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by

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Strong polygamy and monogamy relations for multipartite quantum systems

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Abstract

Monogamy and polygamy are the most striking features of the quantum world. We investigate the monogamy and polygamy relations satisfied by all quantum correlation measures for arbitrary multipartite quantum states. By introducing residual quantum correlations, analytical polygamy inequalities are presented, which are shown to be tighter than the existing ones. Then, similar to polygamy relations, we obtain strong monogamy relations that are better than all the existing ones. Typical examples are presented for illustration.

INTRODUCTION

Quantum correlation is one of the most important properties of quantum physics, which has been extensively studied due to its importance in quantum communication and quantum information processing. One significant property of quantum correlation is known as monogamy. For a tripartite system A, B and C, the usual monogamy of a quantum correlation measure Q implies that the correlation $Q_{A|BC}$ between A and BC satisfies $Q_{A|BC} \geq Q_{AB} + Q_{AC}$. Dually, the polygamy relation is quantitatively displayed as $Q_{A|BC} \leq Q_{AB} + Q_{AC}$. It is shown that while monogamy inequalities provide an upper bound for bipartite sharability of quantum correlations in a multipartite system, the polygamy inequalities give a lower bound. The first monogamy relation was proven for arbitrary three-qubit states based on the squared concurrence. Later, various monogamy inequalities have been established for a number of entanglement measures in multipartite quantum systems [1–8]. Polygamy relations are also generalized to multiqubit systems [9] and arbitrary dimensional multipartite states [3–5].

As is well known, the usual monogamy and polygamy relations are not always satisfied by any correlation measures like entanglement of formation [10] quantifying the amount of entanglement required for preparation of a given bipartite quantum state. It has been shown that the αth ($\alpha \geq 2$) power of concurrence and the αth ($\alpha \geq \sqrt{2}$) power of entanglement of formation do satisfy the monogamy relations for N-qubit states [2, 3]. One may ask whether any measures of quantum correlations satisfy a kind of monogamy or polygamy relations. In this paper, we first show that all quantum correlation measures satisfy some kind of polygamy relations for arbitrary multipartite quantum states. Then we introduce the residual quantum correlations, and present tighter polygamy inequalities that are better than all the existing ones. At last, similar to polygamy relations, we present the strong monogamy relations that are also better than the existing ones.

STRONG POLYGAMY RELATIONS FOR MULTIPARTITE QUANTUM SYSTEMS

Let $\mathcal{Q}$ be an arbitrary quantum correlation measure of bipartite systems. $\mathcal{Q}$ is said to be polygamous for an N-partite quantum state $\rho_{AB_1B_2\cdots B_{N-1}}$, if it satisfies the following inequality,

$$\mathcal{Q}(\rho_{AB_1}) + \mathcal{Q}(\rho_{AB_2}) + \cdots + \mathcal{Q}(\rho_{AB_{N-1}}) \geq \mathcal{Q}(\rho_{A|B_1B_2\cdots B_{N-1}}), \tag{1}$$

where ρ_{AB_i}, $i = 1, ..., N-1$, are the reduced density matrices, $\mathcal{Q}(\rho_{A|B_1B_2\cdots B_{N-1}})$ denotes the quantum correlation $\mathcal{Q}$ of the state $\rho_{AB_1B_2\cdots B_{N-1}}$ under bipartite partition A and $B_1B_2\cdots B_{N-1}$, which keeps invariant under discarding subsystems only for states satisfying monogamy relations. For simplicity, we denote $\mathcal{Q}(\rho_{AB_i})$ by $\mathcal{Q}_{AB_i}$, and $\mathcal{Q}(\rho_{A|B_1B_2\cdots B_{N-1}})$ by $\mathcal{Q}_{A|B_1B_2\cdots B_{N-1}}$. We define the $\mathcal{Q}$-polygamy score for the N-partite state $\rho_{AB_1B_2\cdots B_{N-1}}$,

$$\delta_{\mathcal{Q}} = \sum_{i=1}^{N-1} \mathcal{Q}_{AB_i} - \mathcal{Q}_{A|B_1B_2\cdots B_{N-1}}. \tag{2}$$

Non-negativity of $\delta_{\mathcal{Q}}$ for all quantum states implies the polygamy of $\mathcal{Q}$. For instance, the square of the concurrence in term of the concurrence of assistance has been shown to be polygamous for all multiqubit states [9].

Given any quantum correlation measure that is not polygamous for a multipartite quantum state, it is always possible to find a function of the measure which is polygamous for the same state [11]. It has been proved that for any $d\otimes d_1\otimes\cdots\otimes d_{N-1}$ state $\rho_{AB_1B_2\cdots B_{N-1}}$, there exists $\beta_{\max}(\mathcal{Q}) \in R$ such that for any $0 \leq \gamma \leq \beta_{\max}(\mathcal{Q})$, the quantum correlation measure $\mathcal{Q}$ satisfies the following polygamous relation [11]

$$\mathcal{Q}^{\gamma}_{A|B_1B_2\cdots B_{N-1}} \leq \sum_{i=1}^{N-1} \mathcal{Q}^{\gamma}_{AB_i}. \tag{3}$$

In the following, we denote $\beta = \beta_{\max}(\mathcal{Q})$ the maximal value such that $\mathcal{Q}^{\beta}$ satisfies the above inequality. Similar to the three tangle of concurrence, for tripartite quantum states ρ_{ABC}, we define the residual quantum correlation as a function of α,

$$\mathcal{Q}^{\alpha}_{A|B|C} = \mathcal{Q}^{\alpha}_{AB} + \mathcal{Q}^{\alpha}_{AC} - \mathcal{Q}^{\alpha}_{A|BC}, \quad 0 \leq \alpha \leq \beta. \tag{4}$$

For the class of GHZ states, the equality (4) is valid for $\beta = 0$.

From the original definition in [15], the residual quantum correlation is defined to be $\mathcal{Q}_{A|B|C} = \mathcal{Q}_{A|BC} - \mathcal{Q}_{AB} - \mathcal{Q}_{AC}$ for some quantum correlation measures $\mathcal{Q}$ satisfying the monogamy relations $\mathcal{Q}_{A|BC} \geq \mathcal{Q}_{AB} + \mathcal{Q}_{AC}$. Generally, it is not the quantum correlation measure $\mathcal{Q}$ itself, but the αth power satisfies the monogamy inequality, for instance, the αth ($\alpha \geq 2$) power of concurrence and the αth ($\alpha \geq \sqrt{2}$) power of entanglement of formation

[2]. It is also the case for polygamy relations. Therefore, here we use the αth power of the quantum correlation to define the "residual quantum correlation".

The residual quantum correlations quantify the degree of entanglement distributions among the subsystems: the smaller of α in (4), the greater degree of violation of the polygamy inequality. Let us consider the tripartite systems. The residual quantum correlation is defined by $\mathcal{Q}^{\alpha}_{A|B|C} = \mathcal{Q}^{\alpha}_{AB} + \mathcal{Q}^{\alpha}_{AC} - \mathcal{Q}^{\alpha}_{A|BC}$ $(0 \le \alpha \le \beta)$. For two states ρ_{ABC} and δ_{ABC} such that $\mathcal{Q}^{\alpha_1}_{A|B|C}(\rho_{ABC}) = \mathcal{Q}^{\alpha_2}_{A|B|C}(\delta_{ABC}) = 0$, $\alpha_1 \le \alpha_2$, we have $|\mathcal{Q}(\rho_{AB}) - \mathcal{Q}(\rho_{AC})| \le |\mathcal{Q}(\delta_{AB}) - \mathcal{Q}(\delta_{AC})|$. The distribution of quantum correlation in ρ_{ABC} is more averaged than that in state δ_{ABC}. For example, consider the state $|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle$, where $\lambda_i \ge 0,\ i = 0,\cdots,4$ and $\sum_{i=0}^{4}\lambda_i^2 = 1$. We have the concurrences $C_{A|BC} = 2\lambda_0\sqrt{\lambda_2^2+\lambda_3^2+\lambda_4^2}$, $C_{AB} = 2\lambda_0\lambda_2$, and $C_{AC} = 2\lambda_0\lambda_3$. Taking $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{\sqrt{5}}{5}$, we have $\rho_{ABC} = |\psi_1\rangle\langle\psi_1|$, where $|\psi_1\rangle = \frac{\sqrt{5}}{5}|000\rangle + \frac{\sqrt{5}}{5}e^{i\varphi}|100\rangle + \frac{\sqrt{5}}{5}|101\rangle + \frac{\sqrt{5}}{5}|110\rangle + \frac{\sqrt{5}}{5}|111\rangle$. One gets $C(\rho_{A|BC})^{\alpha} = (\frac{2\sqrt{3}}{5})^{\alpha}$, $C(\rho_{AB})^{\alpha} = C(\rho_{AC})^{\alpha} = (\frac{2}{5})^{\alpha}$ and $\alpha_1 \approx 1.26185$ from $\mathcal{Q}^{\alpha_1}_{A|B|C}(\rho) = 0$. If we take $\lambda_0 = \lambda_2 = \frac{1}{2}$, $\lambda_1 = \lambda_3 = \lambda_4 = \frac{\sqrt{6}}{6}$, then the state becomes $\delta_{ABC} = |\psi_2\rangle\langle\psi_2|$, where $|\psi_2\rangle = \frac{1}{2}|000\rangle + \frac{\sqrt{6}}{6}e^{i\varphi}|100\rangle + \frac{1}{2}|101\rangle + \frac{\sqrt{6}}{6}|110\rangle + \frac{\sqrt{6}}{6}|111\rangle$. One has $\alpha_2 \approx 1.33770$ based on $\mathcal{Q}^{\alpha_2}_{A|B|C}(\delta_{ABC}) = 0$. From above, one can easily get that the entanglement distribution between the subsystems in ρ_{ABC} is more averaged than that in δ_{ABC}.

Consider a $d \otimes d_1 \otimes d_2 \otimes d_3$ state $\rho_{AB_1B_2B_3}$. Define $\mathcal{Q}^{\alpha}_{A|B_1'|B_2'} = \max\{\mathcal{Q}^{\alpha}_{A|B_1|B_2}, \mathcal{Q}^{\alpha}_{A|B_1|B_3}, \mathcal{Q}^{\alpha}_{A|B_2|B_3}\}$, where B_1' and B_2' stand for two of B_1, B_2 and B_3 such that $\mathcal{Q}^{\alpha}_{A|B_1'|B_2'} = \max\{\mathcal{Q}^{\alpha}_{A|B_1|B_2}, \mathcal{Q}^{\alpha}_{A|B_1|B_3}, \mathcal{Q}^{\alpha}_{A|B_2|B_3}\}$.

[**Theorem 1**]. For any $d \otimes d_1 \otimes d_2 \otimes d_3$ state $\rho_{AB_1B_2B_3}$, we have

$$\mathcal{Q}^{\alpha}_{A|B_1B_2B_3} \le \sum_{i=1}^{3}\mathcal{Q}^{\alpha}_{AB_i} - \mathcal{Q}^{\alpha}_{A|B_1'|B_2'}, \tag{5}$$

for $0 \le \alpha \le \beta$.

[Proof]. By definition we have

$$\begin{aligned}\sum_{i=1}^{3}\mathcal{Q}^{\alpha}_{AB_i} - \mathcal{Q}^{\alpha}_{A|B_1'|B_2'} &= \mathcal{Q}^{\alpha}_{AB_3'} + \mathcal{Q}^{\alpha}_{A|B_1'B_2'} \\ &\ge \mathcal{Q}^{\alpha}_{A|B_1B_2B_3},\end{aligned}$$

where B_3' is the complementary of $B_1'B_2'$ in the subsystem $B_1B_2B_3$, the equality is due to the definition of the residual quantum correlation. From (3) we get the inequality. ∎

Concerning the parameter β in Theorem 1, let us consider the following 4-qubit state,

$$\begin{aligned}|\psi\rangle_{AB_1B_2B_3} = \quad & \cos\theta_0|0000\rangle + \sin\theta_0\cos\theta_1 e^{i\varphi}|1000\rangle + \frac{1}{2}\sin\theta_0\sin\theta_1|1010\rangle \\ & + \frac{3}{4}\sin\theta_0\sin\theta_1|1100\rangle + \frac{\sqrt{3}}{4}\sin\theta_0\sin\theta_1|1110\rangle, \end{aligned} \tag{6}$$

where $\theta_0, \theta_1 \in [0, \frac{\pi}{2}]$. We have $C_{A|B_1B_2B_3} = 2\cos\theta_0\sin\theta_0\sin\theta_1$, $C_{AB_1} = \cos\theta_0\sin\theta_0\sin\theta_1$, $C_{AB_2} = \frac{3}{2}\cos\theta_0\sin\theta_0\sin\theta_1$ and $C_{AB_3} = C_{A|B_1'|B_2'} = 0$. From (5) we obtain $(\frac{1}{2})^\alpha + (\frac{3}{4})^\alpha \geq 1$, namely, $\alpha \leq 1.507126$. Therefore, $\beta = 1.507126$ is the largest value saturating the inequality (5) for the state (6).

Inequality (5) presents a tighter polygamy relations for $0 \leq \alpha \leq \beta$. Specially, inequality (5) is satisfied only when $\alpha = 0$ for particular quantum states like the GHZ-class states. Generalizing the conclusion of Theorem 1 to N partite case, we have the following result.

[**Theorem 2**]. For any $d \otimes d_1 \otimes \cdots \otimes d_{N-1}$ state $\rho_{AB_1B_2\cdots B_{N-1}}$, we have

$$\mathcal{Q}^\alpha_{A|B_1B_2\cdots B_{N-1}} \leq \sum_{i=1}^{N-1} \mathcal{Q}^\alpha_{AB_i} - \sum_{k=2}^{N-2} \mathcal{Q}^\alpha_{A|B_1'|B_2'|\cdots|B_k'}, \tag{7}$$

for $0 \leq \alpha \leq \beta$, where $\mathcal{Q}^\alpha_{A|B_1'|B_2'|\cdots|B_k'} = \max_{1\leq l\leq k+1}\{\mathcal{Q}^\alpha_{A|B_1|\cdots|\hat{B}_l|\cdots|B_{k+1}}\}$ (where $\hat{B}_l$ stands for B_l being omitted in the sub-indices), $\mathcal{Q}^\alpha_{A|B_1|B_2|\cdots|B_{k+1}} = \sum_{i=1}^{k+1} \mathcal{Q}^\alpha_{AB_i} - \mathcal{Q}^\alpha_{A|B_1B_2\cdots B_{k+1}} - \sum_{i=2}^{k} \mathcal{Q}^\alpha_{A|B_1'|B_2'|\cdots|B_i'}$, $2 \leq k \leq N-2$, $1 \leq l \leq k+1$, $N \geq 4$.

[Proof]. We prove the theorem by induction. For $N = 4$ it reduces to Theorem 1. Suppose the Theorem 2 holds for $N = n$, i.e.,

$$\mathcal{Q}^\alpha_{A|B_1B_2\cdots B_{n-1}} \leq \sum_{i=1}^{n-1} \mathcal{Q}^\alpha_{AB_i} - \mathcal{Q}^\alpha_{A|B_1'|B_2'} - \cdots - \mathcal{Q}^\alpha_{A|B_1'|B_2'|\cdots|B_{n-2}'}. \tag{8}$$

Then for $N = n+1$, we have

$$\begin{aligned} & \sum_{i=1}^{n} \mathcal{Q}^\alpha_{AB_i} - \mathcal{Q}^\alpha_{A|B_1'|B_2'} - \cdots - \mathcal{Q}^\alpha_{A|B_1'|B_2'|\cdots|B_{n-1}'} \\ & \geq \mathcal{Q}^\alpha_{A|B_1'B_2'\cdots B_{n-1}'} + \mathcal{Q}^\alpha_{AB_n'} \\ & \geq \mathcal{Q}^\alpha_{A|B_1B_2\cdots B_n}, \end{aligned}$$

where B_n' is the complementary of $B_1', B_2', \cdots, B_{n-1}'$ in the subsystem $B_1, B_2, \cdots, B_n$, the first inequality is due to (8). By (3) we get the last inequality. ∎

Since the last term $\sum_{k=2}^{N-2} \mathcal{Q}^\alpha_{A|B_1'|B_2'|\cdots|B_k'}$, $2 \leq k \leq N-2$, $N \geq 4$ in (7) is nonnegative, the inequality (7) is always tighter than (3). Let us consider the following example based on the quantum entanglement measure concurrence. For a bipartite pure state $|\phi\rangle_{AB}$, the concurrence is $C(|\phi\rangle_{AB}) = \sqrt{2\left[1-\mathrm{Tr}(\rho_A^2)\right]}$, where ρ_A is the reduced density matrix by tracing over the subsystem B, $\rho_A = \mathrm{Tr}_B(|\phi\rangle_{AB}\langle\phi|)$. For a mixed state $\rho_{AB} = \sum_i p_i|\phi_i\rangle_{AB}\langle\phi_i|$, the concurrence is defined by the convex roof extension, $C(\rho_{AB}) = \min_{\{p_i,|\phi_i\rangle\}}\sum_i p_i C(|\phi_i\rangle)$, where the minimum is taken over all possible decompositions of $\rho_{AB} = \sum\limits_i p_i|\phi_i\rangle\langle\phi_i|$, with $p_i \geq 0$ and $\sum\limits_i p_i = 1$. The concurrence of assistance is defined by $C_a(\rho_{AB}) = \max_{\{p_i,|\phi_i\rangle\}}\sum_i p_i C(|\phi_i\rangle)$. And the entanglement of assistance τ_a is given by $\tau_a(\rho_{AB}) = \sum_{m=1}^{D_1}\sum_{n=1}^{D_2} C_a((\rho_{AB})_{mn}) = \sum_{m=1}^{D_1}\sum_{n=1}^{D_2}(\max\sum_i p_i|\langle\phi_i|(L_A^m\otimes L_B^n)|\phi_i^*\rangle|)$ [17], where $D_1 = d_1(d_1-1)/2$, $D_2 = d_2(d_2-1)/2$, $L_A^m = P_A^m(-|i\rangle_A\langle j| + |j\rangle_A\langle i|)P_A^m$, $L_B^n = P_B^n(-|k\rangle_B\langle l| + |l\rangle_B\langle k|)P_B^n$, and $P_A^m = |i\rangle_A\langle i| + |j\rangle_A\langle j|$, $P_B^n = |k\rangle_B\langle k| + |l\rangle_B\langle l|$ are the projections onto the subspaces spanned by $\{|i\rangle_A, |j\rangle_A\}$ and $\{|k\rangle_B, |l\rangle_B\}$, respectively. A general polygamy inequality for any multipartite pure state $|\phi\rangle_{A_1\cdots A_n}$ was established as [9], $\tau_a^2(|\phi\rangle_{A_1|A_2\cdots A_n}) \leq \sum_{i=2}^n \tau_a^2(\rho_{A_1A_i})$, where $\rho_{A_1A_k}$ is the reduced density matrix of subsystems A_1A_k for $k = 2,\cdots,n$. It has been further shown that [11],

$$\tau_a^\alpha(|\phi\rangle_{A_1|A_2\cdots A_n}) \leq \sum_{i=2}^n \tau_a^\alpha(\rho_{A_1A_i}), \tag{9}$$

where $0 \leq \alpha \leq 2$

Example 1. Let us consider the entanglement of assistance τ_a of the following 5-qubit pure state,

$$|\psi\rangle_{AB_1B_2B_3B_4} = \frac{1}{\sqrt{5}}(|10000\rangle + |01000\rangle + |00100\rangle + |00010\rangle + |00001\rangle). \tag{10}$$

We have $\beta = 2$, $\tau_a(|\psi\rangle_{A|B_1B_2B_3B_4}) = \frac{4}{5}$, $\tau_a(\rho_{AB_i}) = \frac{2}{5}$, $i = 1,2,3,4$. $\tau_{aA|B_i|B_j|B_k} = 3(\frac{1}{2})^\alpha - (\frac{\sqrt{3}}{2})^\alpha$, $i \neq j \neq k \in \{1,2,3,4\}$. From the result (9) in [11], we get $\tau_a^\alpha(|\psi\rangle_{A|B_1B_2B_3B_4}) \leq 4(\frac{2}{5})^\alpha$. From our inequality (7) in Theorem 2, we have $\tau_a^\alpha(|\psi\rangle_{A|B_1B_2B_3B_4}) \leq 4(\frac{2}{5})^\alpha - 3(\frac{1}{2})^\alpha + (\frac{\sqrt{3}}{2})^\alpha$. Obviously, our result (7) is better than that in [11], see Fig. 1.

In Theorems 1 and 2 we have taken into account the maximum value among $\mathcal{Q}^\alpha_{A|B_1|\cdots|\hat{B}_l|\cdots|B_k}$. If instead of the maximum value, one just considers the mean value of $\mathcal{Q}^\alpha_{A|B_1|\cdots|\hat{B}_l|\cdots|B_k}$, one may have the following corollary.

[**Corollary 1**]. For any $d\otimes d_1\otimes\cdots\otimes d_{N-1}$ state $\rho_{A|B_1B_2\cdots B_{N-1}}$, we have

$$\mathcal{Q}^\alpha_{A|B_1B_2\cdots B_{N-1}} \leq \sum_{i=1}^{N-1}\mathcal{Q}^\alpha_{AB_i} - \sum_{k=3}^{N-1}\left(\frac{1}{k}\sum_{l=1}^{k}\mathcal{Q}^\alpha_{A|B_1|\cdots|\hat{B}_l|\cdots|B_k}\right), \tag{11}$$

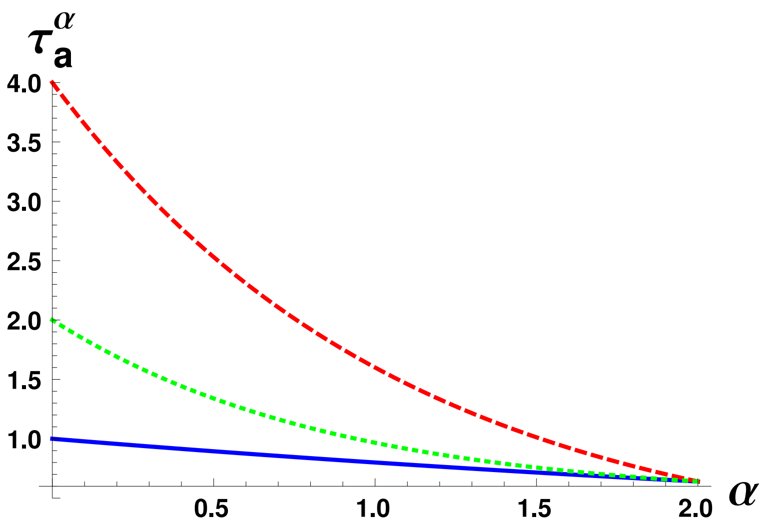

FIG. 1: Solid (blue) line is the αth power of τ_a under bipartition $A|B_1B_2B_3B_4$; Dashed (red) line is the upper bound in (9); Dotted (green) line is the upper bound in (7).

for all $0 \leq \alpha \leq \beta$, $N \geq 4$, where

$$\mathcal{Q}^\alpha_{A|B_1|B_2|\cdots|B_j} = \sum_{i=1}^{j} \mathcal{Q}^\alpha_{AB_i} - \mathcal{Q}^\alpha_{A|B_1B_2\cdots B_j} - \sum_{k=3}^{j}\left(\frac{1}{k}\sum_{l=1}^{k}\mathcal{Q}^\alpha_{A|B_1|\cdots|\hat{B}_l|\cdots|B_k}\right), \tag{12}$$

$3 \leq j \leq N-1$, $3 \leq k \leq N-1$ and $1 \leq l \leq k$.

Next, we adopt an approach used in Ref. [12] to improve further the above results on polygamy relations for multipartite quantum correlation measures. First, we give a Lemma.

[**Lemma 1**]. For any $d_1 \otimes d_2 \otimes d_3$ mixed state ρ_{ABC}, if $\mathcal{Q}_{AB} \geq \mathcal{Q}_{AC}$, we have

$$\mathcal{Q}^\alpha_{A|BC} \leq \mathcal{Q}^\alpha_{AB} + L\mathcal{Q}^\alpha_{AC}, \tag{13}$$

for all $0 \leq \alpha \leq \beta$, where $L = (2^{\frac{\alpha}{\beta}} - 1)$.

[Proof]. For arbitrary $d_1 \otimes d_2 \otimes d_3$ tripartite state ρ_{ABC}. If $\mathcal{Q}_{AB} \geq \mathcal{Q}_{AC}$, we have

$$\begin{aligned} \mathcal{Q}^\alpha_{A|BC} &\leq (\mathcal{Q}^\beta_{AB} + \mathcal{Q}^\beta_{AC})^{\frac{\alpha}{\beta}} = \mathcal{Q}^\alpha_{AB}\left(1 + \frac{\mathcal{Q}^\beta_{AC}}{\mathcal{Q}^\beta_{AB}}\right)^{\frac{\alpha}{\beta}} \\ &\leq \mathcal{Q}^\alpha_{AB}\left[1 + (2^{\frac{\alpha}{\beta}} - 1)\left(\frac{\mathcal{Q}^\beta_{AC}}{\mathcal{Q}^\beta_{AB}}\right)^{\frac{\alpha}{\beta}}\right] \\ &= \mathcal{Q}^\alpha_{AB} + (2^{\frac{\alpha}{\beta}} - 1)\mathcal{Q}^\alpha_{AC}, \end{aligned}$$

where the first inequality is due to (3), the second inequality is due to the inequality $(1+t)^x \leq 1 + (2^x - 1)t^x$ for $0 \leq x \leq 1,\ 0 \leq t \leq 1$. ■

In the above Lemma, without loss of generality, we have assumed that $\mathcal{Q}_{AB} \geq \mathcal{Q}_{AC}$, as the subsystems A and B are equivalent. Moreover, in the proof of the Lemma 1 we have assumed $\mathcal{Q}_{AB} > 0$. If $\mathcal{Q}_{AB} = 0$ and $\mathcal{Q}_{AB} \geq \mathcal{Q}_{AC}$, then $\mathcal{Q}_{AB} = \mathcal{Q}_{AC} = 0$. The upper bound is trivially zero. Generalizing the Lemma 1 to multipartite quantum systems, we have the following Theorem.

[**Theorem 3**]. For any $d \otimes d_1 \otimes \cdots \otimes d_{N-1}$ state $\rho_{AB_1 \cdots B_{N-1}}$, if $\mathcal{Q}_{AB_i} \geq \mathcal{Q}_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \cdots, m$, and $\mathcal{Q}_{AB_j} \leq \mathcal{Q}_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$, $\forall\ 1 \leq m \leq N-3$, $N \geq 4$, we have

$$\begin{aligned} \mathcal{Q}^{\alpha}_{A|B_1B_2\cdots B_{N-1}} \leq \quad & \mathcal{Q}^{\alpha}_{AB_1} + L\mathcal{Q}^{\alpha}_{AB_2} + \cdots + L^{m-1}\mathcal{Q}^{\alpha}_{AB_m} \\ & + L^{m+1}(\mathcal{Q}^{\alpha}_{AB_{m+1}} + \cdots + \mathcal{Q}^{\alpha}_{AB_{N-2}}) + L^m \mathcal{Q}^{\alpha}_{AB_{N-1}}, \end{aligned} \tag{14}$$

for all $0 \leq \alpha \leq \beta$, where $L = (2^{\frac{\alpha}{\beta}} - 1)$.

[Proof]. By using the Lemma 1 repeatedly, one gets

$$\begin{aligned} \mathcal{Q}^{\alpha}_{A|B_1B_2\cdots B_{N-1}} & \leq \mathcal{Q}^{\alpha}_{AB_1} + L\mathcal{Q}^{\alpha}_{A|B_2\cdots B_{N-1}} \\ & \leq \mathcal{Q}^{\alpha}_{AB_1} + L\mathcal{Q}^{\alpha}_{AB_2} + L^2\mathcal{Q}^{\alpha}_{A|B_3\cdots B_{N-1}} \\ & \leq \cdots \leq \mathcal{Q}^{\alpha}_{AB_1} + L\mathcal{Q}^{\alpha}_{AB_2} + \cdots \\ & + L^{m-1}\mathcal{Q}^{\alpha}_{AB_m} + L^m\mathcal{Q}^{\alpha}_{A|B_{m+1}\cdots B_{N-1}}. \end{aligned} \tag{15}$$

As $\mathcal{Q}_{AB_j} \leq \mathcal{Q}_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$, by (13) we get

$$\begin{aligned} \mathcal{Q}^{\alpha}_{A|B_{m+1}\cdots B_{N-1}} & \leq L\mathcal{Q}^{\alpha}_{AB_{m+1}} + \mathcal{Q}^{\alpha}_{A|B_{m+2}\cdots B_{N-1}} \\ & \leq L(\mathcal{Q}^{\alpha}_{AB_{m+1}} + \cdots + \mathcal{Q}^{\alpha}_{AB_{N-2}}) + \mathcal{Q}^{\alpha}_{AB_{N-1}}. \end{aligned} \tag{16}$$

Combining (15) and (16), we have Theorem 3. ■

Similar to the Theorem 2, (14) can be improved by adding a term for residual quantum correlation. By a similar derivation to Theorem 2, we have

[**Theorem 4**]. For any $d \otimes d_1 \otimes \cdots \otimes d_{N-1}$ state $\rho_{AB_1 \cdots B_{N-1}}$, if $\mathcal{Q}_{AB_i} \geq \mathcal{Q}_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \cdots, m$, and $\mathcal{Q}_{AB_j} \leq \mathcal{Q}_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$, $\forall\ 1 \leq m \leq N-3$, $N \geq 4$, we have

$$\mathcal{Q}^{\alpha}_{A|B_1B_2\cdots B_{N-1}} \leq \sum_{i=1}^{N-1} \hat{\mathcal{Q}}^{\alpha}_{AB_i} - \sum_{k=2}^{N-2} \hat{\mathcal{Q}}^{\alpha}_{A|B'_1|B'_2|\cdots|B'_k}, \tag{17}$$

for all $0 \leq \alpha \leq \beta$, where $\hat{\mathcal{Q}}^{\alpha}_{AB_1} = \mathcal{Q}^{\alpha}_{AB_1}$, $\hat{\mathcal{Q}}^{\alpha}_{AB_2} = L\mathcal{Q}^{\alpha}_{AB_2}$, $\cdots$, $\hat{\mathcal{Q}}^{\alpha}_{AB_m} = L^{m-1}\mathcal{Q}^{\alpha}_{AB_m}$, $\hat{\mathcal{Q}}^{\alpha}_{AB_{m+1}} = L^{m+1}\mathcal{Q}^{\alpha}_{AB_{m+1}}$, $\cdots$, $\hat{\mathcal{Q}}^{\alpha}_{AB_{N-2}} = L^{m+1}\mathcal{Q}^{\alpha}_{AB_{N-2}}$, $\hat{\mathcal{Q}}^{\alpha}_{AB_{N-1}} = L^m\mathcal{Q}^{\alpha}_{AB_{N-1}}$, $L = (2^{\frac{\alpha}{\beta}} -$

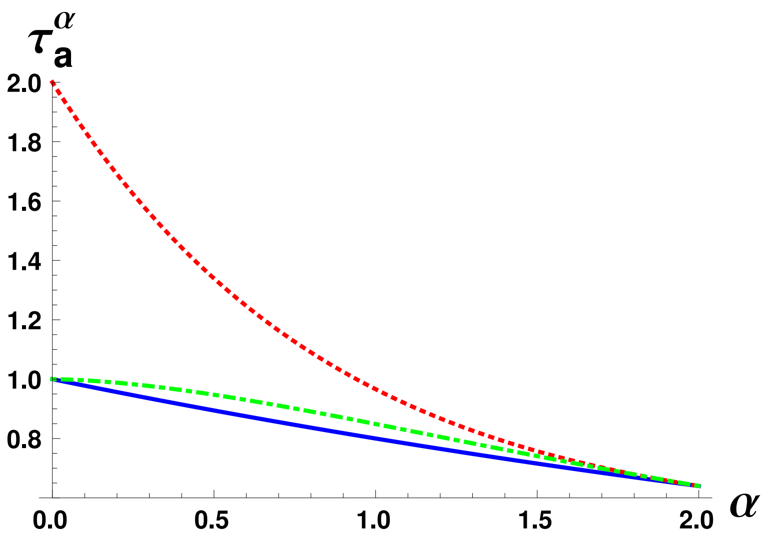

FIG. 2: Solid (blue) line is the αth power of τ_a under bipartition $A|B_1B_2B_3B_4$; Dashed (red) line is the upper bound (7); Dotted (green) line is the upper bound in (17).

1). The residual quantum correlation term $\hat{\mathcal{Q}}^{\alpha}_{A|B'_1|B'_2|\cdots|B'_{k-1}} = \max_{1\leq l\leq k}\{\hat{\mathcal{Q}}_{A|B_1|\cdots|\hat{B}_l|\cdots|B_k}\}$, $\hat{\mathcal{Q}}^{\alpha}_{A|B_1|B_2|\cdots|B_k} = \sum_{i=1}^{k}\hat{\mathcal{Q}}^{\alpha}_{AB_i} - \mathcal{Q}^{\alpha}_{A|B_1B_2\cdots B_k} - \sum_{i=2}^{k-1}\hat{\mathcal{Q}}^{\alpha}_{A|B'_1|B'_2|\cdots|B'_i}$, $2\leq k\leq N-2$, $1\leq l\leq k$.

As an example, let us consider consider again the the concurrence of the state (10). From our inequality (7) in Theorem 2, we have $\tau_a^{\alpha}(|\psi\rangle_{A|B_1B_2B_3B_4}) \leq 4(\frac{2}{5})^{\alpha} - 3(\frac{1}{2})^{\alpha} + (\frac{\sqrt{3}}{2})^{\alpha}$. From the inequality (17) in Theorem 4, we have $\tau_a^{\alpha}(|\psi\rangle_{A|B_1B_2B_3B_4}) \leq 3(\frac{2\sqrt{2}}{5})^{\alpha} - 2(\frac{2}{5})^{\alpha} - 2(\frac{1}{2})^{\frac{\alpha}{2}} + (\frac{1}{2})^{\alpha} + (\frac{\sqrt{3}}{2})^{\alpha}$. Obviously, the inequality (17) is better than the inequality in [11]. We see in Fig. 2 that the bound (7) is improved.

STRONG MONOGAMY RELATIONS FOR MULTIPARTITE QUANTUM SYSTEMS

We now study the monogamy relations for multipartite states. The monogamy relations limit the distributions of quantum correlations among the multipartite systems and play an important role in secure quantum cryptography [13] and in condensed matter physics such as the n-representability problem for fermions [14].

Monogamy and polygamy of entanglement can restrict the possible correlations between the authorized users and the eavesdroppers, thus tightening the security bounds in quantum cryptography. The optimized monogamy and polygamy relations give rise to finer characterizations of the entanglement distributions. Furthermore, to optimize the efficiency of entanglement used in quantum cryptography, finer characterizations of the entanglement distributions are preferred in some physical systems for stronger security in quantum key distribution [16].

Monogamy relations of entanglement for multiqubit some higher-dimensional quantum systems have been investigated in terms of various entanglement measures [2, 3, 5, 15, 18]. However, there are other measures such as quantum discord, quantum deficit, and entanglement of formation, which do not satisfy the monogamy relations for pure three-qubit states [19, 20]. In [21] the authors find a monotonically increasing function of quantum measures, from which a quantum correlation can always be made to be monogamous for given state. It has been proved that for arbitrary dimensional tripartite states, there exists $x_{\min}(\mathcal{Q}) \in R$ such that for any $y \geq x_{\min}(\mathcal{Q})$, a quantum correlation measure $\mathcal{Q}$ satisfies the following monogamy relation [21],

$$\mathcal{Q}^y_{A|BC} \geq \mathcal{Q}^y_{AB} + \mathcal{Q}^y_{AC}. \tag{18}$$

In the following, we denote $x = x_{\min}(\mathcal{Q})$ the minimal value such that $\mathcal{Q}^x$ satisfies the above inequality. Inequality (18) has been generalized to the N partite case for all measures of quantum correlations [22],

$$\mathcal{Q}^y_{A|B_1B_2\cdots B_{N-1}} \geq \sum_{i=1}^{N-1} \mathcal{Q}^y_{AB_i}, \tag{19}$$

for $y \geq x$, $N \geq 3$. (19) has been further improved such that for $y \geq x$, if $\mathcal{Q}_{AB_i} \geq \mathcal{Q}_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \cdots, m$, and $\mathcal{Q}_{AB_j} \leq \mathcal{Q}_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$, $\forall$ $1 \leq m \leq N-3$, $N \geq 4$, then [22],

$$\mathcal{Q}^y_{A|B_1B_2\cdots B_{N-1}} \geq \sum_{i=1}^{N-1} \hat{\mathcal{Q}}^y_{AB_i} + \sum_{k=2}^{N-2} \hat{\mathcal{Q}}^y_{A|B'_1|B'_2|\cdots|B'_k}, \tag{20}$$

for all $y \geq x$, $\hat{\mathcal{Q}}^y_{AB_1} = \mathcal{Q}^y_{AB_1}$, $\hat{\mathcal{Q}}^y_{AB_2} = K\mathcal{Q}^y_{AB_2}$, $\cdots$, $\hat{\mathcal{Q}}^y_{AB_m} = K^{m-1}\mathcal{Q}^y_{AB_m}$, $\hat{\mathcal{Q}}^y_{AB_{m+1}} = K^{m+1}\mathcal{Q}^y_{AB_{m+1}}$, $\cdots$, $\hat{\mathcal{Q}}^y_{AB_{N-2}} = K^{m+1}\mathcal{Q}^y_{AB_{N-2}}$, $\hat{\mathcal{Q}}^y_{AB_{N-1}} = K^m\mathcal{Q}^y_{AB_{N-1}}$ and $K = \frac{y}{x}$. The residual quantum correlation term $\hat{\mathcal{Q}}^y_{A|B'_1|B'_2|\cdots|B'_{k-1}} = \max_{1\leq l\leq k}\{\hat{\mathcal{Q}}_{A|B_1|\cdots|\hat{B}_l|\cdots|B_k}\}$ (where $\hat{B}_l$ stands for B_l being omitted in the sub-indices), $\hat{\mathcal{Q}}^y_{A|B_1|B_2|\cdots|B_k} = \mathcal{Q}^y_{A|B_1B_2\cdots B_k} - \sum_{i=1}^{k} \hat{\mathcal{Q}}^y_{AB_i} - \sum_{i=2}^{k-1} \hat{\mathcal{Q}}^y_{A|B'_1|B'_2|\cdots|B'_i}$, $2 \leq k \leq N-2$, $1 \leq l \leq k$.

In fact, as a kind of characterization of the quantum correlation distribution among the subsystems, the monogamy inequalities satisfied by the quantum correlations can be further refined and become tighter.

[**Lemma 2**]. For any $d_1 \otimes d_2 \otimes d_3$ mixed state ρ_{ABC}, if $\mathcal{Q}_{AB} \geq \mathcal{Q}_{AC}$, we have

$$\mathcal{Q}^y_{A|BC} \geq \mathcal{Q}^y_{AB} + L\mathcal{Q}^y_{AC}, \tag{21}$$

for all $y \geq x$, where $L = (2^{\frac{y}{x}} - 1)$.

[Proof]. For arbitrary $d_1 \otimes d_2 \otimes d_3$ tripartite state ρ_{ABC}. If $\mathcal{Q}_{AB} \geq \mathcal{Q}_{AC}$, we have

$$\begin{aligned}
\mathcal{Q}^y_{A|BC} &\geq (\mathcal{Q}^x_{AB} + \mathcal{Q}^x_{AC})^{\frac{y}{x}} = \mathcal{Q}^y_{AB}\left(1 + \frac{\mathcal{Q}^x_{AC}}{\mathcal{Q}^x_{AB}}\right)^{\frac{y}{x}} \\
&\geq \mathcal{Q}^y_{AB}\left[1 + L\left(\frac{\mathcal{Q}^x_{AC}}{\mathcal{Q}^x_{AB}}\right)^{\frac{y}{x}}\right] \\
&= \mathcal{Q}^y_{AB} + L\mathcal{Q}^y_{AC},
\end{aligned}$$

where the first inequality is due to (18), the second inequality is due to the inequality $(1+t)^x \geq 1 + (2^x - 1)t^x$ for $x \geq 1,\ 0 \leq t \leq 1$ [5]. ■

[**Theorem 5**]. For any $d \otimes d_1 \otimes \cdots \otimes d_{N-1}$ state $\rho_{AB_1 \cdots B_{N-1}}$, if $\mathcal{Q}_{AB_i} \geq \mathcal{Q}_{A|B_{i+1} \cdots B_{N-1}}$ for $i = 1, 2, \cdots, m$, and $\mathcal{Q}_{AB_j} \leq \mathcal{Q}_{A|B_{j+1} \cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$, $\forall\ 1 \leq m \leq N-3$, $N \geq 4$, we have

$$\mathcal{Q}^y_{A|B_1B_2 \cdots B_{N-1}} \geq \sum_{i=1}^{N-1} \widetilde{\mathcal{Q}}^y_{AB_i} + \sum_{k=2}^{N-2} \widetilde{\mathcal{Q}}^y_{A|B'_1|B'_2|\cdots|B'_k}, \tag{22}$$

for all $y \geq x$, where $\widetilde{\mathcal{Q}}^y_{AB_1} = \mathcal{Q}^y_{AB_1}$, $\widetilde{\mathcal{Q}}^y_{AB_2} = L\mathcal{Q}^y_{AB_2}$, $\cdots$, $\widetilde{\mathcal{Q}}^y_{AB_m} = L^{m-1}\mathcal{Q}^y_{AB_m}$, $\widetilde{\mathcal{Q}}^y_{AB_{m+1}} = L^{m+1}\mathcal{Q}^y_{AB_{m+1}}$, $\cdots$, $\widetilde{\mathcal{Q}}^y_{AB_{N-2}} = L^{m+1}\mathcal{Q}^y_{AB_{N-2}}$, $\widetilde{\mathcal{Q}}^y_{AB_{N-1}} = L^m\mathcal{Q}^y_{AB_{N-1}}$, $L = (2^{\frac{y}{x}} - 1)$. The residual quantum correlation term $\widetilde{\mathcal{Q}}^y_{A|B'_1|B'_2|\cdots|B'_{k-1}} = \max_{1 \leq l \leq k}\{\widetilde{\mathcal{Q}}_{A|B_1|\cdots|\hat{B}_l|\cdots|B_k}\}$, $\widetilde{\mathcal{Q}}^y_{A|B_1|B_2|\cdots|B_k} = \mathcal{Q}^y_{A|B_1B_2\cdots B_k} - \sum_{i=1}^k \widetilde{\mathcal{Q}}^y_{AB_i} - \sum_{i=2}^{k-1} \widetilde{\mathcal{Q}}^y_{A|B'_1|B'_2|\cdots|B'_i}$, $2 \leq k \leq N-2$, $1 \leq l \leq k$.

[Proof]. By using the Lemma 2 repeatedly, one gets

$$\begin{aligned}
\mathcal{Q}^y_{A|B_1B_2\cdots B_{N-1}} &\geq \mathcal{Q}^y_{AB_1} + L\mathcal{Q}^\alpha_{A|B_2\cdots B_{N-1}} \\
&\geq \mathcal{Q}^y_{AB_1} + L\mathcal{Q}^\alpha_{AB_2} + L^2\mathcal{Q}^y_{A|B_3\cdots B_{N-1}} \\
&\geq \cdots \geq \mathcal{Q}^y_{AB_1} + L\mathcal{Q}^y_{AB_2} + \cdots \\
&\quad + L^{m-1}\mathcal{Q}^y_{AB_m} + L^m\mathcal{Q}^y_{A|B_{m+1}\cdots B_{N-1}}.
\end{aligned} \tag{23}$$

As $\mathcal{Q}_{AB_j} \leq \mathcal{Q}_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$, by (15) we get

$$\begin{aligned}
\mathcal{Q}^y_{A|B_{m+1}\cdots B_{N-1}} &\geq L\mathcal{Q}^y_{AB_{m+1}} + \mathcal{Q}^y_{A|B_{m+2}\cdots B_{N-1}} \\
&\geq L(\mathcal{Q}^y_{AB_{m+1}} + \cdots + \mathcal{Q}^y_{AB_{N-2})} + \mathcal{Q}^y_{AB_{N-1}}.
\end{aligned} \tag{24}$$

Combining (23) and (24), we have

$$\mathcal{Q}^y_{A|B_1B_2\cdots B_{N-1}} \geq \sum_{i=1}^{N-1} \widetilde{\mathcal{Q}}^y_{AB_i}. \tag{25}$$

Suppose that Theorem 5 holds for $N = n$, i.e.,

$$\mathcal{Q}^y_{A|B_1B_2\cdots B_{n-1}} \geq \sum_{i=1}^{n-1} \widetilde{\mathcal{Q}}^y_{AB_i} + \widetilde{\mathcal{Q}}^y_{A|B_1'|B_2'} + \cdots + \widetilde{\mathcal{Q}}^y_{A|B_1'|B_2'|\cdots|B_{n-2}'}. \tag{26}$$

Then for $N = n+1$, we have

$$\begin{aligned}
&\sum_{i=1}^{n} \widetilde{\mathcal{Q}}^y_{AB_i} + \widetilde{\mathcal{Q}}^y_{A|B_1'|B_2'} + \cdots + \widetilde{\mathcal{Q}}^y_{A|B_1'|B_2'|\cdots|B_{n-1}'} \\
&\leq \widetilde{\mathcal{Q}}^y_{A|B_1'B_2'\cdots B_{n-1}'} + \widetilde{\mathcal{Q}}^y_{AB_n'} \\
&\leq \mathcal{Q}^y_{A|B_1B_2\cdots B_n},
\end{aligned}$$

where B_n' is the complementary of $B_1'B_2', \cdots, B_{n-1}'$ in the subsystem $B_1B_2, \cdots, B_n$. The first inequality is due to (26). By (25) we get the last inequality. ■

Example 2. For the concurrence of the W state,

$$|W\rangle_{A|B_1B_2B_3} = \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle), \tag{27}$$

we have $x = 2$, $C_{AB_i} = \frac{1}{2}$, $i = 1, 2, 3$, and $C_{A|B_1B_2} = C_{A|B_1B_3} = C_{A|B_2B_3} = \frac{\sqrt{2}}{2}$. From the inequality (20), one has $\hat{C}^y_{A|B_1|B_2} = \hat{C}^y_{A|B_1|B_3} = \hat{C}^y_{A|B_2|B_3} = (\frac{\sqrt{2}}{2})^y - (1 + \frac{y}{2})(\frac{1}{2})^y$. Hence the lower bound of $C^y_{A|B_1B_2B_3}$ is $\sum_{i=1}^{3} \hat{C}^y_{AB_i} + \hat{C}^y_{A|B_1|B_2} = (\frac{\sqrt{2}}{2})^y + \frac{y}{2}(\frac{1}{2})^y$. From the inequality (22) in Theorem 5, we have $\widetilde{C}^y_{A|B_1|B_2} = \widetilde{C}^y_{A|B_1|B_3} = \widetilde{C}^y_{A|B_2|B_3} = (\frac{\sqrt{2}}{2})^y - (\frac{1}{2})^{\frac{y}{2}}$. The lower bound of $C^y_{A|B_1B_2B_3}$ is $\sum_{i=1}^{3} \widetilde{C}^y_{AB_i} + \widetilde{C}^y_{A|B_1|B_2} = (\frac{\sqrt{2}}{2})^y + (2^{\frac{y}{2}} - 1)(\frac{1}{2})^y$. One can see that our result is better than (20) in [22], see Fig. 3.

CONCLUSION

Monogamy and polygamy inequalities are the key features of multipartite entanglement, which distinguish the quantum from the classical correlations. We have investigated the monogamy and polygamy relations satisfied by arbitrary quantum correlation measures for arbitrary multipartite quantum states. Similar to the three tangle of concurrence, we have introduced the αth $(0 \leq \alpha \leq \beta)$ power of the residual quantum correlation. In term of the residual quantum correlations, analytical polygamy inequalities have been presented,

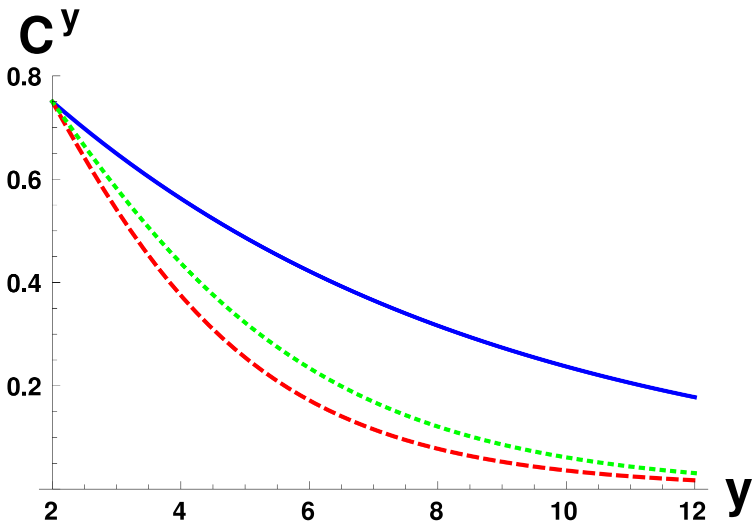

FIG. 3: Solid (blue) line is the yth power of concurrence under bipartition A and $B_1B_2B_3$; Dashed (red) line for the lower bound (20) in [22]; Dotted (green) line for the lower bound in (22).

which are shown to be tighter than the existing ones. Similarly, we have obtained the strong monogamy relations that are also better than all the existing ones. Detailed examples have been given for illustration. The novel residual quantum correlation we introduced may also contribute to improve other relations satisfied by quantum correlation measures.

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