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Improved lower bounds of concurrence and convex-roof extended negativity based on Bloch representations

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Quantum entanglement plays significant roles in quantum information processing. Estimating quantum entanglement is an essential and difficult problem in the theory of quantum entanglement. We study two main measures of quantum entanglement: concurrence and convex-roof extended negativity. Based on the improved separability criterion from the Bloch representation of density matrices, we derive analytical lower bounds of the concurrence and the convex-roof extended negativity for arbitrary dimensional bipartite quantum systems. We show that these bounds are better than the existing ones by detailed examples.

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As a striking feature of quantum systems [1], quantum entanglement plays an essential role in many quantum information processing [2–4] such as quantum computation [5], quantum teleportation[6, 7], dense coding [8], quantum cryptographic schemes [9, 10], entanglement swapping [11–13], remote states preparation [14, 15], and in many pioneering experiments. To quantify entanglement, various measures have been proposed in recent years [16–18, 21, 22]. The concurrence and the convex-roof extended negativity (CREN) are two of the well defined entanglement measures. However, due to the extremization involved in the computation of these entanglement measures, only a few analytic formulas have been found for some very special quantum states. To estimate the concurrence and CREN for general bipartite states, efforts have been made towards the analytical lower bounds of concurrence and CREN [23–28]. In Ref. [17], Vicente provided analytical lower bounds of concurrence in terms of the local uncertainty relations (LUR) and correlation matrix (CM) separability criteria. Recently, we presented a lower bound of concurrence for four-partite systems in terms of the concurrences for $M\,(2 \leq M \leq 3)$ partite sub-systems. And analytical lower bounds for any tripartite quantum states [19] and for four-qubit mixed quantum sates have been derived [20].

Generally, one may expect to derive lower bounds of entanglement from separability criteria. Based on the Bloch representation of a quantum state, a series of separability criteria have been presented recently [17, 18, 24]. In Ref. [18], we presented an improved separability criterion based on Bloch representation of density matrices, which is shown to be more effective in detecting entanglement.

In this paper, we provide analytical lower bounds for both concurrence and CREN based on the improved separability criterion based on Bloch representation. Detailed examples are given to show that these bounds are better than the ones derived in [17] and in [29].

LOWER BOUNDS OF CONCURRENCE

For a bipartite pure state $|\varphi\rangle \in H_{AB} = H_A \otimes H_B$, where H_A (H_B) denotes the M (N)-dimensional vector space associated with the subsystem A (B) such that $M \leq N$, the concurrence is defined by $C\left(|\varphi\rangle\right) = \sqrt{2\left(1 - Tr\rho^2{}_A\right)}$, with the reduced matrix ρ_A obtained by tracing over the subsystem B. The concurrence is then extended to mixed states ρ by the convex roof:

$$C\left(\rho\right) \equiv \min_{\{p_i, |\varphi_i\rangle\}} \sum_i p_i C\left(|\varphi_i\rangle\right), \tag{1}$$

where the minimum is taken over all possible ensemble decompositions of $\rho = \sum\limits_i p_i\,|\varphi_i\rangle\,\langle\varphi_i|$, $p_i \geq 0$ and $\sum\limits_i p_i = 1$.

Under suitable local coordinates, a bipartite pure state $|\varphi\rangle$ can be written in Schmidt form,

$$|\varphi\rangle = \sum_{j=0}^{M-1} \sqrt{u_j}\,|j_A j_B,\rangle \tag{2}$$

where $\sqrt{u_j}$, $j = 1, ..., M$, are the Schmidt coefficients, $|j_A\rangle$ and $|j_B\rangle$ are the orthonormal bases in H_A and H_B, respectively. It has been shown that [16],

$$C^2\left(|\varphi\rangle\right)=2\left(1-\sum_i {u_i}^2\right)=4\sum_{i<j}u_iu_j\geq\frac{8}{M\left(M-1\right)}\left(\sum_{i<j}\sqrt{u_iu_j}\right)^2. \tag{3}$$

For a mixed state ρ, its Bloch representation has the form,

$$\rho=1/MN\left(I\otimes I+\sum\nolimits_i r_i{\lambda_i}^A\otimes I+\sum\nolimits_j s_jI\otimes{\lambda_j}^B+\sum\nolimits_{i,j}t_{ij}{\lambda_i}^A\otimes{\lambda_j}^B\right), \tag{4}$$

where $r_i=\frac{M}{2}Tr\left(\rho{\lambda_i}^A\otimes I_N\right)$, $s_j=\frac{N}{2}Tr\left(\rho I_M\otimes{\lambda_j}^B\right)$, and $t_{ij}=\frac{MN}{4}Tr\left(\rho{\lambda_i}^A\otimes{\lambda_j}^B\right)$, $\left\{{\lambda_i}^A\right\}_{i=1}^{M^2-1}$ and $\left\{{\lambda_i}^B\right\}_{i=1}^{N^2-1}$ are the traceless Hermitian generators of $SU\left(M\right)$ and $SU\left(N\right)$, respectively. Particularly, $\left\{{\lambda_i}^A\right\}$ can be given by $\{\omega_l,u_{jk},v_{jk}\}$ with $\omega_l=\sqrt{\frac{2}{(l+1)(l+2)}}\left(\sum\limits_{i=0}^{l}|i\rangle\langle i|-(l+1)\,|l+1\rangle\langle l+1|\right)$, $u_{jk}=|j\rangle\langle k|+|k\rangle\langle j|$, $v_{jk}=-i\left(|j\rangle\langle k|-|k\rangle\langle j|\right)$, $0\leq l\leq M-2$ and $0\leq j<k\leq M-1$. The matrix T with entries t_{ij} is called the CM.

In [17], Vicente et al presented a lower bound for concurrence as following,

$$C\left(\rho\right)\geq\sqrt{\frac{8}{M^3N^2\left(M-1\right)}}\left(\|T\|_{tr}-K_{MN}\right), \tag{5}$$

where $\|\cdot\|_{tr}$ stands for the trace norm, $K_{MN}=\sqrt{MN\left(M-1\right)\left(N-1\right)}\Big/2$.

In [18] we constructed the following matrix:

$$S^m_{\alpha,\beta}\left(\rho\right)=\begin{bmatrix}\alpha\beta E_{m\times m} & \beta\omega_m(s)^t\\ \alpha\omega_m\left(r\right) & T\end{bmatrix}, \tag{6}$$

where α and β are nonnegative real numbers, m is a given natural number, t stands for transpose, s (r) denotes the column vector with components given by s_i (r_i) in (4), and for any column vector x, $\omega_m\left(x\right)=\underbrace{\left(x,...,x\right)}_{m\ \ columns}$. We have shown that if a state ρ is separable, then [18]

$$\left\|S^m_{\alpha,\beta}\left(\rho\right)\right\|_{tr}\leq\frac{1}{2}\sqrt{\left(2m\beta^2+M^2-M\right)\left(2m\alpha^2+N^2-N\right)}\equiv{K'}_{MN}. \tag{7}$$

Before presenting our main result, we investigate the trace norm of $S^m_{\alpha,\beta}$ by considering the following two types of pure states.

For a general 3×3 pure state $|\varphi\rangle$ in Schmidt form, $|\varphi\rangle=\sum\limits_{j=0}^{2}\sqrt{u_j}\,|j_Aj_B\rangle$, the density matrix $\rho=\sum\limits_{j=0}^{2}\sum\limits_{k=0}^{2}\sqrt{u_ju_k}\,|j_Aj_B\rangle\langle k_Ak_B|$ has the form,

$$\rho=\begin{bmatrix}u_0&0&0&0&\sqrt{u_0u_1}&0&0&0&\sqrt{u_0u_2}\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\\sqrt{u_0u_1}&0&0&0&u_1&0&0&0&\sqrt{u_1u_2}\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\\sqrt{u_0u_2}&0&0&0&\sqrt{u_1u_2}&0&0&0&u_2\end{bmatrix}.$$

The corresponding CM matrix and $S^m_{\alpha,\beta}(\rho)$ are given by (we choose $m=1$ for simplicity), $T(\rho)=\begin{bmatrix} P & 0_{2\times 6} \\ 0_{6\times 2} & D_1 \end{bmatrix}$ and $S^m_{\alpha,\beta}(\rho)=\begin{bmatrix} A_1 & 0_{3\times 6} \\ 0_{6\times 3} & D_1 \end{bmatrix}$, where

$$P=\begin{bmatrix} \frac{9(u_0+u_1)}{4} & \frac{3\sqrt{3}(u_0-u_1)}{4} \\ \frac{3\sqrt{3}(u_0-u_1)}{4} & \frac{3(u_0+u_1+4u_2)}{4} \end{bmatrix},$$

$$A_1=\begin{bmatrix} \alpha\beta & \frac{3\beta(u_0-u_1)}{2} & \frac{\sqrt{3}\beta(u_0+u_1-2u_2)}{2} \\ \frac{3\alpha(u_0-u_1)}{2} & \frac{9(u_0+u_1)}{4} & \frac{3\sqrt{3}(u_0-u_1)}{4} \\ \frac{\sqrt{3}\alpha(u_0+u_1-2u_2)}{2} & \frac{3\sqrt{3}(u_0-u_1)}{4} & \frac{3(u_0+u_1+4u_2)}{4} \end{bmatrix},$$

and

$$D_1=diag\left(\frac{9\sqrt{u_0u_1}}{2},\frac{9\sqrt{u_0u_2}}{2},\frac{9\sqrt{u_1u_2}}{2},\frac{9\sqrt{u_0u_1}}{2},-\frac{9\sqrt{u_0u_2}}{2},-\frac{9\sqrt{u_1u_2}}{2}\right).$$

While for a separable pure state, $\rho^{sep}=\sum_{j=0}^{2}u_j|j_Aj_B\rangle\langle j_Aj_B|$, we get $\rho^{sep}=diag\left(u_0,0,0,0,u_1,0,0,0,u_2\right)$, $T\left(\rho^{sep}\right)=\begin{bmatrix} A_2 & 0_{2\times 6} \\ 0_{6\times 2} & 0_{6\times 6} \end{bmatrix}$, and $S^m_{\alpha,\beta}\left(\rho^{sep}\right)=\begin{bmatrix} A_1 & 0_{3\times 6} \\ 0_{6\times 3} & 0_{6\times 6} \end{bmatrix}$, where

$$A_2=\begin{bmatrix} \frac{9(u_0+u_1)}{4} & \frac{3\sqrt{3}(u_0-u_1)}{4} \\ \frac{3\sqrt{3}(u_0-u_1)}{4} & \frac{3(u_0+u_1+4u_2)}{4} \end{bmatrix},$$

$0_{m\times n}$ denotes an $m\times n$ zero matrix.

Noticing that both the matrices $T(\rho)$ and $S^m_{\alpha,\beta}(\rho)$ are block-diagonal, we have the following relations,

$$\|T\|_{tr}=\|T^{sep}\|_{tr}+9\left(\sqrt{u_0u_1}+\sqrt{u_0u_2}+\sqrt{u_1u_2}\right) \tag{8}$$

and

$$\left\|S^m_{\alpha,\beta}(\rho)\right\|_{tr}=\left\|S^m_{\alpha,\beta}\left(\rho^{sep}\right)\right\|_{tr}+9\left(\sqrt{u_0u_1}+\sqrt{u_0u_2}+\sqrt{u_1u_2}\right). \tag{9}$$

Similar relations hold for higher dimensional case. For 4×4 pure states, one has

$$\|T(\rho)\|_{tr}=\|T^{sep}\|_{tr}+16\sum_{j<k}\sqrt{u_ju_k} \tag{10}$$

and

$$\left\|S^m_{\alpha,\beta}(\rho)\right\|_{tr}=\left\|S^m_{\alpha,\beta}\left(\rho^{sep}\right)\right\|_{tr}+16\sum_{j<k}\sqrt{u_ju_k}\,. \tag{11}$$

With the above analysis, we have the following theorem.

Theorem 1: For any quantum state $\rho\in H_{AB}$, and any α,β,m defined in (6), we have

$$C\left(\rho\right)\geq\sqrt{\frac{8}{M^3N^2\left(M-1\right)}}\left(\left\|S^m_{\alpha,\beta}(\rho)\right\|_{tr}-K'_{MN}\right). \tag{12}$$

Proof. From the Schmidt form (2) of a pure state $|\phi\rangle\in H_{AB}$, we have $\rho_\varphi=|\varphi\rangle\langle\varphi|=\rho^{sep}+\varepsilon=\rho^{sep}+\frac{1}{2}\sum_{j<k}\sqrt{u_ju_k}\left(u_{jk}\otimes u_{jk}-v_{jk}\otimes v_{jk}\right)$, where $\rho^{sep}=\sum_j u_j\left|j_Aj_B\right\rangle\left\langle j_Aj_B\right|$ and $\varepsilon=\sum_{j\neq k}\left|j_Aj_B\right\rangle\left\langle j_Aj_B\right|$. Since ρ^{sep} is diagonal, its Bloch representation is just given by the ω_l's. Therefore, the matrix $S^m_{\alpha,\beta}(\rho)$ is block-diagonal. Thus we get

$$\left\|S^m_{\alpha,\beta}(\rho)\right\|_{tr}=\left\|S^m_{\alpha,\beta}\left(\rho^{sep}\right)\right\|_{tr}+MN\sum\nolimits_{j<k}\sqrt{u_ju_k}\leq K'_{MN}+MN\sum\nolimits_{j<k}\sqrt{u_ju_k}. \tag{13}$$

From (3) we obtain

$$\begin{aligned} C(|\psi\rangle) &\geq \sqrt{\frac{8}{M(M-1)}}\left(\sum_{i<j}\sqrt{u_i u_j}\right) \\ &\geq \sqrt{\frac{8}{M^3 N^2 (M-1)}}\left(\left\|S^m_{\alpha,\beta}(\rho)\right\|_{tr} - K'_{MN}\right). \end{aligned} \tag{14}$$

Assume $\rho = \sum_n p_n |\varphi_n\rangle\langle\varphi_n|$, $\sum p_n = 1$, be the optimal ensemble decomposition such that $C(\rho_\varphi) = \sum_n p_n C(\rho_{\varphi_n})$. We have

$$\begin{aligned} C(\rho) &= \sum_n p_n C(\rho_{\varphi_n}) \\ &\geq \sqrt{\frac{8}{M^3 N^2 (M-1)}}\sum_n p_n\left(\left\|S^m_{\alpha,\beta}(\rho_{\phi_n})\right\|_{tr} - K'_{MN}\right). \\ &\geq \sqrt{\frac{8}{M^3 N^2 (M-1)}}\left(\left\|S^m_{\alpha,\beta}(\rho)\right\|_{tr} - K'_{MN}\right), \end{aligned} \tag{15}$$

which ends the proof. ■

The following two examples show that the lower bound in Theorem 1 is more effective in entanglement detection.

Example 1: Consider the following 2×4 state,

$$\rho_x = x|\xi\rangle\langle\xi| + (1-x)\rho,$$

where ρ is the bound entangled state considered in [18]:

$$\rho = \begin{bmatrix} b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+b}{2} & 0 & 0 & \frac{\sqrt{1-b^2}}{2} \\ b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & \frac{\sqrt{1-b^2}}{2} & 0 & 0 & \frac{1+b}{2} \end{bmatrix}, \tag{16}$$

with $0 < b < 1$, $|\xi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is the singlet state. By choosing $m = 1$, $\alpha = \sqrt{\frac{2}{M(M-1)}}$ and $\beta = \sqrt{\frac{2}{N(N-1)}}$, we can directly show that the lower bound given by Theorem 1 is better than that in [17], see Fig. 1.

Example 2: Consider the 3×3 PPT entangled state constructed in [16]: $\rho = \frac{1}{4}\left(I - \sum_i |\varphi_i\rangle\langle\varphi_i|\right)$, where $|\varphi_0\rangle = \frac{|0\rangle(|0\rangle-|1\rangle)}{\sqrt{2}}$, $|\varphi_1\rangle = \frac{(|0\rangle-|1\rangle)|2\rangle}{\sqrt{2}}$, $|\varphi_2\rangle = \frac{|2\rangle(|1\rangle-|2\rangle)}{\sqrt{2}}$, $|\varphi_3\rangle = \frac{(|1\rangle-|2\rangle)|0\rangle}{\sqrt{2}}$, $|\varphi_4\rangle = \frac{(|0\rangle+|1\rangle+|2\rangle)(|0\rangle+|1\rangle+|2\rangle)}{3}$. By using the Theorem 1 in [17] one gets $C(\rho) \geq 0.052$. From our Theorem 1 we have $C(\rho) \geq 0.116$, which improves the bound in [17].

LOWER BOUND FOR CONVEX-ROOF EXTENDED NEGATIVITY

In this section we consider the convex-roof extended negativity. For a pure state $|\psi\rangle \in H_{AB}$, the CREN is defined to be the negativity [30]:

$$N(|\psi\rangle) = \frac{\left\|(|\psi\rangle\langle\psi|)^{T_B}\right\|_{tr} - 1}{M-1}, \tag{17}$$

where $(|\psi\rangle\langle\psi|)^{T_B}$ is the partial transpose of $|\psi\rangle\langle\psi|$.

For a mixed state, the CREN is defined by the convex-roof extension of the negativity [21]. The CREN gives a perfect discrimination between PPT bound entangled states and separable states. It is a good entanglement measure with

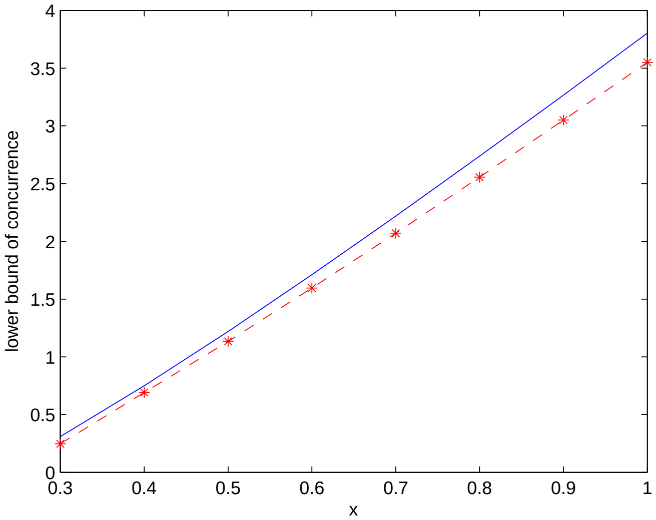

FIG. 1: Lower bounds of concurrence for state ρ_x. The blue solid line is for the bound given by Theorem 1, while the red dashed line for the bound given by CM criterion in [17]

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the property of entanglement monotone: CREN does not increase under local operations and classical communication [21].

For a mixed bipartite quantum state ρ the CREN is defined by

$$\mathcal{N}(\rho)=\min_{\rho=\sum_k p_k|\psi_k\rangle\langle\psi_k|}\sum_k p_k\mathcal{N}(|\psi_k\rangle). \tag{18}$$

$\mathcal{N}(\rho)$ is zero if and only if ρ is separable. Similar to concurrence, the computation of $\mathcal{N}(\rho)$ is also formidably hard.

Based on the improved separability criterion in terms of Bloch representation, we have the following lower bound of the $\mathcal{N}(\rho)$.

Theorem 2: For any quantum state $\rho\in H_{AB}$, we have

$$\mathcal{N}(\rho)\geq\frac{2}{MN\left(M-1\right)}\left(\left\|S^m_{\alpha,\beta}\left(\rho\right)\right\|_{tr}-K'_{MN}\right). \tag{19}$$

Proof. First for a pure state $|\varphi\rangle$ with Schmidt form $|\varphi\rangle=\sum\limits_{j=0}^{M-1}\sqrt{u_j}\,|j_Aj_B\rangle$, one has

$$N\left(|\varphi\rangle\right)=2\left(\sum_{j<k}\sqrt{u_ju_k}\right)/(M-1). \tag{20}$$

By (13) we have

$$\sum_{j<k}\sqrt{u_ju_k}\geq\frac{\left\|S^m_{\alpha,\beta}(|\varphi\rangle)\right\|_{tr}-K'_{MN}}{MN}. \tag{21}$$

Let $\sum\limits_{\alpha}p_\alpha\left|\varphi_\alpha\right\rangle\left\langle\varphi_\alpha\right|$ be the optimal decomposition for ρ such that $\mathcal{N}(\rho)=\sum\limits_{\alpha}p_\alpha N\left(|\varphi_\alpha\rangle\right)$. We obtain

$$\mathcal{N}(\rho)=\sum_\alpha p_\alpha N\left(|\varphi_\alpha\rangle\right)\geq\sum_\alpha p_\alpha\frac{2\left(\left\|S^m_{\alpha,\beta}(|\varphi_\alpha\rangle)\right\|_{tr}-K'_{MN}\right)}{MN(M-1)}\geq\frac{2}{MN\left(M-1\right)}\left(\left\|S^m_{\alpha,\beta}\left(\rho\right)\right\|_{tr}-K'_{MN}\right), \tag{22}$$

which ends the proof of Theorem 2. ■

Example 3: Let us consider the isotropic states [32],

$$\rho_x = \frac{1-x}{d^2} I_d \otimes I_d + x \left|\varphi_+\right\rangle \left\langle\varphi_+\right|, \tag{23}$$

where $d = M = N$, $|\varphi_+\rangle = \frac{1}{\sqrt{d}} \sum\limits_{i=1}^{d-1} |ii\rangle$. In the case of $d = 3$, we choose $m = \alpha = \beta = 1$ for simplicity. Our lower bound (19) shows that $\mathcal{N}_m(\rho) \geq \frac{4}{3}x - \frac{1}{9}$, which is better than the result in [29], see Fig. 2.

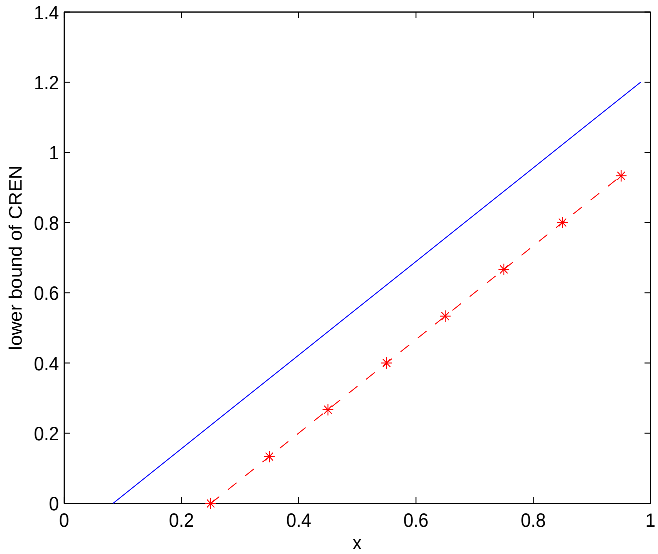

FIG. 2: Lower bounds of CREN for the state ρ_x. The blue solid line is the bound given by our Theorem 2, while the red dashed line is for the bound given by [29]

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CONCLUSIONS AND DISCUSSIONS

It is an essential problem in quantum entanglement theory to estimate the concurrence and the CREN. We have derived analytical lower bounds for both concurrence and CREN by using the extended correlation matrices. The lower bounds are shown to be greater than the ones given in [17] and [29]. In fact, by adjusting the parameters α, β and m in our lower bounds, better optimal lower bounds can be obtained. Moreover, by using the generalized correlation matrices for multipartite quantum systems, our approach can be also applied to the study of entanglement lower bounds for multipartite quantum systems.

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