Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Monoidally graded manifolds

(revised version: December 2022)

by

Shuhan Jiang

Preprint no.: 20 2022

Monoidally Graded Manifolds

Shuhan Jiang[1]

[1]Max Planck Institute for Mathematics in the Sciences, 04103 Leipzig

Abstract

We give a generalization of the theory of $\mathbb{Z}_2$-graded manifolds to a theory of $\mathcal{I}$-graded manifolds, where $\mathcal{I}$ is a commutative semi-ring with some additional properties. We prove Batchelor's theorem in this generalized setting. To our knowledge, such a proof is still missing except for some special cases.

1 Introduction

The notion of a supermanifold appeared in the late 70s when mathematicians tried to understand the concept of supersymmetry proposed by physicists [1]. It extends the notion of a manifold M naturally by attaching Grassmann algebras locally to M. The mysterious anticommutativity property of a fermionic field over M can be then interpreted in terms of the anticommutativity of Grassmann algebras. When multiplying two fermionic fields, one gets a bosonic field. This process can be tracked by assigning $0 \in \mathbb{Z}_2$ to bosonic fields and $1 \in \mathbb{Z}_2$ to fermionic fields. Hence, supermanifolds are also called $\mathbb{Z}_2$-graded manifolds. Though the grading in this case is merely used to distinguish commutative and anticommutative objects.

In 1982, Witten published a seminal paper relating Morse theory to supersymmetric quantum mechanics [2]. It was realized since then that there exists a very deep connection between supersymmetric theories in physics and cohomology theories in mathematics. To establish such a connection, one needs to update the language of $\mathbb{Z}_2$-graded manifolds to the language of $\mathbb{Z}$-graded (or graded) manifolds [3, 4]. One major achievement in that direction is the AKSZ formalism of topological quantum field theories [5], where the topological sigma models [6] are reinterpreted in the language of so-called Q-manifolds.[1]

In cohomological field theories (or topological quantum field theories of Witten type), one can obtain useful invariants of smooth manifolds by studying observables $\mathcal{O}^{(p)}$ satisfying the following descent equations [6, 7]

$$Q(\mathcal{O}^{(p)}) = d(\mathcal{O}^{(p-1)}), \tag{1.1}$$

for $p \geq 1$ with $Q\mathcal{O}^{(0)} = 0$, where d is the de Rham differential. (1.1) is equivalent to saying that $\mathcal{O} := \sum_p \mathcal{O}^{(p)}$ is closed in the total complex of some bicomplex with horizontal differential d and vertical differential Q. Such a bicomplex can be obtained by applying a "change of coordinates" to

[1]A Q-manifold is a graded manifold equipped with a vector field Q of degree 1 satisfying $Q^2 = 0$.

the variational bicomplex of a fiber bundle [8]. It is then also interesting to study $\mathbb{Z} \times \mathbb{Z}$-graded (or bigraded) manifolds.

In this paper, we follow the algebraic-geometric approaches in [1, 9–13] to give a definition of $\mathcal{I}$-graded manifolds, where $\mathcal{I}$ is an arbitrary commutative semi-ring with some additional properties. We also apply the techniques in [10] to give a proof of Batchelor's theorem, i.e., that every $\mathcal{I}$-graded manifold can be obtained from an $\mathcal{I}$-graded vector bundle. To our knowledge, such a proof is still missing except for some special cases [14–16].

2 Commutative Monoids and Parity Functions

Let $(\mathcal{I}, 0, +)$ be a commutative monoid. Let $\mathbb{Z}_q$ denote the cyclic group of order q.

Definition 2.1. A parity function is a (non-trivial) monoid homomorphism $p : \mathcal{I} \to \mathbb{Z}_2$.

Not every $\mathcal{I}$ has a non-trivial parity function. For example, there is no non-trivial homomorphism from $\mathbb{Z}_q$ to $\mathbb{Z}_2$ when q is a odd. Let $\mathcal{I}_a$ denote $p^{-1}(a)$ for $a \in \mathbb{Z}_2$.[2] We have $\mathcal{I}_a + \mathcal{I}_b \subseteq \mathcal{I}_{a+b}$. Recall that an element x in $\mathcal{I}$ is called cancellative if $x + y = x + z$ implies $y = z$ for all y and z in $\mathcal{I}$. Suppose that there is a cancellative element in $\mathcal{I}_1$. It is easy to see that such an element induces an injective map from $\mathcal{I}_a$ to $\mathcal{I}_{a+1}$. It follows from the Cantor-Bernstein theorem that there exists a bijection between $\mathcal{I}_0$ and $\mathcal{I}_1$. A monoid is called cancellative if every element in it is cancellative. We have shown that

Proposition 2.1. *Let $\mathcal{I}$ be an commutative cancellative monoid. If $\mathcal{I}$ has a non-trivial parity function p, then the submonoid $\mathcal{I}_0$ and its complement $\mathcal{I}_1$ have the same cardinality.*

Remark 2.1. In the finite case, proposition 2.1 is no longer true if we drop the cancellative condition. For example, we can consider the commutative monoid defined by the following table. A non-trivial p is defined by setting $p(0) = p(b) = 0$ and $p(a) = 1$.

	0	a	b
0	0	a	b
a	a	b	a
b	b	a	b

Table 2.1: A commutative non-cancellative monoid of order 3.

The question now is, given an appropriate commutative cancellative monoid $\mathcal{I}$, how can one construct a parity function for it? If $\mathcal{I}$ is a finite, it is not hard to show that $\mathcal{I}$ is actually an abelian group. The fundamental theorem of finite abelian groups then tells us that $\mathcal{I}$ is isomorphic to a direct product of cyclic groups of prime-power order. By Proposition 2.1, one of these cyclic groups must be $\mathbb{Z}_{2^k}$, $k \geq 1$. We can write $\mathcal{I} = \mathbb{Z}_{2^k} \times \cdots$ and define p by sending $(x, \cdots) \in \mathcal{I}$ to $a - 1$ (mod 2), where a is the order of $x \in \mathbb{Z}_{2^k}$. If $\mathcal{I}$ is infinite, the construction of p is hard, perhaps not possible in general. However, one can easily work out the case when $\mathcal{I}$ is free. ($\mathcal{I}$ is then cancellative, but not a group.) Let $\mathcal{I}_0$ be the submonoid of elements generated by even number of generators. Let $\mathcal{I}_1$ be the subset of elements generated by odd number of generators. Note that

[2] We say that $\mathcal{I}_0$ is the even part of $\mathcal{I}$, and that $\mathcal{I}_1$ is the odd part of $\mathcal{I}$. We also say that an element of $\mathcal{I}_a$ has parity a for $a = 0, 1$.

$\mathcal{I}_a + \mathcal{I}_b \subseteq \mathcal{I}_{a+b}$. We obtain a parity function which sends elements in $\mathcal{I}_a$ to a. As an example, let $\mathcal{I}$ be $\mathbb{N}$, the monoid of natural numbers under addition. p is then defined by sending even numbers to 0 and odd numbers to 1.

Let $K(\mathcal{I})$ denote the Grothendieck group of $\mathcal{I}$. Recall that it can be constructed as follows. Let $\sim$ be the equivalence relation on $\mathcal{I} \times \mathcal{I}$ defined by $(a_1, a_2) \sim (b_1, b_2)$ if there exists a $c \in \mathcal{I}$ such that $a_1 + b_2 + c = a_2 + b_1 + c$. The quotient $K(\mathcal{I}) = \mathcal{I} \times \mathcal{I}/\sim$ has a group structure by $[(a_1, a_2)] + [(b_1, b_2)] = [(a_1 + b_1, a_2 + b_2)]$.

Proposition 2.2. *Let p be a parity function for $\mathcal{I}$. The map*

$$\begin{aligned} p' : K(\mathcal{I}) &\to \mathbb{Z}_2 \\ [(a_1, a_2)] &\mapsto p(a_1) + p(a_2) \end{aligned}$$

is well-defined and gives a parity function for $K(\mathcal{I})$.

Remark 2.2. When $\mathcal{I}$ is cancellative, it can be seen as a submonoid of $K(\mathcal{I})$ by the embedding

$$\begin{aligned} \iota : \mathcal{I} &\to K(\mathcal{I}) \\ a &\mapsto [(a, 0)]. \end{aligned}$$

For this reason, we sometimes simply write $a-b$ to denote $[(a, b)] \in K(\mathcal{I})$. The cancellative property is not necessary for the proof of Proposition 2.2. But it guarantees the non-triviality of p', since p' restricted to $\mathcal{I}$ must coincide with p.

Proof. Let (a_1, a_2) and (b_1, b_2) represent the same element of $K(\mathcal{I})$, i.e., there exist some c such that $a_1 + b_2 + c = a_2 + b_1 + c$. One then concludes that $a_1 + b_2$ and $a_2 + b_1$ must have the same parity. Note that, for $a, b \in \mathbb{Z}_2$, $a = b$ if and only if $a + b = 0$. We have

$$p'([(a_1, a_2)]) + p'(([b_1, b_2)]) = p(a_1 + b_2) + p(a_2 + b_1) = 0.$$

Hence $p'([a_1, a_2]) = p'([b_1, b_2])$. □

As an example, consider $K(\mathbb{N}) = \mathbb{Z}$, the monoid of integers under addition. The parity function p' induced from the parity function p for $\mathbb{N}$ again sends even numbers to 0 and odd numbers to 1.

3 Monoidally Graded Ringed Spaces

Let R be a commutative ring. Let $\mathcal{I}$ be a countable commutative cancellative monoid equipped with a parity function p.

Definition 3.1. An $\mathcal{I}$-graded R-module is an R-module V with a family of sub-modules $\{V_i\}_{i \in \mathcal{I}}$ indexed by $\mathcal{I}$ such that $V = \bigoplus_{i \in \mathcal{I}} V_i$. $v \in V$ is said to be homogeneous if $v \in V_i$ for some $i \in \mathcal{I}$. We use $d(v)$ to denote the degree of v, $d(v) = i$.

Given two $\mathcal{I}$-graded R-modules V and W, we make the direct sum $V \oplus W$ and the tensor product $V \otimes W$ into $\mathcal{I}$-graded R-modules by setting

$$V \oplus W = \bigoplus_{i \in \mathcal{I}} (V_i \oplus W_i), \quad V \otimes W = \bigoplus_{k \in \mathcal{I}} \left(\bigoplus_{i+j=k} V_i \otimes W_j \right).$$

We can also make the space $\mathrm{Hom}(V,W)$ of R-linear maps from V to W into a $K(\mathcal{I})$-graded R-module by setting

$$\mathrm{Hom}(V,W)=\bigoplus_{\alpha\in K(\mathcal{I})}\mathrm{Hom}(V,W)_\alpha,\quad \mathrm{Hom}(V,W)_\alpha=\{f\in\mathrm{Hom}(V,W)|f(V_i)\subset W_j,[(j,i)]=\alpha\}.$$

A morphism from V to W is just an element of $\mathrm{Hom}(V,W)_0$, i.e., an R-linear map of degree 0.

Remark 3.1. $\mathrm{Hom}(V,W)$ is in general not $\mathcal{I}$-graded. This is because that we should assign degree "$j-i$" to a map f which maps elements in V_i to elements in $w\in W_j$. But the minus operation does make sense for a general monoid $\mathcal{I}$. So we have to work with $K(\mathcal{I})$, the group completion of $\mathcal{I}$. Note that $V^*=\mathrm{Hom}(V,R)$, the dual of V, is in particular $K(\mathcal{I})$-graded. (The degree of elements in V_i^* is $-i$.) Hence $V^*\otimes W$, which is isomorphic to $\mathrm{Hom}(V,W)$, is $K(\mathcal{I})$-graded by assigning degree $j-i$ to elements in $V_i^*\otimes W_j$. Everything is consistent.

Now, suppose that $\mathcal{I}$ also has a commutative multiplicative structure which is compatible with the additive structure. That is, it is a commutative cancellative semi-ring. We write ab as the multiplication of a and b in $\mathcal{I}$.

Definition 3.2. An $\mathcal{I}$-graded R-module A is called an $\mathcal{I}$-graded R-algebra if A is a unital associative R-algebra and if the multiplication $\mu: A\otimes A\to A$ is a morphism of $\mathcal{I}$-graded R-modules. We write $xy=\mu(x\otimes y)$ as the shorthand notation for multiplications of A. A is said to be commutative if

$$xy-(-1)^{p(x)p(y)}yx=0 \tag{3.1}$$

for all homogeneous $x,y\in A$, where we use $p(x)p(y)$ to denote $p(d(x)d(y))\in\mathbb{Z}_2$.

Remark 3.2. Here we have to be careful about the sign factor appearing in the right hand side of (3.1). Although both of $\mathcal{I}$ and $\mathbb{Z}_2$ are semi-rings[3], p is not necessarily a semi-ring homomorphism and we do not have $p(d(x)d(y))=p(d(x))p(d(y))$ in general. To choose which as the sign factor is just a matter of convention.

Morphisms of $\mathcal{I}$-graded algebras are simply linear maps of degree 0 which preserves the algebraic structures. We use $\text{Comm-Alg}_{\mathcal{I}}$ to denote the category of commutative $\mathcal{I}$-graded algebras.

Definition 3.3. The tensor algebra $\mathrm{T}(V)$ is the $\mathcal{I}$-graded R-module $\mathrm{T}(V)=\bigoplus_{n\in\mathbb{N}}V^{\otimes^n}$, together with the tensor product $\otimes$ as the canonical multiplication. The symmetric algebra $\mathrm{S}(V)$ is the quotient algebra of $\mathrm{T}(V)$ by the $\mathcal{I}$-graded two-sided ideal generated by

$$v\otimes w-(-1)^{p(v)p(w)}w\otimes v,$$

where $v,w\in V\subset\mathrm{T}(V)$ are homogeneous.

Remark 3.3. $\mathrm{S}(V)$ has a canonical $\mathbb{N}$-grading inherited from $\mathrm{T}(V)$ which should not be confused with its $\mathcal{I}$-grading. We write $\mathrm{S}(V)=\bigoplus_{n\in\mathbb{N}}\mathrm{S}^n(V)$ to indicate that fact. Note that $\mathrm{S}^0(V)=R$, but $\mathrm{S}(V)_0$, the sub-space of homogeneous elements of degree 0, is in general larger than R.

[3]The multiplicative structure on $\mathbb{Z}_2$ is inherited from the one on $\mathbb{Z}$.

S(V) is universal in the sense that, given a commutative $\mathcal{I}$-graded R-algebra A and a morphism $f : V \to A$. There exists a unique algebraic homomorphism $\tilde{f} : \mathrm{S}(V) \to A$ such that the following diagram commutes

$$\begin{array}{ccc} V & \overset{\iota}{\hookrightarrow} & \mathrm{S}(V) \\ & \searrow^{f} & \downarrow{\scriptstyle \tilde{f}} \\ & & A \end{array}$$

where $\iota : V \to \mathrm{S}(V)$ is the canonical embedding. Note that $\tilde{f}$ preserves the $\mathcal{I}$-grading, i.e., it is a morphism in Comm-Alg$_{\mathcal{I}}$. Choosing A to be R (viewed as an $\mathcal{I}$-graded R-algebra whose components of non-zero degree are 0.) and f to be the zero map, we obtain an R-algebra homomorphism from $\mathrm{S}(V)$ to R. We denote this map by ϵ. Note that $\ker \epsilon = \bigoplus_{n>0} \mathrm{S}^n(V)$.

Let k be a field and R be a commutative k-algebra. Let A be a commutative $\mathcal{I}$-graded k-algebra.

Definition 3.4. A k-algebra epimorphism $\epsilon : A \to R$ is called a body map of A if $\ker \epsilon \supset I$, where I is the ideal in A generated by homogeneous elements of non-zero degree.

By definition, ϵ preserves the $\mathcal{I}$-grading of A.

Definition 3.5. Let ϵ be a body map of A. A is said to be projected if the short exact sequence

$$0 \longrightarrow \ker \epsilon \longrightarrow A \overset{\epsilon}{\longrightarrow} R \longrightarrow 0$$

splits.

The splitting gives A an R-module structure depending on ϵ, with respect to which ϵ becomes an R-algebra homomorphism. Conversely, A is projected if A has an R-module structure and ϵ preserves that structure.

Lemma 3.1. *Let V be an $\mathcal{I}$-graded R-module with $V_0 = 0$. Let ϵ be an R-linear body map of* $\mathrm{S}(V)$*. Then ϵ is unique.*

Proof. In this case, $\mathrm{S}(V) = R \oplus I$ where $I = \bigoplus_{n>0} \mathrm{S}^n(V)$. Since $I \subset \ker \epsilon$ and ϵ is R-linear, the only possible choice of ϵ is the canonical one. □

Remark 3.4. Let V be as in Lemma 3.1. Suppose $A \cong \mathrm{S}(V)$ as $\mathcal{I}$-graded k-algebras. In particular, this implies that A admits a decomposition $A = A' \oplus I$ where $A' \cong R$ and I is the ideal generated by homogeneous elements of non-zero degree. Let ϵ be a body map of A. Since $I \subset \ker \epsilon$, ϵ is determined by $\epsilon|_{A'}$. In other words, ϵ is determined by a k-algebra endomorphism of R.

More can be said if V is free.

Lemma 3.2. *Let V be a free $\mathcal{I}$-graded R-module with $V_0 = 0$. Let ϵ be an R-linear body map of* $\mathrm{S}(V)$*. (By Lemma 3.1, ϵ is the canonical one.) Let I denote the kernel of ϵ. Then there exists an R-algebra isomorphism*

$$\mathrm{S}(V) \cong \mathrm{S}(I/I^2),$$

where I^2 is the square of the ideal I.

Proof. Let $\iota : V \hookrightarrow \mathrm{S}(V)$ be the canonical embedding. Since $I = \bigoplus_{n>0} \mathrm{S}^n(V)$, we have $\iota(V) \subset I$, which yields another embedding $V \hookrightarrow I/I^2 \hookrightarrow \mathrm{S}(I/I^2)$, which induces the desired isomorphic map between $\mathrm{S}(V)$ and $\mathrm{S}(I/I^2)$. $\square$

Definition 3.6. The $\mathcal{I}$-graded algebra of formal power series on V is the R-module

$$\overline{\mathrm{S}(V)} = \prod_{n\in\mathbb{N}} \mathrm{S}^n(V)$$

equipped with the canonical algebraic multiplication.

Remark 3.5. As is in the case of $\mathcal{I} = \mathbb{Z}$ [4], it is actually crucial to work with $\overline{\mathrm{S}(V)}$ instead of $\mathrm{S}(V)$ when the even part of V is non-trivial. The former allows us to have a coordinate description of morphisms between "$\mathcal{I}$-graded domains", a notion of partition of unity for "$\mathcal{I}$-graded manifolds", and more.

Let I be the kernel of the canonical body map of $\mathrm{S}(V)$. One can equip $\mathrm{S}(V)$ with the so-called I-adic topology.[4] Moreover, one can consider the I-adic completion of $\mathrm{S}(V)$ which is defined as the inverse limit

$$\widehat{\mathrm{S}(V)}_I := \varprojlim \mathrm{S}(V)/I^n$$

of the inverse system $((\mathrm{S}(V)/I^n)_{n\in\mathbb{N}}, (\pi_{m,n})_{n\leq m\in\mathbb{N}})$, where $\pi_{m,n} : \mathrm{S}(V)/I^m \to \mathrm{S}(V)/I^n$ is the canonical projection. Note that there is also a canonical projection $\mathrm{S}(V) \to \mathrm{S}(V)/I^n$ for each $n \in \mathbb{N}$. By the universal property of the inverse limit, one obtains a morphism

$$\iota_I : \mathrm{S}(V) \to \widehat{\mathrm{S}(V)}_I$$

with kernel being $\bigcap_{n\geq 0} I^n = \{0\}$. On the other hand, it is easy to see that $\mathrm{S}(V)/I^n \cong \bigoplus_{i=0}^{n-1} \mathrm{S}^i(V)$ for $n \geq 1$. It follows that there is a canonical isomorphism $\widehat{\mathrm{S}(V)}_I \cong \overline{\mathrm{S}(V)}$ under which ι_I coincides with the canonical inclusion $\mathrm{S}(V) \hookrightarrow \overline{\mathrm{S}(V)}$.

In fact, $\mathrm{S}(V)$ can be made into a metric space such that $\overline{\mathrm{S}(V)}$ is the completion of $\mathrm{S}(V)$ with respect to the metric structure [17]. The metric-induced topology on $\overline{\mathrm{S}(V)}$, with a slight abuse of notation, coincides with the I-adic topology on $\overline{\mathrm{S}(V)}$, where $I = \prod_{n>0} \mathrm{S}^n(V)$.

Lemma 3.3. *Let A be a commutative $\mathcal{I}$-graded R-algebra. Let J be an ideal of A such that A is J-adic complete. $\overline{\mathrm{S}(V)}$ is universal in the sense that, given a morphism $f : V \to A$ such that $f(V) \subset J$, there exists a unique (continuous) algebraic homomorphism $\tilde{f} : \overline{\mathrm{S}(V)} \to A$ such that the following diagram commutes*

$$\begin{array}{ccc} V & \overset{\iota}{\hookrightarrow} & \overline{\mathrm{S}(V)} \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle \tilde{f}} \\ & & A \end{array}$$

Proof. We already know that f induces a unique morphism $f' : \mathrm{S}(V) \to A$ such that $f' \circ \iota = f$. By assumption, f' extends naturally to a morphism $\tilde{f} : \overline{\mathrm{S}(V)} \to \widehat{A}_J \cong A$.

[4] To each point x of $\mathrm{S}(V)$ one assigns a collection of subsets $\mathcal{B}(x) = \{x + I^n\}_{x\in A, n>0}$. The I-adic topology is then the unique topology on $\mathrm{S}(V)$ such that $\mathcal{B}(x)$ forms a neighborhood base of x for all x.

Claim: $\tilde{f}$ is continuous.
Proof: It suffices to show that $\tilde{f}^{-1}(J^m)$ is a neighborhood of 0 for any $m \in \mathbb{N}$. By assumption, $I \subset \tilde{f}^{-1}(J)$. It follows that $I^m \subset \tilde{f}^{-1}(J)^m \subset \tilde{f}^{-1}(J^m)$. ■

Since $\mathrm{S}(V)$ is dense in $\overline{\mathrm{S}(V)}$ and $\tilde{f}|_{\mathrm{S}(V)} = f'$, $\tilde{f}$ is also unique. □

Remark 3.6. Likewise, we have a canonical body map of $\overline{\mathrm{S}(V)}$ induced from the zero map $V \to R$. Similar results like Lemma 3.1 and Lemma 3.2 also hold. For example, we have

$$\overline{\mathrm{S}(V)} \cong \overline{\mathrm{S}(I/I^2)},$$

where V and I are as in Lemma 3.2.

Lemma 3.4. *Let ϵ be the canonical body map of $\overline{\mathrm{S}(V)}$. Then for $f \in \overline{\mathrm{S}(V)}$, f is invertible if and only if $\epsilon(f)$ is invertible.*

Proof. "⇒": Trivial.
"⇐": Suppose $\epsilon(f) = c$ where $c \in R$ is invertible. We can write $f = c + f'$ where $f' \in \prod_{n\geq 1} \mathrm{S}^n(V)$. Note that $(f')^k \in \prod_{n\geq k} \mathrm{S}^n(V)$ for all $k > 0$. We can then set the inverse of f to be the formal sum $f^{-1} := c^{-1} \sum_{k\in\mathbb{N}} (-1)^k (c^{-1}f')^k$. ($f^{-1}$ is well-defined because the formal sum restricted to each $\mathrm{S}^n(V)$ is a finite sum.) □

Corollary 3.1. $\overline{\mathrm{S}(V)}$ *is local if R is local.*

Proof. Choose a non-unit $f \in \overline{\mathrm{S}(V)}$. Let $c = \epsilon(f)$. By Lemma 3.4, c is a non-unit. Since R is local, $1 - c$ is invertible. $1 - f$ is then a unit by Lemma 3.4. □

Recall that a ringed space $(X, \mathcal{O})$ is a topological space X with a sheaf of rings $\mathcal{O}$ on X.

Definition 3.7. An $\mathcal{I}$-graded ringed space is a ringed space $(X, \mathcal{O})$ such that

1. $\mathcal{O}(U)$ is an $\mathcal{I}$-graded algebra for any open subset U of X;

2. the restriction morphism $\rho_{V,U} : \mathcal{O}(U) \to \mathcal{O}(V)$ is a morphism of $\mathcal{I}$-graded algebras.

A morphism between two $\mathcal{I}$-graded ringed spaces $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ is just a morphism $\varphi = (\tilde{\varphi}, \varphi^*)$ between ringed spaces such that $\varphi^*_U : \mathcal{O}_2(U) \to \mathcal{O}_1(\tilde{\varphi}^{-1}(U))$ preserves the $\mathcal{I}$-grading for any open subset U of X_2.

Let (X, C) be a ringed space where $C(U)$ are commutative rings. One can define $\mathcal{I}$-graded C-modules and commutative $\mathcal{I}$-graded C-algebras in a similar way. In particular, the structure sheaf $\mathcal{O}$ of an $\mathcal{I}$-graded ringed space can be viewed as an $\mathcal{I}$-graded C-algebra if C is a sub-sheaf of $\mathcal{O}$ such that $C(U)$ are homogeneous sub-algebras of degree 0 of $\mathcal{O}(U)$.

Definition 3.8. Let $\mathcal{F}$ be an $\mathcal{I}$-graded C-module. The formal symmetric power $\overline{\mathrm{S}(\mathcal{F})}$ of $\mathcal{F}$ is the sheafification of the presheaf

$$U \to \overline{\mathrm{S}(\mathcal{F}(U))},$$

where $\overline{\mathrm{S}(\mathcal{F}(U))}$ is the $\mathcal{I}$-graded algebra of formal power series on the $C(U)$-module $\mathcal{F}(U)$.

By definition, $\overline{\mathrm{S}(\mathcal{F})}$ is a commutative $\mathcal{I}$-graded C-algebra.

Lemma 3.5. *Let $\mathcal{A}$ be a commutative $\mathcal{I}$-graded C-algebra. Let $\mathcal{B}$ be a sub-sheaf of $\mathcal{A}$ such that $\mathcal{A}(U)$ is $\mathcal{B}(U)$-adic complete for all open subsets U. $\overline{\mathrm{S}(\mathcal{F})}$ is universal in the sense that, given a morphism of $\mathcal{I}$-graded C-modules $F : \mathcal{F} \to \mathcal{A}$ such that $F(\mathcal{F}(U)) \subset \mathcal{B}(U)$ for all open subsets U, there exists a unique morphism of $\mathcal{I}$-graded C-algebras $\tilde{F} : \overline{\mathrm{S}(\mathcal{F})} \to \mathcal{A}$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{F} & \xhookrightarrow{\iota} & \overline{\mathrm{S}(\mathcal{F})} \\ & \searrow^{F} & \downarrow{\scriptstyle \tilde{F}} \\ & & \mathcal{A} \end{array}$$

where $\iota : \mathcal{F} \to \overline{\mathrm{S}(\mathcal{F})}$ is the canonical monomorphism.

Proof. This follows directly from the universal property of sheafification[5] and the universal property of $\overline{\mathrm{S}(\mathcal{F}(U))}$ stated in Lemma 3.3. □

To end this section, we state the following lemma taken from [10].

Lemma 3.6. *Let*

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow \mathcal{F} \longrightarrow 0. \tag{3.2}$$

be a short exact sequence of C-modules where $\mathcal{F}$ and $\mathcal{G}$ are locally free C-modules. Then the obstruction of the existence of a splitting of (3.2) can be represented as an element in the first sheaf cohomology group $H^1(X, \mathrm{Hom}(\mathcal{F}, \mathcal{G}))$ of $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$.

4 Monoidally Graded Domains

Throughout this section, V is a real $\mathcal{I}$-graded vector space with $V_0 = 0$. The dimension of the homogeneous sub-space V_i of V is m_i. We also assume that only finitely many of m_i are non-zero.

Definition 4.1. Let U be a domain of $\mathbb{R}^n$. An $\mathcal{I}$-graded domain $\mathcal{U}$ of dimension $n|(m_i)_{i\in\mathcal{I}}$ is an $\mathcal{I}$-graded ringed space $(U, \mathcal{O})$, where $\mathcal{O}$ is the sheaf of $\overline{\mathrm{S}(V)}$-valued smooth functions.

Remark 4.1. $\mathcal{U}$ is a locally ringed space by Corollary 3.1.

For example, a domain U with the sheaf C^∞ of smooth functions on U is an $\mathcal{I}$-graded domain of dimension $n|(0, \cdots)$, which is denoted again by U for simplicity.

Lemma 4.1. *Let $F : C^\infty \to C^\infty$ be an endomorphism of sheaves of commutative rings on U. Then F must be the identity.*

Proof. First, we show that F is actually an endomorphism of sheaves of unital $\mathbb{R}$-algebras on U. It suffices to show that F restricted to any open subset of U sends a constant function to itself. We know this is true for $\mathbb{Q}$-valued constant functions. Now, if F sends a constant function f to a non-constant function g, then one can find two rational number b_1 and b_2 such that $g - b_1$ and $g - b_2$ are non-invertible. But then the pre-images $f - b_1$ and $f - b_2$ are non-invertible, which implies that

[5] That is, given a presheaf $\mathcal{F}$, a sheaf $\mathcal{G}$, and a presheaf morphism $F : \mathcal{F} \to \mathcal{G}$, there exists a unique sheaf morphism $\tilde{F} : \mathcal{F}^\sharp \to \mathcal{G}$ such that $\tilde{F} \circ \iota = F$, where $\mathcal{F}^\sharp$ is the sheafification of $\mathcal{F}$ and $\iota : \mathcal{F} \to \mathcal{F}^\sharp$ is the canonical morphism.

f is non-constant: a contradiction. To show that g actually equals f, use the fact that the only field endomorphism of $\mathbb{R}$ is the identity.

Let $p \in U$. F induces a unital ring endomorphism F_p on the stalk C^∞_p. On the other hand, for any open neighborhood $U_p \subset U$ of p, the evaluation map

$$\begin{aligned} \mathrm{ev} : C^\infty(U_p) &\to \mathbb{R} \\ f &\mapsto f(p) \end{aligned}$$

induces a map $\mathrm{ev}_p : C^\infty_p \to \mathbb{R}$. For $f_p \in C^\infty_p$, it is easy to see that f_p is invertible if and only if $\mathrm{ev}_p(f_p) \neq 0$. Let $c = \mathrm{ev}_p(F_p(f_p))$. $f_p - c$ is non-invertible. Hence $\mathrm{ev}_p(f_p) = c$. In other words, for any open subset U' of U, we have $F_{U'}(f)(p) = f(p)$ for all $f \in C^\infty(U')$ and all $p \in U'$. This implies $F = \mathrm{id}$. □

A morphism between $\mathcal{I}$-graded domains is just a morphism of $\mathcal{I}$-graded locally ringed spaces. Recall that we have the canonical body map $\epsilon : C^\infty(U) \otimes \overline{\mathrm{S}(V)} \to C^\infty(U)$.

Proposition 4.1. *There exists a unique monomorphism $\varphi : U \to \mathcal{U}$ with $\tilde{\varphi} = \mathrm{id}$.*

Proof. Existence is guaranteed by ϵ. Uniqueness follows from Remark 3.4 and Lemma 4.1. □

We also have a canonical morphism for the other direction $\mathcal{U} \to U$ induced by the canonical embedding $\iota : C^\infty(U) \to C^\infty(U) \otimes \overline{\mathrm{S}(V)}$.[6] Note that $\epsilon \circ \iota = \mathrm{id}$ on $C^\infty(U)$.

Proposition 4.2. *Let $\varphi = (\tilde{\varphi}, \varphi^*)$ be a morphism from $\mathcal{U}_1 = (U_1, \mathcal{O}_1)$ to $\mathcal{U}_2 = (U_2, \mathcal{O}_2)$. The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{U}_1 & \xrightarrow{\varphi} & \mathcal{U}_2 \\ \uparrow & & \uparrow \\ U_1 & \xrightarrow{\tilde{\varphi}} & U_2 \end{array}$$

Proof. Let U be an open subset of U_2. Let $f \in \mathcal{O}_2(U)$. We need to show that

$$\epsilon(\varphi^*(f)) = \epsilon(f) \circ \tilde{\varphi}.$$

Suppose this does not hold. One can find a $p \in \tilde{\varphi}^{-1}(U)$ such that $\epsilon(\varphi^*(f))(p) = c \neq \epsilon(f)(\tilde{\varphi}(p))$. Then there exists an open neighborhood $U' \subset U$ of $\tilde{\varphi}(p)$ such that $\epsilon(f) - c$ is invertible. By Lemma 3.4, $f - c$ is also invertible on U', which implies that $\varphi^*(f - c)$ is invertible on $\tilde{\varphi}^{-1}(U') \subset \tilde{\varphi}^{-1}(U)$, which contradicts the fact that $\epsilon(\varphi^*(f - c))$ is non-invertible on $\tilde{\varphi}^{-1}(U')$. □

Definition 4.2. A coordinate system of $\mathcal{U}$ is a collection of functions $(x^\mu, \theta_{i,a})$ such that

1. x^μ are elements of $\mathcal{O}(U)_0$ such that $\epsilon(x^\mu)$ form a coordinate system of U;

2. $\theta_{i,a}$ are homogeneous elements of $\mathcal{O}(U)$ of degree $d(\theta_{i,a}) = i$, $i \neq 0$ and $a = 1, \cdots, m_i$, which generate $\mathcal{O}(U)$ as a $C^\infty(U)$-algebra.

[6]There will be no longer such a canonical morphism if we go the category of $\mathcal{I}$-graded manifolds.

Suppose that $\mathcal{I}$ can be given a total order $<$. It follows that any function $f \in \mathcal{O}(U)$ can be written uniquely in the form

$$f = \sum_{\mathcal{J}} \sum_{\beta} f_{\mathcal{J},\beta}(x^\mu) \prod_{j \in \mathcal{J}} \theta_j^{\beta^j}, \tag{4.1}$$

where

- $\mathcal{J} \in \mathrm{Pow}(\mathcal{I})$, $\beta = (\beta^j)_{j \in \mathcal{J}}$, $\beta^j = (\beta^j_1, \dots, \beta^j_{m_j})$, $\beta^j_k \in \{0,1\}$ if $p(j) = 1$, $\beta^j_k \in \mathbb{N}$ if $p(j) = 0$;
- $\theta_j^{\beta^j} = \theta_{j,1}^{\beta^j_1} \cdots \theta_{j,m_j}^{\beta^j_{m_j}}$, the product $\prod_{j \in \mathcal{J}} \theta_j^{\beta^j}$ is arranged in a proper order such that $\theta_j^{\beta^j}$ is on the left of $\theta_{j'}^{\beta^{j'}}$ whenever $j < j'$;
- For a smooth function $g \in C^\infty(U)$, the notation $g(x^\mu)$ should be understood as

$$g(x^\mu) := \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \frac{1}{i_1! \cdots i_n!} \partial_1^{i_1} \cdots \partial_n^{i_n} g(\epsilon(x^\mu))(x^1 - \epsilon(x^1))^{i_1} \cdots (x^n - \epsilon(x^n))^{i_n}. \tag{4.2}$$

  Hence, $g(x^\mu)$ is an element in $\mathcal{O}(U)_0$ instead of $C^\infty(U)$.

The sum in (4.1) is well-defined because, by assumption, only finitely many of m_j are non-zero.

Remark 4.2. One may wonder how we obtain (4.1). In fact, by definition, every function f can be expressed in the form

$$f = \sum_{\mathcal{J}} \sum_{\beta} f_{\mathcal{J},\beta}(\epsilon(x^\mu)) \prod_{j \in \mathcal{J}} \theta_j^{\beta^j}.$$

One can then define a map from $\mathcal{O}(U)$ to itself by sending $g(\epsilon(x^\mu))$ to $g(x^\mu)$. Now consider another map which sends $g(\epsilon(x^\mu))$ to $g^-(x^\mu)$, where

$$g^-(x^\mu) := \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \frac{1}{i_1! \cdots i_n!} \partial_1^{i_1} \cdots \partial_n^{i_n} g(\epsilon(x^\mu))(\epsilon(x^1) - x^1)^{i_1} \cdots (\epsilon(x^n) - x^n)^{i_n}.$$

Using the binomial theorem, it is easy to see that the second map is the inverse of the first. In fact, the reader may notice that the map $g(\epsilon(x^\mu)) \mapsto g(x^\mu)$ is exactly the "Grassmann analytic continuation map" defined in [18].

Corollary 4.1. *Let $\varphi = (\tilde{\varphi}, \varphi^*)$ be as in Proposition 4.2. $\tilde{\varphi}$ is uniquely determined by φ^*.*

Proof. Let $(x^\mu, \theta_{i,a})$ be a coordinate system of $\mathcal{U}_2$. By Proposition 4.2, one has $\tilde{\varphi}^\mu = \epsilon(\varphi^* x^\mu)$, where $(\tilde{\varphi}^\mu)$ is $\tilde{\varphi}$ expressed in the coordinate system $(\epsilon(x^\mu))$ of U_2. ☐

Let ev be the evaluation map of $C^\infty(U)$ at $p \in U$. Let s_p denote $\mathrm{ev} \circ \epsilon$. Let I_p denote the kernel of s_p. We follow [9] to prove the following lemmas.

Lemma 4.2. *For any functions $f \in \mathcal{O}(U)$ and any integer $k \geq 0$, there is a polynomial P_k in the coordinates $(x^\mu, \theta_{i,a})$ such that $f - P_k \in I_p^{k+1}$.*

Proof. Use the classical Hadamard lemma and the decomposition (4.1). □

Lemma 4.3. *Let f and g be functions of $\mathcal{O}(U)$, then $f = g$ if and only if $f - g \in I_p^k$ for all $k \in \mathbb{N}$ and $p \in U$. In other words, $\bigcap_{p\in U} \bigcap_{k\in\mathbb{N}} I_p^k = \{0\}$.*

Proof. Let $h = f - g$. Apply the decomposition (4.1) to h, then by Lemma 4.2, $h_{\mathcal{J},\beta} = 0$ for all $\mathcal{J}$ and β. Hence $h = 0$. □

Lemma 4.4. *Any morphism of $\mathcal{I}$-graded $\mathbb{R}$-algebras $s : \mathcal{O}(U) \to \mathbb{R}$ must take the form $s = s_p$.*

Proof. Since we assume $V_0 = 0$, s can be reduced to a morphism $C^\infty(U) \to \mathbb{R}$. Let x^μ be a coordinate system of U. Let $f^\mu = x^\mu - s(x^\mu)$ and $h = \sum_\mu (f^\mu)^2$. Then $s(h) = 0$, which implies that h is non-invertible. In other words, there exists $p \in U$ such that $x^\mu(p) = s(x^\mu)$ for all μ. Now suppose there exists an $f \in C^\infty(U)$ such that $s(f) \neq s_p(f) = f(p)$. Consider the function $h' = h+(f-s(f))^2$. Since $h > 0$ for all points of $U/\{p\}$. We know $h' > 0$ on U. But this contradicts the fact that $s(h') = 0$. Hence s must equal s_p. □

Theorem 4.1. *Let $\varphi = (\tilde{\varphi}, \varphi^*)$ be a morphism from $\mathcal{U}_1 = (U_1, \mathcal{O}_1)$ to $\mathcal{U}_2 = (U_2, \mathcal{O}_2)$. Let $(x^\mu, \theta_{i,a})$ be a coordinate system of $\mathcal{U}_2$. Then φ^* is uniquely determined by the equations*

$$\varphi^* x^\mu = y^\mu, \quad \varphi^* \theta_{i,a} = \eta_{i,a},$$

where $y^\mu \in \mathcal{O}(U_1)_0$, $\eta_{i,a} \in \mathcal{O}(U_1)_i$ and $(\epsilon(y^\mu))(p) \in U_2$ for all $p \in U_1$.

Proof. Let $f \in \mathcal{O}_2(U_2)$. By (4.1), to construct $\varphi^* f$, we only need to define $\varphi^* f_{\mathcal{J},\beta}$. But this is straightforward: one just replaces x^μ with y^μ and $\theta_{i,a}$ with $\eta_{i,a}$ in (4.2). By construction, we have $\varphi^* 1 = 1$, $\varphi^*(f + g) = \varphi^* f + \varphi^* g$, and $\varphi^*(fg) = \varphi^* f \varphi^* g$, hence φ^* is well-defined.

Now suppose there exists another φ'^* which equals φ^* on coordinates. Then they also equals on all polynomials of $(x^\mu, \theta_{i,a})$. By Lemma 4.2 and Lemma 4.3, $\varphi'^* = \varphi^*$. □

Remark 4.3. Theorem 4.1 can be seen as a generalization of the Global Chart Theorem in the $\mathbb{Z}_2$-graded setting (see Theorem 4.2.5 in [11]).

Corollary 4.2. *Let $\varphi^* : \mathcal{O}_2(U_2) \to \mathcal{O}_1(U_1)$ be a ring homomorphism which preserves the $\mathcal{I}$-grading. Then there exists a unique morphism $\varphi' : \mathcal{U}_1 \to \mathcal{U}_2$ such that $\varphi'^* = \varphi^*$.*

Proof. First, one can easily show that φ^* is actually an $\mathbb{R}$-algebra homomorphism using arguments similar to those in Lemma 4.1. Choose a point $p \in U_1$, by Lemma 4.4, the morphism $s_p \circ \varphi^*$ must take the form $s_{p'}$ for some $p' \in U_2$. It follows that $\varphi^*(I_{p'}) \subset I_p$. Let $(x^\mu, \theta_{i,a})$ be a coordinate system of $\mathcal{U}_2$, we then have $\varphi^* x^\mu - \epsilon(x^\mu)(p') \in I_p$. Hence $(\epsilon(\varphi^* x^\mu))(p) \in U_2$ for all $p \in U_1$. Next, observe that a coordinate system of U_2 restricted to any open subset of it gives a coordinate system of that open subset. Now apply Theorem 4.1 and Corollary 4.1. □

5 Monoidally Graded Manifolds

Definition 5.1. Let M be a n-dimensional manifold. An $\mathcal{I}$-graded manifold $\mathcal{M}$ of dimension $n|(m_i)_{i\in\mathcal{I}}$ is an $\mathcal{I}$-graded ringed space $(M, \mathcal{O}_M)$ which is locally isomorphic to an $\mathcal{I}$-graded domain of dimension $n|(m_i)_{i\in\mathcal{I}}$. That is, for each $x \in M$, there exist an open neighborhood U_x of x, an $\mathcal{I}$-graded domain $\mathcal{U}$, and an isomorphism of locally ringed spaces

$$\varphi = (\tilde{\varphi}, \varphi^*) : (U_x, \mathcal{O}_M|_{U_x}) \to \mathcal{U}.$$

φ is called a chart of $\mathcal{M}$ on U_x.[7]

M with the sheaf C^∞ of smooth functions on M is an $\mathcal{I}$-graded manifold of dimension $n|(0,\cdots)$, which is denoted again by M for simplicity. We call M together with a morphism $\mathcal{O} \to C^\infty$ an underlying manifold of $\mathcal{M}$. Equivalently, an underlying manifold of $\mathcal{M}$ is a morphism $\varphi : M \to \mathcal{M}$ with $\tilde{\varphi} = \mathrm{id}$.

Let $x \in M$. An open neighborhood U of x on which $\mathcal{O}(U) \cong C^\infty(U) \otimes \overline{\mathrm{S}(V)}$ is called a splitting neighborhood. Clearly, every chart is a splitting neighborhood, but not vice versa. The set of splitting neighborhoods form a base of the topology of M. For a splitting U, there exists sub-algebras $C(U)$ and $D(U)$ of $\mathcal{O}(U)$ such that $C(U) \cong C^\infty(U)$, $D(U) \cong \overline{\mathrm{S}(V)}$ and $\mathcal{O}(U) = C(U) \otimes D(U)$. This induces an epimorphism

$$\epsilon : \mathcal{O}(U) \to C^\infty(U)$$

of graded commutative $\mathbb{R}$-algebras, which is a body map of $\mathcal{O}(U)$.

Definition 5.2. A local coordinate system of $\mathcal{M}$ is the data $(U, x^\mu, \theta_{i,a})$ where

1. U is a splitting neighborhood of $\mathcal{M}$;
2. $x^1, \ldots, x^n$ are elements of $C(U)$ such that $\epsilon(x^1), \ldots, \epsilon(x^n)$ are local coordinate functions of M on U;
3. $\theta_{i,a}$ are homogeneous elements of $\mathcal{D}(U)$ of degree $d(\theta_{i,a}) = i$, $i \neq 0$ and $a = 1, \cdots, m_i$, which generate $\mathcal{O}(U)$ as a $C(U)$-algebra.

Remark 5.1. By Theorem 4.1, every local coordinate system determines (non-canonically) a chart.

Now, let U be an arbitrary open subset of M. We can choose a collection of charts $\{U_\alpha\}$ such that $U = \bigcup_\alpha U_\alpha$. For $f \in \mathcal{O}(U)$, one can apply the restriction morphisms to f to get a sequence of sections f_α in $\mathcal{O}(U_\alpha)$. Now, apply ϵ to each of them to get a sequence of smooth functions $\tilde{f}_\alpha$ in $C^\infty(U_\alpha)$. By Proposition 4.1, $\tilde{f}_\alpha$ must be compatible with each other, hence can be glued together to get a smooth function $\tilde{f}$ over U. In this way, we construct a body map for every open subset of M, which are compatible with restrictions. In other words, ϵ can be seen as a sheaf morphism from $\mathcal{O}$ to C^∞.

Proposition 5.1. *There exists a unique monomorphism* $\varphi : M \to \mathcal{M}$ *with* $\tilde{\varphi} = \mathrm{id}$.

Proof. Existence is guaranteed by ϵ. Uniqueness follows from Proposition 4.1. □

Proposition 5.2. *Let* $\varphi = (\tilde{\varphi}, \varphi^*)$ *be a morphism from* $\mathcal{M} = (M, \mathcal{O}_M)$ *to* $\mathcal{N} = (N, \mathcal{O}_N)$. *The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\varphi} & \mathcal{N} \\ \uparrow & & \uparrow \\ M & \xrightarrow{\tilde{\varphi}} & N \end{array}$$

[7]We often refer to U_x as a chart too.

Proof. The proof is essentially the same as the one of Proposition 4.2. □

Lemma 5.1. *Let $\mathcal{O}^1$ be the kernel of ϵ. $\mathcal{O}$ is $\mathcal{O}^1$-adic complete. That is, for any open subset U, $\mathcal{O}(U)$ is $\mathcal{O}^1(U)$-adic complete.*

Proof. Let $\widehat{\mathcal{O}}$ be the $\mathcal{O}^1$-adic completion of $\mathcal{O}$.[8] There exists a canonical morphism $\iota : \mathcal{O} \to \widehat{\mathcal{O}}$. Since $\mathcal{O}$ is locally $\mathcal{O}^1$-adic complete, the induced stalk morphism $\iota_p : \mathcal{O}_p \to \widehat{\mathcal{O}}_p$ is an isomorphism for each $p \in M$. It follows that $\mathcal{O}$ is $\mathcal{O}^1$-adic complete. □

Definition 5.3. An $\mathcal{I}$-graded manifold $\mathcal{M}$ is called projected if there exists a splitting of the short exact sequence of sheaves of rings

$$0 \longrightarrow \mathcal{O}^1 \longrightarrow \mathcal{O} \xrightarrow{\epsilon} C^\infty \longrightarrow 0, \tag{5.1}$$

where $\mathcal{O}^1$ is the kernel of ϵ.

The structure sheaf $\mathcal{O}$ of a projected manifold is a C^∞-module.

Definition 5.4. A projected $\mathcal{I}$-graded manifold $\mathcal{M}$ is called split if there exists a splitting of the short exact sequence of C^∞-modules

$$0 \longrightarrow \mathcal{O}^2 \longrightarrow \mathcal{O}^1 \xrightarrow{\pi} \mathcal{O}^1/\mathcal{O}^2 \longrightarrow 0, \tag{5.2}$$

where $\mathcal{O}^2$ is the square of $\mathcal{O}^1$, π is the canonical quotient map.

Let $\mathcal{O}$ be the structure sheaf of a projected $\mathcal{I}$-graded manifold. Let $\mathcal{F}$ denote the sheaf $\mathcal{O}^1/\mathcal{O}^2$. $\mathcal{F}$ is an $\mathcal{I}$-graded C^∞-module and we can define its formal symmetric power $\overline{\mathrm{S}(\mathcal{F})}$. By construction, the ringed space $\mathcal{M}_S = (M, \overline{\mathrm{S}(\mathcal{F})})$ is also a projected $\mathcal{I}$-graded manifold.

Lemma 5.2. *Let $\mathcal{M} = (M, \mathcal{O})$ be a projected $\mathcal{I}$-graded manifold. $\mathcal{M}$ is split if and only if $\mathcal{M} \cong \mathcal{M}_S$.*

Proof. Let $\iota : \mathcal{F} \to \overline{\mathrm{S}(\mathcal{F})}$ be the canonical monomorphism. ι splits the short exact sequence (5.2). Now if $\mathcal{M}$ is split, one can find a monomorphism $F : \mathcal{F} \to \mathcal{O}$ of C^∞-modules such that $F(\mathcal{F}(U)) \subset \mathcal{O}^1(U)$ for any open subset U. By Lemma 3.5 and Lemma 5.1, there exists a unique C^∞-algebra morphism $\tilde{F} : \overline{\mathrm{S}(\mathcal{F})} \to \mathcal{O}$ such that $\tilde{F} \circ \iota = F$. By Remark 3.6, $\tilde{F}$ induces an isomorphism for each stalk. Hence $\mathcal{M} \cong \mathcal{M}_S$. □

Lemma 5.3. *Every projected $\mathcal{I}$-graded manifold is split.*

Proof. Due to the existence of a smooth partition of unity on M, $H^q(M, \mathrm{Hom}(\mathcal{O}^1/\mathcal{O}^2, \mathcal{O}^2))$ vanishes for $q \geq 1$. By Lemma 3.6, there is no obstruction of the existence of a splitting of (5.2). □

Lemma 5.4. *Every $\mathcal{I}$-graded manifold is projected.*

Proof. Let $\mathcal{O}_{(i)} = \mathcal{O}/\mathcal{O}^{i+1}$. Let $\phi_{(0)} : C^\infty \to \mathcal{O}_{(0)}$ be the identity. (By Proposition 5.1, there is a unique identification $\mathcal{O}_{(0)} \cong C^\infty$.) One can construct by induction on i mappings $\phi_{(i+1)} : C^\infty \to \mathcal{O}_{(i+1)}$ such that $\pi_{i+1,i} \circ \phi_{(i+1)} = \phi_{(i)}$, where $\pi_{i+1,i} : \mathcal{O}_{i+1} \to \mathcal{O}_i$ is the canonical epimorphism. As is shown in [10], one can construct an element

$$\omega(\phi_{(i)}) \in H^1(M, (\mathcal{T} \otimes \mathrm{S}^{i+1}(\mathcal{F}))_0)$$

[8] For each open subset U, one has $\widehat{\mathcal{O}}(U) = \varprojlim \mathcal{O}(U)/\mathcal{O}^n(U)$, where $\mathcal{O}^n(U)$ is the n-th power of $\mathcal{O}^1(U)$.

as the obstruction to the existence of $\phi_{(i+1)}$, where $\mathcal{T}$ is the tangent sheaf of M. Due to the existence of a smooth partition of unity on M, $H^1(M, (\mathcal{T} \otimes \mathrm{S}^{i+1}(\mathcal{F}))_0) = 0$ and $\omega(\phi_{(i)}) = 0$. It follows that there exists a unique morphism $\phi : C^\infty \to \varprojlim \mathcal{O}_{(i)}$ such that $\pi_i \circ \phi = \phi_{(i)}$, where $\pi_i : \varprojlim \mathcal{O}_{(i)} \to \mathcal{O}_i$ is the canonical epimorphism. By Lemma 5.1, ϕ can be seen as a morphism from C^∞ to $\mathcal{O}$. Note that $\pi_0 = \epsilon$ and $\pi_0 \circ \phi = \phi_{(0)} = \mathrm{id}$. ϕ splits (5.1). □

Corollary 5.1. *Every $\mathcal{I}$-graded manifold is split.*

Let V be a (finite dimensional) $\mathcal{I}$-graded vector space. An $\mathcal{I}$-graded vector bundle $\pi : E \to M$ is a vector bundle such that the local trivialization map $\varphi_U : \pi^{-1}(U) \to U \times V$ is a morphism of $\mathcal{I}$-graded vector spaces when restricted to $\pi^{-1}(x)$, $x \in U \subset M$. In other words, $E = \bigoplus_{k\in\mathcal{I}} E_k$ where E_k are vector bundles whose fibers consist of elements of degree k. To any $\mathcal{I}$-graded vector bundle E we can associate an $\mathcal{I}$-graded ringed space with the underlying topological space being M and the structure sheaf being the sheaf of sections of $\overline{\mathrm{S}(\bigoplus_{k\in\mathcal{I}}(E_k)^*)}$. (This is an $\mathcal{I}$-graded manifold in our sense if the fiber of E does not contain elements of degree 0.) Corollary 5.1 can then be rephrased as

Theorem 5.1. *Every $\mathcal{I}$-graded manifold can be obtained from an $\mathcal{I}$-graded vector bundle.*

6 Vector Fields and Tangent Sheaves

Throughout this section, every algebra is assumed to be real.

Let R be a unital associative commutative $\mathcal{I}$-graded algebra. An $\mathcal{I}$-graded algebra A over R is defined to be an $\mathcal{I}$-graded algebra A equipped with a left R-module structure such that $R_iA_j \subset A_{i+j}$, and

$$r(ab) = (ra)b = (-1)^{p(r)p(a)}a(rb)$$

for $r \in R$ and $a, b \in A$. Recall that when we write $p(r)p(a)$, we mean $p(d(r)d(a))$. We also require that $1a = a$, where $1 \in R$ is the identity element.

Definition 6.1. An $\mathcal{I}$-graded Lie algebra over R is an $\mathcal{I}$-graded algebra L over R whose multiplications (denoted by $[\cdot,\cdot]$) satisfy

$$[a, b] = -(-1)^{p(a)p(b)}[b, a], \tag{6.1}$$

$$[a, [b, c]] = [[a, b], c] + (-1)^{p(a)p(b)}[b, [a, c]], \tag{6.2}$$

for all $a, b, c \in L$.

The space of endomorphisms $\mathrm{Hom}(A, A)$ (or $\mathrm{gl}(A)$) of an $\mathcal{I}$-graded R-module A is an associative $K(\mathcal{I})$-graded algebra over R. It can be also viewed as a $K(\mathcal{I})$-graded Lie algebra over R by setting

$$[f, g] = f \circ g - (-1)^{p(f)p(g)}g \circ f$$

for all $f, g \in \mathrm{Hom}(A, A)$. In the case of A being an $\mathcal{I}$-graded algebra, an endomorphism D is said to be a derivation if

$$D(ab) = D(a)b + (-1)^{p(D)p(a)}aD(b). \tag{6.3}$$

It is easy to check that derivations of A form a $K(\mathcal{I})$-graded Lie subalgebra of $\mathrm{gl}(A)$ over R.

Definition 6.2. Let $\mathcal{M} = (M, \mathcal{O})$ be an $\mathcal{I}$-graded manifold. A (local) vector field over $\mathcal{M}$ is a derivation of $\mathcal{O}(U)$, where U is an open subset of M.

Local vector fields over $\mathcal{M}$ actually form a ($K(\mathcal{I})$-graded) sheaf on M. To prove this, we need the partition of unity lemma in the $\mathcal{I}$-graded setting.

Lemma 6.1. *Let $f \in \mathcal{O}(M)$ such that $\epsilon(f)(x) \neq 0$ for all $x \in M$. f is invertible.*

Proof. Choose an open cover $\{U_\alpha\}$ of charts of M. Let f_α denote $\rho_{U_\alpha,M}(f)$. Each f_α is invertible by Lemma 3.4. Let f_α^{-1} denote the inverse of f_α. By uniqueness of the inverse, f_α^{-1} are compatible with each other, hence can be glued to give a section $f^{-1} \in \mathcal{O}(M)$, which is the inverse of f. □

Lemma 6.2. *Let $\{U_\alpha\}$ be an open cover of M. There exists a locally finite refinement $\{V_\beta\}$ of $\{U_\alpha\}$ and a family of functions $\{l_\beta \in \mathcal{O}(M)_0\}$ such that*

1. supp $l_\beta \subset V_\beta$ *is compact and* $\epsilon(l_\beta) \geq 0$ *for all* β*;*
2. $\sum_\beta l_\beta = 1$.

Proof. First, find a partition of unity $\{\tilde{l}_\beta\}$ of M subordinate to $\{V_\beta\}$. Choose $l'_\beta \in \mathcal{O}(V_\beta)$ such that $\epsilon(l'_\beta) = \tilde{l}_\beta$. Since $\tilde{l}_\beta$ are invertible, we can then set l_β to be $(\sum_\beta l'_\beta)^{-1} l'_\beta$. □

Using Lemma 6.2, it is not hard to prove the following lemma.

Lemma 6.3. *Let U and V be open in M such that $V \subset U$. Let D be a derivation of $\mathcal{O}(U)$. Then there exists a unique derivation D' of $\mathcal{O}(V)$ such that $D'(\rho_{V,U}(f)) = \rho_{V,U}(D(f))$ for all $f \in \mathcal{O}(U)$.*

We skip the proof of Lemma 6.3 since it is essentially the same as the one in the $\mathbb{Z}_2$-graded setting [9]. Note that Lemma 6.3 implies that local vector fields over $\mathcal{M}$ form a presheaf $\mathfrak{X}$ on M.

Proposition 6.1. *$\mathfrak{X}$ is a sheaf on M.*

Remark 6.1. $\mathfrak{X}$ is called the tangent sheaf of $\mathcal{M}$.

Proof. Let U be an open subset of M with an open cover $\{U_\alpha\}$. Let $D_\alpha \in \mathfrak{X}(U_\alpha)$ be compatible with each other. We obtain a $D \in \mathfrak{X}(U)$ by setting $D(f)$ to be unique function obtained by gluing $D_\alpha(f_\alpha)$, where $f \in \mathcal{O}(U)$ and $f_\alpha = \rho_{U_\alpha,U}(f)$. $D(f)$ is well defined because $\rho_{U_\alpha\cap U_\beta,U_\alpha}(D_\alpha(f_\alpha)) = \rho_{U_\alpha\cap U_\beta,U_\alpha}(D_\alpha)(\rho_{U_\alpha\cap U_\beta,U_\alpha}(f_\alpha)) = \rho_{U_\alpha\cap U_\beta,U_\beta}(D_\beta)(\rho_{U_\alpha\cap U_\beta,U_\beta}(f_\beta)) = \rho_{U_\alpha\cap U_\beta,U_\beta}(D_\beta(f_\beta))$. □

We are now ready to give the following definition.

Definition 6.3. A QK-manifold is a bigraded manifold equipped with three vector fields Q, K and d of degree $(0,1)$, $(1,-1)$ and $(1,0)$, respectively, satisfying the following relations

$$Q^2 = 0, \quad QK + KQ = d, \quad Kd + dK = 0.$$

QK-manifolds can be used to study the descent equations (1.1) in a cohomological field theory. The physical observables $\mathcal{O}^{(p)}$ can be interpreted as functions over a QK-manifold. Fix a Q-closed function $\mathcal{O}^{(0)}$, a K-sequence is defined by setting $\mathcal{O}^{(p)} = \frac{1}{p!}K^p\mathcal{O}^{(0)}$. A sequence $\{\mathcal{O}^{(p)}\}_{p=0}^n$ is called an exact sequence if there exists another sequence $\{\mathcal{P}^{(p)}\}_{p=0}^n$ such that $\mathcal{O}^{(p)} = Q\mathcal{P}^{(p)} + d\mathcal{P}^{(p-1)}$ for $p \geq 1$ and $\mathcal{O}^{(0)} = Q\mathcal{P}^{(0)}$. In [8], the author proved that

Theorem 6.1. *Every solution to* (1.1) *is a K-sequence up to an exact sequence.*

7 Conclusions

In this paper, we have given a definition of $\mathcal{I}$-graded manifolds, where $\mathcal{I}$ is a countable cancellative commutative semi-ring. Such a definition unifies different objects such as supermanifolds, graded manifolds and colored supermanifolds. We have proved the existence and uniqueness of an underlying manifold of an $\mathcal{I}$-graded manifold. Furthermore, we have also proved Batchelor's theorem in this generalized setting, namely that every $\mathcal{I}$-graded manifold can be obtained from an $\mathcal{I}$-graded vector bundle. At the end of this paper, we have discussed a special class of bigraded manifolds, the QK-manifolds, and their applications to cohomological field theories.

Acknowledgement

The author would like to thank Enno Keßler and Sylvain Lavau for pointing out to him that Batchelor's theorem was proved in both the $\mathbb{Z}_2^n$-graded setting and the $\mathbb{Z}$-graded setting. He would also like to thank an anonymous referee for pointing out the connection between (4.2) and the definition of a supersmooth function in the sense of Rogers-DeWitt. This work was supported by the International Max Planck Research School Mathematics in the Sciences.

References

[1] B. Kostant, *Graded manifolds, graded Lie theory, and prequantization*, Differential Geometrical Methods in Mathematical Physics, 1977, pp. 177–306.

[2] E. Witten, *Supersymmetry and Morse theory*, Journal of Differential Geometry **17** (1982), no. 4, 661–692.

[3] A. S. Cattaneo and F. Schätz, *Introduction to supergeometry*, Reviews in Mathematical Physics **23** (2011), no. 06, 669-690.

[4] M. Fairon, *Introduction to graded geometry*, European Journal of Mathematics **3** (2017), no. 2, 208–222.

[5] M. Alexandrov, A. Schwarz, O. Zaboronsky, and M. Kontsevich, *The geometry of the master equation and topological quantum field theory*, International Journal of Modern Physics A **12** (1997), no. 07, 1405–1429.

[6] E. Witten, *Topological sigma models*, Communications in Mathematical Physics **118** (1988), no. 3, 411–449.

[7] ———, *Topological quantum field theory*, Communications in Mathematical Physics **117** (1988), no. 3, 353–386.

[8] S. Jiang, *Mathematical structures of cohomological field theories* (2022), available at `arxiv:2202.12425`.

[9] D. A. Leites, *Introduction to the theory of supermanifolds*, Russian Mathematical Surveys **35** (1980), no. 1, 1–64.

[10] Y. I. Manin, *Gauge Field Theory and Complex Geometry*, Vol. 289, Springer Berlin, Heidelberg, 1997.

[11] C. Carmeli, L. Caston, and R. Fioresi, *Mathematical Foundations of Supersymmetry*, Vol. 15, European Mathematical Society, 2011.

[12] C. Bartocci, U. Bruzzo, and D. H. Ruipérez, *The Geometry of Supermanifolds*, Springer Dordrecht, 2012.

[13] E. Keßler, *Supergeometry, Super Riemann Surfaces and the Superconformal Action Functional*, Vol. 2230, Springer Cham, 2019.

[14] M. Batchelor, *The structure of supermanifolds*, Transactions of the American Mathematical Society **253** (1979), 329–338.

[15] T. Covolo, J. Grabowski, and N. Poncin, *Splitting theorem for $\mathbb{Z}_2^n$-supermanifolds*, Journal of Geometry and Physics **110** (2016), 393–401.

[16] A. Kotov and V. Salnikov, *The category of $\mathbb{Z}$-graded manifolds: what happens if you do not stay positive* (2021), available at `arxiv:2108.13496`.

[17] B. Singh, *Basic Commutative Algebra*, World Scientific, 2011.

[18] A. Rogers, *Supermanifolds: Theory and Applications*, World Scientific, 2007.