho^\alpha+\rho^\beta}{2},A_0^\dagger\right]\left[\frac{\rho^\alpha+\rho^\beta}{2},A_0\right]\rho^{1-\alpha-\beta}\right),\quad \alpha,\beta\geq 0,\ \alpha+\beta\leq 1. \tag{10}$$

A related quantity $|L_\rho^{\alpha,\beta}|(A)$ is defined as

$$|\mathrm{L}_\rho^{\alpha,\beta}|(A) = \frac{1}{2}\mathrm{Tr}\left(\left\{\frac{\rho^\alpha+\rho^\beta}{2},A_0^\dagger\right\}\left\{\frac{\rho^\alpha+\rho^\beta}{2},A_0\right\}\rho^{1-\alpha-\beta}\right),\quad \alpha,\beta\geq 0,\ \alpha+\beta\leq 1. \tag{11}$$

The properties of the above two quantities are summarized in the following two propositions.

Proposition 4 Let $\alpha,\beta\geq 0$, $\alpha+\beta\leq 1$, $A\in L^2(H)$ and $\rho\in D(H)$. The following statements hold:

(i) $|\mathrm{K}_\rho^{\alpha,\beta}|(A) = |\mathrm{K}_\rho^{\beta,\alpha}|(A)$;

(ii) $|\mathrm{K}_\rho^{\alpha,\beta}|(A^\dagger) = |\mathrm{K}_\rho^{\alpha,\beta}|(A)$;

(iii) $|\mathrm{K}_\rho^{\alpha,\beta}|(A) \geq |\mathrm{I}_\rho^{\alpha,\beta}|(A)$.

Proof. (i) and (ii) can be easily verified from the definition. We now prove (iii). By Proposition 1 (i), we have

$$\begin{aligned}
|\mathrm{K}_\rho^{\alpha,\beta}|(A) &= -\frac{1}{2}\mathrm{Tr}([\frac{\rho^\alpha+\rho^\beta}{2},A_0^\dagger][\frac{\rho^\alpha+\rho^\beta}{2},A_0]\rho^{1-\alpha-\beta}) \\
&= -\frac{1}{8}\mathrm{Tr}([\rho^\alpha,A_0^\dagger][\rho^\alpha,A_0]\rho^{1-\alpha-\beta}+[\rho^\beta,A_0^\dagger][\rho^\beta,A_0]\rho^{1-\alpha-\beta})+\frac{1}{4}(|\mathrm{I}_\rho^{\alpha,\beta}|(A)+|\mathrm{I}_\rho^{\beta,\alpha}|(A)) \\
&= -\frac{1}{8}\mathrm{Tr}([\rho^\alpha,A_0^\dagger][\rho^\alpha,A_0]\rho^{1-\alpha-\beta}+[\rho^\beta,A_0^\dagger][\rho^\beta,A_0]\rho^{1-\alpha-\beta})+\frac{1}{2}|\mathrm{I}_\rho^{\alpha,\beta}|(A).
\end{aligned}$$

Suppose that the spectral decomposition of ρ is $\rho = \sum_m \lambda_m |\psi_m\rangle\langle\psi_m|$ and denote $a_{mn} = \langle\psi_m|A_0|\psi_n\rangle$. Then we obtain

$$\begin{aligned}
\mathrm{Tr}([\rho^\alpha, A_0^\dagger][\rho^\alpha, A_0]\rho^{1-\alpha-\beta}) &= \mathrm{Tr}(\rho^\alpha A_0^\dagger - A_0^\dagger\rho^\alpha)(\rho^\alpha A_0 - A_0\rho^\alpha)\rho^{1-\alpha-\beta} \\
&= \mathrm{Tr}(2\rho^\alpha A_0^\dagger \rho^\alpha A_0 - \rho^{2\alpha} A_0^\dagger A_0 - \rho^{2\alpha} A_0 A_0^\dagger)\rho^{1-\alpha-\beta} \\
&= \mathrm{Tr}(2\rho^{1-\beta} A_0^\dagger \rho^\alpha A_0 - \rho^{1+\alpha-\beta} A_0^\dagger A_0 - \rho^{1+\alpha-\beta} A_0 A_0^\dagger) \\
&= \sum_{mn}(2\lambda_n^{1-\beta}\lambda_m^\alpha - \lambda_n^{1+\alpha-\beta} - \lambda_m^{1+\alpha-\beta})|a_{mn}|^2,
\end{aligned}$$

and

$$\mathrm{Tr}([\rho^\alpha, A_0^\dagger][\rho^\alpha, A_0]\rho^{1-\alpha-\beta}) = \sum_{mn}(2\lambda_n^{1-\alpha}\lambda_m^\beta - \lambda_n^{1-\alpha+\beta} - \lambda_m^{1-\alpha+\beta})|a_{mn}|^2.$$

By Proposition 3 (iii), we get

$$\begin{aligned}
|\mathrm{K}_\rho^{\alpha,\beta}|(A) &= -\frac{1}{8}(\lambda_m^{1+\alpha-\beta} + \lambda_n^{1+\alpha-\beta} - 2\lambda_n^{1-\beta}\lambda_m^\alpha + \lambda_m^{1-\alpha+\beta} + \lambda_n^{1-\alpha+\beta} - 2\lambda_n^{1-\alpha}\lambda_m^\beta) \\
&\quad + \frac{1}{4}\sum_{mn}(\lambda_m^\alpha - \lambda_n^\alpha)(\lambda_m^\beta - \lambda_n^\beta)\lambda_n^{1-\alpha-\beta}|a_{mn}|^2.
\end{aligned}$$

Since $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$ and $0 \leq \lambda_m, \lambda_n \leq 1$, we have $\lambda_m^{1+\alpha-\beta} \geq \lambda_m^{2\alpha}$, $\lambda_m^{1-\alpha+\beta} \geq \lambda_m^{2\beta}$ and $\lambda_n^{1-\alpha-\beta} \leq 1$, and thus

$$\lambda_m^{1+\alpha-\beta} + \lambda_m^{1+\alpha-\beta} \geq (\lambda_m^{2\alpha} + \lambda_m^{2\beta})\lambda_n^{1-\alpha-\beta},$$

which implies that

$$\begin{aligned}
&\lambda_m^{1+\alpha-\beta} + \lambda_n^{1+\alpha-\beta} - 2\lambda_n^{1-\beta}\lambda_m^\alpha + \lambda_m^{1-\alpha+\beta} + \lambda_n^{1-\alpha+\beta} - 2\lambda_n^{1-\alpha}\lambda_m^\beta \\
&\geq \lambda_m^{2\alpha}\lambda_n^{1-\alpha-\beta} + \lambda_n^{1+\alpha-\beta} - 2\lambda_n^{1-\beta}\lambda_m^\alpha + \lambda_m^{2\beta}\lambda_n^{1-\alpha-\beta} + \lambda_n^{1-\alpha+\beta} - 2\lambda_n^{1-\alpha}\lambda_m^\beta \\
&= [(\lambda_m^\alpha - \lambda_n^\alpha)^2 + (\lambda_m^\beta - \lambda_n^\beta)^2]\lambda_n^{1-\alpha-\beta} \\
&\geq 2(\lambda_m^\alpha - \lambda_n^\alpha)(\lambda_m^\beta - \lambda_n^\beta)\lambda_n^{1-\alpha-\beta},
\end{aligned}$$

Again, by Proposition 3 (iii), we conclude that $|\mathrm{K}_\rho^{\alpha,\beta}|(A) \geq |\mathrm{I}_\rho^{\alpha,\beta}|(A)$. □

In a similar way, we can prove the following proposition.

Proposition 5 Let $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, $A \in L^2(H)$ and $\rho \in D(H)$. The following statements hold:

(i) $|\mathrm{L}_\rho^{\alpha,\beta}|(A) = |\mathrm{L}_\rho^{\beta,\alpha}|(A)$;

(ii) $|\mathrm{L}_\rho^{\alpha,\beta}|(A^\dagger) = |\mathrm{L}_\rho^{\alpha,\beta}|(A)$;

(iii) $|\mathrm{L}_\rho^{\alpha,\beta}|(A) \geq |\mathrm{J}_\rho^{\alpha,\beta}|(A)$.

3. Uncertainty relations based on MGWYD and MWGWYD skew information

In this section, we present some new uncertainty relations based on MGWYD skew information and MWGWYD skew information and related quantities defined in the previous section. First of all, imitating the proof of Lemma 1 and Lemma 2 in [44], we can prove the following Lemma.

Lemma 1 For any $x, y \geq 0$, $0 \leq \beta \leq \min\{\alpha, 1-\alpha\}$, we have

$$(x^\alpha + y^\alpha)|x^\beta - y^\beta| \leq |x - y|, \tag{12}$$

$$4\alpha\beta(x-y)^2 \leq (x^{2\alpha} - y^{2\alpha})(x^{2\beta} - y^{2\beta}). \tag{13}$$

Utilizing Lemma 1, the first main result of this paper can be stated as follows.

Theorem 1 Let $0 \leq \beta \leq \min\{\alpha, 1-\alpha\}$, $A, B \in L^2(H)$ and $\rho \in D(H)$. We have

$$|\mathrm{U}_\rho^{\alpha,\beta}|(A) \cdot |\mathrm{U}_\rho^{\alpha,\beta}|(B) \geq 4\alpha\beta||\mathrm{Corr}_\rho^{\alpha,\beta}|(A,B)|^2. \tag{14}$$

Proof. It follows from Proposition 3 (ii) that

$$\begin{aligned}|\mathrm{Corr}_\rho^{\alpha,\beta}|(A,B) &= \frac{1}{2}\sum_{m<n}\lambda_m^\alpha(\lambda_m^\beta - \lambda_n^\beta)[\lambda_m^{1-\alpha-\beta}\overline{a_{nm}}b_{nm} + \lambda_n^{1-\alpha-\beta}a_{mn}\overline{b_{mn}}] \\ &\quad + \frac{1}{2}\sum_{m<n}\lambda_n^\alpha(\lambda_n^\beta - \lambda_m^\beta)[\lambda_n^{1-\alpha-\beta}\overline{a_{mn}}b_{mn} + \lambda_n^{1-\alpha-\beta}a_{nm}\overline{b_{nm}}],\end{aligned}$$

which implies that

$$\begin{aligned}||\mathrm{Corr}_\rho^{\alpha,\beta}|(A,B)| &\leq \frac{1}{2}\sum_{m<n}\lambda_m^\alpha|\lambda_m^\beta - \lambda_n^\beta| \cdot |\lambda_m^{1-\alpha-\beta}\overline{a_{nm}}b_{nm} + \lambda_n^{1-\alpha-\beta}a_{mn}\overline{b_{mn}}| \\ &\quad + \frac{1}{2}\sum_{m<n}\lambda_n^\alpha|\lambda_n^\beta - \lambda_m^\beta| \cdot |\lambda_n^{1-\alpha-\beta}\overline{a_{mn}}b_{mn} + \lambda_n^{1-\alpha-\beta}a_{nm}\overline{b_{nm}}| \\ &= \frac{1}{2}\sum_{m<n}(\lambda_m^\alpha + \lambda_n^\alpha)|\lambda_m^\beta - \lambda_n^\beta| \cdot |\lambda_m^{1-\alpha-\beta}\overline{a_{nm}}b_{nm} + \lambda_n^{1-\alpha-\beta}a_{mn}\overline{b_{mn}}|.\end{aligned}$$

Utilizing (12), we obtain

$$||\mathrm{Corr}_\rho^{\alpha,\beta}|(A,B)| \leq \frac{1}{2}\sum_{m<n}|\lambda_m - \lambda_n| \cdot |\lambda_m^{1-\alpha-\beta}\overline{a_{nm}}b_{nm} + \lambda_n^{1-\alpha-\beta}a_{mn}\overline{b_{mn}}|. \tag{15}$$

Thus, combining Cauchy-Schwarz inequality, Lemma 1 (13), Proposition (iii) and (iv), from Eq. (15) we obtain

$$\begin{aligned}
&4\alpha\beta||\mathrm{Corr}_\rho^{\alpha,\beta}|(A,B)|^2\\
&\le \alpha\beta[\sum_{m<n}|\lambda_m-\lambda_n|\cdot|\lambda_m^{1-\alpha-\beta}\overline{a_{nm}}b_{nm}+\lambda_n^{1-\alpha-\beta}a_{mn}\overline{b_{mn}}|]^2\\
&=\frac{1}{4}[\sum_{m<n}2\sqrt{\alpha\beta}|\lambda_m-\lambda_n|\cdot|\lambda_m^{1-\alpha-\beta}\overline{a_{nm}}b_{nm}+\lambda_n^{1-\alpha-\beta}a_{mn}\overline{b_{mn}}|]^2\\
&\le\frac{1}{4}[\sum_{m<n}[(\lambda_m^\alpha-\lambda_n^\alpha)(\lambda_m^\beta-\lambda_n^\beta)(\lambda_m^\alpha+\lambda_n^\alpha)(\lambda_m^\beta+\lambda_n^\beta)]^{\frac{1}{2}}\cdot|\lambda_m^{1-\alpha-\beta}\overline{a_{nm}}b_{nm}+\lambda_n^{1-\alpha-\beta}a_{mn}\overline{b_{mn}}|]^2\\
&\le\frac{1}{4}[\sum_{m<n}[(\lambda_m^\alpha-\lambda_n^\alpha)(\lambda_m^\beta-\lambda_n^\beta)(\lambda_m^{1-\alpha-\beta}|a_{nm}|^2+\lambda_n^{1-\alpha-\beta}|a_{mn}|^2)]^{\frac{1}{2}}\cdot\\
&[(\lambda_m^\alpha+\lambda_n^\alpha)(\lambda_m^\beta+\lambda_n^\beta)(\lambda_m^{1-\alpha-\beta}|b_{nm}|^2+\lambda_n^{1-\alpha-\beta}|b_{mn}|^2)]^{\frac{1}{2}}]^2\\
&\le\frac{1}{4}\sum_{m<n}(\lambda_m^\alpha-\lambda_n^\alpha)(\lambda_m^\beta-\lambda_n^\beta)(\lambda_m^{1-\alpha-\beta}|a_{nm}|^2+\lambda_n^{1-\alpha-\beta}|a_{mn}|^2)\cdot\\
&\sum_{m<n}(\lambda_m^\alpha+\lambda_n^\alpha)(\lambda_m^\beta+\lambda_n^\beta)(\lambda_m^{1-\alpha-\beta}|b_{nm}|^2+\lambda_n^{1-\alpha-\beta}|b_{mn}|^2)\\
&\le|\mathrm{I}_\rho^{\alpha,\beta}|(A)\cdot|\mathrm{J}_\rho^{\alpha,\beta}|(B).
\end{aligned}$$

From the above deductions, we can also obtain that

$$4\alpha\beta||\mathrm{Corr}_\rho^{\alpha,\beta}|(A,B)|^2\le|\mathrm{I}_\rho^{\alpha,\beta}|(B)\cdot|\mathrm{J}_\rho^{\alpha,\beta}|(A).$$

Therefore, by Proposition 2 (iv), we get

$$|\mathrm{U}_\rho^{\alpha,\beta}|(A)\cdot|\mathrm{U}_\rho^{\alpha,\beta}|(B)\ge4\alpha\beta||\mathrm{Corr}_\rho^{\alpha,\beta}|(A,B)|^2.$$

This completes the proof. □

Imitating the proof of Lemma 3 in [44], we can prove the following lemma.

Lemma 2 For any $x,y\ge0$, $0\le\beta\le\min\{4\alpha,1-\alpha\}$, we have

$$(x^{\alpha+\beta}-x^\alpha y^\beta)^2\le(x^{2\alpha}-y^{2\alpha})(x^{2\beta}-y^{2\beta}).\tag{16}$$

Based on this lemma, we now give the second main result of this paper.

Theorem 2 Let $0\le\beta\le\min\{4\alpha,1-\alpha\}$, $A,B\in L^2(H)$ and $\rho\in D(H)$. We have

$$|\mathrm{U}_\rho^{\alpha,\beta}|(A)\cdot|\mathrm{U}_\rho^{\alpha,\beta}|(B)\ge\frac{1}{4}||\mathrm{Corr}_\rho^{\alpha,\beta}|(A,B)|^2.\tag{17}$$

Proof. It follows from Proposition 3 (ii), (iii), (iv), Lemma 2 and Cauchy-Schwarz inequality that

$$
\begin{aligned}
&||\mathrm{Corr}_\rho^{\alpha,\beta}|(A,B)|^2\\
&=\frac{1}{4}|\sum_{mn}\lambda_m^\alpha(\lambda_m^\beta-\lambda_n^\beta)[\lambda_m^{1-\alpha-\beta}\overline{a_{nm}}b_{nm}+\lambda_n^{1-\alpha-\beta}a_{mn}\overline{b_{mn}}]|^2\\
&\le\frac{1}{4}[\sum_{mn}\lambda_m^\alpha|\lambda_m^\beta-\lambda_n^\beta|\cdot|\lambda_m^{1-\alpha-\beta}\overline{a_{nm}}b_{nm}+\lambda_n^{1-\alpha-\beta}a_{mn}\overline{b_{mn}}]^2\\
&\le\frac{1}{4}[\sum_{mn}[(\lambda_m^\alpha-\lambda_n^\alpha)(\lambda_m^\beta-\lambda_n^\beta)(\lambda_m^\alpha+\lambda_n^\alpha)(\lambda_m^\beta+\lambda_n^\beta)]^{\frac{1}{2}}\cdot\\
&(\lambda_m^{1-\alpha-\beta}|a_{nm}|^2+\lambda_n^{1-\alpha-\beta}|a_{mn}|^2)^{\frac{1}{2}}(\lambda_m^{1-\alpha-\beta}|b_{nm}|^2+\lambda_n^{1-\alpha-\beta}|b_{mn}|^2)^{\frac{1}{2}}]^2\\
&\le\frac{1}{4}[\sum_{mn}[(\lambda_m^\alpha-\lambda_n^\alpha)(\lambda_m^\beta-\lambda_n^\beta)(\lambda_m^\alpha+\lambda_n^\alpha)(\lambda_m^\beta+\lambda_n^\beta)]^{\frac{1}{2}}\cdot\\
&(\lambda_m^{1-\alpha-\beta}|a_{nm}|^2+\lambda_n^{1-\alpha-\beta}|a_{mn}|^2)^{\frac{1}{2}}(\lambda_m^{1-\alpha-\beta}|b_{nm}|^2+\lambda_n^{1-\alpha-\beta}|b_{mn}|^2)^{\frac{1}{2}}]^2\\
&\le\frac{1}{4}\sum_{mn}(\lambda_m^\alpha-\lambda_n^\alpha)(\lambda_m^\beta-\lambda_n^\beta)(\lambda_m^{1-\alpha-\beta}|a_{nm}|^2+\lambda_n^{1-\alpha-\beta}|a_{mn}|^2)\cdot\\
&\sum_{mn}(\lambda_m^\alpha+\lambda_n^\alpha)(\lambda_m^\beta+\lambda_n^\beta)(\lambda_m^{1-\alpha-\beta}|b_{nm}|^2+\lambda_n^{1-\alpha-\beta}|b_{mn}|^2)\\
&=\frac{1}{4}\cdot 4|\mathrm{I}_\rho^{\alpha,\beta}|(A)\cdot 4|\mathrm{J}_\rho^{\alpha,\beta}|(B)\\
&=4|\mathrm{I}_\rho^{\alpha,\beta}|(A)\cdot|\mathrm{J}_\rho^{\alpha,\beta}|(B).
\end{aligned}
$$

Similarly, we have

$$||\mathrm{Corr}_\rho^{\alpha,\beta}|(A,B)|^2\le 4|\mathrm{I}_\rho^{\alpha,\beta}|(B)\cdot|\mathrm{J}_\rho^{\alpha,\beta}|(A).$$

Hence, by Proposition 2 (iv), we conclude that

$$||\mathrm{Corr}_\rho^{\alpha,\beta}|(A,B)|^2\le 4|\mathrm{U}_\rho^{\alpha,\beta}|(A)\cdot|\mathrm{U}_\rho^{\alpha,\beta}|(B).$$

This completes the proof. □

Remark Let us compare the above two theorems. (17) holds when $\alpha,\beta\ge 0$, $\alpha+\beta\le 1$ and $\alpha\le\beta\le 4\alpha$. When $\alpha,\beta\ge 0$, $\alpha+\beta\le 1$ and $\frac{1}{4}\le\beta\le\alpha$, we have $4\alpha\beta\ge\frac{1}{4}$, and thus (17) is better than (14). When $\alpha,\beta\ge 0$, $\alpha+\beta\le 1$ and $\beta\le\alpha\le\frac{1}{4}$, we have $4\alpha\beta\le\frac{1}{4}$ and thus (14) is better than (17).

In the previous section, we have defined the quantities $|\mathrm{K}_\rho^{\alpha,\beta}|(A)$ and $|\mathrm{L}_\rho^{\alpha,\beta}|(A)$. Now we define the quantity $|\mathrm{W}_\rho^{\alpha,\beta}|(A)=\sqrt{|\mathrm{K}_\rho^{\alpha,\beta}|(A)|\mathrm{L}_\rho^{\alpha,\beta}|(A)}$ for any $A\in L^2(H)$. Then it follows from Proposition 2 (iv), Proposition 4 (iii) and Proposition 5 (iii) that

$$|\mathrm{W}_\rho^{\alpha,\beta}|(A)\ge|\mathrm{U}_\rho^{\alpha,\beta}|(A), |\mathrm{W}_\rho^{\alpha,\beta}|(B)\ge|\mathrm{U}_\rho^{\alpha,\beta}|(B),\ \text{for all } A,B\in L^2(H).$$

Therefore, we obtain the following two uncertainty relations as consequences of Theorem 1 and Theorem 2.

Corollary 1 Let $0 \leq \beta \leq \min\{\alpha, 1-\alpha\}$, $A, B \in L^2(H)$ and $\rho \in D(H)$. Then we have

$$|\mathrm{W}_\rho^{\alpha,\beta}|(A) \cdot |\mathrm{W}_\rho^{\alpha,\beta}|(B) \geq 4\alpha\beta||\mathrm{Corr}_\rho^{\alpha,\beta}|(A,B)|^2. \tag{18}$$

Corollary 2 Let $0 \leq \beta \leq \min\{4\alpha, 1-\alpha\}$, $A, B \in L^2(H)$ and $\rho \in D(H)$. Then we have

$$|\mathrm{W}_\rho^{\alpha,\beta}|(A) \cdot |\mathrm{W}_\rho^{\alpha,\beta}|(B) \geq \frac{1}{4}||\mathrm{Corr}_\rho^{\alpha,\beta}|(A,B)|^2. \tag{19}$$

Example 1 Consider the Werner state

$$\rho_w^{ab} = \begin{pmatrix} \frac{1}{3}p & 0 & 0 & 0 \\ 0 & \frac{1}{6}(3-2p) & \frac{1}{6}(4p-3) & 0 \\ 0 & \frac{1}{6}(4p-3) & \frac{1}{6}(3-2p) & 0 \\ 0 & 0 & 0 & \frac{1}{3}p \end{pmatrix},$$

where $p \in [0,1]$. Note that ρ_w^{ab} is separable when $p \in [0, \frac{1}{3}]$. Let A and B be the following non-Hermitian matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & -i \\ 1 & 0 & i & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -i \\ 1 & 0 & i & 0 \end{pmatrix}. \tag{20}$$

Figure 1 illustrates the uncertainty relations of Eq. (14) with different values of α and β.

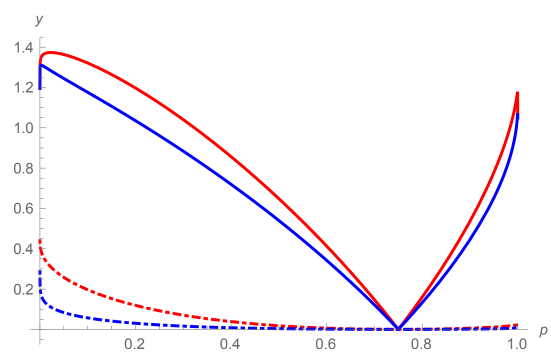
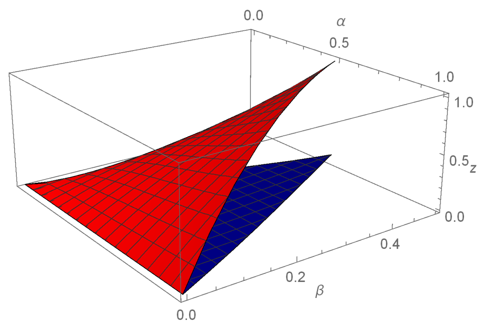
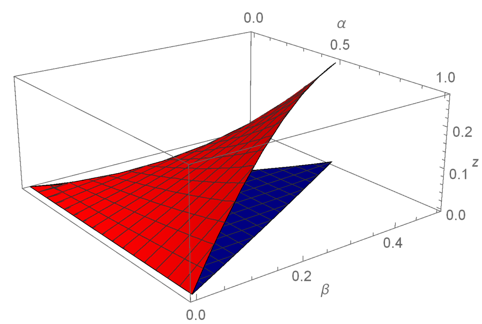


Figure 1: The y-axis shows the uncertainty and its lower bounds. Red solid (dotdashed) line represents the value of the left (right)-hand side of Eq. (14) with $\alpha = \frac{11}{20}$ and $\beta = \frac{2}{5}$ for ρ_w^{ab}; blue solid (dotdashed) line represents the value of the left (right)-hand side of Eq. (14) with $\alpha = \frac{15}{20}$ and $\beta = \frac{1}{5}$ for ρ_w^{ab}.

Moreover, when we fix the value of p, the gap between the left and right hand sides of Eq. (14) for separable states are greater than those for the entangled states. See Figure 2 for an illustration of this fact for $p = 0.3$ and $p = 0.9$.



(a)



(b)

Figure 2: The z-axis shows the uncertainty and its lower bounds. (a) $p = 0.3$ (in this case ρ_w^{ab} is a separable state): Red (blue) surface represents the value of the left (right)-hand side of Eq. (14) for ρ_w^{ab}; (b) $p = 0.9$ (in this case ρ_w^{ab} is an entangled state): Red (blue) surface represents the value of the left (right)-hand side of Eq. (14) for ρ_w^{ab}.

Example 2 Consider the isotropic state

$$\rho_{iso}^{ab} = \begin{pmatrix} \frac{1}{6}(2F+1) & 0 & 0 & \frac{1}{6}(4F-1) \\ 0 & \frac{1}{3}(1-F) & 0 & 0 \\ 0 & 0 & \frac{1}{3}(1-F) & 0 \\ \frac{1}{6}(4F-1) & 0 & 0 & \frac{1}{6}(2F+1) \end{pmatrix},$$

where $F \in [0, 1]$. Note that ρ_{iso}^{ab} is separable when $F \in [0, \frac{1}{2}]$. With A and B being the non-Hermitian matrices given in (20), Figure 3 illustrates the uncertainty relation (14) with different values of α and β.

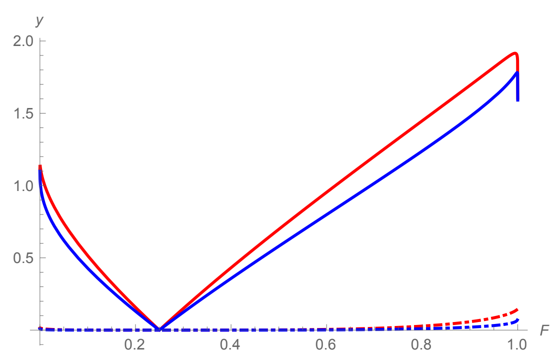
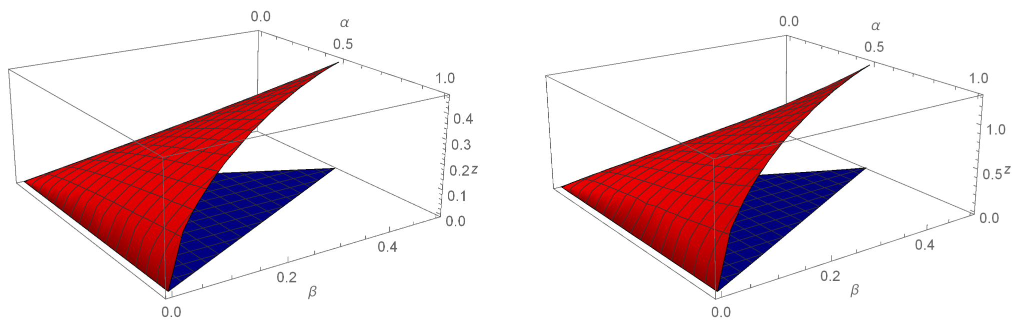


Figure 3: The y-axis shows the uncertainty and its lower bounds. Red solid (dotdashed) line represents the value of the left (right)-hand side of Eq. (14) with $\alpha = \frac{11}{20}$ and $\beta = \frac{2}{5}$ for ρ_{iso}^{ab}; blue solid (dotdashed) line represents the value of the left (right)-hand side of Eq. (14) with $\alpha = \frac{15}{20}$ and $\beta = \frac{1}{5}$ for ρ_{iso}^{ab}.

Moreover, when we fix the value of F, the gap between the left and right hand sides of Eq. (14) for separable states are less than those for the entangled states. See Figure

4 for an illustration of this fact for $F = 0.4$ and $F = 0.7$.



(a) (b)

Figure 4: The z-axis shows the uncertainty and its lower bounds. (a) $F = 0.4$ (in this case ρ_{iso}^{ab} is a separable state): Red (blue) surface represents the value of the left (right)-hand side of Eq. (14) for ρ_{iso}^{ab}; (b) $F = 0.7$ (in this case ρ_{iso}^{ab} is an entangled state): Red (blue) surface represents the value of the left (right)-hand side of Eq. (14) for ρ_{iso}^{ab}.

4. Conclusions

Based on the newly introduced quantities termed modified generalized Wigner-Yanase-Dyson skew information and modified weighted generalized Wigner-Yanase-Dyson skew information, we have derived new uncertainty relations, which turned out to be the generalizations of the main results in [44]. Information based quantum uncertainty relations are of significance for usual Hermitian quantum mechanical systems. Our work shew new light on the study of uncertainty relations for non-Hermitian operators.

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